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Chapter 1

A class of dynamic unilateral contact problems with sub-differential friction law

Oanh CHAU, Adrien PETROV, Arnaud HEIBIG

Abstract We study a class of dynamic unilateral contact problems with sub-differential friction law, and thermal effects, for time depending long memory visco-elastic materials, with or without the clamped condition. We describe the mechanical problem, derive its variational formulation, and after specifying the assumptions on the data and operators, we prove an existence and uniqueness of weak solution on displacement and temperature fields.

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1.1 Introduction

Since the dawn of time, situations of contact between deformable bodies abound in everyday life and engineering applications, where the numerous forces acting on the system may lead to the appearance of microcracks, and deteriorate the mechanical equipments. It is then important to understand the complexity of the contact phenomena, in order to guarantee the safety of the mechanism of the system. There-

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fore considerable efforts have been achieved in modeling, mathematical analysis and numerical simulations, within the weak distributional formulation framework, expressed in terms of evolutionary variational inequalities and hemivariational inequalities. The literature dedicated to this field is increasing day by day. The state of the art can be found in the masterpieces [4], [6], [7].

This work is a continuation of the paper in [1], where the authors studied a frictional unilateral contact problem undergoing the thermal expansion and damage, for viscoelastic solid. The friction is modeled by the Coulomb's dry friction law. Here we investigate a thermal contact with friction, for time depending long memory visco-elastic materials, with or without the clamped condition. The contact conditions are unilateral. The friction is modeled by an inequality of sub-differential type.

The paper is organized as follows. In Section 2 we describe the mechanical problem, its corresponding variational formulation, and then we claim the main existence and uniqueness result under specific assumptions, that we prove in Section 3.

1.2 The contact problem

In this section we describe the mechanical problem, list the assumptions on the data and derive the corresponding variational formulation. Then we state an existence and uniqueness result on displacement field and temperature, which we will prove in the next section.

The physical setting is as follows. A visco-elastic body occupies a domain Ω in \mathbb{R}^d ($d = 1, d = 2$ or $d = 3$) with a Lipschitz boundary Γ that is partitioned into three disjoint measurable parts, Γ_1 , Γ_2 and Γ_3 . We denote by ν the unit outward normal on Γ . Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. Here we suppose that $\text{meas}(\Gamma_1) = 0$ or $\text{meas}(\Gamma_1) > 0$, which means that Γ_1 may be an empty set or reduced to a finite set of points. We assume that a volume force of density f_0 acts in $\Omega \times (0, T)$ and that surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$. The body may come in contact with an obstacle, the foundation, over the contact surface Γ_3 . The contact is unilateral involving here both the normal displacement, the normal velocity and the normal component of the Cauchy stress vector. The model of the friction is specified by a general sub-differential condition, where thermal effects may occur in the frictional contact with the basis. We are interested in the dynamic evolution of the body.

Let us recall now some classical notations, see e.g. [4] for further details. We denote by S_d the space of second order symmetric tensors on \mathbb{R}^d , while “ \cdot ” and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and \mathbb{R}^d . Everywhere in the sequel the indices i and j run from 1 to d , summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We also use the following notation:

$$H = \left(L^2(\Omega) \right)^d, \quad \mathcal{H} = \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \},$$

$$H_1 = \{ u \in H \mid \varepsilon(u) \in \mathcal{H} \}, \quad \mathcal{H}_1 = \{ \sigma \in \mathcal{H} \mid \text{Div } \sigma \in H \}.$$

Here $\varepsilon : H_1 \rightarrow \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \rightarrow H$ are the deformation and the divergence operators, respectively, defined by :

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \sigma = (\sigma_{ij,j}).$$

The spaces H , \mathcal{H} , H_1 and \mathcal{H}_1 are real Hilbert spaces endowed with the canonical inner products given by :

$$(u, v)_H = \int_{\Omega} u_i v_i dx, \quad (\sigma, \tau)_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} dx,$$

$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\text{Div } \sigma, \text{Div } \tau)_H.$$

Recall that $\mathcal{D}(\Omega)$ denotes the set of infinitely differentiable real functions with compact support in Ω ; and $W^{m,p}(\Omega)$, $H^m(\Omega) := W^{m,2}(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq +\infty$ for the classical real Sobolev spaces; $L^p(U; X)$ the classical L^p spaces defined on U with values in X .

To continue, the mechanical problem is then formulated as follows.

