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SHARP ESTIMATES OF THE SOLUTIONS TO BÉZOUT'S POLYNOMIAL EQUATION AND A CORONA THEOREM

EMMANUEL FRICAIN, ANDREAS HARTMANN, WILLIAM T. ROSS,
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ABSTRACT. In this paper, we obtain estimates for the solutions to the classical Bézout equation that are analogous to Carleson's solution to the corona theorem for the bounded analytic functions on the open unit disk. As an application, we extend some results of Luo and obtain a corona theorem for the multipliers of a class of de Branges–Rovnyak spaces.

1. INTRODUCTION

A well-known theorem of Étienne Bézout (1730-1783) says that if A, B belong to $\mathbb{C}[z]$ and have no common roots, then there are $R, S \in \mathbb{C}[z]$ with

$$(1.1) \quad \deg R \leq \deg B - 1 \quad \text{and} \quad \deg S \leq \deg A - 1,$$

such that

$$(1.2) \quad A(z)R(z) + B(z)S(z) = 1 \quad \text{for all } z \in \mathbb{C}.$$

Moreover, if A and B are not both constant polynomials, then R and S are uniquely determined by the degree condition in (1.1) and will be called the *minimal solutions* of (1.2).

The proof of Bézout's theorem often presented to algebra students uses a version of the classical Euclidean algorithm for polynomials (noting of course that the lack of common roots for A and B implies that the greatest common divisor of A and B is the constant polynomial one). A result of Sylvester (see Theorem 2.1 below) gives an

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explicit formula for the minimal solutions R and S in terms of a certain matrix equation involving the *Sylvester resultant*.

Inspired by Carleson's corona theorem (see the discussion below), one central goal of this paper is to estimate the coefficients of the polynomials S and R . To understand what we mean by this, consider the following example. Throughout this paper, we will use

$$(1.3) \quad \|p\| := \max_{0 \leq j \leq n} |p_j|$$

for the norm of a polynomial $p(z) = \sum_{j=0}^n p_j z^j$.

Example 1.4. If $n \in \mathbb{N}$ and $0 < \delta < 1$, set $A(z) = z^n$ and $B(z) = z - \delta$. One can work out the corresponding minimal polynomials R and S that satisfy (1.1) and (1.2) to be

$$R(z) = \frac{1}{\delta^n} \quad \text{and} \quad S(z) = -\frac{1}{\delta^n} \sum_{j=0}^{n-1} \delta^{n-1-j} z^j.$$

Then $\|A\| = \|B\| = 1$, $|B(0)| = \delta$, and

$$\|R\| = \|S\| = \frac{1}{\delta^n}.$$

Notice how n is the maximal order of the zeros of A , which in this case is a single zero at the origin of order n , while the condition $|B(0)| = \delta$ can be interpreted as a lower bound for B at the zero of A .

Our main theorem proves that the type of phenomenon presented in Example 1.4 always occurs. To state this result, let $\mathbb{C}[z]$ denote the vector space of polynomials in the complex variable z with coefficients in \mathbb{C} , and for $N \in \mathbb{N}$, let \mathcal{P}_N be the finite dimensional subspace $\mathcal{P}_N = \{p \in \mathbb{C}[z] : \deg p \leq N\}$.

Theorem 1.5. *Let $A \in \mathcal{P}_N$ be of the form*

$$A(z) = \prod_{j=1}^n (z - \alpha_j)^{m_j},$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ are distinct and $N = \sum_{j=1}^n m_j \geq 1$. Fix $K \in \mathbb{N}$. There is a $C > 0$, depending only on A and K , such that if $B \in \mathcal{P}_K$ satisfies

- (a) $\|B\| \leq 1$, and
- (b) $|B(\alpha_j)| \geq \delta > 0$ for all $1 \leq j \leq n$,

then the minimal solutions R and S of the Bézout equation (1.2) satisfy

$$(1.6) \quad (\|R\|^2 + \|S\|^2)^{\frac{1}{2}} \leq \frac{C}{\delta^{\max m_j}}.$$

Theorem 4.2 below extends this result to multiple polynomials A and B_1, \dots, B_L . Moreover, as shown in Example 1.4, the estimate in (1.6) is sharp in that the exponent $\max\{m_j : 1 \leq j \leq n\}$ on δ can not be lowered. It seems rather surprising to us that such estimates are lacking in the literature.

The statement of Theorem 1.5 is asymmetric in the polynomials A and B : A is fixed (and determines the constant C), while B is chosen freely in the finite dimensional space \mathcal{P}_K . As we will see in Corollary 2.8 below, the formula for the solutions of the Bézout equation involving the Sylvester determinant, given in Section 2, can be used to obtain a more symmetric result for the coefficient estimates. However, the estimate from the symmetric version can be much less precise in terms of the exponent on δ than the asymmetric one presented in Theorem 1.5.

Theorem 1.5 is connected to a related problem, known as the corona problem [6] for H^∞ , the space of bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the standard norm $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$. In a way, the corona theorem can be viewed as a Bézout theorem for H^∞ . In 1962, Carleson [2] answered a conjecture proposed by Kakutani [15] in 1941 and proved that if $(f_j)_{j=1}^N$ is a finite sequence in H^∞ which satisfies

$$0 < \delta \leq \left(\sum_{j=1}^N |f_j(z)|^2 \right)^{\frac{1}{2}} \leq 1 \quad \text{for all } z \in \mathbb{D},$$

then there is a finite sequence $(g_j)_{j=1}^N$ in H^∞ such that

$$\sum_{j=1}^N f_j(z)g_j(z) = 1 \quad \text{for all } z \in \mathbb{D}$$

and

$$\left(\sum_{j=1}^N |g_j(z)|^2 \right)^{\frac{1}{2}} \leq C \quad \text{for all } z \in \mathbb{D}.$$

In the above, the constant C initially depends only on δ and N . Further investigations have shown that $C = C(\delta)$ can be chosen to only

depend on δ . Various estimates of $C(\delta)$ were explored in [14, 25, 28]. The latter paper of Uchiyama contains the sharpest known estimate

$$(1.7) \quad C(\delta) \leq C \frac{1}{\delta^2} \log \frac{1}{\delta},$$

where C is an absolute constant. On the other hand, Treil [26] proved that one can do no better than

$$C \frac{1}{\delta^2} \log \log \frac{1}{\delta}.$$

Carleson's corona theorem has been generalized in many directions. For example, Tolokonnikov [25] and Rosenblum [23] independently proved a version of Carleson's theorem for an *infinite* sequence $(f_j)_{j=1}^\infty$ of H^∞ functions. There are also many generalizations of the corona theorem to matrix and operator-valued functions that are important to control theory and similarity problems. For example, Nikolski's book [21, Ch.9, Sec. 2] relates the operator-valued corona problem to one-sided invertibility of Toeplitz operators.

Another direction of inquiry stems from the fact that H^∞ is the multiplier algebra for the Hardy–Hilbert space H^2 (see Section 5) which inspires one to prove corona type theorems for sequences in the multiplier algebra of other reproducing kernel Hilbert space of analytic functions [19, 20, 27]. A particular class of such reproducing kernel Hilbert spaces, which has received quite a lot of attention over the past several decades, arises from the de Branges–Rovnyak spaces $\mathcal{H}(b)$ [24, 10, 11]. These spaces are an important class of linear submanifolds of the classical Hardy space H^2 that appear when modeling certain Hilbert space contractions. In Theorem 6.5 we use our Bézout estimates to obtain a corona theorem for the multiplier algebras of de Branges–Rovnyak spaces $\mathcal{H}(b)$ in the case where b is any rational function (but not a finite Blaschke product) in the closed unit ball of H^∞ . Our results are related to those from [19, 20].

2. THE CLASSICAL BÉZOUT THEOREM

2.1. Bézout's formula and the Sylvester matrix. The next theorem gives a useful formula for the solution to the Bézout problem involving the Sylvester matrix (see [16, p. 200] or [4, p. 77]).

Theorem 2.1. *If*

$$A(z) = \sum_{j=0}^N A_j z^j \quad \text{and} \quad B(z) = \sum_{j=0}^K B_j z^j$$

are not both constant polynomials, and have no common zeros, then there are unique polynomials

$$R(z) = \sum_{j=0}^{K-1} R_j z^j \quad \text{and} \quad S(z) = \sum_{j=0}^{N-1} S_j z^j$$

such that

$$(2.2) \quad RA + SB \equiv 1.$$

The coefficients R_0, \dots, R_{K-1} and S_0, \dots, S_{N-1} are solutions to the system

$$(2.3) \quad \mathfrak{S} \mathbf{x} = \mathbf{e},$$

where \mathfrak{S} is the $(N + K) \times (K + N)$ matrix

$$\mathfrak{S} := \begin{bmatrix} A_0 & 0 & \cdots & 0 & B_0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & B_1 & B_0 & \cdots & 0 \\ A_2 & A_1 & \ddots & 0 & B_2 & B_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & A_0 & \vdots & \vdots & \ddots & B_0 \\ \vdots & \vdots & \vdots & A_1 & \vdots & \vdots & \vdots & B_1 \\ A_N & \vdots & \ddots & \vdots & B_K & \vdots & \vdots & \vdots \\ 0 & A_N & \cdots & \vdots & 0 & B_K & \cdots & \vdots \\ 0 & 0 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & A_N & 0 & 0 & 0 & B_K \end{bmatrix},$$

$\underbrace{\hspace{15em}}_K \qquad \underbrace{\hspace{15em}}_N$

$$\mathbf{x} = [R_0, R_1, \dots, R_{K-1}, S_0, S_1, \dots, S_{N-1}]^T,$$

and

$$\mathbf{e} = [1, 0, 0, \dots, 0]^T.$$

Moreover, if $\alpha_1, \dots, \alpha_N$ are the roots of A and β_1, \dots, β_K are those for B , counted with multiplicities, then

$$(2.4) \quad \det \mathfrak{S} = A_N^K B_K^N \prod_{i=1}^N \prod_{j=1}^K (\alpha_i - \beta_j).$$

In particular,

$$(2.5) \quad |\det \mathfrak{S}| = |B_K|^N |A(\beta_1) \cdots A(\beta_K)| = |A_N|^K |B(\alpha_1) \cdots B(\alpha_N)|.$$

Example 2.6. Let $\delta > 0$. If

$$A(z) = z^2 = 0 + 0z + 1z^2 \quad \text{and} \quad B(z) = z - \delta,$$

then

$$\mathfrak{S} = \begin{bmatrix} 0 & -\delta & 0 \\ 0 & 1 & -\delta \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} R_0 \\ S_0 \\ S_1 \end{bmatrix} = \mathfrak{S}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\delta^2} \\ -\frac{1}{\delta} \\ -\frac{1}{\delta^2} \end{bmatrix}.$$

Thus,

$$R(z) = \frac{1}{\delta^2} \quad \text{and} \quad S(z) = -\frac{1}{\delta} - \frac{1}{\delta^2}z$$

which is the $n = 2$ case from Example 1.4.

