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# Polarized Linear Logic with Fixpoints

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## Abstract

We introduce and study  $\mu$ LLP, which can be viewed both as an extension of Laurent’s Polarized Linear Logic, LLP, with least and greatest fixpoints, and as a polarized version of Baelde’s Linear Logic with fixpoints ( $\mu$ MALL and  $\mu$ LL). We take advantage of the implicit structural rules of  $\mu$ LLP to introduce a term syntax for this language, in the spirit of the classical lambda-calculus and of system L in the style of Curien, Herbelin and Munch-Maccagnoni. We equip this language with a deterministic reduction semantics as well as a denotational semantics based on the notion of non-uniform totality spaces and the notion of categorical model for linear logic with fixpoint introduced by Ehrhard and Jafarrahmani. We prove an adequacy result for  $\mu$ LLP between these operational and denotational semantics, from which we derive a normalization property for  $\mu$ LLP thanks to the properties of the totality interpretation.

*Keywords:* linear logic, least and greatest fixed points, classical lambda-calculus, categorical semantics, relational semantics

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## 1 Introduction

One major feature of Linear Logic (LL) is that it gives a very clear and clean status to the polarity of formulas and connectives. Logical LL constants and connectives are organized in positive/negative dual pairs  $0/\top$ ,  $\oplus/\&$ ,  $1/\perp$  and  $\otimes/\wp$  which are De Morgan dual pairs swapped by linear negation  $(-)^{\perp}$ . From a proof-search point of view, negative constants and connectives are characterized by the reversibility of the associated sequent calculus inference rules and positive ones by the focusing property, see [5]. From a Curry-Howard point of view polarization is also extremely meaningful. Positive formulas  $P$ , stable under the positive connectives and containing the positive constants as well as all the formulas of shape  $!A$ , are equipped with a canonical proof of  $P \multimap !P$ . This implies that the negative formulas (the  $P^{\perp}$ ’s) have structural rules. This crucial property offers two ways to embed classical logic into LL (*call-by-name* and *call-by-value*) and each of these embeddings equip classical logic with an operationally meaningful cut-elimination<sup>5</sup>. This canonical proof  $P \multimap !P$  means that positive types are discardable and duplicable, that is, have an associated **let** construct, in other words, positive types are *data-types* (see [29] for discussions on this programming viewpoint, notably in connection with *call-by-push-value* [26,27]). This version of LL which features two dual classes of positive and negative formulas exchanged by linear negation is called

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<sup>5</sup> Gentzen’s cut-elimination has the dramatic property that given any two proofs of a given formula there is a proof of the same formula which reduces to the two given proofs. In other words, in this setting, any two proofs of the same formula are equal up to cut-elimination.

*Polarized Linear Logic* (LLP) and has been introduced and studied in [24], together with translations from the  $\lambda\mu$ -calculus [31] to LLP, a sequent calculus, a theory of proof-nets, translations to LL as well as a denotational semantics [25] which is based on the idea of representing positive formulas as objects of the Eilenberg-Moore category of the !-comonad of a categorical model of LL as suggested in [20]. In this system the exponential connectives allow to move from one polarity to the other one: if  $N$  is negative ! $N$  is positive and if  $P$  is positive ? $P$  is negative.

So LLP can be considered as a kind of “classical  $\lambda$ -calculus” (in the sense that the structural rules are kept implicit) where the fine-grained LL resource-handling associated with the exponentials is nevertheless available. In the single-sided version of LLP that we use in this paper, there are two kinds of sequents: those which have only negative formula (called *negative sequents*) and those which have negative formulas and *exactly one* positive formula (called *positive sequents*)<sup>6</sup>. Various proof-term systems have been introduced to account for LLP-like formalisms in a syntactic way that is closer to the  $\lambda$ -calculus than the usual proof systems such as sequent calculus or proof-nets, such systems are often called *system L* see for instance [11,29] and are based on the distinction between three kinds of expressions: (i) negative terms, (ii) positive terms and (iii) commands which are pairings (applications) of a negative term to a positive term representing logical cuts.

Many other polarized calculi have been considered in the literature, ranging from strictly logically-oriented calculi to more programming-oriented calculi. On the programming side of this spectrum, one can refer to Abel *et al*'s copatterns [2,3,33], which account for the definition of codata by observations and which revealed to be flexible enough to be integrated to various programming environment even with rich type-systems, as well as Ariola and Downen computational calculi developed on system L [15,16,17] which allow for the combination of various evaluation strategies. On the logical side, one can refer to Zeilberger's polarized calculi [34,28,35], directly inspired by Andreoli's focusing, or Baelde *et al*'s polarized calculus for  $\mu$ MALL [8] which follows a slightly different tradition on polarity [21]. On more related works from a categorical models point of view, one can mention [23], [22] and [1]. However, those models are more related to the proof-search paradigm and they deal with reversibility and focalization. And we emphasize on the fact that what we will provide as model of  $\kappa\mu$ LLP is related to the Curry-Howard paradigm, that is to say it deals with structural rules (i.e.,negative formulas have canonical contraction and weakening rules).

In many of the above-mentioned works, polarities are instrumental in shaping the dual computational behaviour of inductive versus coinductive types: while inductive data is positive in nature, coinductive data is negative. This observation was made formal from a proof-theoretic point of view in the line of work initiated by Baelde's thesis and his follow up works [6,7,9] that the least fixpoint is a positive logical construct while the greatest fixpoint is negative: in linear logic with fixpoints, the fixpoint operators satisfy the focusing property, both in the finitary setting with Park's rules and in the non-wellfounded and circular setting.

In this paper, following the Curry-Howard-Lambek approach to the linear logic with least and greatest fixpoints  $\mu$ LL initiated in [18], we study  $\mu$ LLP, an extension of LLP with least and greatest fixpoints (we refer to [6,18] for a general introduction to  $\mu$ LL and the associated literature), by introducing and studying a system L calculus for a polarized version of  $\mu$ LL. The search for a maximal syntactic simplicity guided our design of this calculus, see Figure 2 where we use the Greek letter  $\kappa$  for the name binder instead of the more traditional  $\mu/\tilde{\mu}$  which would lead to confusions with the standard notation associated with least and greatest fixpoints ( $\mu/\nu$ ). Related to this syntactical simplicity is that a negative term or a command can be typed by a negative or a positive sequent so that there are actually five kinds of typing judgments and this partitioning is taken into account by the semantics. The polarization of fixpoints means that least fixpoints allow to define data-types (integers, lists, trees *etc.*) while greatest fixpoints allow to define co-data-types, that is types of data-consumers<sup>7</sup>. We refer to [3] for a detailed discussion of the computational duality between data- and co-data-types. As in [7,18] we use the two-premise Park's rule for introducing greatest fixpoints so that our terms are finite trees.

The typed calculus introduced in that way is called  $\kappa\mu$ LLP. It features natural *construction* rules

<sup>6</sup> This simple dichotomy relies on the fact that, in LLP, the introduction of an !-connective is possible only on a negative sequent which corresponds to the fact that in LL promotion requires the context to have only ?-formulas.

<sup>7</sup> This strongly suggests that lists and streams are not of the same nature, streams are not data but data consumers.

associated with the positive connectives and constants (pairing for  $\otimes$ , injections for  $\oplus$  etc). It has a positive promotion construct  $s^!$  for putting a negative term  $s$  in a box (or *think*) which can be used as a piece of data and a negative dereliction  $\text{der}(p)$  which allows to open such a box. Just as the  $\lambda\mu$ -calculus,  $\kappa\mu\text{LLP}$  has names  $\alpha, \beta \dots$  associated with the negative formulas of a sequent<sup>8</sup>. Since at most one positive formula can occur in an LLP sequent, we need only one variable that we denote as  $\bullet$ . There are several binders for names: one general binder  $\kappa\alpha.c$  which allows to select a negative formula in the context, and the other ones are associated with  $\perp$ ,  $\wp$  and Park's rule. All these binders produce a negative term whose type is a negative formula made active for further uses. There is also a binder  $\tilde{\kappa}.c$  associated with the unique variable  $\bullet$  which produces a positive term. One crucial feature of  $\bullet$  is that it can occur only *linearly* in a command or negative term. Again, this is due to the fact that all the formulas in the context of a promotion must be negative. Notice also that  $\bullet$  cannot occur free in a positive term due to the fact that a positive sequent has exactly one positive formula. All these binders apply to *commands* which are cuts  $s \star p$  between a negative term  $s$  and a positive term  $p$ . Our operational semantics provides only reduction relations for commands and can be seen as describing the interactions between positive constructors and negative destructors. One specific critical command is  $\kappa\alpha.c \star \tilde{\kappa}.d$  which could *a priori* lead to  $c[\tilde{\kappa}.d/\alpha]$  or  $d[\kappa\alpha.c/\bullet]$ ; we choose the second option (see Remark 2.9) making our reduction semantics deterministic; we are actually defining a kind of abstract machine whose states are commands  $s \star p$  where  $s$  is the program and  $p$  is the stack (there are no environments because substitutions are executed immediately).

Our goal in this paper is twofold. On the one hand, we provide a categorical semantics of  $\kappa\mu\text{LLP}$  building on [18]. Given a model  $(\mathcal{L}, \vec{\mathcal{L}})$  (simply denoted as  $\mathcal{L}$ ) of  $\mu\text{LL}$  (in the sense of that paper) the main idea is standard: interpret a closed positive formula  $P$  as an object of the Eilenberg-Moore category  $\mathcal{L}^!$ . This requires however to deal also with open positive formulas: we take them into account introducing in Section 4 the notion of *positive functors* which are strong functors (as specified by  $\mathcal{L}$ ) equipped with a distributive law wrt. the comonad  $!$ , they are a functorial generalization of the notion of  $!$ -coalgebra. We illustrate this semantics in the concrete models **Rel** and **Nuts** used in [18].

On the other hand, we also prove some form of normalization (cut-elimination) for  $\kappa\mu\text{LLP}$ . This turned out to be surprisingly difficult. The solution came from understanding that proving a termination property with respect to the relational semantics — saying very roughly that if a command has a non-empty relational semantics its reduction terminates — would be possible because the points of the relational model are *finite* trees on which induction is possible. Concretely this means that we associate sets of terms to points of the relational models and these sets are easily defined by induction. To make this proof of normalization more natural we also provide a presentation of the relational semantics of  $\kappa\mu\text{LLP}$  as an intersection typing system. To derive from this relational normalization a standard normalization property, it is enough to prove that, in sufficiently many meaningful situations, when  $c \vdash \alpha : N$ , the command  $c$  has a non-empty relational semantics. We do that for  $N = ?\text{nat}$  where  $\text{nat}$  is a type of integers defined as a least fixpoint formula. The model **Nuts** of non-uniform *totality* spaces gives us precisely this information: any total subset of the relational interpretation of  $?\text{nat}$  is non-empty and hence, in particular, the interpretation of  $c$  is non-empty.

## 2 The syntax

**Remark 2.1** In this paper we present various deduction systems by means of inference rules. It is essential to keep in mind that, for all of these systems, we consider only *finite deduction trees*. Of course in other  $\mu\text{LL}$ -based logical systems [10,14,19,32] such as  $\mu\text{MALL}^\infty$ , some of these finite trees would be unfolded into infinite deduction trees (satisfying suitable global conditions), but such infinite deductions will never be considered here.