Problem Q : Find a displacement field $u : (0, T) \times \Omega \rightarrow \mathbb{R}^d$ and a stress field $\sigma : (0, T) \times \Omega \rightarrow S_d$ and a temperature field $\theta : (0, T) \times \Omega \rightarrow \mathbb{R}_+$ such that for a.e. $t \in (0, T)$:

$$\begin{aligned} \sigma(t) &= \mathcal{A}(t)\varepsilon(\dot{u}(t)) + \mathcal{G}(t)\varepsilon(u(t)) \\ &+ \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds + C_e(t, \theta(t)) \quad \text{in } \Omega \end{aligned} \quad (1.1)$$

$$\ddot{u}(t) = \text{Div } \sigma(t) + f_0(t) \quad \text{in } \Omega \quad (1.2)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \quad (1.3)$$

$$\sigma(t)v = f_2(t) \quad \text{on } \Gamma_2 \quad (1.4)$$

$$u_v(t) \leq 0, \quad \sigma_v(t) \leq 0, \quad \sigma_v(t) \cdot \dot{u}_v(t) = 0 \quad \text{on } \Gamma_3, \quad (1.5)$$

$$\varphi_c(t, w_\tau) - \varphi_c(t, \dot{u}_\tau(t)) \geq -\sigma_\tau(t) \cdot (w_\tau - \dot{u}_\tau(t)) \quad \forall w \in V \quad \text{on } \Gamma_3 \quad (1.6)$$

$$\dot{\theta}(t) - \text{div}(\mathcal{K}_c(t, \nabla\theta(t))) = D_e(t, \dot{u}(t)) + q(t) \quad \text{in } \Omega, \quad (1.7)$$

$$-\mathcal{K}_c(t, x, \nabla\theta(t, x))v := \varphi'_{thermal}(t, x, \theta(t, x)) \quad \text{a.e. } x \in \Gamma_3, \quad (1.8)$$

$$\theta(t) = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \quad (1.9)$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega \quad (1.10)$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0 \quad \text{in } \Omega \quad (1.11)$$

Here, (1.1) is the Kelving Voigt's time-dependent long memory thermo-visco-elastic constitutive law of the body, where σ represents the stress tensor; \mathcal{A} denotes the viscosity operator depending on the velocity of infinitesimal deformations $\varepsilon(\dot{u})$, with the notation : for $\tau \in S_d$, $\mathcal{A}(t)\tau = \mathcal{A}(t, \cdot, \tau)$ some function defined on Ω ; here a dot above a quantity represents the derivative of the quantity with respect to the time variable; \mathcal{G} is the elastic operator depending on the linearized strain tensor $\varepsilon(u)$ of infinitesimal deformations, with $\mathcal{G}(t)\tau = \mathcal{G}(t, \cdot, \tau)$ which is defined on Ω . The term $\mathcal{B}(t)\tau = \mathcal{B}(t, \cdot, \tau)$ represents the so called relaxation tensor which is time-depending on the linearized strain tensor and is defined on Ω . Recall that the visco-elastic short memory corresponds to the case $\mathcal{B} \equiv 0$. The last tensor $C_e(t, \theta) := C_e(t, \cdot, \theta)$ denotes the thermal expansion tensor depending on time and on the temperature, defined on Ω . For example,

$$C_e(t, \theta) := -\theta C_{exp}(t) \quad \text{in } \Omega,$$

where

$$C_{exp}(t) := (c_{ij}(t, \cdot))$$

is some time-depending expansion tensor, defined on Ω .

In (1.2) is the dynamic equation of motion where the mass density $\rho \equiv 1$. The equation in (1.3) is the clamped condition. In (1.4) is the traction condition, where $\sigma \mathbf{v}$ represents the Cauchy stress vector.

On the contact surface, the general relations in (1.5) are the so-called Signorini's boundary conditions (see [8]), involving here both the normal displacement u_ν , the normal velocity \dot{u}_ν and the normal component of the Cauchy stress vector σ_ν . Recall that the condition $u_\nu \leq 0$ represents a non penetration of the surface asperities into the obstacle : the inequality $u_\nu < 0$ means that there is no contact of the surface with the obstacle, and the equality $u_\nu = 0$ denotes contact.

The last condition stipulates that the product $\sigma_\nu \dot{u}_\nu$ vanishes almost everywhere on $(0, T) \times \Gamma_3$. It's physical meaning can be seen as a natural consequence of the non penetrability, under some regularity assumption. To show that, let fix $(t, x) \in (0, T) \times \Gamma_3$.

Case 1 : $u_\nu(t, x) < 0$. There is no contact, then $\sigma_\nu(t, x) = 0$ and $\sigma_\nu \dot{u}_\nu = 0$ at (t, x) .

Case 2 : $u_\nu(t, x) = 0$.

Consider three sub-cases : (i) $\dot{u}_\nu(t, x) > 0$; (ii) $\dot{u}_\nu(t, x) = 0$; (iii) $\dot{u}_\nu(t, x) < 0$.

In the case (i), as $\frac{u_\nu(t+h, x) - u_\nu(t, x)}{h} \rightarrow \dot{u}_\nu(t, x)$, $h \rightarrow 0$, $h > 0$; then $u_\nu(t+h, x) > 0$ for $h > 0$ small enough, which contradicts the non penetration.

In the case (ii), we have $\sigma_\nu \dot{u}_\nu = 0$ at (t, x) .

In the case (iii), as in (i) we deduce that $u_\nu(t+h, x) < 0$ for any $h > 0$ small enough, thus there is no contact and $\sigma_\nu(t+h, x) = 0$ for any $h > 0$ small enough, and then the continuity regularity implies that $\sigma_\nu(t, x) = 0$ and $\sigma_\nu \dot{u}_\nu = 0$ at (t, x) .