Remark 2.7.

- (a) Requiring that A and B are not both constant polynomials ensures the uniqueness of the solutions R and S to (2.2). We will tacitly make this nontriviality assumption in the sequel.
- (b) From the identity $1 = RA + SB = RA + (S/c)(cB)$, it follows that if we replace B by cB ($c \neq 0$), then S gets replaced by S/c , while R does not change. We will use this rescaling several times.
- (c) As a consequence of the formulas (2.3) and (2.4), along with Cramer's rule, the coefficients of the polynomials R and S are continuous functions of the roots of A, B , as long as A and B do not have any common roots.

Theorem 2.1 yields the following estimate of the coefficients of the polynomials R, S stated in a manner similar to Carleson's corona theorem presented in the introduction.

Corollary 2.8. *With the notation above, let $\|A\|, \|B\| \leq 1$, and*

$$\min\{|A(\beta_j)|, |B(\alpha_i)|, 1 \leq i \leq N, 1 \leq j \leq K\} = \delta > 0.$$

Then

$$(2.9) \quad (\|R\|^2 + \|S\|^2)^{\frac{1}{2}} \leq \frac{\sqrt{2}(N+K-1)!}{\min\{|A_N|^K \delta^N, |B_K|^N \delta^K\}}.$$

Proof. Apply Cramer's rule to the system (2.3). For each coefficient of R and S , the determinant in the numerator of the formula is a homogeneous function, with $(N+K-1)!$ terms, in the coefficients of A and B . By assumption, the latter are all smaller in modulus than 1. The denominator $\det \mathfrak{S}$ can be estimated using (2.5). \square

Compared to Theorem 1.5, the estimate given in (2.9) may be rather rough as seen in the following example.

Example 2.10. Fix $0 < \eta < 1$ and define

$$A(z) = z(z-1) \quad \text{and} \quad B(z) = (z-\eta)(z-1+\eta).$$

Then

$$\delta = \min\{|A(\eta)|, |A(1-\eta)|, |B(0)|, |B(1)|\} = \eta(1-\eta).$$

A computation using the identity $B(z) = A(z) + \eta(1-\eta)$ leads to

$$R(z) = -\frac{1}{\delta} \quad \text{and} \quad S(z) = \frac{1}{\delta}.$$

Hence $(\|R\|^2 + \|S\|^2)^{\frac{1}{2}} = \sqrt{2}\delta^{-1}$, as expected from Theorem 1.5, while the right hand side of (2.9) yields $(\|R\|^2 + \|S\|^2)^{\frac{1}{2}} \leq 6\sqrt{2}\delta^{-2}$.

2.2. Bézout formula and interpolation. Important to our proof of Theorem 1.5 is the following alternate way of finding the solutions R and S to (1.2) via interpolation. In standard Lagrange interpolation, one is given distinct complex numbers x_1, \dots, x_t (nodes) and complex numbers y_1, \dots, y_t (targets) and asked to produce a $p \in \mathcal{P}_{t-1}$ such that $p(x_j) = y_j$ for all $1 \leq j \leq t$. This unique polynomial $p \in \mathcal{P}_{t-1}$ is given by the formula

$$(2.11) \quad p(x) = y_1\delta_1(x) + y_2\delta_2(x) + \dots + y_t\delta_t(x),$$

where

$$\delta_j(x) = \frac{\prod_{i=1; i \neq j}^t (x - x_i)}{\prod_{i=1; i \neq j}^t (x_j - x_i)}.$$

Hermite interpolation extends Lagrange interpolation to include specifying derivatives.

Lemma 2.12 (Hermite interpolation). *Let $x_1, \dots, x_t \in \mathbb{C}$ be distinct, $\ell_1, \ell_2, \dots, \ell_t \in \mathbb{N}$, and for each $1 \leq j \leq t$, let $y_j^0, \dots, y_j^{\ell_j-1} \in \mathbb{C}$.*

(a) *If $L = \ell_1 + \dots + \ell_t$, then there is a unique $p \in \mathcal{P}_{L-1}$, such that*

$$p^{(k)}(x_j) = y_j^k \quad \text{for all } 0 \leq k \leq \ell_j - 1 \text{ and } 1 \leq j \leq t.$$

(b) *If x_j and ℓ_j are fixed, one can write*

$$p = p_1 + \dots + p_t,$$

where for each $1 \leq j \leq t$, the polynomial p_j and its coefficients depend linearly on $(y_j^k)_{k=0}^{\ell_j-1} \in \mathbb{C}^{\ell_j}$. In fact,

$$p_j^{(k)}(x_j) = y_j^k \quad \text{for all } 0 \leq k \leq \ell_j - 1,$$

while, if $i \neq j$, then

$$p_j^{(k)}(x_i) = 0 \quad \text{for all } 0 \leq k \leq \ell_i - 1.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_I$ denote the *distinct* roots of A with corresponding multiplicities $\mu_1, \mu_2, \dots, \mu_I$ and $\beta_1, \beta_2, \dots, \beta_J$ denote the *distinct* roots of B with corresponding multiplicities $\nu_1, \nu_2, \dots, \nu_J$. Below, the roots α_j and β_j are no longer counted with multiplicities (different from the notation from Theorem 2.1). Also recall that

$$N = \deg A = \sum_{i=1}^I \mu_i \quad \text{and} \quad K = \deg B = \sum_{j=1}^J \nu_j.$$

Corollary 2.13. *With the notation above, define:*

– $R \in \mathcal{P}_{K-1}$ with the property that, for each $1 \leq j \leq J$,

$$(2.14) \quad R(\beta_j)A(\beta_j) = 1, \quad (RA)^{(k)}(\beta_j) = 0 \quad \text{for } 1 \leq k \leq \nu_j - 1;$$

– $S \in \mathcal{P}_{N-1}$ with the property that, for each $1 \leq i \leq I$,

$$(2.15) \quad S(\alpha_i)B(\alpha_i) = 1, \quad (SB)^{(\ell)}(\alpha_i) = 0 \quad \text{for } 1 \leq \ell \leq \mu_i - 1.$$

Then,

(a) R, S are the minimal solutions of Bézout's equation (2.2).

(b) One can decompose R as

$$R = \sum_{j=1}^J R_j,$$

where the coefficients of the polynomials R_j are rational functions of $A(\beta_j), \dots, A^{(\nu_j-1)}(\beta_j)$, with denominator $(A(\beta_j))^{\nu_j} \neq 0$.

(c) Similarly, one can decompose S as

$$S = \sum_{i=1}^I S_i,$$

where the coefficients of the polynomials S_i are rational functions of $B(\alpha_i), \dots, B^{(\mu_i-1)}(\alpha_i)$, with denominator $(B(\alpha_i))^{\mu_i} \neq 0$.

Proof. (a) The values of $RA + SB$, and its derivatives up to $\mu_j - 1$ at α_j and up to $\nu_j - 1$ at β_j , coincide with those of the constant function 1. From Lemma 2.12 it follows that (2.2) holds.

(b) By Lemma 2.12, one can write $R = \sum_{j=1}^J R_j$, where the coefficients of the polynomials R_j are linear functions (with coefficients depending on β_j) of the prescribed values at β_j of R and its first $\nu_j - 1$ derivatives. Using Leibnitz's formula and induction on k , the definition of R implies that $R^{(k)}(\beta_j)$ is a rational function of

$$A(\beta_j), \dots, A^{(k)}(\beta_j),$$

with denominator $(A(\beta_j))^{k+1}$. This last value is different from zero, since we are always assuming that A and B have no common zeros.

A similar argument yields (c). \square

3. PROOF OF THEOREM 1.5

The proof requires some preliminary set up. For $p(z) = \sum_{j=0}^K p_j z^j \in \mathcal{P}_K$, we defined the norm

$$\|p\| = \max_{0 \leq j \leq K} |p_j|$$

in (1.3) and used it in the statement of Theorem 1.5. We will also need these two norms on \mathcal{P}_K :

$$(3.1) \quad \|p\|' := \max_{|z|=1} |p(z)| \quad \text{and} \quad \|p\|'' := \max_{|z|=3} |p(z)|.$$

Since \mathcal{P}_K is a finite dimensional vector space, all norms on \mathcal{P}_K are equivalent. Hence there is a $D > 0$, depending on K , such that

$$(3.2) \quad \frac{1}{\sqrt{D}} \|p\| \leq \|p\|', \|p\|'' \leq \sqrt{D} \|p\| \quad \text{for all } p \in \mathcal{P}_K.$$

As a consequence, if $\|T\|$, $\|T\|'$, $\|T\|''$ denote the corresponding operator norms of a linear transformation $T : \mathcal{P}_K \rightarrow \mathcal{P}_K$, where \mathcal{P}_K is endowed with the respective norms $\|p\|$, $\|p\|'$, and $\|p\|''$, then

$$(3.3) \quad \|T\| \leq D\|T\|' \quad \text{and} \quad \|T\| \leq D\|T\|''.$$

The statement and proof of Theorem 1.5 contains constants that will depend on $A \in \mathcal{P}_N$. To define these constants, we proceed as follows. Recall that $\alpha_1, \dots, \alpha_n$ are the zeros of A with corresponding multiplicities m_1, \dots, m_n . Fix positive constants ρ and c_1, c_2, \dots, c_n such that

$$(3.4) \quad |z - \alpha_j| \leq 2\rho \implies |A(z)| \geq c_j |z - \alpha_j|^{m_j}.$$

If $\mathfrak{B}_j = \{z : |z - \alpha_j| \leq \rho\}$, fix η such that

$$(3.5) \quad 0 < \eta \leq \inf \left\{ |A(z)| : z \notin \bigcup_{j=1}^n \mathfrak{B}_j \right\}.$$

Since A is a monic polynomial of degree N , we can choose $M > 0$ such that

$$(3.6) \quad |w| \geq M \implies |A(w)| \geq \frac{1}{2} |w|^N.$$

We can also assume that $0 < \rho, \eta, c_j < 1$ for all $1 \leq j \leq n$ and that $M > 1$. From (3.6) it follows that $M \geq |\alpha_i|$ for all $1 \leq i \leq n$.

