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Measure (ω, c) -pseudo almost periodic functions and Lasota-Ważewska model with ergodic and unbounded oscillating oxygen demand.

James Larrouy* and Gaston M. N'Guérékata †

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Abstract. The primary aim of this work is to introduce a new class of functions called μ - (ω, c) -pseudo almost periodic functions. Using the measure theory, we generalize in a natural way some recent works and study some properties of those μ - (ω, c) -pseudo almost periodic functions including two new composition results which play a crucial role for the existence of some μ - (ω, c) -pseudo almost periodic solutions of certain semilinear differential equations and partial differential equations. We also investigate the existence and uniqueness of the μ - (ω, c) -pseudo almost periodic solutions for some models of Lasota-Ważewska equation with measure (ω, c) -pseudo almost periodic coefficient and mixed delays.

Key words: Periodic oscillations, (ω, c) -periodic, Unboundedness, Lasota-Ważewska model, Composition Theorem

AMS 2020 Subject Classification: 34C27, 34K14, 44A35, 47D06, 92-10

1 Introduction

Most of the natural phenomena we consider as periodic are in fact almost periodic, in other words, they are periodic up to epsilon. The concept of almost periodic functions was introduced in the literature in the mid-1920s by the Danish mathematician Harald Bohr [4]. It was later generalized in various directions by many may researchers [1–3, 7, 8, 10, 12, 14, 18, 24, 25]. As we all know, many phenomena in nature have oscillatory character and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form pseudo almost periodic functions which is a natural generalization of the concept of almost periodicity (in Bohr's sense). In this work we introduce the notion of measure (ω, c) -pseudo almost periodic functions (or μ - (ω, c) -pseudo almost periodic functions) with values in a complex Banach space and enlighten their applications throughout the study of a biological model. This work generalizes the concept of μ -Pseudo almost periodic functions introduced by

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Blot *et al.* [3] which already generalizes the class of weighted pseudo almost periodic functions of Diagana [8,9]. Here, we investigate many interesting properties of this new class of functions and present new and more general results based on measure theory that extend the existing ones.

The concept of (ω, c) -periodicity was introduced by Alvarez *et al.* [1] motivated by the qualitative properties of solutions to the Mathieu linear second-order differential equation

$$y''(t) + [a - 2q \cos(2t)] y(t) = 0,$$

arising in seasonally forced population dynamics. Further on, Alvarez *et al.* proposed a new concept of (ω, c) -pseudo periodicity and proved the existence of positive (ω, c) -pseudo periodic solutions to the Lasota-Ważewska equation with (ω, c) -pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, t \geq 0.$$

This equation describes the survival of red blood cells in the blood of an animal. The works of Khalladi *et al.* [15] have shown that (ω, c) -pseudo periodic functions can be also solutions time varying impulsive differential equations and linear delayed equations.

First, pseudo-almost periodicity was introduced in the literature in the early nineties by Zhang [23–25], as a natural generalization of the classical almost periodicity in the sense of Bohr. Then, Diagana [8,9], introduced the concept of weighted pseudo almost periodicity which generalizes the latter and the author gave some properties of the space of weighted pseudo almost periodic functions such as the completeness and a composition theorem. The concept of weighted pseudo almost periodic functions became an interesting field of dynamical systems that attracted many authors. A few years later, Blot *et al.* [3] came up with a new concept of weighted pseudo almost periodic functions under the light of measure theory. Giving a positive measure μ on \mathbb{R} , they defined the concept of μ -pseudo almost periodic functions as follows : it is said that a function f is μ pseudo almost periodic if

$$f = g + \varphi$$

where g is almost periodic and φ is μ -ergodic in the sense that

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\varphi(t)\| d\mu(t) = 0$$

where $\mu([-r, r])$ is the measure of the set $[-r, r]$. Here, the classical theory of weighted pseudo almost periodicity became a particular case of Blot *et al.* approach. Indeed, one can observe that a weighted pseudo almost periodic function of weight ρ is μ -pseudo almost periodic where the measure μ is absolutely continuous with respect to the Lebesgue measure, and its Radon-Nikodym derivative is ρ :

$$d\mu(t) = \rho(t)dt.$$

In their work, Blot *et al.* have investigated many important results on the theory of μ -pseudo almost periodicity ; they studied the completeness and provided a composition theorem on the functional space of μ -pseudo almost periodic functions. They also gave some applications for evolution equations which included reaction-diffusion systems and partial differential equations.

In this work, we introduce a new class of μ - (ω, c) -ergodic components and we investigate many important results on the new theory of μ - (ω, c) -pseudo almost periodic functions. We study the completeness and the composition theorem on the functional space of μ - (ω, c) -pseudo almost periodic functions.

The organization of this work is as follows : in the next section, we recall the basic definitions and properties of μ -pseudo almost periodic functions. In Section 3, we give the new concept of μ - (ω, c) -pseudo almost periodicity and study the convolution product on the spaces of c -bounded functions, μ - (ω, c) -ergodic functions and μ - (ω, c) -pseudo almost periodic functions. In Section 4, we introduce the concept of (ω, c) -type compactness and then we study a composition theorem which plays a crucial role to study the existence of μ - (ω, c) -pseudo almost periodic solution for a perturbed semilinear system. In Section 5, we propose a more realistic Lasota-Ważewska model than the existing ones due to (ω, c) -periodicity and then we study the existence and uniqueness of μ - (ω, c) -pseudo almost periodic solutions for the model, using the completeness and composition results.

2 Terminology and definitions

In this section we review a few notations, definitions and lemmas which will be utilized throughout this paper.

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|)$ be complex Banach spaces. Throughout this work, $C(\mathbb{R}, \mathbb{X})$ and $BC(\mathbb{R}, \mathbb{X})$ (respectively $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the Banach spaces consisting of all continuous functions and all bounded continuous functions from \mathbb{R} to \mathbb{X} (respectively from $\mathbb{R} \times \mathbb{Y}$ to \mathbb{X}) equipped with the supremum norm

$$\|f\|_{\infty} = \sup_{t \in \mathbb{R}} \|f\|$$

Let's first recall the notion of (ω, c) -periodicity.

Definition 2.1 ([1]). *Let $\omega > 0$ and $c \in \mathbb{C} \setminus \{0\}$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (ω, c) -periodic if*

$$f(t + \omega) = cf(t), \text{ for each } t \in \mathbb{R}.$$

In this case, ω is called a c -period of the function f .

We denote by $P_{(\omega, c)}(\mathbb{R}, \mathbb{X})$ the vector space of all (ω, c) -periodic functions from \mathbb{R} to \mathbb{X} . One can note that the space $P_{(\omega, c)}(\mathbb{R}, \mathbb{X})$ contains the spaces of periodic, antiperiodic and Bloch periodic functions among others (respectively taking $c = 1$, $c = -1$ and $c = e^{ikt}$). (see [17] for more details)

Proposition 2.2 ([1]). *Let $f \in C(\mathbb{R}, \mathbb{X})$. Then, $f \in P_{(\omega, c)}(\mathbb{R}, \mathbb{X})$ if and only if*

$$f(t) = c^{\frac{t}{\omega}} u(t), \text{ } u(t) \in P_{\omega, 1}(\mathbb{R}, \mathbb{X}).$$

Using the principal branch of the complex Logarithm, $c^{\frac{t}{\omega}}$ is defined as

$$c^{\frac{t}{\omega}} := \exp\left(\frac{t}{\omega} \text{Log}(c)\right) = c^{\wedge}(t)$$

and we will use the notation $|c|^\wedge(t) := |c^\wedge(t)| = |c|^\frac{t}{\omega}$.

Now we recall some properties of almost periodic and (ω, c) -pseudo almost periodic functions.

Definition 2.3. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l > 0$, such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ with

$$\sup_{t \in \mathbb{R}} \|f(t + \tau) - f(t)\| < \varepsilon. \quad (1)$$

The vector space consisting of all (Bohr) almost periodic functions is denoted by $AP(\mathbb{X})$.

It is well known that a continuous function $f : \mathbb{R} \rightarrow \mathbb{X}$ is almost periodic if and only if the set

$$\{f_\tau : \tau \in \mathbb{R}\}$$

is relatively compact in $BC(\mathbb{R}, \mathbb{X})$, where the function f_τ is defined by

$$f_\tau(t) = f(t + \tau), t \in \mathbb{R}.$$

Such number τ in (1) is called ε -translation number of $f(t)$ and we denote by $\mathcal{T}_\varepsilon(f)$ the set of all ε -translation numbers of f . This set has the following property :

Given any $f \in AP(\mathbb{X})$,

1. if $\tau \in \mathcal{T}_\varepsilon(f)$, then $-\tau \in \mathcal{T}_\varepsilon(f)$,

This concept has been extended by Khalladi *et al.* [13] as follows :

Definition 2.4 ([13]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (ω, c) -almost periodic if and only if the function

$$f_{(\omega, c)}(t) := c^{-\frac{t}{\omega}} f(t), t \in \mathbb{R}$$

belongs to $AP(\mathbb{X})$.

The vector space consisting of all (ω, c) -almost periodic functions is denoted by $AP_{\omega, c}(\mathbb{X})$.

Unless specified otherwise, in the remainder of the paper, we will always assume that $c \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{R}_+^* \cap \mathcal{T}_\varepsilon(f)$. Furthermore, the principal branches are always used for taking powers of complex numbers.

In the following, we will keep the notation : $f_{(\omega, c)}(t) := c^{-\frac{t}{\omega}} f(t)$.

Remark 2.5. When $c = 1$, $AP_{\omega, c}(\mathbb{X}) = AP(\mathbb{X})$.

Remark 2.6. One can note that in our paper, contrary to the paper [13], ω is not only positive but it belongs also to the set of all ε -translation number of f . This condition yields $AP_{\omega, 1}(\mathbb{X}) := AP_\omega(\mathbb{X}) = AP(\mathbb{X})$.

In order to conserve the periodic structure of (ω, c) -periodic type functions, we need to use an (ω, c) -norm which can be defined as

$$\|f\|_{(\omega, c)} := \sup_{t \in \mathbb{R}} \|c^\wedge(-t)f(t)\|$$

(ω, c) -norms was introduced in the literature by Alvarez et al. taking the supremum norm not on the whole \mathbb{R} but on the principal c -period interval $[0, \omega]$ of the (ω, c) -periodic considered function in order to handle the (ω, c) -periodicity properties of f (see in [1, 2, 17] for more details). We have the following completeness result.

Remark 2.7. We say that f is c -bounded when $\|f\|_{(\omega, c)} < \infty$.

Proposition 2.8 ([15]). $(\text{AP}_{\omega, c}(\mathbb{X}), \|\cdot\|_{(\omega, c)})$ is a Banach space.

Proposition 2.9 ([16]). $\text{AP}_{\omega, c}(\mathbb{X})$ is translation invariant and closed under the multiplication with complex scalars.

Now we recall the concept of μ -pseudo almost periodic functions introduced by Blot et al. [3].

We denote by \mathcal{B} the Lebesgue σ -field of \mathbb{R} and by \mathcal{M} the set of all positive measures μ on \mathcal{B} satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}$ ($a \leq b$).

Definition 2.10 ([3]). Let $\mu \in \mathcal{M}$. A function $f \in \text{BC}(\mathbb{R}, \mathbb{X})$ is said to be μ -ergodic if

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f(t)\| d\mu(t) = 0$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.11 ([3]). Let $\mu \in \mathcal{M}$. A function $f \in \text{C}(\mathbb{R}, \mathbb{X})$ is said to be μ -pseudo almost periodic if f is written in the form

$$f = g + \varphi$$

where $g \in \text{AP}(\mathbb{X})$ and $\varphi \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $\text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$.

Proposition 2.12 ([3]). Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.

We end this section with the following result.

Lemma 2.13. If $f, g \in \text{PAP}(\mathbb{R}, \mathbb{C}, \mu)$, then $fg \in \text{PAP}(\mathbb{R}, \mathbb{C}, \mu)$.