The friction on the contact surface is modeled by the equation in (1.6), which is a sub-differential boundary condition, in the classical framework of convex analysis. Here, V denotes the space of admissible displacements which will be detailed in the following; σ_τ represents the tangential component of the Cauchy stress vector; and $\varphi_c : (0, T) \times \Gamma_3 \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a given sub-differential friction contact function. Various situations may be modelled by such a condition, see below at the end of this Section. The differential equation (1.7) describes the evolution of the temperature field, where $\mathcal{K}_c(t, \nabla \theta) := \mathcal{K}_c(t, \cdot, \nabla \theta)$ is some nonlinear thermal conductivity function defined on Ω , depending on time and on the temperature gradient $\nabla \theta$. For example, denote by

$$K_c(t, \cdot) := (k_{ij}(t, \cdot))$$

the thermal conductivity tensor defined on Ω , we could consider

$$\mathcal{K}_c(t, \cdot, \nabla \theta) = K_c(t, \cdot) \nabla \theta.$$

In the second member, $D_e(t, \dot{u}(t)) := D_e(t, \cdot, \dot{u}(t))$ represents some deformation-viscosity heat nonlinear function defined on Ω and depending on the displacement velocity, whereas $q(t)$ denotes the density of volume heat sources. For example,

$$D_e(t, \dot{u}(t)) = -C_{exp}(t) : \nabla \dot{u}(t) = -c_{ij}(t, \cdot) \frac{\partial \dot{u}_i}{\partial x_j}(t).$$

The associated temperature boundary condition is given by (1.8) and (1.9), where $\varphi_{thermal}$ are some thermal boundary function defined on $(0, T) \times \Gamma_3 \times \mathbb{R}$. Here

$$\varphi'_{thermal}(t, x, r) := [\varphi_{thermal}(t, x, \cdot)]'(r), \quad \forall (t, x, r) \in (0, T) \times \Gamma_3 \times \mathbb{R}$$

denotes the derivative on the third variable of $\varphi_{thermal}$.

Taking the previous example for \mathcal{K}_c , we have

$$\mathcal{K}_c(t, x, \nabla \theta) v = k_{ij}(t, x) \frac{\partial \theta}{\partial x_j} v_i.$$

Let consider the following standard example

$$\varphi_{thermal}(t, x, r) := \frac{1}{2} k_e(t, x) (r - \theta_R(t, x))^2, \quad \forall (t, x, r) \in (0, T) \times \Gamma_3 \times \mathbb{R}, \quad (1.12)$$

where θ_R is the temperature of the foundation, and k_e is the heat exchange coefficient between the body and the obstacle. We obtain

$$\varphi'_{thermal}(t, x, r) = k_e(t, x) (r - \theta_R(t, x)), \quad (t, x, r) \in (0, T) \times \Gamma_3 \times \mathbb{R}.$$

Finally in (1.10) and (1.11), θ_0, u_0, v_0 represent the initial temperature, displacement and velocity respectively.

One may remark that since φ_c is assumed real-valued, then unilateral contact, defined by indicator functions taking infinite values, is excluded. So the body is in fixed contact with the foundation of the body according to a friction law. This is consistent with the linear heat conduction modeled in (1.7).

We insist that the new feature here is that we may have the absence of the usual claimed condition in the case where $\text{meas}(\Gamma_1) = 0$. However, there is coerciveness with regard to the temperature by (1.8).

To derive the variational formulation of the mechanical problems (1.1)–(1.11) we need additional notations. Thus, let consider the space V of admissible displacements defined by the following closed subspace of H_1

$$V = \{ w \in H_1 \mid w = 0 \text{ on } \Gamma_1 \text{ and } w_\nu \leq 0 \}.$$

On V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (u, v)_H \quad \forall u, v \in V,$$

and let $\|\cdot\|_V$ be the associated norm, i.e.

$$\|v\|_V^2 = \|\varepsilon(v)\|_{\mathcal{H}}^2 + \|v\|_H^2 \quad \forall v \in V.$$

It follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, we have a constant

$C_0 > 0$ depending only on Ω , and I_3 such that

$$\|v\|_{L^2(I_3)} \leq C_0 \|v\|_V \quad \forall v \in V.$$

Consider then the following spaces for the temperature field:

$$E = \{\eta \in H^1(\Omega), \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}; \quad F = L^2(\Omega).$$

The spaces E and F , endowed with their respective canonical inner product, are Hilbert spaces.

Identifying then H and F with their own duals, we obtain two Gelfand evolution triples (see e.g. [9] II/A p. 416):

$$V \subset H \equiv H' \subset V', \quad E \subset F \equiv F' \subset E'$$

where the inclusions are continuous and dense.