Since Theorem 1.5 needs to hold for all $B \in \mathcal{P}_K$, we assume from now on that $\deg B = k \leq K$. By Remark 2.7(c), we may suppose that B has simple roots β_1, \dots, β_k and extend the estimate (1.6) by continuity if B has roots of higher multiplicity. Thus, we use the notation

$$(3.7) \quad B(z) = B_0 + \dots + B_k z^k = B_k (z - \beta_1) \cdots (z - \beta_k), \quad \beta_i \neq \beta_j.$$

To summarize, we have fixed positive numbers D (depending only on K) and ρ, η, M, c_j (depending only on A). This notation will be used below without further comment. Our proof of Theorem 1.5 begins with a few preliminary lemmas.

Lemma 3.8. *Suppose $p \in \mathcal{P}_K$ and $p(u) = 0$. If $q(z) = \frac{p(z)}{z - u}$, then*

$$\|q\| \leq D\|p\|.$$

Proof. For at least one of the norms $\|\cdot\|'$ or $\|\cdot\|''$ from (3.1), depending on the position of u with respect to the circle $|z| = 2$, multiplication by $(z - u)^{-1}$ on the subspace $\{p \in \mathcal{P}_N : p(u) = 0\}$ is a contraction. The conclusion is now a consequence of (3.3). \square

The next result follows from Theorem 2.1 and Cramer's rule.

Lemma 3.9. *Suppose $B_k = 1$.*

- (a) *The coefficients of R are quotients $\Delta_i / \det \mathfrak{S}$, where Δ_i is obtained from $\det \mathfrak{S}$ by replacing one of the first k columns with a column consisting of 0s, except a 1 in the first position.*
- (b) $|\det \mathfrak{S}| = |A(\beta_1) \cdots A(\beta_k)|$.
- (c) *For each $1 \leq i \leq k$, Δ_i is a polynomial, with coefficients depending on A , of degree at most N in the coefficients of B .*
- (d) *Alternatively, using Viète's formulas which relate polynomial coefficients to sums of products of its roots, Δ_i is a polynomial, again with coefficients depending on A , of separate degree at most N in each of the roots β_1, \dots, β_k .*

The proof of Theorem 1.5 will depend on the position of the roots β_1, \dots, β_k of B . We begin with a simple case.

Lemma 3.10. *There is a $C > 0$, depending only on A and K , such that if $|\beta_i| \geq M$ for all $1 \leq i \leq k$, then $\|R\| \leq C$.*

Proof. Using Remark 2.7(b), we can assume that $B_k = 1$.

By Lemma 3.9, the coefficients of R are quotients $\Delta_i / \det \mathfrak{S}$, where Δ_i is a polynomial, again with coefficients depending on A and K , of separate degree at most N in each of the roots β_1, \dots, β_k . On the other hand, the definition of M from (3.6) implies that

$$|\det \mathfrak{S}| = |A(\beta_1) \cdots A(\beta_k)| \geq \frac{1}{2^K} |\beta_1|^N \cdots |\beta_k|^N.$$

From here it follows that $|\Delta_i| / |\det \mathfrak{S}| \leq C$ for some $C > 0$ depending only on A and K . \square

The next lemma is central to the proof of Theorem 1.5.

Lemma 3.11. *Let $B \in \mathcal{P}_K$ with $\|B\| \leq 1$, $\deg B = k \leq K$, and B has k distinct zeros $\beta_1, \beta_2, \dots, \beta_k$. If $|\beta_i - \beta_j| \leq \rho$ for all $1 \leq i, j \leq k$, then there is a $C > 0$, depending only on A and K , such that*

$$\|R\| \leq \frac{C}{\delta^{\max m_j}}.$$

Proof. If $|\beta_i| \geq M$ for all $1 \leq i \leq k$, apply Lemma 3.10 to obtain the conclusion.

Otherwise, there is some $1 \leq j \leq k$ with $|\beta_j| < M$, and then the assumption of the lemma implies that $|\beta_i| \leq M + \rho$ for all $1 \leq i \leq k$. Viète's formulas ensure there is some $M_1 > 0$, depending only on A and K , such that $|B_i/B_k| \leq M_1$ for all $1 \leq i \leq k$. Recall from (3.7) that $B(z) = B_0 + B_1z + \cdots + B_kz^k$ and define

$$\tilde{B} := \frac{B}{B_k}.$$

Observe that the leading coefficient of \tilde{B} is 1, while the other coefficients of \tilde{B} have modulus at most M_1 . So condition (a) in the statement of Theorem 1.5 is replaced by $\|\tilde{B}\| \leq M_1$, while (b) is satisfied by \tilde{B} , since the condition $\|B\| \leq 1$ implies that $|B_k| \leq 1$. On the other hand, we know from Remark 2.7(b) that R does not change. Therefore, in the rest of the proof of the lemma, we will assume that $B_k = 1$.

We will consider two cases:

- a) Suppose that $\beta_i \notin \bigcup_j \mathfrak{B}_j$ for all $1 \leq i \leq k$. As stated in Lemma 3.9, the coefficients of R are obtained as quotients $\Delta_i / \det \mathfrak{S}$. Since Δ_i is a polynomial of degree at most N , with coefficients depending on A , in the coefficients of B , which are all in modulus at most M_1 , there is some constant $C > 0$ such that $|\Delta_i| \leq CM_1^N$. On the other hand, (3.5) implies that

$$|\det \mathfrak{S}| = \prod_{i=1}^k |A(\beta_i)| \geq \eta^k \geq \eta^K.$$

Therefore,

$$\|R\| \leq \frac{CM_1^N}{\eta^K},$$

with no dependence on δ .

- b) If a) is not true, then there exists some j such that all β_i satisfy $|\beta_i - \alpha_j| \leq 2\rho$. By condition (3.4) above, and our earlier assumption that $0 < c_j < 1$ for all $1 \leq j \leq n$, we conclude that

$$|\det \mathfrak{S}| = |A(\beta_1) \cdots A(\beta_k)| \geq c_j^k \prod_{i=1}^k |\beta_i - \alpha_j|^{m_j}$$

$$= c_j^k |B(\alpha_j)|^{m_j} \geq c_j^k \delta^{m_j} \geq c_j^K \delta^{m_j}.$$

For the numerators Δ_i , we use the same estimate as in case a) to obtain

$$\|R\| \leq \frac{CM_1^N}{c_j^K \delta^{m_j}} \quad \square$$

We are finally ready for the proof of Theorem 1.5. The coefficients of the polynomials R and S are estimated separately.

Estimating S . From Corollary 2.13, one can decompose S as $S = \sum_{i=1}^n S_i$, where the coefficients of S_i are rational functions in B and its derivatives up to $m_i - 1$ evaluated at α_i , and the denominator of S_i is $B(\alpha_i)^{m_i}$. Since $|B(\alpha_i)| \geq \delta$, it follows that $|B(\alpha_i)|^{m_i} \geq \delta^{m_i}$.

Since for all $0 \leq \ell \leq m_i - 1$ and $0 \leq i \leq n$, the linear functionals $q \mapsto q^{(\ell)}(\alpha_i)$ are bounded on \mathcal{P}_K , the terms appearing in the numerator can be bounded by a fixed polynomial in the norm of B , and the latter is at most 1. Thus, S satisfies the estimate in (1.6).

Estimating R . Here the situation becomes more complicated. We will prove the required estimate by induction on $k = \deg B$. The constant C will change at each step, but this is not an issue since we only require at most $k \leq K$ induction steps.

If $k = 0$, then $B = c$ is a constant, and applying condition (ii) at any root of A (which has degree at least 1) yields that $|c| \geq \delta$. Then (1.6) is satisfied with $R \equiv 0$, $S \equiv 1/c$, and $C = 1$. In fact, in this case, we may replace $\delta^{\max m_j}$ by δ .

If $k = 1$, we can apply Lemma 3.11 since the condition on the β_j is automatically satisfied. We now pass to the induction step from $k - 1$ to k , where $k \geq 2$. We will now take some extra care in denoting the k simple roots $\beta_1, \beta_2, \dots, \beta_k$ of B . First, we can assume that β_1 is the smallest in absolute value; then $|\beta_1| < M$, since otherwise we apply Lemma 3.10 to complete the proof. Second, we can assume there exists a root, that we will call β_k , such that $|\beta_1 - \beta_k| > \rho/2$. Otherwise, for any $1 \leq i, j \leq k$,

$$|\beta_i - \beta_j| \leq |\beta_i - \beta_1| + |\beta_1 - \beta_j| \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho$$

in which case we can apply Lemma 3.11 to complete the proof. The remaining roots will, of course, be labeled $\beta_2, \dots, \beta_{k-1}$.

Since the roots of B are simple, Corollary 2.13 says that R is the unique polynomial of degree at most $k - 1$ that satisfies $R(\beta_i) = 1/A(\beta_i)$ for $i = 1, \dots, k$. Applying (2.11), a direct calculation shows that

$$(3.12) \quad R(z) = \frac{z - \beta_k}{\beta_1 - \beta_k} r(z) - \frac{z - \beta_1}{\beta_1 - \beta_k} s(z),$$

where r and s are the interpolation polynomials corresponding to the targets $1/A(\beta_i)$ and the nodes $\beta_1, \dots, \beta_{k-1}$ and β_2, \dots, β_k respectively. Since r and s have degree $k - 2$, we will apply the induction hypothesis to these polynomials. To obtain a bound for the norm of R , it is enough to estimate the norm of the two terms in the right hand side of (3.12). We will do this in detail only for the first term since the bound on the second follows in a similar way.