We assume to be given an infinite set of literals denoted  $\zeta, \xi, \dots$ . Positive formulas  $P, Q, \dots$  and negative formulas  $M, N, \dots$  are given in Figure 1. Linear negation is defined as usual by induction on formulas, it turns positive formulas into negative ones and conversely. The main cases are  $(\zeta^+)^\perp = \zeta^-$ ,  $(\zeta^-)^\perp = \zeta^+$ ,  $(\mu\zeta.P)^\perp = \nu\zeta.P^\perp$  and  $(\nu\zeta.N)^\perp = \mu\zeta.N^\perp$ .

<sup>8</sup> Indeed in the  $\lambda\mu$ -calculus all formulas are negative and the names are associated with the formulas occurring on the right side of a sequent whereas variables are associated with formulas on the left.

$$\begin{aligned}
 P, Q, \dots &:= \zeta^+ \mid 1 \mid P \otimes Q \mid 0 \mid P \oplus Q \mid \mu\zeta.P \mid !N \\
 M, N, \dots &:= \zeta^- \mid \perp \mid M \wp N \mid \top \mid M \& N \mid \nu\zeta.N \mid ?P
 \end{aligned}$$

Fig. 1. Syntax of formulas

**Remark 2.2** With each literal  $\zeta$  are associated *two distinct variables*  $\zeta^+$  and  $\zeta^-$  which are formulas (a literal is not a formula). In the formula  $\mu\zeta.P$ , only the occurrences of  $\zeta^+$  are bound and dually for  $\nu\zeta.N$  which binds only  $\zeta^-$ . Although not strictly necessary, we adopt the convention that for a given literal  $\zeta$  it is never the case that both  $\zeta^+$  and  $\zeta^-$  occur in a given formula. This property can be enforced by  $\alpha$ -renaming for closed formulas.

An important operation is substitution of formulas in formulas, it is defined in the obvious way (performing as usual  $\alpha$ -renaming of type variables when needed). More precisely we define by induction on a negative or positive formula  $A$ :

- substitution of a positive formula  $P$  for a positive variable  $\zeta^+$  in  $A$ , written  $A[P/\zeta^+]$
- and substitution of a negative formula  $N$  for a negative variable  $\zeta^-$  in  $A$ , written  $A[N/\zeta^-]$ ,

both formulas having the same polarity as  $A$ . We use  $\text{fv}(A)$  for the set of free variable of the type  $A$ .

**Lemma 2.3** *One has  $A[P/\zeta^+]^\perp = A^\perp [P^\perp/\zeta^-]$  and  $A[N/\zeta^-]^\perp = A^\perp [N^\perp/\zeta^+]$ .*

The proof is a simple induction on  $A$ . We shall quite often deal with formulas  $A$  where pairwise distinct positive variables  $\zeta_1^+, \dots, \zeta_k^+$  are substituted by  $P_1, \dots, P_k$  and pairwise distinct negative variables  $\xi_1^-, \dots, \xi_n^-$  are substituted by  $N_1, \dots, N_n$  *in parallel*. Again the definition is a straightforward induction on  $A$  but the use of such parallel substitutions can lead to quite heavy notations. We use letters such as  $\pi, \rho$  to denote such *type valuations*  $\pi = (N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+)$  and write  $A[\pi]$  rather than  $A[N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+]$ . We set  $\pi^\perp = (N_1^\perp/\xi_1^+, \dots, N_n^\perp/\xi_n^+, P_1^\perp/\zeta_1^-, \dots, P_k^\perp/\zeta_k^-)$ . We use  $\text{dom}(\pi)$  for the domain of  $\pi$  which is the set  $\{\zeta_1^+, \dots, \zeta_k^+, \xi_1^-, \dots, \xi_n^-\}$ . We use  $\pi \cdot (P/\zeta^+)$  and  $\pi \cdot (N/\zeta^-)$  to denote extensions of such valuations (assuming of course that  $\zeta^+, \zeta^- \notin \text{dom}(\pi)$ ). We say that  $\pi$  is closed if all formulas  $P_i, N_j$  are closed.

The syntax of terms is given in Figure 2. There are three kinds of terms: (i) positive terms denoted  $p, q, \dots$ , (ii) negative terms denoted  $s, t, \dots$  and (iii) commands (or processes) denoted  $c, d, \dots$ .

A negative context is a sequence  $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_n : N_n)$  and a positive context is a sequence  $\mathcal{P} = (\alpha_1 : N_1, \dots, \alpha_n : N_n, P)$ . We use  $\Gamma$  or  $\mathcal{N}[P]$  for contexts which are negative or positive.

Typing rules are provided in the same figure. Notice that there are actually five kinds of typing judgments:

- Positive term in a negative context  $\vdash \mathcal{N} \mid p : P$ ;
- negative term in a positive context  $\vdash \mathcal{N}, P \mid t : N$ ;
- command in a positive context  $c \vdash \mathcal{N}, P$ ;
- negative term in a negative context  $\vdash \mathcal{N} \mid t : N$ ;
- and command in a negative context  $c \vdash \mathcal{N}$ .

(This distinction will be crucial when dealing with the semantics.) It is also important to observe that in all of these judgments, all formulas are closed. One major difference wrt. the  $\lambda$ -calculus is that we put the context on the right side of the  $\vdash$  symbol, which means that in our setting the formulas of the context are negated wrt. what they would be in a  $\lambda$ -calculus with context on the left. This is why a name  $\alpha$  is a positive term and appears with a negative type in the context (see *e.g.* rule **(t-n)**).

**Remark 2.4** The expression  $\bullet$  should be considered as a variable, the unique variable of negative type. Because of this uniqueness we do not need to mention it in the context and  $\mathcal{N}, P$  should be read as  $\mathcal{N}, \bullet : P$ .

**Lemma 2.5** *If  $\alpha$  does not occur in negative context  $\mathcal{N}$ , the following holds. If  $c \vdash \mathcal{N}[P]$  then  $c \vdash \mathcal{N}, \alpha : N[\bullet]$ . If  $\vdash \mathcal{N}[P] \mid t : M$  then  $\vdash \mathcal{N}, \alpha : N[\bullet] \mid t : M$ . If  $\vdash \mathcal{N} \mid p : P$  then  $\vdash \mathcal{N}, \alpha : N[\bullet] \mid p : P$ .*

$$\begin{array}{l}
 p, q, \dots := \alpha \mid () \mid (p_1, p_2) \mid \text{in}_i(p) \mid \tilde{\kappa}.c \mid \text{fd}(p) \mid t^\dagger \\
 s, t, \dots := \bullet \mid \langle \rangle \mid \langle t_1, t_2 \rangle \mid \kappa\alpha.c \mid \kappa_\perp.c \mid \kappa(\alpha_1, \alpha_2).c \mid \bar{\kappa}_{N, \zeta}\alpha.(c; s) \mid \text{der}(p) \\
 c, d, \dots := t \star p \\
 \\
 \frac{}{\vdash \mathcal{N}, \alpha : P^\perp \mid \alpha : P} \text{ (t-n)} \quad \frac{}{\vdash \mathcal{N} \mid () : 1} \text{ (t-1)} \quad \frac{\vdash \mathcal{N} \mid p_1 : P_1 \quad \vdash \mathcal{N} \mid p_2 : P_2}{\vdash \mathcal{N} \mid (p_1, p_2) : P_1 \otimes P_2} \text{ (t-}\otimes\text{)} \\
 \frac{\vdash \mathcal{N} \mid p : P_i \quad i \in \{1, 2\}}{\vdash \mathcal{N} \mid \text{in}_i(p) : P_1 \oplus P_2} \text{ (t-}\oplus\text{)} \\
 \frac{c \vdash \mathcal{N}, P}{\vdash \mathcal{N} \mid \tilde{\kappa}.c : P} \text{ (t-}\tilde{\kappa}\text{)} \quad \frac{\vdash \mathcal{N} \mid p : P[\mu\zeta.P/\zeta^+]}{\vdash \mathcal{N} \mid \text{fd}(p) : \mu\zeta.P} \text{ (t-}\mu\text{)} \quad \frac{\vdash \mathcal{N} \mid t : N}{\vdash \mathcal{N} \mid t^\dagger : !N} \text{ (t-!)} \quad \frac{}{\vdash \mathcal{N}, P \mid \bullet : P^\perp} \text{ (t-}\bullet\text{)} \\
 \frac{}{\vdash \Gamma \mid \langle \rangle : \top} \text{ (t-}\top\text{)} \\
 \frac{\vdash \Gamma \mid t_1 : N_1 \quad \vdash \Gamma \mid t_2 : N_2}{\vdash \Gamma \mid \langle t_1, t_2 \rangle : N_1 \& N_2} \text{ (t-}\&\text{)} \quad \frac{c \vdash \mathcal{N}, \alpha : N[P]}{\vdash \mathcal{N}[P] \mid \kappa\alpha.c : N} \text{ (t-}\kappa\text{)} \quad \frac{c \vdash \Gamma}{\vdash \Gamma \mid \kappa_\perp.c : \perp} \text{ (t-}\perp\text{)} \\
 \frac{c \vdash \mathcal{N}, \alpha_1 : N_1, \alpha_2 : N_2[P]}{\vdash \mathcal{N}[P] \mid \kappa(\alpha_1, \alpha_2).c : N_1 \wp N_2} \text{ (t-}\wp\text{)} \\
 \frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-] \quad \vdash \mathcal{N}[Q] \mid s : P^\perp}{\vdash \mathcal{N}[Q] \mid \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; s) : \nu\zeta.R^\perp} \text{ (t-}\nu\text{)} \quad \frac{\vdash \mathcal{N} \mid p : P}{\vdash \mathcal{N} \mid \text{der}(p) : ?P} \text{ (t-?)} \\
 \frac{\vdash \mathcal{N}[P] \mid t : N \quad \vdash \mathcal{N} \mid p : N^\perp}{t \star p \vdash \mathcal{N}[P]} \text{ (t-cut)}
 \end{array}$$

Fig. 2. Syntax of terms and typing rules

In other words, we can use freely weakening on negative formulas in the context. Assume that  $\vdash \mathcal{N}, P \mid s : N$ . We have  $\vdash \mathcal{N}, \alpha : N \mid \alpha : N^\perp$  and hence  $s \star \alpha \vdash \mathcal{N}, \alpha : N, P$  so that  $\vdash \mathcal{N}, \alpha : N \mid \tilde{\kappa}.(s \star \alpha) : P$ , we shall often use this kind of *change of active formula*.

### 2.1 Substitution of terms in terms

Substitution for a variable  $\alpha$  and for  $\bullet$  are defined in the obvious way.

**Lemma 2.6** *Assume that  $\vdash \mathcal{N} \mid p : P$ .*

- If  $c \vdash \mathcal{N}, \alpha : P^\perp[Q]$  then  $c[p/\alpha] \vdash \mathcal{N}[Q]$ ,
- if  $\vdash \mathcal{N}, \alpha : P^\perp[Q] \mid t : N$  then  $\vdash \mathcal{N}[Q] \mid t[p/\alpha] : N$
- and if  $\vdash \mathcal{N}, \alpha : P^\perp \mid q : Q$  then  $\vdash \mathcal{N} \mid q[p/\alpha] : Q$ .

**Lemma 2.7** *Assume that  $\vdash \mathcal{N}[Q] \mid t : N$ .*

- If  $c \vdash \mathcal{N}, N^\perp$  then  $c[t/\bullet] \vdash \mathcal{N}[Q]$  and
- if  $\vdash \mathcal{N}, N^\perp \mid s : M$  then  $\vdash \mathcal{N}[Q] \mid s[t/\bullet] : M$ .