Finally, we use the notation $\langle \cdot, \cdot \rangle_{V' \times V}$ and $\langle \cdot, \cdot \rangle_{E' \times E}$ to represent the duality pairing between V' and V , and respectively between E' and E , which means :

$$\langle u, v \rangle_{V' \times V} = \langle u, v \rangle_H, \quad \forall u \in H, \forall v \in V.$$

and

$$\langle \eta, \xi \rangle_{E' \times E} = \langle \eta, \xi \rangle_F, \quad \forall \eta \in F, \forall \xi \in E.$$

In the study of the mechanical problem (1.1)-(1.11), we assume that the viscosity operator $\mathcal{A} : (0, T) \times \Omega \times S_d \rightarrow S_d$, $(t, x, \tau) \mapsto \mathcal{A}(t, x, \tau)$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } \mathcal{A}(\cdot, \cdot, \tau) \text{ is measurable on } (0, T) \times \Omega, \forall \tau \in S_d; \\ \text{(ii) } \mathcal{A}(t, x, \cdot) \text{ is continuous on } S_d \text{ for a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(iii) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(t, x, \tau_1) - \mathcal{A}(t, x, \tau_2)) \cdot (\tau_1 - \tau_2) \geq m_{\mathcal{A}} |\tau_1 - \tau_2|^2, \\ \quad \forall \tau_1, \tau_2 \in S_d, \text{ for a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(iv) there exists } c_0^{\mathcal{A}} \in L^2((0, T) \times \Omega; \mathbb{R}^+), c_1^{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(t, x, \tau)| \leq c_0^{\mathcal{A}}(t, x) + c_1^{\mathcal{A}} |\tau|, \\ \quad \forall \tau \in S_d, \text{ for a.e. } (t, x) \in (0, T) \times \Omega. \end{array} \right. \quad (1.13)$$

In this paper for every $t \in (0, T)$, $\tau \in S_d$ we denote by $\mathcal{A}(t) = \mathcal{A}(t, \cdot, \cdot)$ a functional which is defined on $\Omega \times S_d$ and $\mathcal{A}(t) \tau = \mathcal{A}(t, \cdot, \tau)$ some function defined on Ω .

The elasticity operator $\mathcal{G} : (0, T) \times \Omega \times S_d \rightarrow S_d$ satisfies :

$$\left\{ \begin{array}{l} \text{(i) there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(t, x, \varepsilon_1) - \mathcal{G}(t, x, \varepsilon_2)| \leq L_{\mathcal{G}} |\varepsilon_1 - \varepsilon_2| \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(ii) } \mathcal{G}(\cdot, \cdot, \varepsilon) \text{ is Lebesgue measurable on } (0, T) \times \Omega, \forall \varepsilon \in S_d; \\ \text{(iii) the mapping } \mathcal{G}(\cdot, \cdot, 0) \in \mathcal{H}. \end{array} \right. \quad (1.14)$$

We put again $\mathcal{G}(t)\tau = \mathcal{G}(t, \cdot, \tau)$ some function defined on Ω for every $t \in (0, T)$, $\tau \in S_d$.

The relaxation tensor $\mathcal{B} : (0, T) \times \Omega \times S_d \rightarrow S_d$, $(t, x, \tau) \mapsto (B_{ijkh}(t, x) \tau_{kh})$ satisfies

$$\begin{cases} \text{(i)} & B_{ijkh} \in L^\infty((0, T) \times \Omega); \\ \text{(ii)} & \mathcal{B}(t)\sigma \cdot \tau = \sigma \cdot \mathcal{B}(t)\tau \\ & \forall \sigma, \tau \in S_d, \text{ a.e. } t \in (0, T), \text{ a.e. in } \Omega \end{cases} \quad (1.15)$$

where we denote by $\mathcal{B}(t)\tau = \mathcal{B}(t, \cdot, \tau)$ which is defined on Ω for every $t \in (0, T)$, $\tau \in S_d$.

We suppose the body forces and surface tractions satisfy

$$f_0 \in L^2(0, T; H), \quad f_2 \in L^2(0, T; L^2(\Gamma_2)^d) \quad (1.16)$$

On the contact surface, the following frictional contact function $\psi_c : (0, T) \times V \rightarrow \mathbb{R}$,

$$\psi_c(t, w) := \int_{\Gamma_3} \varphi_c(t, w) da, \quad \forall (t, w) \in (0, T) \times V,$$

verifies

$$\begin{cases} \text{(i)} & t \in (0, T) \mapsto \psi_c(t, w) \text{ is Lebesgue measurable } \forall w \in V; \\ \text{(ii)} & |\psi_c(t, w)| \leq c(t) + d \|w\|_V, \forall w \in V, \text{ a.e. } t \in (0, T); \\ \text{(iii)} & \psi_c(t, \cdot) \text{ is convex on } V \text{ a.e. } t \in (0, T), \end{cases} \quad (1.17)$$

where $d > 0$ is some constant and $c \in L^2(0, T; \mathbb{R}^+)$.