We separate our discussion into two cases.

(a): $|\beta_k| \leq 2M$: Let

$$\tilde{B} := \frac{B}{D(z - \beta_k)}$$

and note that Lemma 3.8 implies that $\|\tilde{B}\| \leq \|B\| \leq 1$. On the other hand, since $|\beta_k - \alpha_i| \leq 2M + M = 3M$ for all i ,

$$|\tilde{B}(\alpha_i)| \geq \frac{\delta}{3DM}.$$

Since r is an interpolation polynomial corresponding to the nodes $\beta_1, \dots, \beta_{k-1}$ and the targets

$$\frac{1}{A(\beta_1)}, \frac{1}{A(\beta_2)}, \dots, \frac{1}{A(\beta_{k-1})},$$

Corollary 2.13 implies that r is the polynomial of degree at most $k - 2$ that satisfies the Bézout equation

$$Ar + \tilde{B}\tilde{S} \equiv 1$$

for some \tilde{S} with $\deg \tilde{S} < \deg A$.

Now apply the induction hypothesis to obtain the estimate

$$\|r\| \leq \frac{C}{\left(\frac{\delta}{3DM}\right)^{\max m_j}} = \frac{C(3DM)^{\max m_j}}{\delta^{\max m_j}}.$$

When $|z| = 1$ we have

$$\frac{|z - \beta_k|}{|\beta_1 - \beta_k|} \leq \frac{(2M + 1)2}{\rho}.$$

Therefore, applying (3.3),

$$\begin{aligned} \left\| \frac{z - \beta_k}{\beta_1 - \beta_k} r(z) \right\| &\leq 2D \frac{2M + 1}{\rho} \|r\| \\ &\leq 2 \frac{CD(2M + 1)(3DM)^{\max m_j}}{\rho \delta^{\max m_j}}. \end{aligned}$$

This ends the proof of case (a).

(b): $|\beta_k| > 2M$. Instead of \tilde{B} , we now consider the polynomial

$$\hat{B} := \frac{\beta_k B}{2D(z - \beta_k)},$$

and we view the polynomial r as satisfying the Bézout equation

$$Ar + \hat{B}\hat{S} \equiv 1$$

for some polynomial \hat{S} with $\deg \hat{S} < \deg A$. For any $|z| = 1$ we have, remembering that $M > 1$ and writing

$$\frac{\beta_k}{z - \beta_k} = \frac{z}{z - \beta_k} - 1,$$

that

$$(3.13) \quad \left| \frac{\beta_k}{z - \beta_k} \right| \leq \frac{1}{2M - 1} + 1 = \frac{2M}{2M - 1} \leq 2.$$

Therefore, multiplication by

$$\frac{\beta_k}{2(z - \beta_k)}$$

is a contraction operator in the norm $\|\cdot\|'$, and from (3.3) it follows that $\|\hat{B}\| \leq 1$. On the other hand, since $|\alpha_i| \leq M$ and $|\beta_k| > 2M$, we have

$$\left| \frac{\beta_k}{\alpha_i - \beta_k} \right| \geq \frac{|\beta_k|}{|\beta_k| + M} \geq \frac{|\beta_k|}{\frac{3}{2}|\beta_k|} = \frac{2}{3},$$

whence

$$|\hat{B}(\alpha_i)| = \frac{|\beta_k| |B(\alpha_i)|}{2D|\alpha_i - \beta_k|} \geq \frac{1}{3D} |B(\alpha_i)| \geq \frac{\delta}{3D}.$$

We now apply the induction hypothesis to obtain the bound

$$\|r\| \leq \frac{C(3D)^{\max m_j}}{\delta^{\max m_j}}.$$

Moreover, under the assumptions of this case, since $|\beta_1| < M$ and $|\beta_k| \geq 2M$, we see that

$$|\beta_1 - \beta_k| \geq |\beta_k| - M \geq \frac{|\beta_k|}{2}$$

and thus

$$\frac{1}{|\beta_1 - \beta_k|} \leq \frac{1}{M} \quad \text{and} \quad \left| \frac{\beta_k}{\beta_1 - \beta_k} \right| \leq 2.$$

It now follows that

$$\left| \frac{z - \beta_k}{\beta_1 - \beta_k} \right| \leq \frac{1}{M} + 2 = \frac{2M + 1}{M} \quad \text{for all } |z| = 1.$$

Therefore, applying (3.3) yields

$$\begin{aligned} \left\| \frac{z - \beta_k}{\beta_1 - \beta_k} r(z) \right\| &\leq D \frac{2M + 1}{M} \|r\| \\ &\leq \frac{CD(2M + 1)(3D)^{\max m_j + 1}}{M \delta^{\max m_j}}. \end{aligned}$$

This ends the proof of case (b), concluding the induction argument needed to estimate the coefficients of R . The proof of the theorem is now complete. \square

We remind the reader that Example 1.4 shows that the exponent in δ from the estimate (1.6) is best possible.

4. EXTENSION TO SEVERAL POLYNOMIALS

Extending Theorem 1.5 to several polynomials involves the *plank theorem* [1].

Lemma 4.1. *For vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in a Hilbert space \mathcal{H} that satisfy*

$$\|\mathbf{v}_i\|_{\mathcal{H}} \geq 1 \quad \text{for all } 1 \leq i \leq n,$$

there exists a unit vector $\mathbf{y} \in \mathcal{H}$ such that

$$|\langle \mathbf{v}_i, \mathbf{y} \rangle_{\mathcal{H}}| \geq \frac{1}{\sqrt{n}} \quad \text{for all } 1 \leq i \leq n.$$

Here is our extension of Theorem 1.5 to several polynomials.

Theorem 4.2. *Let $A \in \mathcal{P}_N$ be of the form*

$$A(z) = \prod_{j=1}^n (z - \alpha_j)^{m_j},$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are distinct and $N = \sum_{j=1}^n m_j \geq 1$. Fix $K \in \mathbb{N}$. Then there is a $C > 0$, depending only on A and K , such that, if $B_1, \dots, B_L \in \mathcal{P}_K$ satisfy the conditions

- (a) $\sum_{j=1}^L \|B_j\|^2 \leq 1$ and
 (b) $\sum_{j=1}^L |B_j(\alpha_i)|^2 \geq \delta^2 > 0$ for all $1 \leq i \leq n$,

then there is an $R \in \mathcal{P}_{K-1}$, and $S_1, \dots, S_L \in \mathcal{P}_{N-1}$, such that

$$(4.3) \quad RA + S_1 B_1 + \dots + S_L B_L \equiv 1,$$

and

$$(4.4) \quad \left(\|R\|^2 + \sum_{j=1}^L \|S_j\|^2 \right)^{\frac{1}{2}} \leq \frac{C}{\delta^{\max m_i}}.$$

Proof. Consider the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{C}^L$ defined by

$$\mathbf{v}_i = \frac{1}{\delta} (B_1(\alpha_i), B_2(\alpha_i), \dots, B_L(\alpha_i))$$

and note that $\|\mathbf{v}_i\|_{\mathbb{C}^L} \geq 1$ for all $1 \leq i \leq n$. Lemma 4.1 produces a unit vector $\mathbf{y} = (y_1, \dots, y_L) \in \mathbb{C}^L$ such that

$$\frac{1}{\sqrt{n}} \leq |\langle \mathbf{v}_i, \mathbf{y} \rangle_{\mathbb{C}^L}| = \frac{1}{\delta} \left| \sum_{j=1}^L B_j(\alpha_i) \bar{y}_j \right| \quad \text{for all } 1 \leq i \leq n.$$

If $B(z) = \sum_{j=1}^L \bar{y}_j B_j(z)$, it follows that

$$|B(\alpha_i)| \geq \frac{\delta}{\sqrt{n}} \geq \frac{\delta}{\sqrt{N}}.$$

Furthermore,

$$\|B\| \leq \left(\sum_{j=1}^L |y_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^L \|B_j\|^2 \right)^{\frac{1}{2}} \leq 1.$$

Theorem 1.5 produces an $R \in \mathcal{P}_{K-1}$ and an $S \in \mathcal{P}_{N-1}$ that satisfy the conditions

$$(4.5) \quad RA + SB \equiv 1 \quad \text{and} \quad (\|R\|^2 + \|S\|^2)^{\frac{1}{2}} \leq \frac{C\sqrt{N}^{\max m_i}}{\delta^{\max m_i}}.$$

The Bézout identity $RA + SB \equiv 1$ can be written as

$$RA + \sum_{j=1}^L \bar{y}_j SB_j \equiv 1.$$

Moreover, if we define $S_j = \bar{y}_j S$, then (4.3) holds along with

$$\|R\|^2 + \sum_{j=1}^L \|S_j\|^2 = \|R\|^2 + \sum_{j=1}^L |y_j|^2 \|S\|^2 = \|R\|^2 + \|S\|^2.$$

The estimate in (4.4) now follows from the estimate in (4.5). \square

Remark 4.6. In Theorem 1.5 the polynomials R, S are uniquely determined by the degree condition from (1.1). However, Theorem 4.2 only yields the existence of *some* polynomials R and S_1, \dots, S_L that satisfy the desired estimates.

5. DE BRANGES–ROVNYAK SPACES

The second main theorem of this paper (Theorem 6.5) extends results from [20] and establishes a corona theorem for the multipliers of certain de Branges–Rovnyak spaces. In fact, this corona theorem is what originally drew us to investigate coefficient estimates of Bézout’s identity. In this section we present some of the basics of de Branges–Rovnyak spaces [10, 11, 24], along with some additional results which seem to be interesting on their own. The next section will contain our corona theorem.