Proof. Since $f, g \in \text{PAP}(\mathbb{R}, \mathbb{C}, \mu)$ then they have following decompositions $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1, g_1 \in \text{AP}(\mathbb{C})$ and $f_2, g_2 \in \mathcal{E}(\mathbb{R}, \mathbb{C}, \mu)$. Then we have

$$fg = f_1g_1 + f_1g_2 + g_1f_2 + f_2g_2$$

First we show that the product $f_1g_1 \in \text{AP}(\mathbb{C})$. If we take $f_1 = g_1$ we have

$$\|(f_1)^2(t + \tau) - (f_1)^2(t)\| = \|(f_1)(t + \tau) + (f_1)(t)\| \times \|(f_1)(t + \tau) - (f_1)(t)\|$$

It can be easily seen that since f_1 is bounded, then there exists $M \in \mathbb{R}_+$ such that

$$\|f_1\| \leq M$$

Thus, it comes the following

$$\|(f_1)^2(t + \tau) - (f_1)^2(t)\| \leq 2M\varepsilon \leq \varepsilon'$$

Then $(f_1)^2 \in \text{AP}(\mathbb{C})$. Now, one can note that $f_1 g_1 = \frac{1}{4} \left((f_1 + g_1)^2 - (f_1 - g_1)^2 \right)$. Since $(f_1 + g_1)^2 \in \text{AP}(\mathbb{C})$ and $(f_1 - g_1)^2 \in \text{AP}(\mathbb{C})$, then $f_1 g_1 \in \text{AP}(\mathbb{C})$. Now, for $(f_1 g_2 + g_1 f_2 + f_2 g_2)$ one has that :

$$\begin{aligned} & \frac{1}{\mu([-r, r])} \int_{-r}^r \left(|(f_1)(g_2) + (g_1)(f_2) + (f_2)(g_2)|(t) \right) d\mu(t) \\ & \leq \frac{1}{\mu([-r, r])} \int_{-r}^r \left(\|f_1\|_\infty |(g_2)(t)| + \|g_1\|_\infty |(f_2)(t)| + \|f_2\|_\infty |(g_2)(t)| \right) d\mu(t) \end{aligned}$$

And consequently, since $f_2, g_2 \in \mathcal{E}(\mathbb{R}, \mathbb{C}, \mu)$ we have

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \left(|(f_1)(g_2) + (g_1)(f_2) + (f_2)(g_2)|(t) \right) d\mu(t) = 0$$

The proof is complete. \square

3 Measure (ω, c) -pseudo almost periodic functions

In this section, we introduce the new concepts of μ - (ω, c) -ergodic functions and the μ - (ω, c) -pseudo almost periodic functions. The notion of μ - (ω, c) -pseudo almost periodic functions is a generalization of μ -pseudo almost periodic functions introduced by Blot et al. [3] which now becomes the particular case $c = 1$ of our work. It is also a generalization of the concept of weighted pseudo almost periodicity given by Diagana [8, 9] and consequently this work generalizes that of Zhang [23–25] on the classical pseudo almost periodicity.

Here we introduce the space $\text{BC}(\mathbb{R}, \mathbb{X}, c)$ (resp. $\text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$), where $\text{BC}(\mathbb{R}, \mathbb{X}, c)$ (resp. $\text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$) denotes the Banach space consisting of all c -bounded continuous functions from \mathbb{R} to \mathbb{X} (resp. from $\mathbb{R} \times \mathbb{Y}$ to \mathbb{X}) equipped with the (ω, c) -norm $\|\cdot\|_{(\omega, c)}$ defined in Section 2.

Remark 3.1. One can note that in the case $c = 1$,

$$(\text{BC}(\mathbb{R}, \mathbb{X}, c), \|\cdot\|_{(\omega, c)}) = (\text{BC}(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty),$$

$$(\text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c), \|\cdot\|_{(\omega, c)}) = (\text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}), \|\cdot\|_\infty).$$

Moreover we have the following results.

Theorem 3.2. Let $f \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$. Then $f \in \text{BC}(\mathbb{R}, \mathbb{X})$ is and only if $c = 1$.

We begin this part with the following helpful convolution theorem for c -bounded functions.

Let $\mathcal{L}(\mathbb{X})$ be the space of bounded linear maps from the complex Banach space \mathbb{X} into itself. We denote $L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$ the Lebesgue space with respect to the Lebesgue measure on \mathbb{R} .

Remark 3.3. One can note that if $h \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then $\phi(\cdot) := c^\wedge(\cdot)h(\cdot) \notin h \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$ but $\tilde{\phi} := (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$.

Theorem 3.4. Let $f \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$ and $\tilde{\phi} := (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product of $f \star \phi$ defined by

$$\begin{aligned} (f \star \phi)(t) &= \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds, \text{ for } t \in \mathbb{R}. \\ &= \int_{-\infty}^{+\infty} \tilde{\phi}(s)c^\wedge(s)f(t-s)ds, \text{ for } t \in \mathbb{R}. \end{aligned}$$

is c -bounded.

Proof. Let $f \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$. In order to state that $(f \star \phi) \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$, we consider the function $\varrho_n : \mathbb{R} \rightarrow \mathbb{X}$ ($n \in \mathbb{N}, n \geq 1$) defined by

$$\varrho_n(t) = \int_{-n}^n \phi(s)f(t-s)ds.$$

Observing that

$$c^\wedge(-t)\varrho_n(t) = \int_{-n}^n c^\wedge(-s)\phi(s)c^\wedge(-t+s)f(t-s)ds.$$

it is clear that ϱ_n is c -bounded on \mathbb{R} . We deduce that $\varrho_n(t)$ is continuous by using the uniform continuity of f on all compact subsets of \mathbb{R} . Consequently, $\varrho_n(t) \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$ and from the following inequality :

$$\left\| c^\wedge(-t) \left((f \star \phi)(t) - \varrho_n(t) \right) \right\| \leq \|f\|_{(\omega, c)} \left(\int_{-\infty}^{-n} \|c^\wedge(-s)\phi(s)\| ds + \int_n^{+\infty} \|c^\wedge(-s)\phi(s)\| ds \right),$$

we deduce that $\lim_{n \rightarrow +\infty} \varrho_n(t) = f \star \phi$ uniformly on \mathbb{R} , therefore $f \star \phi \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$. \square

3.1 On μ - (ω, c) -ergodicity.

First, we introduce the new concept of μ - (ω, c) -ergodic functions.

Definition 3.5. Let $\mu \in \mathcal{M}$. A function $f \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$ is said to be μ - (ω, c) -ergodic if

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega, c)}(t)\| d\mu(t) = 0$$

We denote the space of all such functions by $\mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We give a completeness result and a characterization of μ - (ω, c) -ergodic functions.

Proposition 3.6. Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_{(\omega, c)})$ is a Banach space.

Proof. It is clear that $\mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a vector subspace of $\text{BC}(\mathbb{R}, \mathbb{X}, c)$. We show that $\mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is closed in $\text{BC}(\mathbb{R}, \mathbb{X}, c)$. Let $(f_n) \subset \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ be a Cauchy sequence converging to f uniformly in \mathbb{R} . From $\mu(\mathbb{R}) = +\infty$, it follows that $\mu([-r, r]) > 0$ for r sufficiently large. We have that

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t) \leq \|f - f_n\|_{(\omega,c)} + \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|(f_n)_{(\omega,c)}(t)\| d\mu(t)$$

then

$$\limsup_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t) \leq \|f - f_n\|_{(\omega,c)}, \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{n \rightarrow +\infty} \|f - f_n\|_{(\omega,c)} = 0$, we deduce that

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t) = 0$$

□

Theorem 3.7. *Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$).*

Let $f \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$, then following assertions are equivalent :

1. $f \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$,
2. $\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \|f_{(\omega,c)}(t)\| d\mu(t) = 0$,
3. For any $\varepsilon > 0$, $\frac{\mu(\{t \in [-r, r] \setminus I : \|f_{(\omega,c)}(t)\| > \varepsilon\})}{\mu([-r, r] \setminus I)} = 0$.

Proof. (1 \Leftrightarrow 2) Let $\Gamma := \mu(I)$ and $\Lambda := \int_I \|f_{(\omega,c)}\| d\mu(t)$ then $\Gamma, \Lambda \in \mathbb{R}$ since I is bounded and f is c -bounded and continuous. For $r > 0$ such that $I \subset [-r, r]$ and $\mu([-r, r] \setminus I) > 0$, we have

$$\begin{aligned} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \|f_{(\omega,c)}(t)\| d\mu(t) &= \frac{1}{\mu([-r, r]) - \Gamma} \left(\int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t) - \Lambda \right) \\ &= \frac{\mu([-r, r])}{\mu([-r, r]) - \Gamma} \left(\frac{1}{\mu([-r, r])} \int_{\mu([-r, r])} \|f_{(\omega,c)}(t)\| d\mu(t) - \frac{\Lambda}{\mu([-r, r])} \right) \end{aligned}$$

Assume that 2 holds, then using the first equality and since $\mu(\mathbb{R}) = +\infty$, we have that

$$0 = \lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r]) - \Gamma} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t) = \lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t)$$

then 1 is true.

Conversely, if assumption 1 holds, using the second equality and since $\lim_{r \rightarrow \infty} \frac{\mu([-r, r])}{\mu([-r, r]) - \Gamma} = 1$

then assumption 2 is obviously holds.

(2 \Leftrightarrow 3) We set :

$$\Gamma_r^\varepsilon := \{t \in [-r, r] \setminus I : \|f_{(\omega, c)}(t)\| > \varepsilon\} \quad \text{and} \quad \Lambda_r^\varepsilon := \{t \in [-r, r] \setminus I : \|f_{(\omega, c)}(t)\| \leq \varepsilon\}$$

Using that

$$\int_{[-r, r]} \|f_{(\omega, c)}(t)\| d\mu(t) = \int_{\Gamma_r^\varepsilon} \|f_{(\omega, c)}(t)\| d\mu(t) + \int_{\Lambda_r^\varepsilon} \|f_{(\omega, c)}(t)\| d\mu(t)$$

we have for sufficiently large $r > 0$

$$\frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \|f_{(\omega, c)}(t)\| d\mu(t) \leq \|f\|_{(\omega, c)} \frac{\mu(\Gamma_r^\varepsilon)}{\mu([-r, r] \setminus I)} + \varepsilon$$

Now assume that assumption 3 holds. Then from previous inequality, we have that for all $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \|f_{(\omega, c)}(t)\| d\mu(t) \leq \varepsilon$$

and consequently, assumption 2 holds.

In order to prove the last implication, we use the following inequality

$$\begin{aligned} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r, r] \setminus I} \|f_{(\omega, c)}(t)\| d\mu(t) &\geq \frac{1}{\mu([-r, r] \setminus I)} \int_{\Gamma_r^\varepsilon} \|f_{(\omega, c)}(t)\| d\mu(t) \\ &\geq \varepsilon \frac{\mu(\Gamma_r^\varepsilon)}{\mu([-r, r] \setminus I)} \end{aligned}$$

assume that assumption 2 holds, we obtain assumption 3 when making $r \rightarrow +\infty$.

The proof is complete. \square

We recall here the notion of equivalent measures.

Definition 3.8 ([3]). Let $\mu_1, \mu_2 \in \mathcal{M}$. μ_1 is equivalent to μ_2 (or $\rho_1 \sim \rho_2$) if there exists constants $\alpha, \beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha \mu_1(A) \leq \mu_2(A) \leq \beta \mu_1(A), \quad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset$$

We have following result:

Theorem 3.9. Let $\mu_1, \mu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ then $\mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu_1) = \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu_2)$.

Proof. Assume that $\mu_1 \sim \mu_2$ and \mathcal{B} is the Lebesgue σ -field. Then we obtain for r sufficiently large

$$\begin{aligned} \frac{\alpha \mu_1(\{t \in [-r, r] \setminus I : \|f_{(\omega, c)}(t)\| > \varepsilon\})}{\beta \mu_1([-r, r] \setminus I)} &\leq \frac{\mu_2(\{t \in [-r, r] \setminus I : \|f_{(\omega, c)}(t)\| > \varepsilon\})}{\mu_2([-r, r] \setminus I)} \\ &\leq \frac{\beta \mu_1(\{t \in [-r, r] \setminus I : \|f_{(\omega, c)}(t)\| > \varepsilon\})}{\alpha \mu_1([-r, r] \setminus I)} \end{aligned}$$

We deduce that $\mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu_1) = \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu_2)$. \square

Now, we intend to prove that $\mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant.

For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}), \text{ for } A \in \mathcal{B}. \quad (2)$$

We need to formulate the following hypotheses for $\mu \in \mathcal{M}$ and to recall some lemmas.

(H1) For all $a, b, c \in \mathbb{R}$ such that $0 \leq a < b \leq c$, there exists $\tau_0 \geq 0$ and α_0 such that

$$|\tau| \geq \tau_0 \Rightarrow \mu((a + \tau, b + \tau)) \geq \alpha_0 \mu([\tau, \tau + c]).$$

(H2) For all $\in \mathbb{R}$, there exists $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta \mu(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Lemma 3.10 ([3]). (H2) implies (H1).

Lemma 3.11 ([3]). Let $m \in \mathcal{M}$ satisfying (H2). Then the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.