The thermal expansion tensor $C_e : (0, T) \times \Omega \times \mathbb{R} \rightarrow S_d$ verifies

$$\begin{cases} \text{(i)} & C_e(\cdot, \cdot, \vartheta) \text{ is measurable on } (0, T) \times \Omega, \forall \vartheta \in \mathbb{R}; \\ \text{(ii)} & \text{there exists } L_e > 0 \text{ such that} \\ & |C_e(t, x, \vartheta_1) - C_e(t, x, \vartheta_2)| \leq L_e |\vartheta_1 - \vartheta_2| \\ & \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \text{ a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(iii)} & \text{there exists } c_0^{C_e} \in L^\infty((0, T) \times \Omega; \mathbb{R}^+), c_1^{C_e} \geq 0 \text{ such that} \\ & |C_e(t, x, \vartheta)| \leq c_0^{C_e}(t, x) + c_1^{C_e} |\vartheta|, \quad \forall \vartheta \in \mathbb{R}, \text{ for a.e. } (t, x) \in (0, T) \times \Omega. \end{cases} \quad (1.18)$$

Here we recall the notation $C_e(t, \vartheta) = C_e(t, \cdot, \vartheta)$ some function defined on Ω , for all $t \in (0, T)$ and $\vartheta \in \mathbb{R}$.

The nonlinear function $\mathcal{K}_c : (0, T) \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies :

$$\left\{ \begin{array}{l} \text{(i) } \mathcal{K}_c(\cdot, \cdot, \xi) \text{ is measurable on } (0, T) \times \Omega, \forall \xi \in \mathbb{R}^d; \\ \text{(ii) } \mathcal{K}_c(t, x, \cdot) \text{ is continuous on } \mathbb{R}^d, \text{ a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(iii) there exists } c_0^{\mathcal{K}_c} \in L^2((0, T) \times \Omega; \mathbb{R}^+), c_1^{\mathcal{K}_c} \geq 0, \text{ such that} \\ \quad |\mathcal{K}_c(t, x, \xi)| \leq c_0^{\mathcal{K}_c}(t, x) + c_1^{\mathcal{K}_c} |\xi|, \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(iv) there exists } m_{\mathcal{K}_c} > 0 \text{ such that} \\ \quad (\mathcal{K}_c(t, x, \xi_1) - \mathcal{K}_c(t, x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq m_{\mathcal{K}_c} |\xi_1 - \xi_2|^2, \\ \quad \forall \xi_1, \xi_2 \in \mathbb{R}^d, \text{ a.e. } (t, x) \in (0, T) \times \Omega; \\ \text{(v) there exists } n_{\mathcal{K}_c} > 0 \text{ such that } \mathcal{K}_c(t, x, \xi) \cdot \xi \geq n_{\mathcal{K}_c} |\xi|^2, \\ \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. } (t, x) \in (0, T) \times \Omega. \end{array} \right. \quad (1.19)$$

We suppose that the deformation-viscosity heat $D_e : (0, T) \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } D_e(\cdot, \cdot, v) \text{ is measurable on } (0, T) \times \Omega, \forall v \in \mathbb{R}^d; \\ \text{(ii) there exists } L_{D_e} > 0 \text{ such that} \\ \quad |D_e(t, x, v_1) - D_e(t, x, v_2)| \leq L_{D_e} |v_1 - v_2|, \\ \quad \forall v_1, v_2 \in \mathbb{R}^d, \text{ a.e. } (t, x) \in (0, T) \times \Omega. \end{array} \right. \quad (1.20)$$

We assume for the heat sources density, that

$$q \in L^2(0, T; L^2(\Omega)) \quad (1.21)$$

The nonlinear function $\varphi_{thermal} : (0, T) \times \Gamma_c \times \mathbb{R} \rightarrow \mathbb{R}$ verifies :

$$\left\{ \begin{array}{l} \text{(i) } \varphi_{thermal}(\cdot, \cdot, r) \text{ is measurable on } (0, T) \times \Gamma_c, \forall r \in \mathbb{R}; \\ \text{(ii) } \varphi_{thermal}(t, x, \cdot) \text{ is convex derivable on } \mathbb{R} \text{ for a.e. } (t, x) \in (0, T) \times \Gamma_c; \\ \text{(iii) there exists } c_0^\varphi \in L^2((0, T) \times \Gamma_c; \mathbb{R}^+), c_1^\varphi \geq 0, \text{ such that} \\ \quad |\varphi'_{thermal}(t, x, r)| \leq c_0^\varphi(t, x) + c_1^\varphi |r|, \\ \quad \forall r \in \mathbb{R}, \text{ a.e. } (t, x) \in (0, T) \times \Gamma_c. \end{array} \right. \quad (1.22)$$

We notice that these assumptions are verified for the example (1.12).

Finally we assume that the initial data satisfy the conditions

$$u_0 \in H, \quad v_0 \in V, \quad \theta_0 \in E. \quad (1.23)$$

To continue, using Green's formula, we obtain the variational formulation of the mechanical problem Q in abstract form as follows.