As a particular case of (t- $\nu$ ), if  $c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-]$  then  $\vdash \mathcal{N}, P \mid \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; \bullet) : \nu\zeta.R^\perp$  since  $\vdash \mathcal{N}, P \mid \bullet : P^\perp$ . We use (t- $\nu^1$ ) for this most important derived rule and use the notation  $\bar{\kappa}_{R^\perp, \zeta}\alpha.c = \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; \bullet)$ .

### 2.2 Substitution of terms in formulas

Assume that  $\vdash \mathcal{N}, P \mid s : N$ , let  $Q$  be a positive formula and let  $\zeta$  be a variable. Let also  $\pi$  be a type valuation such that  $\zeta^+, \zeta^- \notin \text{dom}(\pi)$ . Then we define two negative terms  $Q[\pi, s/\zeta^+]$  and  $Q[\pi, s/\zeta^-]$  in

$$\begin{array}{ll}
 s \star \tilde{\kappa}.c \rightarrow c[s/\bullet] & \kappa\alpha.c \star p \rightarrow c[p/\alpha] \quad \text{if } p \notin \tilde{\kappa} \\
 \langle s_1, s_2 \rangle \star \text{in}_i(p) \rightarrow s_i \star p & \kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rightarrow c[p_1/\alpha_1, p_2/\alpha_2] \\
 \kappa_{\perp}.c \star () \rightarrow c & \text{der}(p) \star s^! \rightarrow s \star p \\
 \bar{\kappa}_{R^{\perp}, \zeta\alpha}.(c; s) \star \text{fd}(p) \rightarrow (R[\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c/\zeta^+] \star p)[\kappa\alpha.(c[s/\bullet])/\bullet] & 
 \end{array}$$

Fig. 3. Reduction of commands

such a way that

$$\begin{array}{l}
 \vdash \mathcal{N}, Q[\pi, P/\zeta^+] \mid Q[\pi, s/\zeta^+] : Q^{\perp}[\pi^{\perp}, N/\zeta^-] \\
 \vdash \mathcal{N}, Q[\pi, N/\zeta^-] \mid Q[\pi, s/\zeta^-] : Q^{\perp}[\pi^{\perp}, P/\zeta^+] .
 \end{array}$$

In the first case (called *positive substitution*) we assume that  $\text{fv}(Q) \subseteq \text{dom}(\pi) \cup \{\zeta^+\}$  and in the second case (*negative substitution*), that  $\text{fv}(Q) \subseteq \text{dom}(\pi) \cup \{\zeta^-\}$ . The definition is by induction on  $Q$ . We give only the most interesting cases, the others are in Appendix 6.1.

▷ Assume that  $Q = \mu\xi.R$  and let us set  $R_P = R[P/\zeta^+]$  and  $R_N = R[N^{\perp}/\zeta^+]$ . Let  $\rho = \pi \cdot (\mu\xi.R_P[\pi]/\xi^+)$ . By inductive hypothesis we have defined  $t = R[\rho, s/\zeta^+]$  with  $\vdash \mathcal{N}, R_P[\rho] \mid t : R_N^{\perp}[\rho^{\perp}]$ . We have  $\vdash \mathcal{N}, \alpha : R_N^{\perp}[\rho^{\perp}] \mid \tilde{\kappa}.(t \star \alpha) : R_P[\rho]$ . Notice that, due to the definition of  $\rho$ , we have  $R_S[\rho] = R_S[\pi][\mu\xi.R_P[\pi]/\xi^+]$  for  $S = P, N$ . It follows that  $\vdash \mathcal{N}, \alpha : R_N^{\perp}[\rho^{\perp}] \mid \text{fd}(\tilde{\kappa}.(t \star \alpha)) : \mu\xi.R_P[\pi]$ . Therefore  $(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha))) \vdash \mathcal{N}, \mu\xi.R_P[\pi], \alpha : R_N^{\perp}[\rho^{\perp}] = (R_N[\pi][\mu\xi.R_P[\pi]/\xi^+])^{\perp}$  so we define  $(\mu\xi.R)[\pi, s/\zeta^+]$  as  $\bar{\kappa}_{R_N^{\perp}[\pi^{\perp}], \zeta\alpha}(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha)))$  by  $(\mathbf{t}\nu^1)$ .

▷ Assume last that  $Q = !R^{\perp}$ . By inductive hypothesis we have defined  $t = R[\pi^{\perp}, s/\zeta^-]$  which satisfies  $\vdash \mathcal{N}, R[\pi^{\perp}, N/\zeta^-] \mid t : R^{\perp}[\pi, P/\zeta^+]$ . It follows that  $\vdash \mathcal{N}, \alpha : R^{\perp}[\pi, P/\zeta^+] \mid \tilde{\kappa}.(t \star \alpha) : R[\pi^{\perp}, N/\zeta^-]$  and hence  $\vdash \mathcal{N}, \alpha : R^{\perp}[\pi, P/\zeta^+] \mid \text{der}(\tilde{\kappa}.(t \star \alpha)) : ?R[\pi^{\perp}, N/\zeta^-]$  so that  $\vdash \mathcal{N}, \beta : ?R[\pi^{\perp}, N/\zeta^-] \mid \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta) : R^{\perp}[\pi, P/\zeta^+]$ . Next we obtain  $\vdash \mathcal{N}, \beta : ?R[\pi^{\perp}, N/\zeta^-] \mid \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^! : !R^{\perp}[\pi, P/\zeta^+]$  so that  $\vdash \mathcal{N}, !R^{\perp}[\pi, P/\zeta^+] \mid \kappa\beta.(\bullet \star \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^!) : ?R[\pi^{\perp}, N/\zeta^-]$  and we define  $!R^{\perp}[\pi, s/\zeta^+]$  as the term  $\kappa\beta.(\bullet \star \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^!)$ .

In the same induction we define completely similarly the negative substitution  $Q[\pi, s/\zeta^-]$ .

### 2.3 Reduction relation

We equip commands with a deterministic rewriting relation  $\rightarrow$  specified in Figure 3. We use  $\tilde{\kappa}$  for the set of all positive terms of shape  $\tilde{\kappa}.c$ . The reduction of  $\bar{\kappa}_{R^{\perp}, \zeta\alpha}.(c; s) \star \text{fd}(p)$  deserves some typing. We must have  $\vdash \mathcal{N} \mid p : R[\mu\zeta.R/\zeta^+]$  so that  $\vdash \mathcal{N} \mid \text{fd}(p) : \mu\zeta.R$ ,  $c \vdash \mathcal{N}, P, \alpha : R^{\perp}[P^{\perp}/\zeta^-]$  and  $\vdash \mathcal{N}[Q] \mid s : P^{\perp}$  so  $\vdash \mathcal{N}[Q] \mid \bar{\kappa}_{R^{\perp}, \zeta\alpha}.(c; s) : \nu\zeta.R^{\perp}$  by  $(\mathbf{t}\nu)$  and hence  $\bar{\kappa}_{R^{\perp}, \zeta\alpha}.(c; s) \star \text{fd}(p) \vdash \mathcal{N}[Q]$ .

We also have  $\vdash \mathcal{N}, P \mid \bar{\kappa}_{R^{\perp}, \zeta\alpha}.c : \nu\zeta.R^{\perp}$  by  $(\mathbf{t}\nu^1)$ . By positive substitution we get  $\vdash \mathcal{N}, R[P/\zeta^+] \mid R[\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c/\zeta^+] : R^{\perp}[\nu\zeta.R^{\perp}/\zeta^-]$  and hence  $R[\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c/\zeta^+] \star p \vdash \mathcal{N}, R[P/\zeta^+]$ . Observe last that  $\vdash \mathcal{N}, P \mid \kappa\alpha.c : R^{\perp}[P^{\perp}/\zeta^-]$  thus  $\vdash \mathcal{N}[Q] \mid \kappa\alpha.(c[s/\bullet]) : R^{\perp}[P^{\perp}/\zeta^-]$  so that  $(R[\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c/\zeta^+] \star p)[\kappa\alpha.(c[s/\bullet])/\bullet] \vdash \mathcal{N}[Q]$ .

As a special case of this reduction rule we have the following one that we shall often use:  
 $\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c \star \text{fd}(p) \rightarrow (R[\bar{\kappa}_{R^{\perp}, \zeta\alpha}.c/\zeta^+] \star p)[\kappa\alpha.c/\bullet]$ .

**Theorem 2.8** *If  $c \vdash \Gamma$  and  $c \rightarrow c'$  then  $c' \vdash \Gamma$ .*

The proof is an easy verification, using Lemmas 2.6 and 2.7.

We use SN for the set of all  $c$ 's which are normalizing for this reduction relation.

$$a, b, \dots := * \mid (j, \zeta) \mid (a, b) \mid (i, a) \mid [a_1, \dots, a_n] \mid \sigma(a) \quad \text{with } j \in I \text{ and } i \in \{1, 2\}$$

$$\text{sz}(\ast) = \text{sz}(j, \zeta) = 1 \quad \text{sz}(a, b) = \text{sz}(a) + \text{sz}(b) \quad \text{sz}(i, a) = \text{sz}(\sigma(a)) = 1 + \text{sz}(a) \quad \text{sz}([a_1, \dots, a_n]) = 1 + \sum_{i=1}^n \text{sz}(a_i)$$

$$\begin{array}{c}
 \frac{}{(j, \zeta) : \zeta} \text{ (p-var)} \quad \frac{}{\ast : 1} \text{ (p-1)} \quad \frac{a : P \quad b : Q \quad \forall \xi (\text{rg}_\xi a \cap \text{rg}_\xi b = \emptyset)}{(a, b) : P \otimes Q} \text{ (p-}\otimes\text{)} \\
 \\
 \frac{(a_i : N^\perp)_{i=1}^n \quad \forall \xi \forall i \neq i' (\text{rg}_\xi a_i \cap \text{rg}_\xi a_{i'} = \emptyset)}{[a_1, \dots, a_n] : !N} \text{ (p-!)} \quad \frac{a : P_i}{(i, a) : P_1 \oplus P_2} \text{ (p-}\oplus\text{)} \quad \frac{a : P [\mu\zeta.P/\zeta^+]}{\sigma(a) : \mu\zeta.P} \text{ (p-}\mu\text{)} \\
 \\
 \frac{k \in \mathbb{N}}{\ast \tilde{1} k[\ast]} \text{ (s-1)} \quad \frac{a^i \tilde{P}_i [a_1^i, \dots, a_n^i] \text{ for } i = 1, 2}{(a^1, a^2) \widetilde{P_1 \otimes P_2} [(a_1^1, a_1^2), \dots, (a_n^1, a_n^2)]} \text{ (s-}\otimes\text{)} \quad \frac{\underline{k} = \bigsqcup_{i=1}^m J_i \quad (a_j : N^\perp)_{j=1}^k}{[a_1, \dots, a_k] \widetilde{!N} [[a_j \mid j \in J_i] \mid i = 1, \dots, m]} \text{ (s-!)} \\
 \\
 \frac{i \in \{1, 2\} \quad a \tilde{P}_i [a_1, \dots, a_n]}{(i, a) \widetilde{P_1 \oplus P_2} [(i, a_1), \dots, (i, a_n)]} \text{ (s-}\oplus\text{)} \quad \frac{a R [\mu\zeta.R/\zeta^+] [a_j \mid j \in J]}{\sigma(a) \widetilde{\mu\zeta.R} [\sigma(a_j) \mid j \in J]} \text{ (s-}\mu\text{)}
 \end{array}$$

Fig. 4. Syntax, size, typing rules and structural relation for points

**Remark 2.9** Our choice of reducing the “critical pair”  $\kappa\alpha.c \star \tilde{\kappa}.d$  to  $d[\kappa\alpha.c/\bullet]$  and not to  $c[\tilde{\kappa}.d/\alpha]$  will be essential in our proof of normalization. However this choice is not as critical as in classical systems such as system L [11] or the  $\lambda\mu$ -calculus [31,30]: our denotational semantics is compatible with both choices.