Lemma 3.12 ([3]). (H2) implies that for all $\alpha > 0$

$$\limsup_{r \rightarrow +\infty} \frac{\mu([-r - \alpha, r + \alpha])}{\mu([-r, r])} < +\infty.$$

We can prove the following result.

Theorem 3.13. Let $\mu \in \mathcal{M}$ satisfying (H2). Then $\mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant.

Proof. Let $f \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\tau \in \mathbb{R}$. Since $\mu(\mathbb{R}) = +\infty$, there exists $r_0 > 0$ such that $\mu([-r - |\tau|, r + |\tau|]) > 0$ for all $r \geq r_0$. In this proof, we always assume that $r \geq r_0$. Let us denote by

$$M_\tau(r) = \frac{1}{\mu_\tau([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu(t), \text{ for } r > 0 \text{ and } \tau \in \mathbb{R} \quad (3)$$

where μ_τ is the positive measure defined by (2). By using Lemma 3.11, it follows that μ_τ and μ are equivalent, then by using Theorem 3.9 we have $\mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu_\tau) = \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$, therefore $f \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu_\tau)$ that is

$$\lim_{r \rightarrow +\infty} M_\tau(r) = 0 \text{ for all } \tau \in \mathbb{R}. \quad (4)$$

For all $A \in \mathcal{B}$, we denote by χ_A the characteristic functions of A ; By using definition of μ_τ we have that

$$\int_{[-r, r]} \chi_A(t) d\mu_\tau(t) = \int_{[-r+\tau, r+\tau]} \chi_A(t - \tau) d\mu(t)$$

for all $A \in \mathcal{B}$ and since $t \rightarrow \|f_{(\omega,c)}(t)\|$ is the pointwise limit of an increasing sequence of linear combination of characteristic functions (see [20] Theorem 1.17), we deduce that

$$\int_{[-r, r]} \|f_{(\omega,c)}(t)\| d\mu_\tau(t) = \int_{[-r+\tau, r+\tau]} \|f_{(\omega,c)}(t - \tau)\| d\mu(t) \quad (5)$$

From (2), (3), (5) we obtain

$$M_\tau(r) = \frac{1}{\mu([-r + \tau, r + \tau])} \int_{[-r + \tau, r + \tau]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t).$$

If we denote by $\tau^+ := \max(\tau, 0)$ and $\tau^- := \max(-\tau, 0)$, we have that $|\tau| + \tau = 2\tau^+$ and $|\tau| - \tau = 2\tau^-$ and then $[-r + \tau - |\tau|, r + \tau = |\tau|] = [-r - 2\tau^-, r + 2\tau^+]$. Therefore we obtain

$$M_\tau(r + |\tau|) = \frac{1}{\mu([-r - 2\tau^-, r + 2\tau^+])} \int_{[-r - 2\tau^-, r + 2\tau^+]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t). \quad (6)$$

From (6) and the following inequality :

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t) \leq \frac{1}{\mu([-r, r])} \int_{[-r - 2\tau^-, r + 2\tau^+]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t)$$

it comes that

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t) \leq \frac{\mu([-r - 2\tau^-, r + 2\tau^+])}{\mu([-r, r])} M_\tau(r + |\tau|),$$

that implies

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t) \leq \frac{\mu([-r - 2|\tau|, r + 2|\tau|])}{\mu([-r, r])} M_\tau(r + |\tau|). \quad (7)$$

From (4), (7) and using Lemma 3.12, we deduce that

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega, c)}(t - \tau)\| d\mu(t) = 0.$$

The proof is complete. \square

We end this section by giving a convolution theorem for μ - (ω, c) -ergodic functions.

Theorem 3.14. *Let $\mu \in \mathcal{M}$ satisfying (H2). If $f \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\tilde{\phi}(\cdot) := (c^\wedge(\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product of $f \star \phi$ defined by*

$$\begin{aligned} (f \star \phi)(t) &= \int_{-\infty}^{+\infty} \phi(s) f(t - s) ds, \quad \text{for } t \in \mathbb{R}, \\ &= \int_{-\infty}^{+\infty} \tilde{\phi}(s) c^\wedge(s) f(t - s) ds, \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

is μ - (ω, c) -ergodic.

Proof. Let $f \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. According to Theorem 3.4, $(f \star \phi) \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$.

From $\mu(\mathbb{R}) = +\infty$, we deduce the existence of $r_0 \geq 0$ such that $\mu([-r, r]) > 0$ for all $r \geq r_0$.

Using the inequality :

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|(f \star \phi)_{(\omega, c)}(t)\| d\mu(t) \leq \frac{1}{\mu([-r, r])} \int_{[-r, r]} \int_{-\infty}^{+\infty} \|\tilde{\phi}(s)\| \|c^\wedge(-t+s) f(t-s)\| ds d\mu(t)$$

where $\tilde{\phi} := \phi_{(\omega,c)} \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, In view of the Fubini's Theorem, we deduce

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|(f \star \phi)_{(\omega,c)}(t)\| d\mu(t) \leq \int_{-\infty}^{+\infty} \frac{\|\tilde{\phi}(s)\|}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t-s)\| d\mu(t) ds$$

Invoking Theorem 3.13, we have that

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t-s)\| d\mu(t) = 0, \text{ for all } s \in \mathbb{R}.$$

And since

$$0 \leq \frac{\|\tilde{\phi}(s)\|}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t-s)\| d\mu(t) \leq \|\tilde{\phi}(s)\| \|f\|_{(\omega,c)}$$

in view of the Lebesgue dominated convergence Theorem, we obtain

$$\lim_{r \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\|\tilde{\phi}(s)\|}{\mu([-r, r])} \int_{[-r, r]} \|f_{(\omega,c)}(t-s)\| d\mu(t) ds = 0$$

It comes that

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|(f \star \phi)_{(\omega,c)}(t)\| d\mu(t) = 0$$

and consequently, $(f \star \phi) \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$. □

Example 3.2. *The unique solution of the heat equation*

$$u_t(x, t) = u_{xx}(x, t), \quad x \in \mathbb{R}, t \geq 0,$$

with the initial condition $u(x, 0) = f(x)$ is given by

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s) ds, \quad x \in \mathbb{R}, t \geq 0,$$

If $c^\wedge(-\cdot)e^{-\frac{(\cdot)^2}{4t_0}} \in L^1(\mathbb{R})$ and $f \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{R}, \mu)$, then by Theorem 3.14, the solution

$$[x \mapsto u(x, t_0), x \in \mathbb{R}] \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{R}, \mu).$$

Now we are ready to define measure (ω, c) -pseudo almost periodic functions.

3.3 Measure (ω, c) -pseudo almost periodic function

In this subsection, we introduce the new class of measure (ω, c) -pseudo almost periodic function and we study some properties of such functions. Let us define this new notion.

Definition 3.15. *Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be measure (ω, c) -pseudo almost periodic (or μ - (ω, c) -pseudo almost periodic) if f can be written in the form*

$$f = g + \varphi$$

where $g \in \text{AP}_{\omega,c}(\mathbb{X})$ and $\varphi \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $\text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We will say that g is the (ω, c) -almost periodic part of f and φ the μ - (ω, c) -pseudo ergodic perturbation of f .

We have the following spaces inclusions

$$\text{AP}_{\omega, c}(\mathbb{X}) \subset \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu) \subset \text{BC}(\mathbb{R}, \mathbb{X}, c)$$

Remark 3.16. Observe that $\text{AP}_{\omega, c}(\mathbb{X})$ is a proper subspace of $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ since the function

$$\phi(t) = (0.2)^t (\sin^2(\pi t) + \sin^2(\sqrt{5}t) + e^{-t \cos^2(t)}) \in \text{PAP}_{(1, 0.2)}(\mathbb{R}, \mathbb{X}, \mu)$$

but $\phi \notin \text{AP}_{(1, 0.2)}(\mathbb{X})$ since $[t \mapsto \sin^2(\pi t) + \sin^2(\sqrt{5}t) + e^{-t \cos^2(t)}] \notin \text{AP}(\mathbb{X})$.

The following theorem gives a characterization of the measure (ω, c) -pseudo almost periodic functions.

Theorem 3.17. Let $f \in \text{C}(\mathbb{R}, \mathbb{X})$. Then, $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ if and only if

$$f(t) \equiv c^\wedge(t)u(t), \text{ with } c^\wedge(t) := c^{\frac{t}{\omega}} \text{ and } u \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu).$$

Proof. Obviously, if $f(t) = c^\wedge(t)u(t)$ with $u \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ then $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. Conversely, let $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. Then $\exists (g, \varphi) \in \text{AP}_{\omega, c}(\mathbb{X}) \times \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ such that $f = g + \varphi$. Therefore, taking $u(t) := c^\wedge(-t)f(t)$ it comes that $u \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. \square

In view of Definition (3.15), for any $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ we say that $c^\wedge(t)u(t)$ is the c -factorization of f .

We give the first basic result

Proposition 3.18. Let $\mu \in \mathcal{M}$. Then $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a vector space.

Proof. Obvious. \square

Now we intend to show that $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu) = \text{AP}_{\omega, c}(\mathbb{X}) \oplus \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. In order to prove Proposition 3.20, we will need following lemma.

Lemma 3.19. Assume $f \in \text{AP}_{\omega, c}(\mathbb{X})$, write

$$B_\varepsilon := \{\tau \in \mathbb{R} : \|f_{(\omega, c)}(t_0 + \tau) - f_{(\omega, c)}(t_0)\| < \varepsilon\}$$

where $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ is fixed. Then there exists $s_1, s_2, \dots, s_m \in \mathbb{R}$ such that

$$\bigcup_{i=1}^m (s_i + B_\varepsilon) = \mathbb{R}.$$

We have following result.

Proposition 3.20. *Let $\mu \in \mathcal{M}$ satisfying (H1) and $f \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ be such that*

$$f = g + \varphi$$

where g is its (ω, c) -almost periodic component, then we have

$$g(\mathbb{R}) \subset \overline{f(\mathbb{R})} \quad (8)$$

Therefore, $\|f\|_{(\omega,c)} \geq \|g\|_{(\omega,c)} \geq \inf_{t \in \mathbb{R}} |g_{(\omega,c)}(t)| \geq \inf_{t \in \mathbb{R}} |f_{(\omega,c)}(t)|$.

Proof. Suppose that (8) is not true. Then there exists $t_0 \in \mathbb{R}$, $\varepsilon > 0$ such that

$$\|g_{(\omega,c)}(t_0) - f_{(\omega,c)}(t_0)\| \geq 2\varepsilon, \quad t \in \mathbb{R}. \quad (9)$$

Let s_1, s_2, \dots, s_m be as in Lemma 3.19 and write

$$\tau_i = s_i - t_0, \quad i = 1, 2, \dots, m, \quad \eta = \max_{1 \leq i \leq m} |\tau_i|.$$

For $r \in \mathbb{R}$ with $|r| > \eta$, we let

$$B_{\varepsilon,r}^{(i)} := [-r + \eta - \tau_i, r - \eta - \tau_i] \cap (t_0 + B_\varepsilon), \quad i = 1, 2, \dots, m.$$

where B_ε is as in Lemma 3.19. It is clear that

$$\bigcup_{i=1}^m (\tau_i + B_{\varepsilon,r}^{(i)}) = [-r + \eta, r - \eta].$$

Thus, we obtain

$$\begin{aligned} 2(r - \eta) &= \mu([-r + \eta, r - \eta]) \leq \sum_{i=1}^m \mu(\tau_i + B_{\varepsilon,r}^{(i)}) = \sum_{i=1}^m \mu(B_{\varepsilon,r}^{(i)}) \\ &\leq m \cdot \max_{1 \leq i \leq m} \{\mu(B_{\varepsilon,r}^{(i)})\} \\ &\leq m \cdot \mu([-r, r] \cap (t_0 + B_\varepsilon)) \end{aligned} \quad (10)$$

since for each $i = 1, 2, \dots, m$

$$B_{\varepsilon,r}^{(i)} \subset \left([-r, r] \cap (t_0 + B_\varepsilon) \right)$$

Using inequality (9), we have

$$\|\phi_{(\omega,c)}(t)\| = \|f_{(\omega,c)}(t) - g_{(\omega,c)}(t)\| \geq \|g_{(\omega,c)}(t_0) - f_{(\omega,c)}(t_0)\| - \|g_{(\omega,c)}(t) - g_{(\omega,c)}(t_0)\| > \varepsilon$$

for any $t \in t_0 + B_\varepsilon$.