Denote

$$\text{ball}(H^\infty) := \left\{ b \in H^\infty : \|b\|_\infty = \sup_{z \in \mathbb{D}} |b(z)| \leq 1 \right\}.$$

For $b \in \text{ball}(H^\infty)$, the *de Branges–Rovnyak space* $\mathcal{H}(b)$ is the reproducing kernel Hilbert space associated with the positive definite kernel

$$(5.1) \quad k_\lambda^b(z) = \frac{1 - b(z)\overline{b(\lambda)}}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

It is known that $\mathcal{H}(b)$ is contractively contained in the well-studied Hardy space H^2 of analytic functions f on \mathbb{D} for which

$$\|f\|_{H^2} := \left(\sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi) \right)^{\frac{1}{2}} < \infty,$$

where m is normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ [7, 13]. For $f \in H^2$, the radial limit $\lim_{r \rightarrow 1^-} f(r\xi) =: f(\xi)$ exists for m -almost every $\xi \in \mathbb{T}$ and

$$(5.2) \quad \|f\|_{H^2} = \left(\int_{\mathbb{T}} |f(\xi)|^2 dm(\xi) \right)^{\frac{1}{2}}.$$

Furthermore, Parseval's theorem says that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belongs to H^2 , then

$$(5.3) \quad \|f\|_{H^2}^2 = \sum_{k=0}^{\infty} |a_k|^2.$$

Though $\mathcal{H}(b)$ is contractively contained in H^2 , it is generally not closed in the H^2 norm. In fact, for the b explored in this section, $\mathcal{H}(b)$ is dense in H^2 .

Throughout this section, we will assume that $b \in \text{ball}(H^\infty)$ is a rational function that is not a finite Blaschke product. We exclude the finite Blaschke products from our discussion since we will be exploring a corona theorem for the multiplier algebra of $\mathcal{H}(b)$. When b is a finite Blaschke product, $\mathcal{H}(b)$ becomes the usual model space $H^2 \ominus bH^2$ and in that case it is well-known that the multiplier algebra of $\mathcal{H}(b)$ is formed just by the constant functions [12]. Thus, when b is a finite Blaschke product, any corona theorem concerning the multipliers of $\mathcal{H}(b)$ becomes a triviality.

Although, for a general $b \in \text{ball}(H^\infty)$ the contents of $\mathcal{H}(b)$ seem mysterious, when $b \in \text{ball}(H^\infty)$ is a rational function (and not a finite Blaschke product) the description of $\mathcal{H}(b)$ is quite explicit. For such a b there exists a unique nonconstant rational function a with no zeros on \mathbb{D} such that $a(0) > 0$ and $|a(\xi)|^2 + |b(\xi)|^2 = 1$ for all $|\xi| = 1$. This function a is called the *Pythagorean mate* of b . In fact, one can obtain a from the Fejér–Riesz theorem (see [8]). Let ξ_1, \dots, ξ_n denote the *distinct* roots of a on \mathbb{T} , with corresponding multiplicities m_1, \dots, m_n , and define the polynomial a_1 by

$$(5.4) \quad a_1(z) := \prod_{j=1}^n (z - \xi_j)^{m_j}.$$

Results from [3, 8] show that $\mathcal{H}(b)$ has an explicit description as

$$(5.5) \quad \mathcal{H}(b) = a_1 H^2 \dot{+} \mathcal{P}_{N-1},$$

where $N = m_1 + \cdots + m_n$ and $\dot{+}$ above denotes the algebraic direct sum in that $a_1 H^2 \cap \mathcal{P}_{N-1} = \{0\}$. Moreover, if $f \in \mathcal{H}(b)$ is decomposed with respect to (5.5) as

$$(5.6) \quad f = a_1 \tilde{f} + p, \quad \text{where } \tilde{f} \in H^2 \text{ and } p \in \mathcal{P}_{N-1},$$

an equivalent norm on $\mathcal{H}(b)$ (to the natural one induced by the positive definite kernel $k_\lambda^b(z)$ above) is

$$(5.7) \quad \|a_1 \tilde{f} + p\|_b^2 := \|\tilde{f}\|_{H^2}^2 + \|p\|_{H^2}^2.$$

It is important to note that $\|\cdot\|_b$ is only equivalent to the original norm corresponding to the kernel in (5.1), and its scalar product as well as the reproducing kernels and the adjoints of operators defined on $\mathcal{H}(b)$ will be different. With the norm $\|\cdot\|_b$ and the corresponding inner product in mind, we need to introduce a new notation for the associated reproducing kernels, different from (5.1), namely \mathbf{k}_λ^b (note the bold face). By the term reproducing kernel we mean that $\mathbf{k}_\lambda^b \in \mathcal{H}(b)$ for all $\lambda \in \mathbb{D}$ and

$$\langle f, \mathbf{k}_\lambda^b \rangle_b = f(\lambda) \quad \text{for all } f \in \mathcal{H}(b) \text{ and } \lambda \in \mathbb{D}.$$

Using (5.6) and the standard estimate that any $g \in H^2$ satisfies

$$(5.8) \quad |g(z)| \leq \frac{\|g\|_{H^2}^2}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D},$$

we see that for fixed $1 \leq k \leq n$ and for each $f \in \mathcal{H}(b)$ we have

$$(5.9) \quad f(\xi_k) = \lim_{r \rightarrow 1^-} f(r\xi_k) = p(\xi_k),$$

where $f = a_1 \tilde{f} + p$ with $\tilde{f} \in H^2$ and $p \in \mathcal{P}_{N-1}$. In the spirit of (5.8), the next lemma (interesting in its own right and useful later) yields more precise information on the boundary behavior of $\mathcal{H}(b)$ functions. In particular, it shows that $\mathcal{H}(b)$ functions admit tangential limits in suitable approach regions at each point ξ_k (see Remark 5.13 below).

Lemma 5.10. *For each fixed $1 \leq k \leq n$, there is a $c_k > 0$, depending only on b , such that for each $f \in \mathcal{H}(b)$, $\eta > 0$, and $z \in \mathbb{D}$, we have*

$$|f(z)|^2 \leq (1 + \eta)|f(\xi_k)|^2 + c_k \left(1 + \frac{1}{\eta}\right) \frac{|z - \xi_k|^2}{1 - |z|^2} \|f\|_b^2.$$

Proof. For fixed $1 \leq k \leq n$, remembering from (5.4) that ξ_k is a root of a_1 , we define the polynomial $a_k^\#(z)$ by

$$a_k^\#(z) := \frac{a_1(z)}{z - \xi_k}.$$

Write $f \in \mathcal{H}(b)$ as $f = a_1 \tilde{f} + p$ as in (5.6). By (5.9) we have $f(\xi_k) = p(\xi_k)$, and so

$$\begin{aligned} f(z) &= (z - \xi_k) a_k^\#(z) \tilde{f}(z) + (p(z) - p(\xi_k)) + p(\xi_k) \\ &= (z - \xi_k) \left(a_k^\#(z) \tilde{f}(z) + \frac{p(z) - p(\xi_k)}{z - \xi_k} \right) + p(\xi_k) \\ &= (z - \xi_k) f_k(z) + f(\xi_k), \end{aligned}$$

where

$$(5.11) \quad f_k(z) = a_k^\#(z) \tilde{f}(z) + \frac{p(z) - p(\xi_k)}{z - \xi_k} \in H^2.$$

Given $\eta > 0$, for any $a, b > 0$ we have

$$2ab \leq \eta a^2 + \frac{1}{\eta} b^2$$

and so

$$\begin{aligned} |f(z)|^2 &\leq |(z - \xi_k) f_k(z) + f(\xi_k)|^2 \\ &\leq |f(\xi_k)|^2 + |z - \xi_k|^2 |f_k(z)|^2 + 2|(z - \xi_k) f_k(z)| |f(\xi_k)| \\ &\leq (1 + \eta) |f(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right) |z - \xi_k|^2 |f_k(z)|^2 \\ &\leq (1 + \eta) |f(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right) \frac{|z - \xi_k|^2}{1 - |z|^2} \|f_k\|_{H^2}^2. \end{aligned}$$

In the last inequality above, note the use of (5.8).

To finish the proof, it suffices to show there exists a $c_k > 0$, depending only on b and k , such that

$$(5.12) \quad \|f_k\|_{H^2}^2 \leq c_k \|f\|_b^2.$$

The definition of f_k from (5.11) says that

$$\begin{aligned} \|f_k\|_{H^2}^2 &\leq 2 \left(\|a_k^\# \tilde{f}\|_{H^2}^2 + \left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_{H^2}^2 \right) \\ &\leq 2 \left(\|a_k^\#\|_\infty \|\tilde{f}\|_{H^2}^2 + \left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_{H^2}^2 \right). \end{aligned}$$

Since the map

$$p(z) \mapsto \frac{p(z) - p(\xi_k)}{z - \xi_k}$$

is a linear transformation from \mathcal{P}_{N-1} to itself and \mathcal{P}_{N-1} is a finite dimensional space (and hence all norms on \mathcal{P}_{N-1} are equivalent), this map is continuous and hence there is a constant $\tilde{c}_k > 0$ such that

$$\left\| \frac{p - p(\xi_k)}{z - \xi_k} \right\|_{H^2}^2 \leq \tilde{c}_k \|p\|_{H^2}^2 \quad \text{for all } p \in \mathcal{P}_{N-1}.$$

Thus,

$$\begin{aligned} \|f_k\|_{H^2}^2 &\leq 2(\|a_k^\# \|_\infty^2 \|\tilde{f}\|_{H^2}^2 + \tilde{c}_k \|p\|_{H^2}^2) \\ &\leq c_k (\|\tilde{f}\|_{H^2}^2 + \|p\|_{H^2}^2) \\ &= c_k \|f\|_b^2, \end{aligned}$$

where $c_k = 2 \max(\|a_k^\# \|_\infty^2, \tilde{c}_k)$. This verifies (5.12) and thus completes the proof. \square