Problem QV : Find $u : [0, T] \rightarrow V, \theta : [0, T] \rightarrow E$ satisfying a.e. $t \in (0, T)$:

$$\left\{ \begin{array}{l} \langle \ddot{u}(t) + A(t)\dot{u}(t) + B(t)u(t) + C(t)\theta(t), w - \dot{u}(t) \rangle_{V' \times V} \\ + \left(\int_0^t \mathcal{B}(t-s) \varepsilon(u(s)) ds, \varepsilon(w) - \varepsilon(\dot{u}(t)) \right)_{\mathcal{H}} + \Psi_c(t, w_\tau) - \Psi_c(t, \dot{u}_\tau(t)) \\ \geq \langle f(t), w - \dot{u}(t) \rangle_{V' \times V} \quad \forall w \in V. \end{array} \right. \quad (1.24)$$

$$\left\{ \begin{array}{l} \langle \dot{\theta}(t), \eta \rangle_{E' \times E} + \langle K(t)\theta(t), \eta \rangle_{E' \times E} + \langle \Psi_{thermal}(t)\theta(t), \eta \rangle_{E' \times E} \\ = \langle R(t)\dot{u}(t), \eta \rangle_{E' \times E} + \langle Q(t), \eta \rangle_{E' \times E}, \quad \forall \eta \in E. \end{array} \right. \quad (1.25)$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0. \quad (1.26)$$

Here, the operators and functions $A(t), B(t) : V \rightarrow V'$, $C(t) : E \rightarrow V'$, $f : [0, T] \rightarrow V'$, $K(t) : E \rightarrow E'$, $\Psi_{thermal}(t) : E \rightarrow E'$, $R(t) : V \rightarrow E'$ and $Q : [0, T] \rightarrow E'$ are defined by $\forall v \in V, \forall w \in V, \forall \zeta \in E, \forall \eta \in E$, a.e. $t \in (0, T)$:

$$\langle A(t)v, w \rangle_{V' \times V} = (\mathcal{A}(t) \varepsilon(v), \varepsilon(w))_{\mathcal{H}}; \quad (1.27)$$

$$\langle B(t)v, w \rangle_{V' \times V} = (\mathcal{G}(t) \varepsilon(v), \varepsilon(w))_{\mathcal{H}}; \quad (1.28)$$

$$\langle C(t)\zeta, w \rangle_{V' \times V} = (C_e(t, \zeta), \varepsilon(w))_{\mathcal{H}}; \quad (1.29)$$

$$\langle f(t), w \rangle_{V' \times V} = (f_0(t), w)_H + (f_2(t), w)_{(L^2(\Gamma_2))^d}; \quad (1.30)$$

$$\langle K(t)\zeta, \eta \rangle_{E' \times E} = \int_{\Omega} \mathcal{K}_c(t, \nabla \zeta) \cdot \nabla \eta \, dx; \quad (1.31)$$

$$\langle \Psi_{thermal}(t)\zeta, \eta \rangle_{E' \times E} = \int_{\Gamma_3} \Phi'_{thermal}(t, x, \zeta(x)) \eta(x) \, da(x); \quad (1.32)$$

$$\langle R(t)v, \eta \rangle_{E' \times E} = \int_{\Omega} D_e(t, v) \eta \, dx; \quad (1.33)$$

$$\langle Q(t), \eta \rangle_{E' \times E} = \int_{\Omega} q(t) \eta \, dx. \quad (1.34)$$

Our main existence and uniqueness result is stated as follows, that we prove in the next Section.

Theorem 1. *Assume that (1.13)–(1.23) hold, then there exists an unique solution $\{u, \theta\}$ to the problem QV with the regularity :*

$$\left\{ \begin{array}{l} u \in W^{1,2}(0, T; V) \cap W^{2,2}(0, T; V') \cap C^1(0, T; H) \\ \theta \in L^2(0, T; E) \cap W^{1,2}(0, T; E') \cap C(0, T; F). \end{array} \right. \quad (1.35)$$

Before proving the main theorem, we present here some examples with sub-differential friction laws of the form (1.6), see e.g. the monograph [6] or the Habilitation thesis [2] p. 117.

Example 1. *Contact with Tresca's friction law.* This contact condition can be found in [4, 7]. It is in the form of the following boundary condition:

$$\begin{cases} u_\nu = 0, & |\sigma_\tau| \leq g, \\ |\sigma_\tau| < g \implies \dot{u}_\tau = 0, \\ |\sigma_\tau| = g \implies \dot{u}_\tau = -\lambda \sigma_\tau, & \lambda \geq 0 \end{cases} \quad \text{on } \Gamma_3 \times (0, T). \quad (1.36)$$

Here $g \in L^\infty(\Gamma_3; \mathbb{R}^+)$ represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. The contact is assumed to be bilateral, i.e., there is no loss of contact during the process.

We choose here

$$\varphi_c(v) = g|v|, \forall v \in V.$$

Example 2. *Contact with viscoelastic friction condition.* We consider the problems with the boundary conditions

$$u_\nu = 0, \quad \sigma_\tau = -\mu|\dot{u}_\tau|^{p-1}\dot{u}_\tau \quad \text{on } \Gamma_3 \times (0, T), \quad (1.37)$$

where $\mu \in L^\infty(\Gamma_3; \mathbb{R}^+)$ is the coefficient of friction and $0 < p \leq 1$. Here, the tangential shear is proportional to the power p of the tangential speed, which is the case when the contact surface is lubricated with a thin layer of non-Newtonian fluid.