This and inequality (10) together give

$$\frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\phi_{(\omega,c)}(t)\| d\mu(t) \geq \frac{r - \eta}{mr} \varepsilon \longrightarrow \frac{\varepsilon}{m}, \quad \text{as } r \longrightarrow \infty.$$

This is a contradiction since $h \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and establishes our claim (8). \square

We can now establish the uniqueness of the decomposition in Definition 3.15.

Theorem 3.21. *Let $\mu \in \mathcal{M}$ satisfying (H1). Then the decomposition of a μ - (ω, c) -pseudo almost periodic function in the form $f = g + \varphi$, where $g \in \text{AP}_{\omega, c}(\mathbb{X})$ and $\varphi \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ is unique.*

Proof. Assume that $f = g_1 + \varphi_1$ and $f = g_2 + \varphi_2$. Then, $0 = (g_1 - g_2) + (\varphi_1 - \varphi_2)$. Since $g_1 - g_2 \in \text{AP}_{\omega, c}(\mathbb{X})$ and $\varphi_1 - \varphi_2 \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$, in view of proposition 3.20, we deduce that $g_1 - g_2 = 0$ and consequently, $\varphi_1 = \varphi_2$ which proves the uniqueness of the decomposition. \square

From above it is clear that

$$\text{AP}_{\omega, c}(\mathbb{X}) \cap \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu) = \{0\}$$

Furthermore, we have following results

Theorem 3.22. *The space $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a translation invariant.*

Proof. This is a direct consequence of Proposition 2.9, Theorem 3.13 and Theorem 3.21. \square

Theorem 3.23. *The space $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a translation invariant C^* -subalgebra of $\text{BC}(\mathbb{R}, \mathbb{X}, c)$ without the constant functions. Furthermore,*

$$\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu) / \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu) \cong \text{AP}_{\omega, c}(\mathbb{X})$$

Proof. We show that $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $\text{BC}(\mathbb{R}, \mathbb{X}, c)$.

Let $(f_n) \subset \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ be Cauchy. By proposition 3.20, the sequence $(g_n) \subset \text{AP}_{\omega, c}(\mathbb{X})$ is Cauchy too and so is $(\varphi_n) \subset \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. Since $\text{AP}_{\omega, c}(\mathbb{X})$ and $\mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ are closed in $\text{BC}(\mathbb{R}, \mathbb{X}, c)$, there are $g \in \text{AP}_{\omega, c}(\mathbb{X})$ and $\varphi \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ such that $\|(g_n)_{(\omega, c)} - g_{(\omega, c)}\| \rightarrow 0$ and $\|(\varphi_n)_{(\omega, c)} - \varphi_{(\omega, c)}\| \rightarrow 0$ as $n \rightarrow \infty$. Set $f = g + \varphi$, then $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\|(f_n)_{(\omega, c)} - f_{(\omega, c)}\| \rightarrow 0$ as $n \rightarrow \infty$.

The rest of the proof is clear. \square

Now we show the completeness of $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ with the following result.

Theorem 3.24. *Let $\mu \in \mathcal{M}$ satisfying (H1). Then $(\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_{(\omega, c)})$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$. Then, given $\varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$,

$$\|(f_p)(t) - (f_q)(t)\|_{(\omega, c)} < \varepsilon.$$

Invoking Theorem 3.17, $\exists (u_p, u_q) \in (\text{PAP}(\mathbb{R}, \mathbb{X}, \mu))^2$ such that $f_p(t) = c^\wedge(t)u_p(t)$ and $f_q(t) = c^\wedge(t)u_q(t)$ for all $t \in \mathbb{R}$ and since $p, q \geq n_0$ we have

$$\|u_p(t) - u_q(t)\| \leq \|f_p(t) - f_q(t)\|_{(\omega, c)} < \varepsilon.$$

Consequently, (u_n) be a Cauchy sequence in $\text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. Using the completeness of $\text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$, we know that $\exists u \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ such that $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$.

We take $f(t) := c^\wedge(t)u(t)$. We claim that $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. And it can be easily seen that :

$$\|f_n - f\|_{(\omega, c)} = \sup_{t \in \mathbb{R}} \|u_n - u\| \rightarrow 0 \quad (n \rightarrow \infty)$$

Which completes the proof. \square

We end this subsection giving a general convolution theorem for our new class of functions. In their paper, Blot et al. obtained the following convolution result.

Theorem 3.25 ([3]). *Let $\mu \in \mathcal{M}$ satisfying (H2). If $f \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ and $\phi(\cdot) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product $f \star \phi$ defined by*

$$(f \star \phi)(t) = \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds, \quad \text{for } t \in \mathbb{R}.$$

is μ -pseudo almost periodic.

We generalize this result as follows.

Theorem 3.26. *Let $\mu \in \mathcal{M}$ satisfying (H2). If $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\tilde{\phi} := (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product of $f \star \phi$ defined by*

$$\begin{aligned} (f \star \phi)(t) &= \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds, \quad \text{for } t \in \mathbb{R}. \\ &= \int_{-\infty}^{+\infty} \tilde{\phi}(s)c^\wedge(s)f(t-s)ds, \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{11}$$

is μ - (ω, c) -pseudo almost periodic.

Proof. Let $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\tilde{\phi} := (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$.

First, note that using Theorem 3.4, $(f \star \phi) \in \text{BC}(\mathbb{R}, \mathbb{X}, c)$. Furthermore, according to Theorem 3.17, there exists a $u \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ such that $f(t) := c^\wedge(t)u(t)$, for any $t \in \mathbb{R}$. It comes that

$$\begin{aligned} (f \star \phi)(t) &= \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds \\ &= c^\wedge(t) \int_{-\infty}^{+\infty} c^\wedge(-s)\phi(s)c^\wedge(-t+s)f(t-s)ds \\ &= c^\wedge(t) \int_{-\infty}^{+\infty} \tilde{\phi}(s)u(t-s)ds \end{aligned} \tag{12}$$

Invoking successively Theorems 3.25 and 3.17, we have that (12) is μ - (ω, c) -pseudo almost periodic. The proof is complete. \square

Example 3.4. *The unique solution of the heat equation*

$$u_t(x, t) = u_{xx}(x, t), \quad x \in \mathbb{R}, t \geq 0,$$

with the initial condition $u(x, 0) = f(x)$ is given by

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s) ds, \quad x \in \mathbb{R}, t \geq 0,$$

If $c^\wedge(-\cdot)e^{-\frac{(\cdot)^2}{4t_0}} \in L^1(\mathbb{R})$ and $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}, \mu)$, then by Theorem 3.14, the solution

$$[x \mapsto u(x, t_0), x \in \mathbb{R}] \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}, \mu).$$

4 Jointly continuous case.

This section is devoted to the study of a composition result well-suited for the introduced (ω, c) -periodicity concept. The main results of this section are Theorems 4.7 and 4.16. But first, let's define some new notions.

4.1 Basic definitions and properties.

First of all, reader should be aware that the already known concept of compactness for subsets seems to be irrelevant when it comes to deal with (ω, c) -periodicity where $c \neq 1$ since (ω, c) periodic type functions are not bounded on \mathbb{R} (i.e. 1-bounded on \mathbb{R}) but c -bounded on \mathbb{R} . With the following definition, we propose a new concept of compactness for subset well-suited for (ω, c) -periodic calculus.

Definition 4.1. Let \mathcal{K} be a non-empty set. We say that \mathcal{K} is an (ω, c) -type compact subset of \mathbb{Y} if and only if following assumptions are satisfied :

1. \mathcal{K} is compact,
2. for any $\bar{t} \in \mathbb{R}$, $c^\wedge(\bar{t})y \in \mathcal{K}$ where $y \in \mathbb{Y}$.

One can note that a compact subset of $\text{AP}(\mathbb{X})$ is in fact a (ω, c) -type subset of $\text{AP}_{\omega, c}(\mathbb{X})$ since if \mathcal{K} is a compact subset of $\text{AP}(\mathbb{X})$ we have the following equality

$$\begin{aligned} \Sigma &:= \{u(t) : t \in \mathbb{R}, u \in \mathcal{K}\} \\ &= \{(c^\wedge(t)u(t))_{(\omega, c)} : t \in \mathbb{R}, (c^\wedge(t)u(t)) \in \text{AP}_{\omega, c}(\mathbb{X}), u \in \mathcal{K}\} \end{aligned}$$

for any $c \in \mathbb{C} - \{0\}$.

Definition 4.2 ([22]). A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (Yoshizawa) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $L > 0$, such that for all $\beta \in \mathbb{R}$, there exists $\tau \in [\beta, \beta + L]$ with

$$\sup_{t \in \mathbb{R}} \sup_{y \in K} \|F(t + \tau, y) - F(t, y)\| < \varepsilon. \quad (13)$$

for all $t \in \mathbb{R}$ and all $y \in K$.

The collection of such functions will be denoted by $\text{AP}(\mathbb{Y}, \mathbb{X})$.

Such number τ in (13) is called ε -translation number of $F(t, y)$ and we denote by $\mathcal{T}_\varepsilon(F, K)$ the set of all ε -translation numbers of F for $y \in K$. This set has the following properties :
For a fixed compact set K ,

1. an ε -translation number is also an ε' -translation number if $\varepsilon' > \varepsilon$, and hence we have the inclusion $\mathcal{T}_\varepsilon(F, K) \subset \mathcal{T}_{\varepsilon'}(F, K)$,
2. if $\tau \in \mathcal{T}_\varepsilon(F, K)$, then $-\tau \in \mathcal{T}_\varepsilon(F, K)$,
3. if $(\tau_1, \tau_2) \in \mathcal{T}_{\varepsilon_1}(F, K) \times \mathcal{T}_{\varepsilon_2}(F, K)$ then $\tau_1 + \tau_2 \in \mathcal{T}_{\varepsilon_1 + \varepsilon_2}(F, K)$.

In what follows, we assume that $\omega \in \mathbb{R}_+^* \cap \mathcal{T}_\varepsilon(F, K)$.

In [13] authors have introduced two concepts of (ω, c) -almost periodic functions in the case of jointly continuous functions, but in this paper use a novel approach.

Definition 4.3. A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (ω, c) -almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any (ω, c) -type compact subset \mathcal{K} of \mathbb{Y} there exists $L > 0$, such that for all $\beta \in \mathbb{R}$, there exists $\tau \in [\beta, \beta + L]$ with

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \sup_{y \in \mathcal{K}} \|c^\wedge(-t - \tau)F(t + \tau, y) - c^\wedge(-t)F(t, y)\| < \varepsilon. \\ \iff & \sup_{t \in \mathbb{R}} \sup_{y \in \mathcal{K}} \|F_{(\omega, c)}(t + \tau, y) - F_{(\omega, c)}(t, y)\| < \varepsilon \end{aligned}$$

for all $t \in \mathbb{R}$ and all $y \in K$, where $F_{(\omega, c)}(t, \cdot) := c^{-\frac{t}{\omega}} F(t, \cdot)$.

The space of all such functions will be denoted by $\text{AP}_{\omega, c}(\mathbb{Y}, \mathbb{X})$.

In the following, we use the notation : $F_{(\omega, c)}(t, \cdot) := c^{-\frac{t}{\omega}} F(t, \cdot)$.

Remark 4.4. When $c = 1$, $\text{AP}_{\omega, c}(\mathbb{Y}, \mathbb{X}) = \text{AP}(\mathbb{Y}, \mathbb{X})$.

Proposition 4.5. $(\text{AP}_{\omega, c}(\mathbb{Y}, \mathbb{X}), \|\cdot\|_{(\omega, c)})$ is a Banach space.

We need to develop some tools in order to propose a composition theorem for measure (ω, c) -pseudo almost periodic functions.

We give the following results

Lemma 4.6. If \mathcal{K} is an (ω, c) -type compact subset of $\text{AP}_{\omega, c}(\mathbb{Y})$, then

$$\Sigma := \{u_{(\omega, c)}(t) : t \in \mathbb{R}, u \in \text{AP}_{\omega, c}(\mathbb{Y}), u_{(\omega, c)} \in \mathcal{K}\}$$

is a relatively compact subset of \mathbb{Y} .

Proof. Let $\varepsilon > 0$.