Remark 5.13. In the spirit of the above proof, one can write

$$f(z) = (z - \xi_k)^{m_k} a_k^\dagger(z) \tilde{f}(z) + p(z),$$

where

$$a_k^\dagger(z) = \prod_{j \neq k} (z - \xi_j)^{m_j},$$

to prove that each $f \in \mathcal{H}(b)$ admits a boundary limit at ξ_k in the approach regions

$$\left\{ z \in \mathbb{D} : \frac{|z - \xi_k|^{2m_k}}{1 - |z|} \leq c \right\}, \quad c > 1,$$

which are larger than the standard nontangential (Stolz) regions

$$\left\{ z \in \mathbb{D} : \frac{|z - \xi_k|}{1 - |z|} \leq c \right\}, \quad c > 1.$$

Let

$$\mathfrak{M}(\mathcal{H}(b)) := \{\varphi \in \mathcal{H}(b) : \varphi \mathcal{H}(b) \subseteq \mathcal{H}(b)\}$$

denote the *multiplier algebra* of $\mathcal{H}(b)$. Standard results for multiplier algebras, true for any reproducing kernel Hilbert space of analytic functions, say that if $\varphi \in \mathfrak{M}(\mathcal{H}(b))$, then $\varphi \in H^\infty$, and the multiplication operator $M_\varphi f = \varphi f$ is bounded on $\mathcal{H}(b)$ and satisfies

$$(5.14) \quad M_\varphi^* \mathbf{k}_\lambda^b = \overline{\varphi(\lambda)} \mathbf{k}_\lambda^b \quad \text{for all } \lambda \in \mathbb{D}.$$

For general $\mathcal{H}(b)$ spaces, the multiplier algebra $\mathfrak{M}(\mathcal{H}(b))$ lacks a complete description [5, 17, 18]. In our case, where $b \in \text{ball}(H^\infty)$ is rational and not a finite Blaschke product, things again become much simpler. Indeed, [9, Proposition 3.1] says that

$$(5.15) \quad \mathfrak{M}(\mathcal{H}(b)) = \mathcal{H}(b) \cap H^\infty,$$

and (5.5) implies that $\varphi \in \mathfrak{M}(\mathcal{H}(b))$ if and only if

$$\varphi = a_1 \tilde{\varphi} + r, \quad \text{where } \tilde{\varphi} \in H^2, r \in \mathcal{P}_{N-1}, \text{ and } a_1 \tilde{\varphi} \in H^\infty.$$

In particular, it follows easily from (5.6) that every polynomial is a multiplier of $\mathcal{H}(b)$ (this is also a consequence of more general facts from [24, Ch. IV]). Since all norms on the finite dimensional space \mathcal{P}_{N-1} are equivalent, we fix a $C_1 > 0$ that satisfies

$$(5.16) \quad \max\{\|p\|, \|p\|_{\mathfrak{M}(\mathcal{H}(b))}\} \leq C_1 \|p\|_b = C_1 \|p\|_{H^2}, \quad p \in \mathcal{P}_{N-1}.$$

6. A CORONA THEOREM FOR DE BRANGES–ROVNYAK SPACES

Our corona theorem for $\mathfrak{M}(\mathcal{H}(b))$ will be stated in terms of column multipliers. For a sequence $\Phi = (\varphi_j)_{j \geq 1}$ of functions in $\mathfrak{M}(\mathcal{H}(b))$, define the column multiplier

$$(6.1) \quad \mathbf{M}_\Phi : \mathcal{H}(b) \rightarrow \bigoplus_{j=1}^{\infty} \mathcal{H}(b), \quad \mathbf{M}_\Phi f = (\varphi_j f)_{j \geq 1}.$$

When \mathbf{M}_Φ is bounded, its adjoint is given by $\mathbf{M}_\Phi^* = (M_{\varphi_1}^*, M_{\varphi_2}^*, \dots)$. As is standard,

$$\bigoplus_{j=1}^{\infty} \mathcal{H}(b) := \left\{ (f_j)_{j \geq 1} : f_j \in \mathcal{H}(b), \sum_{j=1}^{\infty} \|f_j\|_b^2 < \infty \right\}$$

with

$$\|(f_j)_{j \geq 1}\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)} := \left(\sum_{j=1}^{\infty} \|f_j\|_b^2 \right)^{\frac{1}{2}}$$

(recall the norm $\|\cdot\|_b$ on $\mathcal{H}(b)$ from (5.7)). The next lemma generalizes [19, Lemma 3.2.3].

Lemma 6.2. \mathbf{M}_Φ is a bounded (column) operator if and only if

$$(a) \quad C_2 := \left(\sum_{j=1}^{\infty} \|\varphi_j\|_b^2 \right)^{\frac{1}{2}} < \infty, \text{ and}$$

$$(b) \quad C_3 := \sup_{z \in \mathbb{D}} \left(\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \right)^{\frac{1}{2}} < \infty.$$

Furthermore, $\max(C_2, C_3) \leq \|\mathbf{M}_\Phi\| \leq \sqrt{2} \max(C_3, C_1 C_2)$, where C_1 was defined in (5.16).

Proof. Suppose that \mathbf{M}_Φ is a bounded operator. Since $1 \in \mathcal{H}(b)$ and (5.7) shows that $\|1\|_b = 1$, we conclude that

$$C_2^2 = \sum_{j=1}^{\infty} \|\varphi_j\|_b^2 = \|\mathbf{M}_\Phi 1\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)}^2 \leq \|\mathbf{M}_\Phi\|^2 \cdot \|1\|_b^2 = \|\mathbf{M}_\Phi\|^2.$$

This proves (a). To prove (b), let $N \in \mathbb{N}$ and $(\gamma_j)_{j \geq 1}$ be a complex sequence such that $\gamma_j = 0$ when $j \geq N + 1$. For every $z \in \mathbb{D}$ and $N \in \mathbb{N}$, it follows from (5.14) that

$$\mathbf{M}_\Phi^*((\gamma_j \mathbf{k}_z^b)_{j \geq 1}) = \sum_{j=1}^N \mathbf{M}_{\varphi_j}^*(\gamma_j \mathbf{k}_z^b) = \left(\sum_{j=1}^N \gamma_j \overline{\varphi_j(z)} \right) \mathbf{k}_z^b,$$

and so

$$\begin{aligned} \left| \sum_{j=1}^N \gamma_j \overline{\varphi_j(z)} \right| \|\mathbf{k}_z^b\|_b &\leq \|\mathbf{M}_\Phi^*\| \|(\gamma_j \mathbf{k}_z^b)_{j \geq 1}\|_{\bigoplus_{j=1}^N \mathcal{H}(b)} \\ &= \|\mathbf{M}_\Phi\| \left(\sum_{j=1}^N |\gamma_j|^2 \right)^{\frac{1}{2}} \|\mathbf{k}_z^b\|_b. \end{aligned}$$

Therefore,

$$\left| \sum_{j=1}^N \gamma_j \overline{\varphi_j(z)} \right| \leq \|\mathbf{M}_\Phi\| \left(\sum_{j=1}^N |\gamma_j|^2 \right)^{\frac{1}{2}} \quad \text{for all } N \in \mathbb{N}.$$

Since the inequality above is true for any $(\gamma_j)_{j \geq 1}$, the Riesz representation theorem implies that

$$\sum_{j=1}^N |\varphi_j(z)|^2 \leq \|\mathbf{M}_\Phi\|^2 \quad \text{for all } z \in \mathbb{D}.$$

The inequality above is true for all N , which proves (b).

Conversely, assume that conditions (a) and (b) are satisfied. For any $f \in \mathcal{H}(b)$ we have

$$\|\mathbf{M}_\Phi f\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)}^2 = \sum_{j=1}^{\infty} \|\varphi_j f\|_b^2.$$

From (5.6) we can write $f = a_1 \tilde{f} + p$, with $\tilde{f} \in H^2$ and $p \in \mathcal{P}_{N-1}$. The definition of the norm on $\mathcal{H}(b)$ from (5.7) yields

$$\begin{aligned} \|\varphi_j f\|_b^2 &= \|a_1 \tilde{f} \varphi_j + \varphi_j p\|_b^2 \\ &\leq 2(\|a_1 \tilde{f} \varphi_j\|_b^2 + \|\varphi_j p\|_b^2) \\ &= 2(\|\tilde{f} \varphi_j\|_{H^2}^2 + \|\varphi_j p\|_b^2). \end{aligned}$$

Hence,

$$(6.3) \quad \|\mathbf{M}_\Phi f\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)}^2 \leq 2 \left(\sum_{j=1}^{\infty} \|\tilde{f} \varphi_j\|_{H^2}^2 + \sum_{j=1}^{\infty} \|\varphi_j p\|_b^2 \right).$$

To estimate the first sum on the right hand side of (6.3), we can use (5.2) and Fubini's theorem to obtain

$$(6.4) \quad \begin{aligned} \sum_{j=1}^{\infty} \|\tilde{f} \varphi_j\|_{H^2}^2 &= \sum_{j=1}^{\infty} \int_{\mathbb{T}} |\varphi_j(\zeta)|^2 |\tilde{f}(\zeta)|^2 dm(\zeta) \\ &= \int_{\mathbb{T}} \sum_{j=1}^{\infty} |\varphi_j(\zeta)|^2 |\tilde{f}(\zeta)|^2 dm(\zeta). \end{aligned}$$

From the facts that $\varphi_j(r\zeta) \rightarrow \varphi_j(\zeta)$ for almost every $\zeta \in \mathbb{T}$ as $r \rightarrow 1^-$, and

$$\sum_{j=1}^N |\varphi_j(r\xi)|^2 \leq C_3^2 \quad \text{for all } N \in \mathbb{N},$$

one sees that

$$\sum_{j=1}^N |\varphi_j(\xi)|^2 \leq C_3^2 \quad \text{for almost every } \xi \in \mathbb{T} \text{ and every } N \in \mathbb{N}.$$

Now let $N \rightarrow \infty$ to conclude that

$$\sum_{j=1}^{\infty} |\varphi_j(\xi)|^2 \leq C_3^2 \quad \text{for almost every } \xi \in \mathbb{T}.$$