**Lemma 2.10** *If  $c \vdash \mathcal{N}, P, c \rightarrow c'$  and  $\vdash \mathcal{N} \mid s : P^\perp$  then  $c[s/\bullet] \rightarrow c'[s/\bullet]$ . If  $c \vdash \mathcal{N}, \alpha : P^\perp, Q, c \rightarrow c'$  and  $\vdash \mathcal{N} \mid p : P$  with  $p \notin \tilde{\kappa}$  then  $c[p/\alpha] \rightarrow c'[p/\alpha]$ .*

The proof is a simple inspection of the reduction rules.

## 2.4 The example of integers

We define a type of integers as  $\text{nat} = \mu\zeta.(1 \oplus \zeta^+)$ . We set  $\underline{0} = \text{fd}(\text{in}_1())$  so that  $\vdash \mathcal{N} \mid \underline{0} : \text{nat}$  and given a positive term  $p$  such that  $\vdash \mathcal{N} \mid p : \text{nat}$  we set  $\text{succ } p = \text{fd}(\text{in}_2(p))$  so that  $\vdash \mathcal{N} \mid \text{succ } p : \text{nat}$ . Let  $p, t$  and  $s$  be such that  $\vdash \mathcal{N} \mid p : P, \vdash \mathcal{N}, P \mid t : P^\perp$  and  $\vdash \mathcal{N}[Q] \mid s : P^\perp$ . Then  $\vdash \mathcal{N}, P \mid \langle \kappa_\perp.(\bullet \star p), t \rangle : \perp \& \text{nat}^\perp$  and we set  $\text{it}(p, t; s) = \bar{\kappa}_{\perp \& \zeta^-, \zeta} \alpha.(\langle \kappa_\perp.(\bullet \star p), t \rangle \star \alpha; s)$  so that  $\vdash \mathcal{N}[Q] \mid \text{it}(p, t; s) : \text{nat}^\perp$  and  $\text{it}(p, t; s)$  should be understood as iterating the function  $t$  with  $p$  as initial value and  $s$  as continuation. Indeed one can check that  $(1 \oplus \zeta^+) [u/\zeta^+] = \langle \kappa\alpha.(\bullet \star \text{in}_1 \alpha), \kappa\alpha.(\bullet \star \text{in}_2(\tilde{\kappa}.(u \star \alpha))) \rangle$  from which it follows by simple computations that  $\text{it}(p, t; s) \star \underline{0} \rightarrow^* s \star p$  and  $\text{it}(p, t; s) \star \text{succ } q \rightarrow^* \text{it}(p, t; t[s/\bullet]) \star q$ .

## 3 An intersection typing system

### 3.1 The syntax of points

Let  $I$  be an infinite and countable set of indices (we can take  $I = \mathbb{N}$ ). The syntax of *relational types* or *points* is given in Figure 4, as well as the size  $\text{sz}(a)$  of a point, which is an integer  $\geq 1$ . Given a point  $a$  and a literal  $\xi$ , we define a finite subset  $\text{rg}_\xi(a)$  of  $I$  as follows:  $\text{rg}_\xi \ast = \emptyset$ ,  $\text{rg}_\xi(a, b) = \text{rg}_\xi a \cup \text{rg}_\xi b$ ,  $\text{rg}_\xi(j, \xi) = \{j\}$ ,  $\text{rg}_\xi(j, \zeta) = \emptyset$  if  $\zeta \neq \xi$ ,  $\text{rg}_\xi(i, a) = \text{rg}_\xi a$ ,  $\text{rg}_\xi[a_1, \dots, a_n] = \bigcup_{i=1}^n \text{rg}_\xi a_i$  and  $\text{rg}_\xi \sigma(a) = \text{rg}_\xi a$ . We give a typing system for these points in Figure 4. Its main purpose is to enforce that, when  $a : P$ , given a literal  $\zeta$ , the indices of  $I$  associated with  $\zeta$  in  $a$  are pairwise distinct. Given  $a^0 : R$ , a literal  $\zeta$  and a family of points  $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$  such that  $b_j : P$  for all  $j$  and such that for any literal  $\xi$  the sets  $\text{rg}_\xi b_j$  are pairwise disjoint and disjoint from  $\text{rg}_\xi a^0$  (when these disjointness conditions hold we say that the pair  $(a^0, \vec{b})$  is *adapted*) then we define in the obvious way the point  $a^0 \{b_j/(j, \zeta)\}_{j \in J}$  for  $J = \text{rg}_\zeta a^0$  such that

$$\begin{array}{c}
 \frac{(a_i \tilde{P}_i [])_{i \in \mathbb{N} \setminus \{j\}} \quad (\mathbf{i-n}) \quad \frac{(a_i \tilde{P}_i [])_{i=1}^n \quad (\mathbf{i-1})}{\vdash \Phi \mid () : * : 1}}{\vdash \Phi \mid \alpha_j : a_j : P_j} \quad (\mathbf{i-1}) \\
 \frac{(\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid p_j : b_j : Q_j)_{j=1,2} \quad (a_i \tilde{P}_i [a_i^1, a_i^2])_{i=1}^n \quad (\mathbf{i-\otimes})}{\vdash \Phi \mid (p_1, p_2) : (b_1, b_2) : Q_1 \otimes Q_2} \\
 \frac{\vdash \Phi \mid p : a : Q_i \quad (\mathbf{i-\oplus}) \quad \frac{c \vdash \Phi, a : P}{\vdash \Phi \mid \tilde{\kappa}.c : a : P} \quad (\mathbf{i-\tilde{\kappa}}) \quad \frac{\vdash \Phi \mid p : a : P [\mu\zeta.P/\zeta^+]}{\vdash \Phi \mid \text{fd}(p) : \sigma(a) : \mu\zeta.P} \quad (\mathbf{i-\mu})}{\vdash \Phi \mid \text{in}_i(p) : (i, a) : Q_1 \oplus Q_2} \\
 \frac{(\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid s : b_j : N)_{j \in J} \quad (a_i \tilde{P}_i [a_i^j \mid j \in J])_{i=1}^n \quad (\mathbf{i-!}) \quad \frac{(a_i \tilde{P}_i [])_{i=1}^n}{\vdash \Phi, a : P \mid \bullet : a : P^\perp} \quad (\mathbf{i-\bullet})}{\vdash \Phi \mid s^! : [b_j \mid j \in J] : !N} \\
 \frac{c \vdash \Phi, \alpha : a : N [b : P] \quad (\mathbf{i-\kappa}) \quad \frac{c \vdash \Phi, \alpha_1 : a_1 : N_1, \alpha_2 : a_2 : N_2 [b : P]}{\vdash \Phi [b : P] \mid \kappa(\alpha_1, \alpha_2).c : (a_1, a_2) : N_1 \wp N_2} \quad (\mathbf{i-\wp})}{\frac{c \vdash \Phi [a : P]}{\vdash \Phi [a : P] \mid \kappa_\perp.c : * : \perp} \quad (\mathbf{i-\perp})} \\
 \frac{\vdash \Phi [a : P] \mid s_i : b : N_i \quad \vdash \mathcal{N} [P] \mid s_{3-i} : N_{3-i} \quad (\mathbf{i-\&}) \quad \frac{\vdash \Phi \mid p : a : P}{\vdash \Phi \mid \text{der}(p) : [a] : ?P} \quad (\mathbf{i-?})}{\vdash \Phi [a : P] \mid \langle s_1, s_2 \rangle : (i, b) : N_1 \& N_2} \\
 \frac{\vdash (\alpha_i : a_i^1 : P_i^\perp) [a : P] \mid s : b : P^\perp \quad \vdash (\alpha_i : a_i^2 : ) \mid p : b : P \quad (a_i \tilde{P}_i [a_i^1, a_i^2])_{i=1}^n \quad (\mathbf{i-cut})}{(t \star p) \vdash \Phi [a : P]}
 \end{array}$$

Fig. 5. Point deduction system — the rule (i-ν) is given in the body of Section 3.2.

$a^0 \{b_j / (j, \zeta)\}_{j \in J} : R [P/\zeta^+]$ . One proves easily that, for any literal  $\xi$

$$\text{rg}_\xi a^0 \{b_j / (j, \zeta)\}_{j \in J} = \text{rg}_\xi a^0 \uplus \bigsqcup_{j \in \text{rg}_\zeta a^0} \text{rg}_\xi b_j. \quad (1)$$

Crucially, this point substitution is in some sense reversible.

**Lemma 3.1** *Let  $R$  and  $P$  be positive formulas and let  $\zeta$  be a literal. Let  $a$  be a point such that  $a : R [P/\zeta^+]$ . Let  $J \subseteq I$  be an infinite set. There is a point  $a^0$  such that  $a^0 : R$  and  $\text{rg}_\zeta a^0 \subseteq J$  and there is a family of points  $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$  such that  $b_j : P$  for all  $j \in \text{rg}_\zeta a^0$ ,  $(a^0, \vec{b})$  is adapted and  $a = a^0 \{b_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a^0}$ .*

Given a closed positive type  $P$ , we define a binary relation  $\tilde{P}$  between points  $a$  and multisets of points  $[a_1, \dots, a_n]$  where  $a, a_1, \dots, a_n : P$ . The definition is provided as a deduction system in Figure 4. Notice that in each of the deduction rules the sum of the sizes of the points occurring on the left in the premises is strictly smaller than the size of the point occurring on the left in the conclusion. So the size of such a deduction tree is upper-bounded by the size of the point occurring on the left in its conclusion.

### 3.2 The point typing system

A *negative point typing context* is a sequence  $\Phi = (\alpha_1 : a_1 : N_1, \dots, \alpha_k : a_k : N_k)$  where the  $\alpha_i$ 's are pairwise distinct and  $a_i : N_i^\perp$  for each  $i$ . A *positive point typing context* is a sequence  $\Phi, a : P$  with  $a : P$ . In these rules we use  $\Phi$  to denote the context  $(\alpha_i : a_i : P_i^\perp)_{i=1}^n$  and  $\mathcal{N}$  for the ordinary typing context  $(\alpha_i : P_i^\perp)_{i=1}^n$ . All rules but (i-ν) are given in Figure 5. Notice that there are two instances of the rules (i-⊕) and (i-⋆), one for  $i = 1$  and one for  $i = 2$ . We give now the lacking (i-ν) inference rule: if  $h : Q$ ,  $d : R$ ,  $(b_l : P)_{l \in L}$  where  $L = \text{rg}_\zeta d$ , and

- $\vdash (\alpha_i : a_i'' : P_i^\perp)_{i=1}^n [h : Q] \mid s : b : P^\perp$
- $c \vdash (\alpha_i : a_i' : P_i^\perp)_{i=1}^n, b : P, \alpha : d \{b_l / (l, \zeta)\}_{l \in L} : R^\perp [P^\perp / \zeta^-]$
- $(\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, b_l : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : f_l : \nu \zeta.R^\perp)_{l \in L}$

- $a_i \tilde{P}_i [a'_i, a''_i] + [a_i^l \mid l \in L]$  for  $i = 1, \dots, n$

then  $\vdash \Phi, h : Q \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s) : \sigma(d)\{f_l/(l, \zeta)\}_{l \in L} : \nu \zeta.R^\perp$ .