Since \mathcal{K} is compact, it is also precompact, thus there exists $\{(u_1)_{(\omega,c)}, \dots, (u_m)_{(\omega,c)}\}$ a finite (ω, c) -type subset of $\text{AP}_{\omega,c}(\mathbb{Y})$ (i.e. a finite subset of $\text{AP}(\mathbb{Y})$) such that

$$\mathcal{K} \subset \bigcup_{1 \leq i \leq m} \left\{ z \in \text{AP}_{\omega,c}(\mathbb{Y}) : \|z - u_i\|_{(\omega,c)} \leq \frac{\varepsilon}{2} \right\}.$$

Since, $(u_i)_{(\omega,c)}(\mathbb{R})$ is relatively compact in \mathbb{X} for all $i = 1, \dots, m$, then $\bigcup_{1 \leq i \leq m} (u_i)_{(\omega,c)}(\mathbb{R})$ is also relatively compact and consequently, there exists a finite subset of \mathbb{R} $\{t_1, \dots, t_k\}$ such that

$$\bigcup_{1 \leq i \leq m} (u_i)_{(\omega,c)}(\mathbb{R}) \subset \bigcup_{1 \leq i \leq m} \bigcup_{1 \leq j \leq k} \left\{ y \in \mathbb{Y} : \|y - (u_i)_{(\omega,c)}(t_j)\| \leq \frac{\varepsilon}{2} \right\}.$$

If $y \in \mathbb{Y}$ there exists $z \in \mathcal{K}$ and $t \in \mathbb{R}$ such that $y = z(t)$, and there exists $i \in \{1, \dots, m\}$ such that $\|z - (u_i)_{(\omega,c)}\| \leq \frac{\varepsilon}{2}$ and consequently $\|y - (u_i)_{(\omega,c)}(t)\| \leq \frac{\varepsilon}{2}$.

Now, using the previous inclusion, there exists $p \in \{1, \dots, m\}$ and $j \in \{1, \dots, k\}$ such that

$$\|(u_i)_{(\omega,c)}(t) - (u_p)_{(\omega,c)}(t_j)\| \leq \frac{\varepsilon}{2}$$

It comes that

$$\|y - (u_i)_{(\omega,c)}(t_j)\| \leq \|y - (u_i)_{(\omega,c)}(t)\| + \|(u_i)_{(\omega,c)}(t) - (u_p)_{(\omega,c)}(t_j)\| \leq \frac{2\varepsilon}{2} = \varepsilon$$

This proves that $\Sigma \subset \bigcup_{1 \leq i \leq m} \bigcup_{1 \leq j \leq k} \left\{ y \in \mathbb{Y} : \|y - (u_i)_{(\omega,c)}(t_j)\| \leq \frac{\varepsilon}{2} \right\}$ or in other words, Σ is precompact, and since \mathbb{Y} is complete, we obtain that Σ is relatively compact. \square

Now, for a given function $F \in \text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$, we define the Nemytskii's superposition operator $\mathcal{N}_F : \text{AP}_{\omega,c}(\mathbb{Y}) \rightarrow \text{AP}_{\omega,c}(\mathbb{X})$ such that $[t \mapsto u(t)] \mapsto \mathcal{N}_F(u) := [t \mapsto F(t, u(t))]$.

The first main result of this section is the following theorem.

Theorem 4.7. *Let $F \in \text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$. Then the Nemytskii superposition operator \mathcal{N}_F is continuous from $\text{AP}_{\omega,c}(\mathbb{Y})$ into $\text{AP}_{\omega,c}(\mathbb{X})$.*

Proof. Let \mathcal{K} be an (ω, c) -type compact subset of $\text{AP}_{\omega,c}(\mathbb{Y})$ let $\tilde{u} \in \mathcal{K}$ and $\varepsilon > 0$.

We set $\Sigma := \{u_{(\omega,c)}(t) : t \in \mathbb{R}, u \in \text{AP}_{\omega,c}(\mathbb{Y}), u_{(\omega,c)} \in \mathcal{K}\}$. According to Lemma 4.6, the closure $\bar{\Sigma}$ is compact.

Since $F \in \text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$, there exists $l > 0$ such that for $\alpha \in \mathbb{R}$, there exists $-\tau \in [\alpha, \alpha + l]$ satisfying :

$$\|F_{(\omega,c)}(t + \tau, y) - F_{(\omega,c)}(t, y)\| \leq \frac{\varepsilon}{3}, \text{ for all } (y, t) \in \Sigma \times \mathbb{R}. \quad (14)$$

Since $\bar{\Sigma} \times [0, l]$ is compact then $F_{(\omega,c)}$ is uniformly continuous on it and consequently there exists $\delta > 0$ such that, for all $t_1, t_2 \in [0, l]$ and for all $y_1, y_2 \in \bar{\Sigma}$

$$\left(\|y_1 - y_2\| \leq \delta, |t_1 - t_2| \leq \delta \right) \Rightarrow \|F_{(\omega,c)}(t_1, y_1) - F_{(\omega,c)}(t_2, y_2)\| \leq \frac{\varepsilon}{3}.$$

And this implies that :

$$\|y_1 - y_2\| \leq \delta \implies \|F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2)\| \leq \frac{\varepsilon}{3}, \text{ for all } t \in [0, l]. \quad (15)$$

If $\tilde{v} \in \mathcal{K}$ satisfies $\|\tilde{v} - \tilde{u}\|_\infty \leq \delta \iff \|c^\wedge(t)\tilde{v}(t) - c^\wedge(t)\tilde{u}(t)\|_{(\omega,c)} \leq \delta$.

We set $u(t) := c^\wedge(t)\tilde{u}(t)$ and $v(t) := c^\wedge(t)\tilde{v}(t)$ for all $t \in \mathbb{R}$.

Then we have $u, v \in \text{AP}_{\omega,c}(\mathbb{Y})$, and using (14) and (15), we obtain, for all $t \in \mathbb{R}$

$$\begin{aligned} \|F_{(\omega,c)}(t, u(t)) - F_{(\omega,c)}(t, v(t))\| &\leq \|F_{(\omega,c)}(t, u(t)) - F_{(\omega,c)}(t - \tau, u(t))\| \\ &\quad + \|F_{(\omega,c)}(t - \tau, u(t)) - F_{(\omega,c)}(t - \tau, v(t))\| \\ &\quad + \|F_{(\omega,c)}(t - \tau, v(t)) - F_{(\omega,c)}(t, v(t))\| \\ &\leq 3\frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

And so, by taking the supremum on the $t \in \mathbb{R}$, we obtain $\|\mathcal{N}_F(u) - \mathcal{N}_F(v)\|_{(\omega,c)} \leq \varepsilon$.

This proves that the restriction of \mathcal{J}_F to \mathcal{K} is continuous for all (ω, c) -type compact subset \mathcal{K} of $\text{AP}_{\omega,c}(\mathbb{Y})$. And since $(\text{AP}_{\omega,c}(\mathbb{Y}), \|\cdot\|_{(\omega,c)})$ and $(\text{AP}_{\omega,c}(\mathbb{X}), \|\cdot\|_{(\omega,c)})$ are Banach Spaces, this proves the continuity of \mathcal{J}_F on $\text{AP}_{\omega,c}(\mathbb{Y})$. \square

The following proposition is a generalization of Cieutat, Fatajou and N'Guérékata's Theorem in [6] which becomes the particular case $c = 1$ of our result.

Proposition 4.8. *Let $F : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X}$ be a continuous function. Then $F \in \text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$ if and only if the following conditions hold :*

1. for all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in \text{AP}(\mathbb{X})$,
2. $F_{(\omega,c)}$ is uniformly continuous on each (ω, c) -type compact set \mathcal{K} in \mathbb{Y} with respect to the second variable, namely, for each (ω, c) -type compact set \mathcal{K} in \mathbb{Y} , for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in \mathcal{K}$, one has

$$\|y_1 - y_2\| \leq \delta \implies \sup_{t \in \mathbb{R}} \|F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2)\| \leq \varepsilon.$$

Here, we propose a concept of μ - (ω, c) -ergodicity for the jointly continuous functions case.

Definition 4.9. *Let $\mu \in \mathcal{M}$. A function $F \in \text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$ is said to be μ - (ω, c) -ergodic in t uniformly with respect to $y \in \mathbb{Y}$ if the two following conditions are true :*

1. for all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$,
2. $F_{(\omega,c)}$ is uniformly continuous on each (ω, c) -type compact set \mathcal{K} in \mathbb{Y} with respect to the second variable, namely, for each (ω, c) -type compact set \mathcal{K} in \mathbb{Y} , for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in \mathcal{K}$, one has

$$\|y_1 - y_2\| \leq \delta \implies \sup_{t \in \mathbb{R}} \|F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2)\| \leq \varepsilon.$$

We denote the space of all such functions by $\mathcal{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Remark 4.10. When $c = 1$, we write $\mathcal{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ instead of $\mathcal{E}_{\omega,1,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Definition 4.11. Let $\mu \in \mathcal{M}$. A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be μ - (ω, c) -pseudo almost periodic in t uniformly with respect to $y \in \mathbb{Y}$ if F is written in the form

$$F = G + \Phi$$

where $G \in \text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$ and $\Phi \in \mathcal{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

$\text{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ denotes the set of such that functions.

The following inclusion hold

$$\text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X}) \subset \text{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) \subset \text{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$$

Remark 4.12. When $c = 1$, we write $\text{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ instead of $\text{PAP}_{\omega,1}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

As in the previous section, we propose a characterization result which hold for (ω, c) -almost periodic, μ - (ω, c) -ergodic and μ - (ω, c) -pseudo almost periodic functions in t uniformly with respect to $y \in \mathbb{Y}$.

Theorem 4.13. Let $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Then, $F \in \text{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ (resp. $\text{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$ or $\mathcal{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$) if and only if

$$F(t, y) \equiv c^\wedge(t)u(t, y),$$

with $c^\wedge(t) := c^{\frac{t}{\omega}}$ and $u \in \text{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ (resp. $\text{AP}(\mathbb{Y}, \mathbb{X})$ or $\mathcal{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$).

Proof. The proof is similar to the one of Theorem 3.17. □

We end this section with this result which is a consequence of Proposition 4.8 and definition 4.9.

Theorem 4.14. Let $\mu \in \mathcal{M}$ and $F : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X}$ be μ - (ω, c) -ergodic (of second kind) in t uniformly with respect to $y \in \mathbb{Y}$. Then

1. for all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in \text{PAP}(\mathbb{R}, \mathbb{X}, \mu)$,
2. $F_{(\omega,c)}$ is uniformly continuous on each (ω, c) -type compact set \mathcal{K} in \mathbb{Y} with respect to the second variable.

We are now in a position to give a composition theorem for measure (ω, c) -pseudo almost periodic functions.

4.2 Composition of measure (ω, c) -pseudo almost periodic functions.

The proof of our result of composition of μ - (ω, c) -pseudo almost periodic functions is based on the following lemma due to Schwartz [19].

Lemma 4.15. *If $\Psi \in C(\mathbb{X}, \mathbb{Y})$, then for each compact set \mathcal{K} in \mathbb{X} and all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in \mathbb{X}$, one has*

$$x_1 \in \mathcal{K} \text{ and } \|x_1 - x_2\| \leq \delta \Rightarrow \|\Psi(x_1) - \Psi(x_2)\| \leq \varepsilon.$$

We now can state and prove our composition result.

Theorem 4.16. *Let $\mu \in \mathcal{M}$, $F \in \text{PAP}_{\omega, c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ and $y \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{Y}, \mu)$. Assume that the following hypothesis holds*

1. *For all bounded subset Ω of \mathbb{Y} , F is c -bounded on $\mathbb{R} \times \Omega$ (i.e. $F_{(\omega, c)}$ is bounded on $\mathbb{R} \times \Omega$).*

Then $[t \mapsto F(t, y_{(\omega, c)}(t))] \in \text{PAP}_{\omega, c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Proof. First note that the function $[t \mapsto F(t, y_{(\omega, c)}(t))]$ is continuous and by Hypothesis (1) it is c -bounded. Since $y \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{Y}, \mu)$ there exists $y_1 \in \text{AP}_{\omega, c}(\mathbb{Y})$ and $y_2 \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{Y}, \mu)$ such that

$$y = y_1 + y_2.$$

Moreover, since $F \in \text{PAP}_{\omega, c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$, there exists $F_1 \in \text{AP}_{\omega, c}(\mathbb{Y}, \mathbb{X})$ and $F_2 \in \mathcal{E}_{\omega, c, 3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ such that

$$F = F_1 + F_2.$$

and there exists

$$\tilde{F} \in \text{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) \text{ such that } F(t, y(t)) := c^\wedge(t) \tilde{F}(t, y(t)), \forall t \in \mathbb{R}.$$

$$\tilde{F}_1 \in \text{AP}(\mathbb{Y}, \mathbb{X}) \text{ such that } F_1(t, y(t)) := c^\wedge(t) \tilde{F}_1(t, y(t)), \forall t \in \mathbb{R}.$$

$$\tilde{F}_2 \in \mathcal{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) \text{ such that } F_2(t, y(t)) := c^\wedge(t) \tilde{F}_2(t, y(t)), \forall t \in \mathbb{R}.$$

So we have the following decomposition for $F(t, y(t))$

$$\begin{aligned} F(t, y(t)) &= c^\wedge(t) \left(\tilde{F}_1(t, y_1(t)) + [\tilde{F}(t, y(t)) - \tilde{F}(t, y_1(t))] + [\tilde{F}(t, y_1(t)) - \tilde{F}_1(t, y_1(t))] \right) \\ &= c^\wedge(t) \left(\tilde{F}_1(t, y_1(t)) + [\tilde{F}(t, y(t)) - \tilde{F}(t, y_1(t))] + \tilde{F}_2(t, y_1(t)) \right) \end{aligned}$$

Invoking Theorem 4.7 and keeping in mind that

$$(y_1)_{(\omega, c)} \in \text{AP}(\mathbb{Y}) \text{ then } \tilde{F}_1(t, (y_1)_{(\omega, c)}(t)) \in \text{AP}(\mathbb{X}) \tag{16}$$

and consequently we have that $[t \mapsto F_1(t, (y_1)_{(\omega, c)}(t))] \in \text{AP}_{\omega, c}(\mathbb{Y}, \mathbb{X})$.