Here we choose

$$\varphi_c(v) = \frac{\mu}{p+1}|v|^{p+1}, \forall v \in V.$$

1.3 Proof of Theorem 1

The idea is to bring the second order inequality to a first order inequality, using monotone operator, convexity and fixed point arguments, and will be carried out in several steps.

Let us introduce the velocity variable

$$v = \dot{u}.$$

The system in Problem QV is then written for a.e. $t \in (0, T)$:

$$\left\{ \begin{array}{l} u(t) = u_0 + \int_0^t v(s) ds; \\ \langle \dot{v}(t) + A(t)v(t) + B(t)u(t) + C(t)\theta(t), w - v(t) \rangle_{V' \times V} \\ + \left(\int_0^t \mathcal{B}(t-s) \varepsilon(u(s)) ds, \varepsilon(w) - \varepsilon(v(t)) \right)_{\mathcal{H}} + \Psi_c(t, w) - \Psi_c(t, v(t)) \\ \geq \langle f(t), w - v(t) \rangle_{V' \times V} \quad \forall w \in V; \\ \langle \dot{\theta}(t), \eta \rangle_{E' \times E} + \langle K(t)\theta(t), \eta \rangle_{E' \times E} + \langle \Psi_{thermal}(t)\theta(t), \eta \rangle_{E' \times E} \\ = \langle R(t)\dot{u}(t), \eta \rangle_{E' \times E} + \langle Q(t), \eta \rangle_{E' \times E}, \quad \forall \eta \in E; \\ v(0) = v_0, \quad \theta(0) = \theta_0, \end{array} \right.$$

with the regularity

$$\left\{ \begin{array}{l} v \in L^2(0, T; V) \cap W^{1,2}(0, T; V') \cap C(0, T; H) \\ \theta \in L^2(0, T; E) \cap W^{1,2}(0, T; E') \cap C(0, T; F). \end{array} \right.$$

To continue, we assume in the sequel that the conditions (1.13)–(1.17) of the Theorem 1 are satisfied. Let define

$$\mathcal{W} := L^2(0, T; \mathcal{H}).$$

We begin by

Lemma 1. For all $\eta \in \mathcal{W}$, there exists an unique

$$v_\eta \in L^2(0, T; V) \cap W^{1,2}(0, T; V') \cap C(0, T; H)$$

satisfying

$$\left\{ \begin{array}{l} \langle \dot{v}_\eta(t) + A(t)v_\eta(t), w - v_\eta(t) \rangle_{V' \times V} + (\eta(t), \varepsilon(w) - \varepsilon(v_\eta(t)))_{\mathcal{H}} \\ + \Psi_c(t, w) - \Psi_c(t, v_\eta(t)) \geq \langle f(t), w - v_\eta(t) \rangle_{V' \times V}, \\ \quad \forall w \in V, \quad \text{a.e. } t \in (0, T); \\ v_\eta(0) = v_0. \end{array} \right. \quad (1.38)$$

Moreover, $\exists c > 0$ such that $\forall \eta_1, \eta_2 \in \mathcal{W}$:

$$\|v_{\eta_2}(t) - v_{\eta_1}(t)\|_H^2 + \int_0^t \|v_{\eta_1} - v_{\eta_2}\|_V^2 \leq c \int_0^t \|\eta_1 - \eta_2\|_{\mathcal{H}}^2, \quad \forall t \in [0, T]. \quad (1.39)$$

Proof. Let $\eta \in \mathcal{W}$. Using [9] II/B p. 893, we deduce the existence and uniqueness of v_η .

Now let $\eta_1, \eta_2 \in \mathcal{W}$. In (1.38) we take $(\eta = \eta_1, w = v_{\eta_2}(t))$, then $(\eta = \eta_2, w = v_{\eta_1}(t))$. Adding the two inequalities, we deduce that for a.e. $t \in (0, T)$:

$$\begin{aligned} & \langle \dot{v}_{\eta_2}(t) - \dot{v}_{\eta_1}(t), v_{\eta_2}(t) - v_{\eta_1}(t) \rangle_{V' \times V} + \langle A(t)v_{\eta_2}(t) - A(t)v_{\eta_1}(t), v_{\eta_2}(t) - v_{\eta_1}(t) \rangle_{V' \times V} \\ & \leq -(\eta_2(t) - \eta_1(t), \varepsilon(v_{\eta_2}(t)) - \varepsilon(v_{\eta_1}(t)))_{\mathcal{H}}. \end{aligned}$$

Then integrating over $(0, t)$, from (1.13)(iii) and from the initial condition on the velocity, we obtain:

$$\begin{aligned} & \forall t \in [0, T], \quad \|v_{\eta_2}(t) - v_{\eta_1}(t)\|_H^2 + m_A \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_V^2 ds \\ & \leq - \int_0^t (\eta_2(s) - \eta_1(s), \varepsilon(v_{\eta_2}(s)) - \varepsilon(v_{\eta_1}(s)))_{\mathcal{H}} ds + m_A \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_H^2 ds. \end{aligned}$$