Upon taking  $s = \bullet$  we obtain the following derived rule ( $\mathbf{i}\text{-}\nu^\perp$ ) (with the same notations as above). If

- $c \vdash (\alpha_i : a'_i : P_i^\perp)_{i=1}^n, b : P, \alpha : d\{b_l/(l, \zeta)\}_{l \in L} : R^\perp [P^\perp/\zeta^-]$
  - $(\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, b_l : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : f_l : \nu \zeta.R^\perp)_{l \in L}$
  - $a_i \tilde{P}_i [a'_i] + [a_i^l \mid l \in L]$  for  $i = 1, \dots, n$
- then  $\vdash \Phi, b : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : \sigma(d)\{f_l/(l, \zeta)\}_{l \in L} : \nu \zeta.R^\perp$ .

### 3.3 Interpretation of points

Given a set  $\mathcal{P}$  of positive terms  $p$  such that  $\vdash \mid p : P$ , we set  $\mathcal{P}^\bullet = \{c \mid c \vdash P \text{ and } \exists p \in (\mathcal{P} \setminus \tilde{\kappa}) c \rightarrow^* \bullet \star p\}$  and  $\mathcal{P}^{\tilde{\kappa}} = \mathcal{P} \cup \{\tilde{\kappa}.c \mid c \in \mathcal{P}^\bullet\}$  so that any  $p \in \mathcal{P}^{\tilde{\kappa}}$  satisfies  $\vdash \mid p : P$ . We set  $\perp = \text{SN}$ . Given  $a : P$  we define  $\perp(a : P) = |a|_{\mathcal{P}^\bullet}$  and we set  $|a|_P = \|\!|a|\!\|_{\tilde{\mathcal{P}}}$  where

- $\|\!*\!\|_1 = \{()\}$
- $\|\!(a_1, a_2)\!\|_{P_1 \otimes P_2} = \{(p_1, p_2) \mid p_i \in |a_i|_{P_i} \text{ for } i = 1, 2\}$
- $\|\!(i, a)\!\|_{P_1 \oplus P_2} = \{\text{in}_i(p) \mid p \in |a|_{P_i}\}$  for  $i = 1, 2$
- $\|\!|a_j \mid j \in J|\!\|_{\perp N} = \{t^l \mid \forall j \in J \forall p \in |a_j|_{N^\perp} t \star p \in \perp\}$  – notice that in that formula  $\vdash \mid t : N$ .
- $\|\!\sigma(a)\!\|_{\mu \zeta.P} = \{\text{fd}(p) \mid p \in |a|_{P[\mu \zeta.P/\zeta^+]}\}$ .

Notice that  $\|\!a|\!\|_P = |a|_P \setminus \tilde{\kappa}$  since  $\|\!a|\!\|_P \cap \tilde{\kappa} = \emptyset$ . As an auxiliary notion, given  $a : P$  and  $b : N^\perp$  we set  $|b|_N(a : P) = \{s \mid \vdash P \mid s : N \text{ and } \forall p \in |b|_{N^\perp} s \star p \in \perp(a : P)\}$ . We will also use  $|b|_N$  for the set of  $s$  such that  $\vdash \mid s : N$  and  $\forall p \in |b|_{N^\perp} s \star p \in \perp$ .

**Lemma 3.2** *Let  $a : P, b : Q$  and  $s$  be such that  $\vdash P \mid s : Q^\perp$ . If  $\forall p \in \|b\|_Q s \star p \in \perp(a : P)$  then  $s \in |b|_{Q^\perp}(a : P)$ .*

**Lemma 3.3** *If  $a \tilde{P} [a_1, \dots, a_n]$ . Then  $\|\!a|\!\|_P \subseteq \|\!a_i|\!\|_P$  and  $|a|_P \subseteq |a_i|_P$  for  $1 \leq i \leq n$ .*

**Lemma 3.4** *Let  $P$  be a closed positive formula and let  $a : P$ . If  $b : P$ , one has  $a \tilde{P} [b]$  iff  $a = b$ . Let  $(b_j : P)_{j \in J}, (J_k)_{k \in K}$  be such that  $\biguplus_{k \in K} J_k = J$ . Then one has  $a \tilde{P} [b_j \mid j \in J]$  iff there is  $(c_k : P)_{k \in K}$  such that  $a \tilde{P} [c_k \mid k \in K]$  and  $c_k \tilde{P} [b_j \mid j \in J_k]$  for each  $k \in K$ .*

**Proof.** Straightforward induction on the size of  $a$ . □

Upon taking  $K = J = \emptyset$  this implies in particular  $\exists e : P e \tilde{P} []$ . Such an  $e$  is called a *coneutral point* of  $P$ . By Lemma 3.4, if  $a \tilde{P} [e, b]$  and  $e$  is coneutral then  $a = b$ , and if  $a : P$  there is a coneutral  $e$  such that  $a \tilde{P} [e, a]$  (which depends generally on  $a$ ). Coneutral points are generally not unique: for instance in  $1 \oplus 1$ , both  $(1, *)$  and  $(2, *)$  are coneutral.

**Lemma 3.5** *Let  $Q$  be a positive formula,  $\eta$  a literal,  $\pi$  a closed type valuation such that all free type variables of  $Q$ , but possibly  $\eta^+, \eta^-$ , are in  $\text{dom}(\pi)$ . Let  $d : Q[\pi]$ , assume that  $\vdash (\alpha_i : a_i^j : N_i)_{i=1}^n, b_j : P \mid s : c_j : N$  for each  $j \in \text{rg}_\eta d$ . If  $(a_i : N_i^\perp)_{i=1}^n$  are such that for all  $i \in \underline{n}$  one has  $a_i \widetilde{N_i^\perp} [a_i^j \mid j \in \text{rg}_\eta d]$  then  $\vdash (\alpha_i : a_i : N_i), d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q[\pi, P/\eta^+] \mid Q[\pi, s/\eta^+] : d\{c_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q^\perp [\pi^\perp, N/\eta^-]$  and  $\vdash (\alpha_i : a_i : N_i), d\{c_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q[\pi, N/\eta^-] \mid Q[\pi, s/\eta^-] : d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q^\perp [\pi^\perp, P/\eta^+]$ .*

**Lemma 3.6** *Let  $Q$  be a positive formula,  $\eta$  a literal,  $\pi$  a closed type valuation such that all free type variables of  $Q$ , but possibly  $\eta^+, \eta^-$ , are in  $\text{dom}(\pi)$  and let  $d : Q[\pi]$ . If  $s \in |c_j|_N(b_j : P)$  for all  $j \in J = \text{rg}_\eta d$  then  $Q[\pi, s/\eta^+] \in |d(\vec{c})|_{Q^\perp[\pi^\perp, N/\eta^-]}(d(\vec{b})) : Q[\pi, P/\eta^+]$  where  $d(\vec{b}) = d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d}$  and similarly for  $d(\vec{c})$ . And we have  $Q[\pi, s/\eta^-] \in |d(\vec{b})|_{Q^\perp[\pi^\perp, P/\eta^+]}(d(\vec{c})) : Q[\pi, N/\eta^-]$ .*

### 3.4 Example: the integers

We continue the example initiated in Section 2.4. There is a bijection from  $\mathbb{N}$  to the points  $a : \mathbf{nat}$  which maps 0 to  $\bar{0} = \sigma(1, *)$  and  $n + 1$  to  $\bar{n + 1} = \sigma(2, \bar{n})$ . With these notations the following rules are derivable in the point typing system.

- If  $(a_i \widetilde{N}_i [])_{i=1}^n$  then  $\vdash (\alpha_i : a_i : P_i)_{i=1}^n \mid \bar{0} : \bar{0} : \mathbf{nat}$ .
- If  $\vdash \Phi \mid p : \bar{k} : \mathbf{nat}$  then  $\vdash \Phi \mid \mathbf{succ} p : \bar{k + 1} : \mathbf{nat}$ .
- If  $\vdash (\alpha_i : a_i^1 : N_i)_{i=1}^n \mid p : a : P, \vdash (\alpha_i : a_i^2 : N_i)_{i=1}^n \mid [b : Q] \mid a : s : P^\perp$  and  $(a_i \widetilde{N}_i^\perp [a_i^1, a_i^2])_{i=1}^n$  then  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n \mid [b : Q] \mid \mathbf{it}(p, t; s) : \bar{0} : \mathbf{nat}^\perp$ .
- If  $\vdash (\alpha_i : a_i^1 : N_i)_{i=1}^n \mid [b : Q] \mid s : a : P^\perp, \vdash (\alpha_i : a_i^2 : N_i)_{i=1}^n, a : P \mid t : a' : P^\perp, \vdash (\alpha_i : a_i^3 : N_i)_{i=1}^n, a' : P \mid \mathbf{it}(p, t; \bullet) : \bar{n} : \mathbf{nat}^\perp$  and  $(a_i \widetilde{N}_i^\perp [a_i^1, a_i^2, a_i^3])_{i=1}^n$  then  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n \mid [b : Q] \mid \mathbf{it}(p, t; s) : \bar{n + 1} : \mathbf{nat}^\perp$ .

### 3.5 Interpretation Theorem

Using the previous lemmas it is not very hard to relate the point typing system with normalization.

**Theorem 3.7** *Let  $\Phi = (\alpha_i : a_i : P_i)_{i=1}^n$  be a point context and let  $a : P$ , let  $p_i \in |a_i|_{P_i}$  for  $i = 1, \dots, n$ .*

- (i) *If  $c \vdash \Phi$  then  $c[p_i/\alpha_i]_{i=1}^n \in \perp$ .*
- (ii) *If  $c \vdash \Phi, a : P$  then  $c[p_i/\alpha_i]_{i=1}^n \in \perp(a : P)$ .*
- (iii) *If  $\vdash \Phi \mid s : b : N$  then  $s[p_i/\alpha_i]_{i=1}^n \in |b|_N$ .*
- (iv) *If  $\vdash \Phi, a : P \mid t : b : N$  then  $t[p_i/\alpha_i]_{i=1}^n \in |b|_N(a : P)$ .*
- (v) *If  $\vdash \Phi \mid p : a : P$  then  $p[p_i/\alpha_i]_{i=1}^n \in |a|_P$ .*

### 3.6 Application: a normalization property

Anticipating on Section 5, we explain now how one can use Theorem 3.7, combined with the **Nuts** denotational semantics of Section 5.1, to prove “normalization” properties for  $\kappa\mu\text{LLP}$  commands. More precisely we show that the reduction of programs of type integer in  $\kappa\mu\text{LLP}$  terminates but of course many more applications are possible. We explain first why the type of such a program should be  $?\mathbf{nat}$  and not  $\mathbf{nat}$ .