Now denote \mathcal{K} the closure of the range of $(y_1)_{(\omega, c)} : \mathcal{K} = \overline{\{(y_1)_{(\omega, c)}(t) : t \in \mathbb{R}\}}$. One can easily

note that since $y_1 \in \text{AP}_{\omega,c}(\mathbb{Y})$, using Theorem 3.17, we have $(y_1)_{(\omega,c)} \in \text{AP}(\mathbb{Y})$ and consequently, \mathcal{K} is a compact subset of \mathbb{Y} [7].

In what follows, we denote by Ψ the function defined by

$$\begin{aligned}\Psi : \mathbb{Y} &\longrightarrow \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu) \\ y &\longmapsto \Psi(y) = c^\wedge(t)\tilde{F}(\cdot, y)\end{aligned}$$

Since $F(\cdot, y(\cdot)) := c^\wedge(\cdot)\tilde{F}(\cdot, y(\cdot)) \in \text{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$, by using Theorem 4.14 we deduce that the restriction of Ψ on all (ω, c) -type compact subset \mathcal{K} of \mathbb{Y} is uniformly continuous, which is equivalent to say that the function Ψ is continuous on \mathbb{Y} . From Lemma 4.15 applied to Ψ , we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, $u_1, u_2 \in \mathbb{Y}$, one has

$$u_1 \in \mathcal{K} \text{ and } \|u_1 - u_2\| \leq \delta \Rightarrow \|\tilde{F}(t, u_1(t)) - \tilde{F}(t, u_2(t))\| \leq \varepsilon.$$

Since, $y(t) = y_1(t) + y_2(t)$ and $(y_1)_{(\omega,c)}(t) \in \mathcal{K}$, it comes that

$$t \in \mathbb{R} \text{ and } \|(y_2)_{(\omega,c)}(t)\| \leq \delta \Rightarrow \|\tilde{F}(t, y_{(\omega,c)}(t)) - \tilde{F}(t, (y_1)_{(\omega,c)}(t))\| \leq \varepsilon.$$

therefore, the following inequality holds

$$\frac{\mu(\{t \in [-r, r] : \|\tilde{F}(t, y_{(\omega,c)}(t)) - \tilde{F}(t, (y_1)_{(\omega,c)}(t))\| > \varepsilon\})}{\mu([-r, r])} \leq \frac{\mu(\{t \in [-r, r] : \|(y_2)_{(\omega,c)}(t)\| > \delta\})}{\mu([-r, r])}$$

Since $y_2 \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{Y}, \mu)$ and according to Theorem 3.7 we have for the above-mentioned δ

$$\lim_{r \rightarrow +\infty} \frac{\mu(\{t \in [-r, r] : \|(y_2)_{(\omega,c)}(t)\| > \delta\})}{\mu([-r, r])} = 0,$$

and consequently

$$\lim_{r \rightarrow +\infty} \frac{\mu(\{t \in [-r, r] : \|\tilde{F}(t, y_{(\omega,c)}(t)) - \tilde{F}(t, (y_1)_{(\omega,c)}(t))\| > \varepsilon\})}{\mu([-r, r])} = 0,$$

Invoking Theorem 3.7 with $c = 1$ and previous inequality it is proved that

$$[t \longmapsto \tilde{F}(t, y_{(\omega,c)}(t)) - \tilde{F}(t, (y_1)_{(\omega,c)}(t))] \in \mathcal{E}_{\omega,1}(\mathbb{R}, \mathbb{Y}, \mu) = \mathcal{E}(\mathbb{R}, \mathbb{Y}, \mu).$$

which is equivalent to say that (according to Theorem 4.13)

$$[t \longmapsto F(t, y_{(\omega,c)}(t)) - F(t, (y_1)_{(\omega,c)}(t))] \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{Y}, \mu).$$

In order to complete the proof, we prove that $[t \longmapsto F_2(t, (y_1)_{(\omega,c)}(t))] \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{Y}, \mu)$.

Since $F_2 := c^\wedge(\cdot)\tilde{F}_2$ is uniformly continuous on the (ω, c) -type compact set $\mathcal{K} = \{(y_1)_{(\omega,c)}(t) : t \in \mathbb{R}\}$ with respect to the second variable, we deduce that for given $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $t \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathcal{K}$, one has

$$\|\xi_1 - \xi_2\| \leq \eta \implies \|\tilde{F}_2(t, \xi_1(t)) - \tilde{F}_2(t, \xi_2(t))\| \leq \varepsilon.$$

Therefore, there exists ϑ_ε and $\{(y_i)_{(\omega,c)}\}_{i=1}^{\vartheta_\varepsilon} \subset \mathcal{K}$, such that $\mathcal{K} \subset \bigcup_{i=1}^{\vartheta_\varepsilon} B((y_i)_{(\omega,c)}, \eta)$ and then

$$\|\tilde{F}_2(t, (y_1)_{(\omega,c)}(t))\| \leq \varepsilon + \sum_{i=1}^{\vartheta_\varepsilon} \|\tilde{F}_2(t, (y_i)_{(\omega,c)}(t))\|.$$

Since,

$$\forall i \in \{1, \dots, \vartheta_\varepsilon\}, \quad \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\tilde{F}_2(t, (y_i)_{(\omega,c)}(t))\| d\mu(t) = 0.$$

we deduce that

$$\forall \varepsilon > 0, \quad \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\tilde{F}_2(t, (y_1)_{(\omega,c)}(t))\| d\mu(t) \leq \varepsilon,$$

which implies that

$$\forall \varepsilon > 0, \quad \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \|\tilde{F}_2(t, (y_1)_{(\omega,c)}(t))\| d\mu(t) = 0.$$

Therefore, using Theorem 3.7 with $c = 1$, we have that $[t \mapsto \tilde{F}_2(t, (y_1)_{(\omega,c)}(t))] \in \mathcal{E}(\mathbb{R}, \mathbb{Y}, \mu)$. In conclusion, invoking Theorem 4.13, we showed that

$$[t \mapsto F_2(t, (y_1)_{(\omega,c)}(t))] \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{Y}, \mu).$$

Finally, it comes that

$$[t \mapsto F(t, y_{(\omega,c)}(t))] \in \text{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu).$$

The proof is complete. \square

We end this section with the following result.

Corollary 4.17. *Let $\mu \in \mathcal{M}$, $\Psi \in C(\mathbb{X}, \mathbb{Y})$, $\omega > 0$ and $c \in \mathbb{C} - \{0\}$.*

Assume that for all bounded subset \mathbf{B} of \mathbb{X} , Ψ is c -bounded on \mathbf{B} , (i.e. $\Psi(\cdot) := c^\wedge(t)\tilde{\Psi}(\cdot)$ where $\tilde{\Psi}$ is bounded on \mathbf{B}). Then if $x \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$,

$$[t \mapsto \Psi(x_{(\omega,c)}(t)) := c^\wedge(t)\tilde{\Psi}(x_{(\omega,c)}(t))] \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu).$$

Proof. It is a consequence of Theorem 4.16 with $F(t, x_{(\omega,c)}(t)) = \Psi(x_{(\omega,c)}(t))$. \square

5 Application : Measure (ω, c) -Pseudo Almost Periodic solutions to a Lasota-Ważewska model.

First, Ważewska-Czyżewska and Lasota [21] proposed in 1976 the delay logistic equations with one constant concentrated delay

$$N'(t) = -\mu N(t) + pe^{-rN(t-\tau)}$$

in order to describe the survival of red blood cells in an animal. Here $N(t)$ denotes the number of red blood cells at time t , μ is the probability of death of a red blood cell, p and r are positive constants related to the production of red blood cells per unit time and τ is the time required to produce a red blood cell. Few years later, Gopalsamy and Trofimchuk [11] obtained that the Lasota-Ważewska model with one discrete delay

$$x'(t) = -\alpha(t)x(t) + \beta(t)e^{-\nu x(t-\tau)}$$

has a globally attractive almost periodic solution under some additional assumptions. The aim here is to study the existence and uniqueness of a generalized Lasota-Ważewska model with μ - (ω, c) -pseudo almost periodic coefficients and with mixed delay which is in the form :

$$y'(t) = -\alpha(t)y(t) + \sum_{j=1}^m a_j(t)e^{-c \wedge (-t)\omega_j(t)} \int_{-\infty}^t K_j(t-s)y(s)ds + \sum_{i=1}^n b_i(t)e^{-c \wedge (-t)\beta_i(t) \sum_{j=1}^p y(t-\tau_{ij})}, \quad t \in \mathbb{R} \quad (17)$$

where $y(t)$ stands for the number of red blood cells at time t and $\alpha(t)$ is the average part of red blood cells population being destroyed in time t . For all $1 \leq j \leq m$ and $1 \leq i \leq n$, $a_j(t)$ and $b_i(t)$ are the connected with demand for oxygen at time t , $\omega_j(t)$ and $\beta_i(t)$ characterize excitability of haematopoietic system at time t , K_j is the probability kernel of the distributed delays and τ_{ij} is the time required to produce a red blood cell. One can note that we consider in our new approach the μ - (ω, c) -pseudo almost periodic for the connected with demand for oxygen at time t and the μ -pseudo almost periodic for the excitability of haematopoietic system at time t since it is more realistic for the description of the physical and biological phenomena.

The method consists to reduce the existence of the unique solution for the Lasota-Ważewska model (17) to the search for the existence of the unique fixed point of an appropriate operator on the Banach space $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$. Notice that we restrict our selves to \mathbb{R}_+ -valued functions since only non-negative solutions are biologically meaningful.

5.1 Existence and uniqueness of μ - (ω, c) -pseudo almost periodic solution to the model.

In what follows, given a c -bounded continuous function f defined on \mathbb{R} , $\overline{f_{(\omega, c)}}$ and $\underline{f_{(\omega, c)}}$ are defined by

$$\overline{f_{(\omega, c)}} = \sup_{t \in \mathbb{R}} f_{(\omega, c)}(t) = \sup_{t \in \mathbb{R}} c \wedge (-t)f(t), \quad \text{and} \quad \underline{f_{(\omega, c)}} = \inf_{t \in \mathbb{R}} f_{(\omega, c)}(t) = \inf_{t \in \mathbb{R}} c \wedge (-t)f(t)$$

Remark 5.1. If $c = 1$ we use the notations

$$\overline{f} := \overline{f_{(\omega, 1)}} = \sup_{t \in \mathbb{R}} f(t) \quad \text{and} \quad \underline{f} := \underline{f_{(\omega, 1)}} = \inf_{t \in \mathbb{R}} f(t)$$

First, we give sufficient conditions which ensures existence and uniqueness of μ - (ω, c) -pseudo almost periodic solution of (17).

- (C1) $0 < c \leq 1$,
(C2) $\alpha \in \text{AP}(\mathbb{R}_+)$,
(C3) $\beta_i, \omega_j \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$, for all $(i, j) \in [[1, n]] \times [[1, m]]$,
(C4) $a_j, b_i \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, for all $(i, j) \in [[1, n]] \times [[1, m]]$,
(C5) $\frac{\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} \omega_j + p\xi \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}} \beta_i}{\alpha} < 1$,
(C6) For all $1 \leq j \leq m$, $(K_j)_{(\omega, c)} : [0, +\infty) \rightarrow \mathbb{R}_+$ are continuous, integrable and

$$\int_0^\infty (K_j)_{(\omega, c)}(u) du = 1, \quad \text{and} \quad \int_0^\infty (K_j)_{(\omega, c)}(u) e^{\lambda u} du < +\infty,$$

where λ is a sufficiently non negative small constant.