Thus, continuing the estimate from (6.4), we obtain

$$\sum_{j=1}^{\infty} \|\tilde{f} \varphi_j\|_{H^2}^2 \leq C_3^2 \int_{\mathbb{T}} |\tilde{f}(\zeta)|^2 dm(\zeta) = C_3^2 \|\tilde{f}\|_{H^2}^2.$$

To estimate the second term on the right hand side of (6.3), observe that

$$\|\varphi_j p\|_b \leq \|p\|_{\mathfrak{M}(\mathcal{H}(b))} \|\varphi_j\|_b \leq C_1 \|p\|_{H^2} \|\varphi_j\|_b,$$

where C_1 is defined by (5.16). Therefore,

$$\sum_{j=1}^{\infty} \|\varphi_j p\|_b^2 \leq C_1^2 \|p\|_{H^2}^2 \left(\sum_{j=1}^{\infty} \|\varphi_j\|_b^2 \right) = C_1^2 C_2^2 \|p\|_{H^2}^2.$$

It follows from (6.3) that

$$\|\mathbf{M}_{\Phi} f\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)}^2 \leq 2(C_3^2 \|\tilde{f}\|_{H^2}^2 + C_1^2 C_2^2 \|p\|_{H^2}^2).$$

Since $\|f\|_b^2 = \|\tilde{f}\|_{H^2}^2 + \|p\|_{H^2}^2$ (see (5.7)), we see that

$$\|\mathbf{M}_{\Phi} f\|_{\bigoplus_{j=1}^{\infty} \mathcal{H}(b)}^2 \leq 2 \max(C_3^2, C_1^2 C_2^2) \|f\|_b^2.$$

Therefore, M_{Φ} is bounded and $\|\mathbf{M}_{\Phi}\| \leq \sqrt{2} \max(C_3, C_1 C_2)$, which finishes the proof of the lemma. \square

Here is the second main result of this paper, a corona theorem for $\mathfrak{M}(\mathcal{H}(b))$. We remind the reader that for rational $b \in \text{ball}(H^{\infty})$ (and not a finite Blaschke product), there exists a Pythagorean mate a to which we can associate the polynomial $a_1(z) = \prod_{j=1}^n (z - \xi_j)^{m_j}$ as explained in (5.4).

Theorem 6.5. *Let $b \in \text{ball}(H^{\infty})$ be rational, but not a finite Blaschke product. Suppose that $\Phi = (\varphi_j)_{j \geq 1}$ is a sequence in $\mathfrak{M}(\mathcal{H}(b))$ that satisfies the conditions*

- (i) $\|\mathbf{M}_{\Phi}\| \leq 1$, and
- (ii) $0 < \delta^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2$ for all $z \in \mathbb{D}$.

Then there is a sequence $\mathbf{B} = (b_j)_{j \geq 1}$ in $\mathfrak{M}(\mathcal{H}(b))$ such that

- (a) $\sum_{j=1}^{\infty} \varphi_j(z) b_j(z) = 1$ for all $z \in \mathbb{D}$, and
- (b) $\|\mathbf{M}_{\mathbf{B}}\| \leq \frac{C}{\delta^{\max m_j}} \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right)$,

where $C > 0$ depends only on b .

Proof. We begin by decomposing each $\varphi_j \in \mathfrak{M}(\mathcal{H}(b))$ as

$$(6.6) \quad \varphi_j = a_1 \widetilde{\varphi}_j + p_j, \quad \widetilde{\varphi}_j \in H^2, \quad p_j \in \mathcal{P}_{N-1},$$

where a_1 is the polynomial associated with the Pythagorean mate of b as defined by (5.4). Furthermore,

$$(6.7) \quad a_1 \widetilde{\varphi}_j \in H^{\infty} \quad \text{for all } j \geq 1.$$

Conditions (i), (ii), and Lemma 6.2 imply that

$$(6.8) \quad \delta^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1 \quad \text{for all } z \in \mathbb{D}.$$

Apply Lemma 5.10 to see that for fixed $1 \leq k \leq n$ there is a $c_k > 0$, depending only on b , such that for every $\eta > 0$, every $z \in \mathbb{D}$, and every $j \geq 1$,

$$|\varphi_j(z)|^2 \leq (1 + \eta)|\varphi_j(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right) c_k \frac{|z - \xi_k|^2}{1 - |z|^2} \|\varphi_j\|_b^2.$$

By summing over j in the previous inequality, it follows from Lemma 6.2 that

$$\begin{aligned} \delta^2 &\leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right) c_k \frac{|z - \xi_k|^2}{1 - |z|^2} \sum_{j=1}^{\infty} \|\varphi_j\|_b^2 \\ &\leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 + \left(1 + \frac{1}{\eta}\right) c_k \frac{|z - \xi_k|^2}{1 - |z|^2}. \end{aligned}$$

In the above, note the use of the fact that $\|\mathbf{M}_\Phi\| \leq 1$ and so $C_2 \leq 1$. Now let $z \rightarrow \xi_k$ radially to see that the second term above goes to zero and thus

$$\delta^2 \leq (1 + \eta) \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2.$$

Letting $\eta \rightarrow 0^+$ yields

$$\delta^2 \leq \sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2.$$

The estimate in (6.8) and Fatou's lemma yield

$$\sum_{j=1}^{\infty} |\varphi_j(\xi_k)|^2 \leq 1.$$

Finally, use $p_j(\xi_k) = \varphi_j(\xi_k)$ to obtain

$$\delta^2 \leq \sum_{j=1}^{\infty} |p_j(\xi_k)|^2 \leq 1.$$

In particular, for every $1 \leq k \leq n$ there exists an $\ell_k \in \mathbb{N}$ such that

$$\sum_{j=\ell_k+1}^{\infty} |p_j(\xi_k)|^2 \leq \frac{\delta^2}{2}.$$

Define $L = \max\{\ell_k : 1 \leq k \leq n\}$. Then for every $1 \leq k \leq n$, we have

$$\begin{aligned} \sum_{j=1}^L |p_j(\xi_k)|^2 &= \sum_{j=1}^{\infty} |p_j(\xi_k)|^2 - \sum_{j=L+1}^{\infty} |p_j(\xi_k)|^2 \\ &\geq \delta^2 - \sum_{j=\ell_k+1}^{\infty} |p_j(\xi_k)|^2 \\ &\geq \delta^2 - \frac{\delta^2}{2} = \frac{\delta^2}{2} > 0. \end{aligned}$$

On the other hand, by assumption (i) and Lemma 6.2,

$$\sum_{j=1}^{\infty} \|\varphi_j\|_b^2 = C_2^2 \leq \|\mathbf{M}_\Phi\|^2 \leq 1,$$

whence the norm (5.7) implies that $\sum_{j=1}^{\infty} \|p_j\|_{H^2} \leq 1$. Using the constant C_1 defined in (5.16), we see that

$$\sum_{j=1}^L \|p_j\|^2 \leq C_1^2 \sum_{j=1}^L \|p_j\|_{H^2}^2 \leq C_1^2 \sum_{j=1}^{\infty} \|p_j\|_{H^2}^2 \leq C_1^2.$$

From now on, all constants will be denoted by C and may change from line to line.

Now apply Theorem 4.2 with $K = N - 1$, $A = a_1$, and $B_j = p_j$ for $1 \leq j \leq L$, to produce a $q \in \mathcal{P}_{N-2}$ and $q_1, \dots, q_L \in \mathcal{P}_{N-1}$ such that

$$(6.9) \quad \begin{aligned} qa_1 + q_1p_1 + \dots + q_Lp_L &\equiv 1 \quad \text{and} \\ \left(\|q\|_{H^2}^2 + \sum_{j=1}^L \|q_j\|_{H^2}^2 \right)^{\frac{1}{2}} &\leq \frac{C}{\delta^{\max m_i}} \end{aligned}$$

for some $C > 0$, depending only on a_1 and hence depending only on b . (Since all norms on \mathcal{P}_{N-1} are equivalent, we may replace the norm used in (4.4) by the H^2 -norm). We set $q_k \equiv 0$ for all $k \geq L + 1$.

By Tolokonnikov's theorem [25] (mentioned in the introduction), there is a universal $C > 0$ and a sequence $(e_j)_{j \geq 1}$ in H^∞ such that

$$(6.10) \quad \begin{aligned} \sum_{j=1}^{\infty} \varphi_j(z)e_j(z) &= 1 \text{ for all } z \in \mathbb{D}, \\ \left(\sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \right)^{\frac{1}{2}} &\leq C \frac{1}{\delta^2} \log \frac{1}{\delta} \end{aligned}$$

For each $j \geq 1$ define

$$(6.11) \quad b_j := q_j + \left(1 - \sum_{k=1}^L \varphi_k q_k\right) e_j.$$

We will now show that $(b_j)_{j \geq 1}$ is the required sequence satisfying conditions (a) and (b).

First we check that $b_j \in \mathfrak{M}(\mathcal{H}(b))$ for all $j \geq 1$. Using (6.6) and (6.9), one obtains

$$\begin{aligned} \left(1 - \sum_{k=1}^L \varphi_k q_k\right) e_j &= \left(1 - \sum_{k=1}^L q_k (a_1 \widetilde{\varphi}_k + p_k)\right) e_j \\ &= \left(1 - \sum_{k=1}^L q_k p_k - a_1 \sum_{k=1}^L q_k \widetilde{\varphi}_k\right) e_j \\ &= a_1 \left(q - \sum_{k=1}^L q_k \widetilde{\varphi}_k\right) e_j. \end{aligned}$$

Therefore (6.11) can be written as

$$(6.12) \quad b_j = a_1 \left(q - \sum_{k=1}^L q_k \widetilde{\varphi}_k\right) e_j + q_j.$$

Since $q_j \in \mathcal{P}_{N-1}$ and $q - \sum_{k=1}^L q_k \widetilde{\varphi}_k \in H^2$, this is precisely the decomposition of b_j from (5.6). Therefore, $b_j \in \mathcal{H}(b)$ and it follows from (6.7) that $b_j \in H^\infty$. Thus, from (5.15), $b_j \in \mathcal{H}(b) \cap H^\infty = \mathfrak{M}(\mathcal{H}(b))$.