Imagine that we want to encode in  $\kappa\mu\text{LLP}$  a typed  $\lambda$ -calculus extended with primitive recursion on some data types, in the spirit of Gödel’s  $\mathbb{T}$ ; we shall simply call this system  $\mathbb{T}$ . This calculus will have two kinds of types: *data-types* which are associated with positive formulas of  $\kappa\mu\text{LLP}$  and *function types*  $\sigma \Rightarrow \tau$ . In  $\mathbb{T}$  there will be at least a data-type of integers  $\iota$  associated with  $\mathbf{nat}$  and possibly other ones, for instance a type of binary trees with leaves labeled by natural numbers associated with  $\mu\zeta.(\mathbf{nat} \oplus (\zeta \otimes \zeta))$ . We would like to use the Girard encoding  $(\sigma \Rightarrow \tau)^- = !\sigma^- \multimap \tau^-$  where as usual  $A \multimap B = A^\perp \wp B$ . In other words  $(\sigma \Rightarrow \tau)^- = ?(\sigma^-)^\perp \wp \tau^-$ . The polarity constraints of  $\kappa\mu\text{LLP}$  require  $\sigma^-$  to be negative, this prevents us from setting simply  $\iota^- = \mathbf{nat}$  since  $\mathbf{nat}$  is positive. For benefiting from the structural rules available for free on all negative formulas in  $\kappa\mu\text{LLP}$  we introduce also a positive translation defined by  $\iota^+ = \mathbf{nat}$  (and similarly for all data-types of  $\mathbb{T}$ ) and  $\sigma^+ = !\sigma^-$  if  $\sigma$  is not a data-type and then the negative translation can be defined by  $\iota^- = ?\mathbf{nat}$  (and similarly for all data-types) and  $(\sigma \Rightarrow \tau)^- = (\sigma^+)^\perp \wp \tau^-$ . For instance  $((\iota \Rightarrow \iota) \Rightarrow \iota)^- = ?(\mathbf{nat} \otimes !\mathbf{nat}^\perp) \wp ?\mathbf{nat}$ . A closed term of type  $\iota$  of  $\mathbb{T}$  will therefore be translated into a  $\kappa\mu\text{LLP}$  negative term  $t$  such that  $\vdash t : ?\mathbf{nat}$  and we can form  $c = t \star \alpha$ . Now we explain how we can extract effectively a value  $\in \mathbb{N}$  from such a  $c$  which satisfies  $c \vdash ?\mathbf{nat}$ .

First we define two mutually recursive partial functions  $\mathbf{val}$  (on terms  $p$  such that  $\vdash p : \mathbf{nat}$ ) and  $\mathbf{val}^\oplus$  (on terms  $q$  such that  $\vdash q : 1 \oplus \mathbf{nat}$ ) with values in  $\mathbb{N}$ . If  $\vdash p : \mathbf{nat}$  then

- if  $p = \mathbf{fd}(q)$  then we have  $\vdash q : 1 \oplus \mathbf{nat}$  and we take  $\mathbf{val} p = \mathbf{val}^\oplus q$ ;
- else if  $p = \tilde{\kappa}.c$  then if  $c \rightarrow^* \bullet \star p_0 \in \perp$  (that is  $p_0 \notin \tilde{\kappa}$ ) then  $\vdash p_0 : \mathbf{nat}$  and we take  $\mathbf{val} p = \mathbf{val} p_0$ .

And if  $\vdash q : 1 \oplus \mathbf{nat}$  then

- if  $q = \text{in}_1 r$  then if  $r = ()$  or if  $r = \tilde{\kappa}.c$  and  $c \rightarrow^* \bullet \star ()$ , then  $\text{val}^\oplus q = 0$ ;
- if  $q = \text{in}_2 p'$  then we have  $\vdash | p' : \text{nat}$  and we take  $\text{val}^\oplus q = 1 + \text{val} p'$ ;
- if  $q = \tilde{\kappa}.c$  and  $c \rightarrow^* \bullet \star q_0 \in \perp$  (that is  $q_0 \notin \tilde{\kappa}$ ) then we have  $\vdash | q_0 : 1 \oplus \text{nat}$  and we take  $\text{val}^\oplus q = \text{val}^\oplus q_0$ .

**Lemma 3.8** *If  $\vdash | p : \text{nat}$  and  $p \in |n|_{\text{nat}}$  for some  $n \in \mathbb{N}$  then  $\text{val} p$  is defined and has  $n$  as value.*

The proof is a straightforward application of the definition of  $|n|_{\text{nat}}$ . This lemma means that when we know that  $p \in |n|_{\text{nat}}$  for some  $n$  we can extract algorithmically the value of  $n$  from  $p$ .

Then we extend  $\kappa\mu\text{LLP}$  with a constant  $\circ$  which is a new command, typed by  $\circ \vdash \mathcal{N}$  where  $\mathcal{N}$  is an arbitrary *negative* context<sup>9</sup>. We also extend the point typing system with the rule  $\circ \vdash (\alpha_i : a_i : N_i)_{i=1}^n$  under the proviso that  $a_i \in \widehat{N_i^\perp} []$  for all  $i \in \underline{n}$ . The benefit of this extension is that now  $\perp$  contains closed commands.

We have  $\vdash | () : 1, \vdash | 1 | \bullet : \perp$  and  $\vdash | \kappa_\perp.\circ : \perp$ . Therefore  $\vdash | \text{rd} = \text{it}(\bullet, \bullet; \kappa_\perp.\circ) : \text{nat}^\perp$ . This negative term is a “reader of integer” which behaves as follows:  $\text{rd} \star \underline{0} \rightarrow^* \circ$  and  $\text{rd} \star \text{succ } p \rightarrow^* \text{rd} \star p$ . By induction on  $n \in \mathbb{N}$  it is not hard to check that  $\forall n \in \mathbb{N} \vdash | \text{rd} : \bar{n} : \text{nat}^\perp$ . Let  $m = [\bar{n}_1, \dots, \bar{n}_k] : ?\text{nat}$ , we have therefore  $\vdash | \text{rd}^! : m : !\text{nat}^\perp$  and hence  $\text{rd}^! \in \|m\|_{!\text{nat}^\perp}$  by Theorem 3.7.

If  $c \vdash \alpha : m = [\bar{n}_1, \dots, \bar{n}_k] : ?\text{nat}$  then by Theorem 3.7 we have  $c [\text{rd}^!/\alpha] \in \perp$ . By Lemma 2.10 this implies that the reduction of  $c$  terminates, so that  $c \rightarrow^* t \star \alpha$  where  $\vdash \alpha : m_1 : ?\text{nat} \mid t : m_2 : ?\text{nat}$ ,  $t$  is not of shape  $\kappa\beta.d$  and  $m = m_1 + m_2$ . So we must have  $t = \text{der}(p)$  with  $\vdash \alpha : m_1 : ?\text{nat} \mid p : \bar{n} : \text{nat}$  for some  $n \in \mathbb{N}$  and  $m_2 = [\bar{n}]$ . So by Theorem 3.7 we have  $p [\text{rd}^!/\alpha] \in |\bar{n}|_{\text{nat}}$  and hence by Lemma 3.8  $\text{val } p = n$ . As a whole we have described an algorithm which, under the assumption that  $c \vdash \alpha : m : ?\text{nat}$  for some  $m$ , produces an integer  $n$  that we denote as  $\text{val}_\alpha c$ , and we have  $m = m_1 + [\bar{n}]$ .

By the results of Section 4 and 5 we know that the interpretation of  $\llbracket c \rrbracket_{?\text{nat}}$  in **Rel** belongs to  $\mathcal{T}(\llbracket ?\text{nat} \rrbracket)$  where  $\llbracket ?\text{nat} \rrbracket$  is the interpretation of  $?\text{nat}$  in **Nuts** (that is  $\llbracket c \rrbracket_{?\text{nat}}$  is *total* in the interpretation of that type). As explained in Section 5.4  $\llbracket \text{nat}^\perp \rrbracket = (\mathbb{N}, \{\mathbb{N}\})$  and hence  $\mathcal{T}(\llbracket !\text{nat}^\perp \rrbracket) = \{\mathcal{M}_{\text{fin}}(\mathbb{N})\}$ . Therefore  $\llbracket c \rrbracket_{?\text{nat}} \cap \mathcal{M}_{\text{fin}}(\mathbb{N}) \neq \emptyset$  that is  $\llbracket c \rrbracket_{?\text{nat}} \neq \emptyset$  so by Theorem 5.13 there is  $m : ?\text{nat}$  such that  $c \vdash \alpha : m : ?\text{nat}$ .

Assume  $c \vdash ?\text{nat}$  and  $c$  does not contain  $\circ$ . By the considerations above the integer  $n = \text{val}_\alpha c$  is well defined (and we have given an algorithm to compute it consisting in executing  $\kappa\mu\text{LLP}$  commands). Moreover  $c \vdash \alpha : m_1 + [\bar{n}] : ?\text{nat}$  for some  $m_1 : ?\text{nat}$ . Using the model **RelW** introduced in [4] (it is a variation on the relational model where each object is a set  $E$  equipped with a function  $E \rightarrow \mathbb{Z}$ ) it is possible to prove that, because  $c$  does not contain  $\circ$ , one has  $m_1 = []$ . So we actually have  $c \vdash \alpha : [\bar{n}] : ?\text{nat}$ , that is, the value obtained by execution coincides with the value provided by the semantics.

## 4 Categorical semantic of $\kappa\mu\text{LLP}$

We use the following conventions: The identity morphism on an object  $A$  is written as  $A$ , and we very often simply write  $\text{Id}$  for the sake of readability, when the object can easily be retrieved from the context.  $\text{Alg}_{\mathcal{A}}(\mathbb{F})$  (resp.  $\text{Coalg}_{\mathcal{A}}(\mathbb{F})$ ) is the category of algebra (resp. co-algebra) of the endofunctor  $\mathbb{F}$  on  $\mathcal{A}$ . Other notations are based on [18].

Let  $(\mathcal{L}, \vec{\mathcal{L}} = (\mathcal{L}_n)_{n \in \mathbb{N}})$  be a categorical model of  $\mu\text{LL}$  in the sense of [18], Definition 7. We recall that  $\mathcal{L}$  is a categorical model of **LL** and that the elements  $\mathbb{F}$  of  $\mathcal{L}_n$  are strong functors  $\mathcal{L}^n \rightarrow \mathcal{L}$ , that is  $\mathbb{F}$  is a pair  $(\overline{\mathbb{F}}, \widehat{\mathbb{F}})$  where  $\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$  is a functor, and  $\widehat{\mathbb{F}}_{A, \vec{B}} \in \mathcal{L}(!A \otimes \overline{\mathbb{F}}(\vec{B}), \overline{\mathbb{F}}(!A \otimes \vec{B}))$  is a natural transformation, called *strength*, satisfying monoidality and compatibility with **dig**, see [18] Figure 1. Some additional closedness properties must be satisfied by the  $\mathcal{L}_n$ 's, see [18], Definition 7.

**Definition 4.1** A  $(n, p)$ -positive functor  $\mathbb{P}$  is a pair  $(\underline{\mathbb{P}}, \widetilde{\mathbb{P}})$  where  $\underline{\mathbb{P}} \in \mathcal{L}_{n+p}$  and  $\widetilde{\mathbb{P}}_{\vec{A}, \vec{B}} \in \mathcal{L}(\underline{\mathbb{P}}(\vec{A}, !\vec{B}), !\underline{\mathbb{P}}(\vec{A}, \vec{B}))$  is a natural transformation called the distributive law of  $\mathbb{P}$ . It is assumed moreover that the diagrams of Figure 6 commute, expressing the compatibility of  $\widetilde{\mathbb{P}}$  with **der**, **dig**, and  $\widehat{\underline{\mathbb{P}}}$ .