Lemma 5.2. Let $f \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ and $g \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$. If $c > 0$, then $fg \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$.

Proof. According to Theorem 3.17 there exists a unique $u \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$ such that

$$f(t) := c^\wedge(t)u(t)$$

for all $t \in \mathbb{R}$. Using Lemma 2.13 it is clear that $u \times g \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$. Then,

$$(fg)(t) := c^\wedge(t)(u(t)g(t)), \quad \text{where } ug \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu).$$

Invoking Theorem 3.17, we complete the proof. □

Lemma 5.3. Let $\mu \in \mathcal{M}$ satisfying (H2). For all $x(\cdot) \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, the function

$$\begin{aligned} \psi_i : t \mapsto \psi_i(x_{(\omega, c)}(t)) &= a(t)e^{-c^\wedge(-t)\omega_i(t) \sum_{j=1}^p x(t-\tau_{ij})} \\ &= c^\wedge(t)a_{(\omega, c)}(t)e^{-\omega_i(t) \sum_{j=1}^p c^\wedge(-\tau_{ij})x_{(\omega, c)}(t-\tau_{ij})} \end{aligned}$$

belongs to $\text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ for all $1 \leq i \leq n$.

Proof. First, by Lemma 3.22, the function

$$t \mapsto x(t - \tau_{ij}) \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$$

for all $1 \leq i \leq n$ and $1 \leq j \leq p$. Then according to Proposition 3.18

$$t \mapsto \sum_{j=1}^p x(t - \tau_{ij}) \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$$

Furthermore, by Lemma 5.2

$$t \mapsto \omega_i(t) \sum_{j=1}^p x(t - \tau_{ij}) \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu).$$

for all $1 \leq i \leq n$. Now, using the fact that the function $(x \mapsto e^{-x})$ is Lipschitzian and bounded, and $a_{(\omega,c)} \in \text{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$ is also bounded then invoking the Corollary 4.17 it is clear that

$$\psi_i : t \mapsto c^\wedge(t) a_{(\omega,c)}(t) e^{-c^\wedge(-t)\omega_i(t) \sum_{j=1}^p x(t-\tau_{ij})} \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$$

for all $1 \leq i \leq n$. □

By using condition **(C6)** and Theorem 3.26, we can deduce the following Lemma.

Lemma 5.4. *Suppose that (H2) and (C6) hold. If $x \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, then the function defined by*

$$t \mapsto \int_{-\infty}^t K_j(t-s)x(s)ds \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$$

for all $1 \leq j \leq m$.

Theorem 5.5. *Suppose that (H2) and (C1) – (C4) are satisfied. Then the nonlinear operator Γ defined for each $x \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ by*

$$(\Gamma x)(t) = \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s) e^{-c^\wedge(-s)\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma} + \sum_{i=1}^n b_i(s) e^{-c^\wedge(-s)\beta_i(s) \sum_{j=1}^p x(s-\tau_{ij})} \right] ds$$

maps $\text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ into itself.

Proof. Using Lemmas 2.13, 5.2, 5.3, 5.4 and Corrolary 4.17, then the function χ defined by

$$\begin{aligned} \chi(s) &= \sum_{j=1}^m a_j(s) e^{-c^\wedge(-s)\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma} + \sum_{i=1}^n b_i(s) e^{-c^\wedge(-s)\beta_i(s) \sum_{j=1}^p x(s-\tau_{ij})} \\ &= c^\wedge(s) \left(\sum_{j=1}^m (a_j)_{(\omega,c)}(s) e^{-c^\wedge(-s)\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma} + \sum_{i=1}^n (b_i)_{(\omega,c)}(s) e^{-c^\wedge(-s)\beta_i(s) \sum_{j=1}^p x(s-\tau_{ij})} \right) \end{aligned}$$

is measure (ω, c) -pseudo almost periodic.

Consequently, we can write $\chi = \chi_1 + \chi_2$ where $\chi_1 \in \text{AP}_{\omega,c}(\mathbb{R}_+)$ and $\chi_2 \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$. It follows that

$$(\Gamma\chi)(t) := \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \chi(s)ds = (\Gamma\chi_1)(t) + (\Gamma\chi_2)(t)$$

Let us show that $(\Gamma\chi_1) \in \text{AP}_{\omega,c}(\mathbb{R}_+)$.

We recall that by applying condition **(C1)** to the model (17), α is almost periodic (i.e. (ω, c) -almost periodic with constant $c = 1$). Now, in the view of the almost periodicity of the function α and the (ω, c) -almost periodicity of the function χ_1 , there exists a number l_ε such that in any interval $[\delta, \delta + l_\varepsilon]$ one finds a number η , such that

$$\sup_{\xi \in \mathbb{R}} |\alpha(\xi + \eta) - \alpha(\xi)| < \varepsilon \quad \text{and} \quad \sup_{\xi \in \mathbb{R}} |(\chi_1)_{(\omega,c)}(\xi + \eta) - (\chi_1)_{(\omega,c)}(\xi)| < \varepsilon$$

It comes that

$$\begin{aligned}
(\Gamma\chi_1)(t+\eta) - (\Gamma\chi_1)(t) &= \int_{-\infty}^{t+\eta} e^{-\int_s^{t+\eta} \alpha(\xi) d\xi} \chi_1(s) ds - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s) ds \\
&= \int_{-\infty}^{t+\eta} e^{-\int_s^{t+\eta} \alpha(\xi) d\xi} \chi_1(s) ds - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s) ds \\
&= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s+\eta) ds - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s) ds \\
&= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s+\eta) ds - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s+\eta) ds \\
&\quad + \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s+\eta) ds - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_1(s) ds
\end{aligned}$$

So there exists $\delta \in (0, 1)$ such that

$$\begin{aligned}
&|(\Gamma\chi_1)_{(\omega,c)}(t+\eta) - (\Gamma\chi_1)_{(\omega,c)}(t)| \\
&\leq |\chi_1|_{(\omega,c)} \int_{-\infty}^t \left| e^{-\int_s^t \alpha(\xi) d\xi} - e^{-\int_s^t \alpha(\xi) d\xi} \right| ds + \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \left| (\chi_1)_{(\omega,c)}(s+\eta) - (\chi_1)_{(\omega,c)}(s) \right| ds \\
&\leq |\chi_1|_{(\omega,c)} \int_{-\infty}^t \left| e^{-\int_s^t \alpha(\xi) d\xi} - e^{-\int_s^t \alpha(\xi) d\xi} \right| ds + \varepsilon \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} ds \\
&\leq |\chi_1|_{(\omega,c)} \int_{-\infty}^t \left| e^{-\int_s^t \alpha(\xi) d\xi} - e^{-\int_s^t \alpha(\xi) d\xi} \right| ds + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq |\chi_1|_{(\omega,c)} \int_{-\infty}^t \left\{ e^{-\left[\int_s^t \alpha(\xi) d\xi + \delta \left(\int_s^t \alpha(\xi) d\xi - \int_s^t \alpha(\xi+\eta) d\xi \right) \right]} \times \left| \int_s^t \alpha(\xi+\eta) - \alpha(\xi) d\xi \right| ds \right\} \\
&\quad + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq |\chi_1|_{(\omega,c)} \int_{-\infty}^t \left\{ e^{-\int_s^t \alpha(\xi) d\xi} e^{-\delta \left(\int_s^t \alpha(\xi) d\xi - \int_s^t \alpha(\xi+\eta) d\xi \right)} \int_s^t \left| \alpha(\xi+\eta) - \alpha(\xi) \right| d\xi ds \right\} + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq \varepsilon |\chi_1|_{(\omega,c)} \int_{-\infty}^t [e^{-(t-s)\underline{\alpha}} e^{-\delta \left(\int_s^t \alpha(\xi) d\xi - \int_s^t \alpha(\xi+\eta) d\xi \right)} (t-s)] ds + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq \varepsilon |\chi_1|_{(\omega,c)} \int_{-\infty}^t [e^{-(t-s)\underline{\alpha}} (t-s)] ds + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq \frac{\varepsilon |\chi_1|_{(\omega,c)}}{\underline{\alpha}^2} + \varepsilon \int_{-\infty}^t e^{-(t-s)\underline{\alpha}} ds \\
&\leq \frac{\varepsilon |\chi_1|_{(\omega,c)}}{\underline{\alpha}^2} + \frac{\varepsilon}{\underline{\alpha}} = \left(\frac{|\chi_1|_{(\omega,c)}}{\underline{\alpha}^2} + \frac{1}{\underline{\alpha}} \right) \varepsilon
\end{aligned}$$

This proves that $(\Gamma\chi_1) \in \text{AP}_{\omega,c}(\mathbb{R}_+)$. Now, let us show that $(\Gamma\chi_2) \in \mathcal{E}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$. We have

that

$$\begin{aligned}
& \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \left| \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_2(s) ds \right|_{(\omega, c)} d\mu(t) \\
& \leq \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \int_{-\infty}^t e^{-(t-s)\alpha} \left| \chi_2(s) \right|_{(\omega, c)} ds d\mu(t) \\
& \leq \lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \int_0^\infty e^{-\alpha\xi} \left| \chi_2(t-\xi) \right|_{(\omega, c)} d\xi d\mu(t) \\
& = \lim_{r \rightarrow +\infty} \int_0^\infty e^{-\alpha\xi} \left(\frac{1}{\mu([-r, r])} \int_{-r}^r \left| \chi_2(t-\xi) \right|_{(\omega, c)} d\mu(t) \right) d\xi.
\end{aligned}$$

By the Lebesgue dominated convergence Theorem and (H2), we obtain that

$$\lim_{r \rightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^r \left| \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \chi_2(s) ds \right|_{(\omega, c)} d\mu(t) = 0.$$

Then, $(\Gamma\chi_2) \in \mathcal{E}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ and consequently, $\Gamma \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu)$. \square

Theorem 5.6. *Assume that (H2) and (C1) – (C6) hold, then the Lasota-Ważewska model with mixed delays (17) possess a unique measure (ω, c) -pseudo almost periodic solution y and we have $y_{(\omega, c)}$ in the region*

$$\mathbf{R} = \{ \psi \in \text{PAP}_{(\omega, c)}(\mathbb{R}, \mathbb{R}_+, \mu), \mathcal{LB} \leq |\psi_{(\omega, c)}| \leq \mathcal{UB} \}$$

where

$$\mathcal{UB} = \frac{\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}}}{\underline{\alpha}} \quad \text{and} \quad \mathcal{LB} = \frac{\sum_{j=1}^m (a_j)_{(\omega, c)} e^{-\overline{\omega_j} \mathcal{UB}} + \sum_{i=1}^n (b_i)_{(\omega, c)} e^{-\overline{p_i} \mathcal{UB}}}{\overline{\alpha}}$$

Proof. First, we prove that the operator Γ is a mapping from \mathbf{R} to \mathbf{R} . We set

$$\begin{aligned}
\gamma_j(s) &= a_j(s) e^{-c \wedge (-s) \omega_j(s)} \int_{-\infty}^s K_j(s-\sigma) x(\sigma) d\sigma \\
\theta_i(s) &= b_i(s) e^{-c \wedge (-s) \beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij})
\end{aligned}$$

In fact, we have

$$\begin{aligned}
|\Gamma x(t)|_{(\omega, c)} & \leq \int_{-\infty}^t \left| e^{-\int_s^t \alpha(\xi) d\xi} \left[\sum_{j=1}^m \gamma_j(s) + \sum_{i=1}^n \theta_i(s) \right] \right|_{(\omega, c)} ds = \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \left[\left| \sum_{j=1}^m \gamma_j(s) + \sum_{i=1}^n \theta_i(s) \right|_{(\omega, c)} \right] ds \\
& \leq \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \left[\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}} \right] ds \\
& \leq \int_{-\infty}^t e^{-\alpha(t-s)} \left[\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}} \right] ds \\
& = \frac{\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}}}{\underline{\alpha}}
\end{aligned}$$