Second, we observe that

$$\begin{aligned} \sum_{j=1}^{\infty} b_j \varphi_j &= \sum_{j=1}^{\infty} \left(q_j + \left(1 - \sum_{k=1}^L \varphi_k q_k\right) e_j\right) \varphi_j \\ &= \sum_{j=1}^{\infty} \varphi_j q_j + \left(1 - \sum_{k=1}^L \varphi_k q_k\right) \sum_{j=1}^{\infty} e_j \varphi_j \\ &= \sum_{j=1}^{\infty} \varphi_j q_j + \left(1 - \sum_{k=1}^L \varphi_k q_k\right) \cdot 1 && \text{(by (6.10))} \\ &= \sum_{j=1}^L \varphi_j q_j + \left(1 - \sum_{k=1}^L \varphi_k q_k\right) \\ &= 1. \end{aligned}$$

Thus (a) is proved.

In order to prove (b) of Theorem 6.5, we need to show that M_B satisfies inequalities (a) and (b) in Lemma 6.2. Apply (5.7) and (6.12) to obtain

$$(6.13) \quad \sum_{j=1}^{\infty} \|b_j\|_b^2 = \sum_{j=1}^{\infty} \|q_j\|_{H^2}^2 + \sum_{j=1}^{\infty} \left\| \left(q - \sum_{k=1}^L q_k \widetilde{\varphi}_k \right) e_j \right\|_{H^2}^2.$$

By (6.9), the first term on the right hand side of the above is bounded by $\frac{C}{\delta^{2 \max m_j}}$. To bound the second term, we have

$$\begin{aligned} & \sum_{j=1}^{\infty} \left\| \left(q - \sum_{k=1}^L q_k \widetilde{\varphi}_k \right) e_j \right\|_{H^2}^2 \\ &= \int_{\mathbb{T}} \sum_{j=1}^{\infty} \left| q(\zeta) - \sum_{k=1}^L q_k(\zeta) \widetilde{\varphi}_k(\zeta) \right|^2 |e_j(\zeta)|^2 dm(\zeta) \\ &= \int_{\mathbb{T}} \left| q(\zeta) - \sum_{k=1}^L q_k(\zeta) \widetilde{\varphi}_k(\zeta) \right|^2 \left(\sum_{j=1}^{\infty} |e_j(\zeta)|^2 \right) dm(\zeta) \\ &\leq \left(\frac{C}{\delta^2} \log \frac{1}{\delta} \right)^2 \int_{\mathbb{T}} \left| q(\zeta) - \sum_{k=1}^L q_k(\zeta) \widetilde{\varphi}_k(\zeta) \right|^2 dm(\zeta) \quad \text{by (6.10)} \\ &= \left(\frac{C}{\delta^2} \log \frac{1}{\delta} \right)^2 \left\| q - \sum_{k=1}^L q_k \widetilde{\varphi}_k \right\|_{H^2}^2 \\ &\leq 2 \left(\frac{C}{\delta^2} \log \frac{1}{\delta} \right)^2 \left(\|q\|_{H^2}^2 + \left\| \sum_{k=1}^L q_k \widetilde{\varphi}_k \right\|_{H^2}^2 \right). \end{aligned}$$

Again (6.9) yields $\|q\|_{H^2}^2 \leq \frac{C}{\delta^{2 \max m_j}}$. On the other hand, (5.7) implies that $\|\widetilde{\varphi}_k\|_{H^2} \leq \|\varphi_k\|_b$, which, together with the Cauchy–Schwarz inequality, yield

$$\begin{aligned} \left\| \sum_{k=1}^L q_k \widetilde{\varphi}_k \right\|_{H^2}^2 &\leq \left(\sum_{k=1}^L \|q_k \widetilde{\varphi}_k\|_{H^2} \right)^2 \leq \left(\sum_{k=1}^L \|q_k\|_{\infty} \|\varphi_k\|_b \right)^2 \\ &\leq C \left(\sum_{k=1}^L \|q_k\|_{H^2} \|\varphi_k\|_b \right)^2 \\ &\leq C \left(\sum_{k=1}^L \|q_k\|_{H^2}^2 \right) \left(\sum_{k=1}^L \|\varphi_k\|_b^2 \right). \end{aligned}$$

Using (6.9) once more, we see that the first factor in the last formula is bounded above by $\frac{C}{\delta^{2 \max m_j}}$, while, by condition (i) in the statement of the theorem and Lemma 6.2, the second factor is bounded above by 1. Consequently,

$$\sum_{j=1}^{\infty} \left\| \left(q - \sum_{k=1}^L q_k \widetilde{\varphi}_k \right) e_j \right\|_{H^2}^2 \leq \frac{C}{\delta^{2 \max m_j}} \left(\frac{1}{\delta^2} \log \frac{1}{\delta} \right)^2.$$

Returning to (6.13), it follows that

$$(6.14) \quad \sum_{j=1}^{\infty} \|b_j\|_b^2 \leq \frac{C}{\delta^{2 \max m_j}} \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta} \right)^2.$$

proving that the inequality (a) in Lemma 6.2 is satisfied.

In order to prove (b), fix $z \in \mathbb{D}$. From the definition of b_j from (6.11), we see that

$$|b_j(z)|^2 \leq 2 \left(|q_j(z)|^2 + \left| 1 - \sum_{k=1}^L \varphi_k(z) q_k(z) \right|^2 |e_j(z)|^2 \right),$$

whence

$$(6.15) \quad \sum_{j=1}^{\infty} |b_j(z)|^2 \leq 2 \sum_{j=1}^{\infty} |q_j(z)|^2 + 2 \left| 1 - \sum_{k=1}^L \varphi_k(z) q_k(z) \right|^2 \left(\sum_{j=1}^{\infty} |e_j(z)|^2 \right).$$

From (6.10) we obtain the estimate

$$\sum_{j=1}^{\infty} |e_j(z)|^2 \leq \left(\frac{C}{\delta^2} \log \frac{1}{\delta} \right)^2,$$

while from (6.9) we have the estimate

$$\sum_{j=1}^{\infty} |q_j(z)|^2 = \sum_{j=1}^L |q_j(z)|^2 \leq \sum_{j=1}^L \|q_j\|_{\infty}^2 \leq C \sum_{j=1}^L \|q_j\|_{H^2}^2 \leq \frac{C}{\delta^{2 \max m_j}}$$

(note that $q_j \equiv 0$ for all $j > L$). Finally, using condition (b) in Lemma 6.2 (valid for \mathbf{M}_{Φ} by assumption) as well as (6.9), we see

that

$$\begin{aligned}
\left|1 - \sum_{k=1}^L \varphi_k(z)q_k(z)\right| &\leq 1 + \sum_{k=1}^L |\varphi_k(z)q_k(z)| \\
&\leq 1 + \left(\sum_{k=1}^L |\varphi_k(z)|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^L |q_k(z)|^2\right)^{\frac{1}{2}} \\
&\leq 1 + \left(\sum_{k=1}^L |\varphi_k(z)|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^L \|q_k\|_\infty^2\right)^{\frac{1}{2}} \\
&\leq 1 + C \left(\sum_{k=1}^L |\varphi_k(z)|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^L \|q_k\|_{H^2}^2\right)^{\frac{1}{2}} \\
&\leq 1 + \frac{C}{\delta^{\max m_j}}.
\end{aligned}$$

Gathering up all the last estimates and plugging them into (6.15) yields

$$(6.16) \quad \sum_{j=1}^{\infty} |b_j(z)|^2 \leq \frac{C}{\delta^{2\max m_j}} \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta}\right)^2.$$

Together, (6.14) and (6.16) show that \mathbf{M}_B satisfies conditions (a) and (b) in Lemma 6.2. It is therefore a bounded operator whose norm satisfies

$$(6.17) \quad \|\mathbf{M}_B\| \leq \frac{C}{\delta^{\max m_j}} \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta}\right),$$

for some constant $C > 0$, which ends the proof of the theorem. \square

If $\|b\|_\infty < 1$, it is known that $\mathcal{H}(b) = H^2$, with an equivalent norm. Furthermore, in this case the Pythagorean mate a will have no zeros and so the exponent on δ in (6.17) will be $\max m_j = 0$. This corresponds to the estimate

$$\|\mathbf{M}_B\| \leq C \left(1 + \frac{1}{\delta^2} \log \frac{1}{\delta}\right)$$

which is the Uchiyama estimate from (1.7). Of course, Uchiyama's result was used in our proof.

7. FINAL REMARKS

As noted in the introduction, Theorem 6.5 is related to some of the results in [19, 20]. Here is how one makes the connection. If μ is a finite positive Borel measure on \mathbb{T} and P_μ is its Poisson integral

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} d\mu(\xi), \quad z \in \mathbb{D},$$

the *harmonically weighted Dirichlet space* \mathcal{D}_μ , introduced by Richter [22], is the space of $f \in H^2$ satisfying

$$\int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where dA is area measure. In [3], Costara and Ransford proved that a de Branges–Rovnyak space $\mathcal{H}(b)$, where $b \in \text{ball}(H^\infty)$ is rational and not a finite Blaschke product, coincides (with equivalent norms) with a \mathcal{D}_μ space if and only if the zeros on \mathbb{T} of the Pythagorean mate a of b are simple, that is, with our notation from (5.4), when all the zeros of a_1 are simple. If this happens, then the support of μ is precisely this set of (simple) zeros.

For this class of \mathcal{D}_μ spaces, Luo [19] proved a corona theorem, including estimates of the norm of the solutions. This turns out to be, when translated in the context of de Branges spaces, the particular case of Theorem 6.5 when $m_j = 1$ for all $1 \leq j \leq n$. One sees that our Theorem 6.5 covers the general case when the roots of a on \mathbb{T} have arbitrary multiplicities, where $\mathcal{H}(b)$ no longer coincides with a $\mathcal{D}(\mu)$ space. It should also be noted that part of the argument in the proof of Theorem 6.5 is similar to an argument from [19, Theorem 3.2.4].

Finally, note that in [20] Luo obtained a general corona theorem for harmonically weighted Dirichlet spaces \mathcal{D}_μ .

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