<sup>9</sup> This may seem surprising at first sight but remember that weakening is freely available for all negative formulas. So the real meaning of this rule is  $\circ \vdash$  which is the familiar 0-ary mix rule of **LL**. It is easy to check that all properties of  $\kappa\mu\text{LLP}$  proven so far are still valid for this extension.

$$\begin{array}{ccc}
 & & \mathbb{P}(\vec{A}, !\vec{B}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} !\mathbb{P}(\vec{A}, \vec{B}) \\
 \mathbb{P}(\vec{A}, !\vec{B}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} !\mathbb{P}(A, B) & \mathbb{P}(\vec{A}, \text{dig}_{\vec{B}}) \downarrow & \mathbb{P}(\vec{A}, !\vec{B}) \downarrow \text{dig}_{\mathbb{P}(\vec{A}, \vec{B})} \\
 \mathbb{P}(\vec{A}, \text{der}_{\vec{B}}) \searrow & \downarrow \text{der}_{\mathbb{P}(A, B)} & \mathbb{P}(\vec{A}, !!\vec{B}) \\
 & \mathbb{P}(\vec{A}, \vec{B}) & \mathbb{P}(\vec{A}, !\vec{B}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} !!\mathbb{P}(\vec{A}, \vec{B}) \\
 & & \mathbb{P}(\vec{A}, !\vec{B}) \downarrow \tilde{\mathbb{P}}_{\vec{A}, !\vec{B}} \\
 & & !\mathbb{P}(\vec{A}, !\vec{B}) \xrightarrow{!\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} !!\mathbb{P}(\vec{A}, \vec{B}) \\
 \\ 
 !!C \otimes \mathbb{P}(\vec{A}, !\vec{B}) & \xrightarrow{\hat{\mathbb{P}}_{!C, (\vec{A}, \vec{B})}} & \mathbb{P}(!!C \otimes \vec{A}, !!C \otimes \vec{B}) \\
 !!C \otimes \tilde{\mathbb{P}}_{\vec{A}, \vec{B}} \downarrow & & \downarrow \mathbb{P}(\text{der}_{!C} \otimes \vec{A}, \mu_{!C, \vec{B}}^2) \\
 !!C \otimes !\mathbb{P}(\vec{A}, \vec{B}) & & \mathbb{P}(!C \otimes \vec{A}, !(C \otimes \vec{B})) \\
 \mu_{!C, \mathbb{P}(\vec{A}, \vec{B})}^2 \downarrow & & \downarrow \tilde{\mathbb{P}}_{!C \otimes \vec{A}, !C \otimes \vec{B}} \\
 !(C \otimes \mathbb{P}(\vec{A}, \vec{B})) & \xrightarrow{\hat{\mathbb{P}}_{C, (\vec{A}, \vec{B})}} & !\mathbb{P}(!C \otimes \vec{A}, !C \otimes \vec{B})
 \end{array}$$

 Fig. 6. Compatibility of  $\tilde{\mathbb{P}}$  with  $\text{der}$ ,  $\text{dig}$ , and  $\hat{\mathbb{P}}$ 

**Lemma 4.2** Any  $(n, p)$ -positive functor  $\mathbb{P}$  induces a functor  $\mathbb{P}^+ : \mathcal{L}^n \times (\mathcal{L}^!)^p \rightarrow \mathcal{L}^!$ .

As a consequence of Lemma 4.2, a  $(0, 0)$ -positive functor  $\mathbb{P}$  induces an object of  $\mathcal{L}^!$ , and in that case  $\tilde{\mathbb{P}} = h_{\mathbb{P}}$ .

We write  $X, Y, \dots$  for objects of the category  $\mathcal{L}^!$ , and  $A, B, \dots$  for those of  $\mathcal{L}$ . We recall that  $\mathcal{L}[Z]$  is the Kleisli category of the comonad  $(\text{fc}_Z, \text{W}_Z, \text{C}_Z)$  where  $\text{fc}_Z : \mathcal{L} \rightarrow \mathcal{L}$  is the functor which maps an object  $A$  to  $\underline{Z} \otimes A$  and a morphism  $f$  to  $\underline{Z} \otimes f$ , and  $\text{W}_Z, \text{C}_Z$  are weakening and contraction morphism respectively.

Given a strong functor  $\mathbb{F} \in \mathcal{L}_k$  and object  $Z = (\underline{Z}, h_Z) \in \mathcal{L}^!$ , one can extend  $\mathbb{F}$  to a functor  $\mathbb{F}[Z] : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$ . On objects, one sets  $\mathbb{F}[Z](\vec{A}) = \mathbb{F}(\vec{A})$ . And given a morphism  $\vec{f} \in \mathcal{L}[Z]^k(\vec{A}_1, \vec{A}_2)$ , we define  $\mathbb{F}[Z](\vec{f})$  as  $(\mathbb{F}(\vec{f})) \circ (\mathbb{F}(\text{der}_{\underline{Z}} \otimes \vec{A}_1)) \circ ((\hat{\mathbb{F}}_{\underline{Z}, \vec{A}_1}) \circ (h_Z \otimes \mathbb{F}(A_1)))$ :

**Lemma 4.3** Let  $\mathbb{P} = (\mathbb{P}, \tilde{\mathbb{P}})$  be a positive functor and  $f \in \mathcal{L}^!(Z \otimes X, Y)$ . If  $\mathbb{P}$  is a  $(n, p+1)$ -positive functor, then

$\mathbb{P}[Z](\vec{\text{Id}}, \vec{\text{Id}}, f) \in \mathcal{L}^!(Z \otimes \mathbb{P}^+(\vec{A}, \vec{Y}_1, X), \mathbb{P}^+(\vec{A}, \vec{Y}_1, Y))$ . And if  $\mathbb{P}$  is a  $(n+1, p)$ -positive functor, then  $\mathbb{P}[Z](f^\perp, \vec{\text{Id}}, \vec{\text{Id}}) \in \mathcal{L}^!(Z \otimes \mathbb{P}^+(\underline{Y}^\perp, \vec{A}, \vec{Y}_1), \mathbb{P}^+(\underline{X}^\perp, \vec{A}, \vec{Y}_1))$ .

## 4.1 Operations on positive functors

### 4.1.1 LL operations on positive functors

Given a  $(n, p)$ -positive functor  $\mathbb{P}$ , strong functors  $\mathbb{F}_1, \dots, \mathbb{F}_n$  in  $\mathcal{L}_k$ , and  $(n', p')$ -positive functors  $\mathbb{Q}_1, \dots, \mathbb{Q}_p$  such that  $n' + p' = k$ , one can define a  $(n', p')$ -positive functor  $\mathbb{R} = \mathbb{P} \circ (\mathbb{F}_1, \dots, \mathbb{F}_n, \mathbb{Q}_1, \dots, \mathbb{Q}_p)$ : the strong functor  $\mathbb{R}$  is just  $\mathbb{P} \circ (\mathbb{F}_1, \dots, \mathbb{F}_n, \underline{\mathbb{Q}}_1, \dots, \underline{\mathbb{Q}}_p)$  as [18]. The distributive law is defined as follows, and satisfies commutations of Fig. 6.

$$\begin{aligned}
 \underline{\mathbb{R}}(\vec{A}, \vec{B}) &= \underline{\mathbb{P}}((\underline{\mathbb{F}}_i(\vec{A}, \vec{B}))_{i=1}^n, (\underline{\mathbb{Q}}_i(\vec{A}, \vec{B}))_{i=1}^p) \\
 &\quad \downarrow \underline{\mathbb{P}}((\underline{\mathbb{F}}_i(\vec{A}, \text{der}_{\vec{B}}))_{i=1}^n, (\underline{\mathbb{Q}}_i)_{i=n+1}^p) \\
 &= \underline{\mathbb{P}}((\underline{\mathbb{F}}_i(\vec{A}, \vec{B}))_{i=1}^n, (\underline{\mathbb{Q}}_i(\vec{A}, \vec{B}))_{i=1}^p) \\
 &\quad \downarrow \underline{\mathbb{P}}_{(\underline{\mathbb{F}}_i(\vec{A}, \vec{B}))_{i=1}^n, \underline{\mathbb{Q}}_i(\vec{A}, \vec{B})} \\
 &= \underline{\mathbb{R}}(\vec{A}, \vec{B})
 \end{aligned}$$

The bifunctor  $\otimes$  can be turned into a  $(0, 2)$ -positive functor: the distributive law is  $\mu_{A,B}^2 \in \mathcal{L}(!A \otimes !B, !(A \otimes B))$ , and it satisfies commutations of Fig. 6.

The bifunctor  $\oplus$  can be turned into a  $(0, 2)$ -positive functor: the distributive law is  $[\bar{\pi}_1, \bar{\pi}_2] \in \mathcal{L}(!A \oplus !B, !(A \oplus B))$ , and it satisfies commutations of Fig. 6.

The functor  $!$  is a  $(1, 0)$ -positive functor: the distributive law is  $\text{dig}_A \in \mathcal{L}(!A, !!A)$ , and satisfies commutations of Fig. 6.

Let  $\mathbb{P}$  be a  $(n, p)$ -positive functor. One can define the De Morgan dual of it, denoted as  $\mathbb{P}^\perp$ , as a strong functor:  $\mathbb{P}^\perp = \underline{\mathbb{P}}^\perp$  and the last term is just  $\underline{\mathbb{P}}^\perp$ .

#### 4.1.2 Fixpoint of positive functors

Let  $\mathbb{P}$  be a  $(n, p+1)$ -positive functor. We must define a  $(n, p)$ -positive functor  $\mu\mathbb{P}$ . We set  $\underline{\mu\mathbb{P}} = \underline{\mu\mathbb{P}}$ . The distr. law  $\widetilde{\mu\mathbb{P}}$  must be a natural transf.  $\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}} \in \mathcal{L}(\underline{\mu\mathbb{P}}(\vec{A}, \vec{B}), \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}))$ . To define it, we first notice that  $(\underline{\mu\mathbb{P}}(\vec{A}, \vec{B}), \widetilde{\mathbb{P}}_{\vec{A}, (\vec{B}, \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}))})$  is an object of  $\mathbf{Coalg}_{\mathcal{L}}(\underline{\mathbb{P}}_{\vec{A}, \vec{B}})$ . So, by the universal property of  $\underline{\mu\mathbb{P}}(\vec{A}, \vec{B})$ , there is a unique morphism  $\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}}$  such that the following diagram commutes:

$$\begin{array}{ccc}
 \underline{\mathbb{P}}(\vec{A}, \vec{B}, \underline{\mu\mathbb{P}}(\vec{A}, \vec{B})) & \xrightarrow{\underline{\mathbb{P}}(\vec{A}, \vec{B}, \widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}})} & \underline{\mathbb{P}}(\vec{A}, \vec{B}, \underline{\mu\mathbb{P}}(\vec{A}, \vec{B})) \\
 \simeq \downarrow & & \downarrow \widetilde{\mathbb{P}}_{\vec{A}, (\vec{B}, \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}))} \\
 \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}) & \xrightarrow{\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}}} & \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}) \xleftarrow{\simeq} \underline{\mathbb{P}}(\vec{A}, \vec{B}, \underline{\mu\mathbb{P}}(\vec{A}, \vec{B}))
 \end{array}$$

**Lemma 4.4** Given a strong functor  $\mathbb{F} \in \mathcal{L}_{k+1}$  and an object  $Z = (\underline{Z}, h_Z)$  in  $\mathcal{L}^!$ , there is a unique functor  $\mu(\mathbb{F}[Z]) : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$  such that  $\mu(\mathbb{F}[Z])(\vec{A})$  is the initial object of the category  $\mathbf{Alg}_{\mathcal{L}[Z]}(\underline{\mathbb{F}}_{\vec{A}})$  for any object  $\vec{A} \in \mathcal{L}^k$ , and  $\mu(\mathbb{F}[Z])(\vec{f})$  is the unique morphism satisfying commutation of the following diagram for any  $\vec{f} \in \mathcal{L}[Z]^k(\vec{B}_1, \vec{B}_2)$ :

$$\begin{array}{ccccc}
 \mu(\mathbb{F}[Z])(\vec{B}_1) & \xrightarrow{\mu(\mathbb{F}[Z])(\vec{f})} & \mu(\mathbb{F}[Z])(\vec{B}_2) & & \\
 \simeq \downarrow & & & \swarrow \simeq & \\
 \underline{\mathbb{F}}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{B}_1)) & \xrightarrow{\underline{\mathbb{F}}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{f}))} & \underline{\mathbb{F}}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{B}_2)) & \xrightarrow{\underline{\mathbb{F}}(\vec{f}, \text{Id})} & \underline{\mathbb{F}}_{\vec{B}_2}(\mu(\mathbb{F}[Z])(\vec{B}_2))
 \end{array}$$

where the composition is considered in category  $\mathcal{L}[Z]$ .