In the other hand, if we set

$$\begin{aligned}\xi &= \max \{c^\wedge(\tau_{i,j})\} \text{ for } (i,j) \in [[1,n]] \times [[1,p]], \\ \delta_j(s) &= c^\wedge(-s)\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma, \text{ for } j \in [[1,m]]. \\ \phi_i(s) &= c^\wedge(-s)\beta_i(s) \sum_{j=1}^p x(s-\tau_{ij}) \text{ for } i \in [[1,n]].\end{aligned}$$

then, we have for $x \in \mathbf{R}$

$$\begin{aligned}|\Gamma x(t)|_{(\omega,c)} &= \left| \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s)e^{-\delta_j(s)} + \sum_{i=1}^n b_i(s)e^{-\phi_i(s)} \right] ds \right|_{(\omega,c)} \\ &\geq c^\wedge(-t) \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s) e^{-\bar{\omega}_j \int_{-\infty}^s (K_j)_{(\omega,c)}(s-\sigma)x_{(\omega,c)}(\sigma)d\sigma} \right. \\ &\quad \left. + \sum_{i=1}^n b_i(s) e^{-\bar{\beta}_i \sum_{j=1}^p c^\wedge(-\tau_{ij})x_{(\omega,c)}(s-\tau_{ij})} \right] ds \\ &\geq c^\wedge(-t) \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s) e^{-\bar{\omega}_j \mathcal{UB} \int_{-\infty}^s (K_j)_{(\omega,c)}(s-\sigma)d\sigma} + \sum_{i=1}^n b_i(s) e^{-\bar{\beta}_i \mathcal{UB} \sum_{j=1}^p c^\wedge(-\tau_{ij})} \right] ds \\ &\geq c^\wedge(-t) \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s) e^{-\bar{\omega}_j \mathcal{UB}} + \sum_{i=1}^n b_i(s) e^{-p\xi \bar{\beta}_i \mathcal{UB}} \right] ds \\ &\geq \int_{-\infty}^t c^\wedge(-s) e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m a_j(s) e^{-\bar{\omega}_j \mathcal{UB}} + \sum_{i=1}^n b_i(s) e^{-p\xi \bar{\beta}_i \mathcal{UB}} \right] ds \\ &= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^m (a_j)_{(\omega,c)}(s) e^{-\bar{\omega}_j \mathcal{UB}} + \sum_{i=1}^n (b_i)_{(\omega,c)}(s) e^{-p\xi \bar{\beta}_i \mathcal{UB}} \right] ds \\ &\geq \int_{-\infty}^t e^{-\bar{\alpha}(t-s)} \left[\sum_{j=1}^m (a_j)_{(\omega,c)} e^{-\bar{\omega}_j \mathcal{UB}} + \sum_{i=1}^n (b_i)_{(\omega,c)} e^{-p\xi \bar{\beta}_i \mathcal{UB}} \right] ds \\ &= \frac{\sum_{j=1}^m (a_j)_{(\omega,c)} e^{-\bar{\omega}_j \mathcal{UB}} + \sum_{i=1}^n (b_i)_{(\omega,c)} e^{-p\xi \bar{\beta}_i \mathcal{UB}}}{\bar{\alpha}}\end{aligned}$$

which implies that the operator Γ is a mapping from \mathbf{R} to itself. To end the proof, it suffices

to prove that Γ is a contraction mapping. Let $x, y \in \mathbf{R}$. Then

$$\begin{aligned}
|\Gamma x(t) - \Gamma y(t)|_{(\omega, c)} &= \left| \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \right. \\
&\quad \left[\sum_{j=1}^m a_j(s) \left(e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)x(\sigma) d\sigma - e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)y(\sigma) d\sigma \right) \right. \\
&\quad \left. + \sum_{i=1}^n b_i(s) \left(e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij}) - e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p y(s-\tau_{ij}) \right) \right] ds \Big|_{(\omega, c)} \\
&\leq \int_{-\infty}^t \left| e^{-\int_s^t \alpha(\xi) d\xi} \right. \\
&\quad \left[\sum_{j=1}^m a_j(s) \left(e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)x(\sigma) d\sigma - e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)y(\sigma) d\sigma \right) \right. \\
&\quad \left. + \sum_{i=1}^n b_i(s) \left(e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij}) - e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p y(s-\tau_{ij}) \right) \right] \Big|_{(\omega, c)} ds \\
&= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \\
&\quad \left[\sum_{j=1}^m (a_j)_{(\omega, c)}(s) \left(e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)x(\sigma) d\sigma - e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)y(\sigma) d\sigma \right) \right. \\
&\quad \left. + \sum_{i=1}^n (b_i)_{(\omega, c)}(s) \left(e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij}) - e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p y(s-\tau_{ij}) \right) \right] \Big| ds \\
&\leq \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \\
&\quad \left[\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} \left(e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)x(\sigma) d\sigma - e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)y(\sigma) d\sigma \right) \right. \\
&\quad \left. + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}} \left(e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij}) - e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p y(s-\tau_{ij}) \right) \right] \Big| ds \\
&\leq \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \\
&\quad \left[\sum_{j=1}^m \overline{(a_j)_{(\omega, c)}} \left| e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)x(\sigma) d\sigma - e^{-c^\wedge(-s)\omega_j(s)} \int_{-\infty}^s K_j(s-\sigma)y(\sigma) d\sigma \right| \right. \\
&\quad \left. + \sum_{i=1}^n \overline{(b_i)_{(\omega, c)}} \left| e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p x(s-\tau_{ij}) - e^{-c^\wedge(-s)\beta_i(s)} \sum_{j=1}^p y(s-\tau_{ij}) \right| \right] ds
\end{aligned}$$

Obviously, for $u, v \in [0, +\infty)$

$$|e^{-u} - e^{-v}| \leq |u - v|$$

then it comes that

$$\begin{aligned}
|\Gamma x(t) - \Gamma y(t)|_{(\omega,c)} &\leq \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \\
&\quad \left[\sum_{j=1}^m \overline{(a_j)_{(\omega,c)}} \left| c^\wedge(-s) \omega_j(s) \int_{-\infty}^s K_j(s-\sigma) (x(\sigma) - y(\sigma)) d\sigma \right| \right. \\
&\quad \left. + \sum_{i=1}^n \overline{(b_i)_{(\omega,c)}} \left| c^\wedge(-s) \beta_i(s) \sum_{j=1}^p (x(s-\tau_{ij}) - y(s-\tau_{ij})) \right| \right] ds \\
&\leq \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \\
&\quad \left[\sum_{j=1}^m \overline{(a_j)_{(\omega,c)}} \omega_j \int_{-\infty}^s (K_j)_{(\omega,c)}(s-\sigma) d\sigma |x-y|_{(\omega,c)} \right. \\
&\quad \left. + p\xi |x-y|_{(\omega,c)} \sum_{i=1}^n \overline{(b_i)_{(\omega,c)}} \beta_i \right] ds \\
&= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi) d\xi} \left[\sum_{j=1}^m \overline{(a_j)_{(\omega,c)}} \omega_j + p\xi \sum_{i=1}^n \overline{(b_i)_{(\omega,c)}} \beta_i \right] ds |x-y|_{(\omega,c)} \\
&\leq \frac{\sum_{j=1}^m \overline{(a_j)_{(\omega,c)}} \omega_j + p\xi \sum_{i=1}^n \overline{(b_i)_{(\omega,c)}} \beta_i}{\alpha} |x-y|_{(\omega,c)}
\end{aligned}$$

which implies (invoking **(C5)**) that the mapping Γ is a contraction mapping of \mathbf{R} . Consequently, Γ possess a unique fixed point $x^* \in \mathbf{R}$. Hence, x^* is the unique measure (ω, c) -pseudo almost periodic solution of Equation (17) in \mathbf{R} . \square

5.2 Example

In order to illustrate some features of our theoretical study, we will apply our main results to a special system and demonstrate the efficiencies of our criteria.

We consider the following Lasota-Ważewska model with mixed delays

$$y'(t) = -\alpha(t)y(t) + \sum_{j=1}^3 a_j(t) e^{-c^\wedge(-t)\omega_j(t)} \int_{-\infty}^t K_j(t-s)y(s)ds + \sum_{i=1}^3 b_i(t) e^{-c^\wedge(-t)\beta_i(t)} y(t-\tau_i)$$

where $\alpha(t) = 8 + \cos^2(\sqrt{5}t) + \cos^2(t)$, $c = 0.9$, $\omega = 1$,

$$\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix} = \begin{pmatrix} (0.9)^\wedge(t) \left(1 + 0.25 \cos^2(\sqrt{2}t) + 0.25 \cos^2(\pi t) + \frac{0.5}{1+t^2} \right) \\ (0.9)^\wedge(t) \left(0.5 + 0.25 \cos^2(\sqrt{3}t) + 0.25 \cos^2(\pi t) + \frac{1}{1+t^2} \right) \\ (0.9)^\wedge(t) \left(0.5 + 0.25 \cos^2(\sqrt{5}t) + 0.25 \cos^2(\sqrt{2}t) + e^{-t^2 \cos^2(t)} \right) \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{pmatrix} &= \begin{pmatrix} 0.125 \cos^2(\sqrt{2}t) + 0.125 \cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125 \cos^2(\sqrt{2}t) + 0.125 \cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.250 \cos^2(\sqrt{2}t) + 0.25e^{-t^2 \cos^2(t)} \end{pmatrix} \\ \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} &= \begin{pmatrix} (0.9)^\wedge(t) \left(1 + 0.25 \cos^2(\sqrt{5}t) + 0.25 \cos^2(\pi t) + 0.5e^{-t^2 \cos^2(t)} \right) \\ (0.9)^\wedge(t) \left(1 + 0.25 \cos^2(\sqrt{5}t) + 0.25 \cos^2(\pi t) + 0.5e^{-t^2 \cos^2(t)} \right) \\ (0.9)^\wedge(t) \left(1 + 0.25 \cos^2(\sqrt{5}t) + 0.25 \cos^2(\pi t) + 0.5e^{-t^2 \cos^2(t)} \right) \end{pmatrix} \\ \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix} &= \begin{pmatrix} 0.125 \cos^2(\sqrt{2}t) + 0.125 \cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125 \cos^2(\sqrt{2}t) + 0.125 \cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125 \cos^2(\sqrt{2}t) + 0.125 \cos^2(\pi t) + 0.25e^{-t^2 \cos^2(t)} \end{pmatrix} \end{aligned}$$

$\tau_1 = \tau_2 = \tau_3 = 1$, $K_j = (0.9)^\wedge(t)e^{-t}$. Then

$$\frac{\sum_{j=1}^m \overline{(a_j)_{(\omega,c)} \omega_j} + p\xi \sum_{i=1}^n \overline{(b_i)_{(\omega,c)} \beta_i}}{\alpha} \leq \frac{3}{4} < 1$$

If the Radon-Nikodym derivative ρ of the measure μ is $\rho(t) = e^{\sin t}$ with respect to the Lebesgue measure on \mathbb{R} (i.e. $d\mu = \rho(t)dt$), then $\mu \in \mathcal{M}$ and satisfies (H1), since

$$\mu([-r, r]) = \int_{-r}^r e^{\sin(t)} dt \rightarrow +\infty, \text{ if } r \rightarrow +\infty,$$

and

$$\mu(\tau + a) \leq e^2 \mu(A), \forall \tau \in \mathbb{R} \text{ and } a \in A.$$

Hence, conditions (C1) – (C6) and (H2) are satisfied then according to the Theorem 5.6, the Lasota-Ważewska model with a mixed delays (18) has a unique μ - (ω, c) -pseudo almost periodic solution in the region $\mathbf{R} = \{y \in \text{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu), \mathcal{LB} \leq |y_{(\omega,c)}| \leq \mathcal{UB}\}$ where

$$\mathcal{UB} = \frac{\sum_{j=1}^m \overline{(a_j)_{(\omega,c)}} + \sum_{i=1}^n \overline{(b_i)_{(\omega,c)}}}{\alpha} = \frac{3}{2}$$

and

$$\begin{aligned} \mathcal{LB} &= \frac{\sum_{j=1}^m \overline{(a_j)_{(\omega,c)} e^{-\overline{\omega_j} \mathcal{UB}}} + \sum_{i=1}^n \overline{(b_i)_{(\omega,c)} e^{-p\xi \overline{\beta_i} \mathcal{UB}}}}{\overline{\alpha}} \leq \frac{e^{-\frac{1}{2} \cdot \frac{3}{2}} + 0.5e^{-\frac{1}{2} \cdot \frac{3}{2}} + 0.5e^{-\frac{1}{2} \cdot \frac{3}{2}} + e^{-0.9 \cdot \frac{1}{2} \cdot \frac{3}{2}} + e^{-0.9 \cdot \frac{1}{2} \cdot \frac{3}{2}} + e^{-0.9 \cdot \frac{1}{2} \cdot \frac{3}{2}}}{10} \\ &= \frac{e^{-\frac{3}{4}}}{5} + \frac{3e^{-\frac{0.9}{4}}}{10} \approx 0.246 \end{aligned}$$

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