We conclude that $\exists c > 0$ such that $\forall \eta_1, \eta_2 \in \mathcal{W}, \forall t \in [0, T]$:

$$\begin{aligned} & \|v_{\eta_2}(t) - v_{\eta_1}(t)\|_H^2 + \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V^2 ds \\ & \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 ds + c \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_H^2 ds. \end{aligned} \quad (1.40)$$

Now let fix $\tau \in [0, T]$. We have $\forall t \in [0, \tau]$:

$$\|v_{\eta_2}(t) - v_{\eta_1}(t)\|_H^2 \leq c \int_0^\tau \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 ds + c \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_H^2 ds.$$

Using then Gronwall's inequality, we obtain $\forall \tau \in [0, T]$:

$$\|v_{\eta_2}(\tau) - v_{\eta_1}(\tau)\|_H^2 \leq \left(c \int_0^\tau \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 ds \right) e^{cT}.$$

Finally, integrating the last inequality and reporting the result in (1.40), we get (1.39).

Here and below, we denote by $c > 0$ a generic constant, which value may change from lines to lines.

Lemma 2. For all $\eta \in \mathcal{W}$, there exists an unique

$$\theta_\eta \in L^2(0, T; E) \cap W^{1,2}(0, T; E') \cap C(0, T; F)$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t) + K(t)\theta_\eta(t) + \Psi_{thermal}(t)\theta_\eta(t) \\ \quad = R(t)v_\eta(t) + Q(t), \quad \forall \eta \in E, \text{ a.e. } t \in (0, T); \\ \theta_\eta(0) = \theta_0. \end{cases} \quad (1.41)$$

Moreover, $\exists c > 0$ such that $\forall \eta_1, \eta_2 \in \mathcal{W}$:

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 \leq c \int_0^t \|v_{\eta_1} - v_{\eta_2}\|_V^2, \quad \forall t \in [0, T]. \quad (1.42)$$

Proof. The existence and uniqueness result verifying (1.41) follows from standard result on first order evolution equation (see e.g. [5]). Indeed we verify that from the expression of the operator R , we have

$$v_\eta \in L^2(0, T; V) \implies Rv_\eta \in L^2(0, T; E'),$$

as $Q \in L^2(0, T; E')$ then $Rv_\eta + Q \in L^2(0, T; E')$.

Using the assumptions (1.19) and (1.22), the operator

$$K(t) + \Psi_{thermal}(t) : E \longrightarrow E'$$

for a.e. $t \in (0, T)$ is strongly monotone.

Now for $\eta_1, \eta_2 \in \mathscr{W}$, we have for a.e. $t \in (0, T)$:

$$\begin{aligned} & \langle \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} + \langle K(t) \theta_{\eta_1}(t) - K(t) \theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} \\ & \leq \langle R(t) v_{\eta_1}(t) - R(t) v_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E}. \end{aligned}$$

Then integrating the last property over $(0, t)$, using the strong monotonicity of $K(t)$ and the Lipschitz continuity of $R(t) : V \longrightarrow E'$, we deduce (1.42).

Proof of Theorem 1.

We have now all the ingredients to prove the Theorem 1.

Consider the operator $\Lambda : \mathscr{W} \rightarrow \mathscr{W}$ defined by for all $\eta \in \mathscr{W}$:

$$\Lambda \eta(t) = \mathcal{G}(\varepsilon(u_\eta(t))) + \int_0^t B(t-s) \varepsilon(u_\eta(s)) ds + C_e(t, \theta_\eta(t)), \quad \forall t \in [0, T],$$

where

$$u_\eta(t) = u_0 + \int_0^t v_\eta(s) ds, \quad \forall t \in [0, T]; \quad u_\eta \in W^{1,2}(0, T; V) \cap W^{2,2}(0, T; V') \cap C^1(0, T; H).$$

Then from (1.14), (1.15), and Lemma 2, we deduce that for all $\eta_1, \eta_2 \in \mathscr{W}$, for all $t \in [0, T]$:

$$\begin{aligned} & \|\Lambda \eta_1(t) - \Lambda \eta_2(t)\|_{\mathscr{H}}^2 \leq c \|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 + c \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V^2 ds \\ & \leq c \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V^2 ds. \end{aligned} \tag{1.43}$$

Now using (1.43), after some algebraic manipulations, we have for any $\beta > 0$:

$$\int_0^T e^{-\beta\tau} \|\Lambda \eta_1(\tau) - \Lambda \eta_2(\tau)\|_{\mathscr{H}}^2 d\tau \leq \frac{c}{\beta} \int_0^T e^{-\beta\tau} \|\eta_1(\tau) - \eta_2(\tau)\|_{\mathscr{H}}^2 d\tau.$$

We conclude from the last inequality by contracting principle that the operator Λ has a unique fixed point $\eta^* \in \mathscr{W}$. We verify then that the functions

$$u(t) := u_0 + \int_0^t v_{\eta^*}, \quad \forall t \in [0, T], \quad \theta := \theta_{\eta^*}$$

are solutions to problem QV with the regularity (1.35), the uniqueness follows from the uniqueness in Lemma 1 and Lemma 2.

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