**Lemma 4.5** Given a morphism  $g \in \mathcal{L}[Z](\underline{\mathbb{Q}}(\underline{Y}), \underline{Y})$  where  $\mathbb{Q}$  is a  $(0, 1)$ -positive functor, there is a unique morphism  $\tilde{g} \in \mathcal{L}[Z](\underline{\mu\mathbb{Q}}, \underline{Y})$  such that  $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\underline{\mathbb{Q}}[Z])(\underline{\mu\mathbb{Q}}, \underline{Y})$ .

**Lemma 4.6** Given a  $(0, 1)$ -positive functor  $\mathbb{Q}$ , and a morphism  $g \in \mathcal{L}^!(Z \otimes \mathbb{Q}^+(Y), Y)$ , then there is a unique morphism  $\tilde{g} \in \mathcal{L}^!(Z \otimes (\mu\mathbb{Q})^+, Y)$  such that  $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\underline{\mathbb{Q}}[Z])(\underline{\mu\mathbb{Q}}, \underline{Y})$ .

## 4.2 Interpretation of proofs and formulas

**Definition 4.7**  $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$  is adapted to  $P$  (resp.  $N$ ) if  $(\vec{\zeta}^-, \vec{\xi}^+)$  is repetition-free and all the free variables of  $P$  (resp.  $N$ ) appear in that list.

Given a positive formula  $P$  with an adapted list  $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$ , its interpretation is a  $(n, p)$ -positive functor  $\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}$ . If  $n = p = 0$  we simply write  $\llbracket P \rrbracket$ .

For a negative formula  $N$  with an adapted list  $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$ , we define its interpretation  $\llbracket N \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}$  as  $(\llbracket N^\perp \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})^\perp$ .

One can define  $\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}$  in the obvious way by induction on formulas and using the construction in the section 4.1:  $\llbracket P_1 \otimes P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \otimes \circ (\llbracket P_1 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}, \llbracket P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})$ ,  $\llbracket P_1 \oplus P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \oplus \circ (\llbracket P_1 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}, \llbracket P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})$ ,  $\llbracket !N \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = ! \circ (\llbracket N^\perp \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})^\perp$ , and  $\llbracket \mu Z.P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \mu(\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+, Z})$ .

**Lemma 4.8** *Let  $A$  be a formula and  $\pi$  be a type valuation  $N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+$ . Then*

$$\llbracket A[\pi] \rrbracket = \llbracket A \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} \circ (\llbracket N_1 \rrbracket, \dots, \llbracket N_n \rrbracket, \llbracket P_1 \rrbracket, \dots, \llbracket P_k \rrbracket).$$

We define the interpretation of terms, as usual, by induction on proofs based on Figure 2. As we have different judgments, we must have different ways of interpreting terms which is explained as follows:

- if  $\vdash \mathcal{N} \mid p : P$ , then  $\llbracket p \rrbracket_{\mathcal{N}}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket, \llbracket P \rrbracket)$ ;
- if  $\vdash \mathcal{N}, P \mid t : N$ , then  $\llbracket t \rrbracket_{\mathcal{N}, P}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket, \llbracket P \rrbracket)$ ;
- if  $c \vdash \mathcal{N}, P$ , then  $\llbracket c \rrbracket_{\mathcal{N}, P}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket, \llbracket P \rrbracket)$ ;
- if  $\vdash \mathcal{N} \mid t : N$ , then  $\llbracket t \rrbracket_{\mathcal{N}} \in \mathcal{L}(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket, \perp)$ ;
- if  $c \vdash \mathcal{N}$ , then  $\llbracket c \rrbracket_{\mathcal{N}} \in \mathcal{L}(\llbracket \mathcal{N}^\perp \rrbracket, \perp)$ .

We only give the details of the interpretation of  $\bar{\kappa}_{N, \zeta} \alpha.(c; s)$ , and the other cases are defined in 6.13.

By induction hypothesis, we know that  $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp[P/\zeta^+] \rrbracket, \llbracket P \rrbracket)$ , and  $\llbracket s \rrbracket_{\mathcal{N}}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket, \llbracket Q \rrbracket)$ .

By Lemma 4.8,  $\llbracket N^\perp[P/\zeta^+] \rrbracket = (\llbracket N^\perp \rrbracket_{\zeta^+}) \circ (\llbracket P \rrbracket)$ . So, we have  $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket_{\zeta^+}(\llbracket P \rrbracket), \llbracket P \rrbracket)$ . Hence, by Lemma 4.6, we have a morphism  $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \mu \llbracket N^\perp \rrbracket_{\zeta^+}, \llbracket P \rrbracket)$ . And by the interpretation of formula, we know that  $\llbracket \mu \zeta^+. N^\perp \rrbracket = \mu \llbracket N^\perp \rrbracket_{\zeta^+}$ . So, we have  $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mu \zeta^+. N^\perp \rrbracket, \llbracket P \rrbracket)$ . Notice that  $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+$  is a unique morphism by lemma 4.6, so, we just take the following morphism of category  $\mathcal{L}^!$  as  $\llbracket \bar{\kappa}_{N, \zeta} \alpha.(c; s) \rrbracket_{\mathcal{N}, Q}^+$ :

$$\begin{array}{ccc} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mu \zeta^+. N^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \otimes \llbracket \mu \zeta^+. N^\perp \rrbracket \\ & & \downarrow \text{Id} \otimes \llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \\ \llbracket Q \rrbracket & \xleftarrow{\llbracket s \rrbracket_{\mathcal{N}, Q}^+} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \end{array}$$

## 4.3 Soundness

We first state the substitution lemmas (Lemmas 4.9, 4.10 & 4.11).

**Lemma 4.9** *Assume that  $\vdash \mathcal{N} \mid p : P$ .*

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \text{If } c \vdash \mathcal{N}, \alpha : P^\perp, \text{ then} & \downarrow \llbracket [c/p/\alpha] \rrbracket_{\mathcal{N}} & & \downarrow \text{Id} \otimes \llbracket [p] \rrbracket_{\mathcal{N}}^+ \\
 & \perp & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \\
 & & & \llbracket c \rrbracket_{\mathcal{N}, P^\perp}^+
 \end{array}$$

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \text{If } c \vdash \mathcal{N}, \alpha : P^\perp, Q, \text{ then} & \downarrow \llbracket [c/p/\alpha] \rrbracket_{\mathcal{N}}^+ & & \downarrow \text{Id} \otimes \llbracket [p] \rrbracket_{\mathcal{N}}^+ \\
 & \llbracket Q \rrbracket & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \\
 & & & \llbracket c \rrbracket_{\mathcal{N}, P^\perp, Q}^+
 \end{array}$$

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 \text{If } t \vdash \mathcal{N}, \alpha : P^\perp \mid t : N, \text{ then} & \downarrow \llbracket [t/p/\alpha] \rrbracket_{\mathcal{N}} & & \downarrow \text{Id} \otimes \llbracket [p] \rrbracket_{\mathcal{N}}^+ \otimes \text{Id} \\
 & \perp & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 & & & \text{cur}^{-1}(\llbracket t \rrbracket_{\mathcal{N}, P})
 \end{array}$$

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 \text{If } t \vdash \mathcal{N}, \alpha : P^\perp, Q \mid t : N, \text{ then} & \downarrow \llbracket [t/p/\alpha] \rrbracket_{\mathcal{N}}^+ & & \downarrow \text{Id} \otimes \llbracket [p] \rrbracket_{\mathcal{N}}^+ \otimes \text{Id} \\
 & \llbracket Q \rrbracket & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 & & & \llbracket t \rrbracket_{\mathcal{N}, P^\perp, Q}^+
 \end{array}$$

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \text{if } t \vdash \mathcal{N}, \alpha : P^\perp \mid q : Q, \text{ then} & \downarrow \llbracket [q/p/\alpha] \rrbracket_{\mathcal{N}}^+ & & \downarrow \text{Id} \otimes \llbracket [p] \rrbracket_{\mathcal{N}}^+ \\
 & \llbracket Q \rrbracket & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \\
 & & & \llbracket q \rrbracket_{\mathcal{N}, P^\perp}^+
 \end{array}$$

**Lemma 4.10** Assume that  $\vdash \mathcal{N}, Q \mid t : N$ .

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \text{If } c \vdash \mathcal{N}, N^\perp, \text{ then} & \downarrow \llbracket [c/t/\bullet] \rrbracket_{\mathcal{N}, Q}^+ & & \downarrow \text{Id} \otimes \llbracket [c] \rrbracket_{\mathcal{N}, N^\perp}^+ \\
 & \llbracket Q \rrbracket & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 & & & \llbracket t \rrbracket_{\mathcal{N}, Q}^+
 \end{array}$$

$$\begin{array}{ccc}
 & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \\
 \text{If } t \vdash \mathcal{N}, N^\perp \mid s : M, \text{ then} & \downarrow \llbracket [s/t/\bullet] \rrbracket_{\mathcal{N}, Q}^+ & & \downarrow \text{Id} \otimes \llbracket [s] \rrbracket_{\mathcal{N}, N^\perp}^+ \\
 & \llbracket Q \rrbracket & \longleftarrow & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 & & & \llbracket t \rrbracket_{\mathcal{N}, Q}^+
 \end{array}$$

**Lemma 4.11** Assume that  $\vdash \mathcal{N} \mid t : N$ .

$$\begin{array}{ccc}
 \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \text{If } c \vdash \mathcal{N}, N^\perp, \text{ then} & \downarrow \llbracket [c[t/\bullet]]_{\mathcal{N}} \rrbracket & \downarrow \text{Id} \otimes [c]_{\mathcal{N}, N^\perp}^+ \\
 \perp & \longleftarrow \llbracket [t]_{\mathcal{N}} \rrbracket & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \\
 \\
 \text{If } \vdash \mathcal{N}, N^\perp \mid s : M, \text{ then} & & \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \\
 \downarrow \llbracket [s[t/\bullet]]_{\mathcal{N}} \rrbracket & & \downarrow \text{Id} \otimes [s]_{\mathcal{N}, N^\perp}^+ \\
 \perp & \longleftarrow \llbracket [t]_{\mathcal{N}} \rrbracket & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket
 \end{array}$$

The following lemma relates syntactic functoriality (section 2.2) and the semantical one (Lemma 4.3).

**Lemma 4.12** *Assume that  $\vdash \mathcal{N}, P \mid s : N$ , let  $Q$  be a positive formula and let  $\zeta$  be a variable. Let also  $\pi$  be a type valuation such that  $\zeta^+, \zeta^- \notin \text{dom}(\pi)$ . Then*

$$\begin{aligned}
 \llbracket Q[\pi, s/\zeta^+] \rrbracket_{\mathcal{N}, Q[\pi, P/\zeta^+]}^+ &= \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}(\llbracket \mathcal{N}^\perp \rrbracket)(\overrightarrow{\text{Id}}, \overrightarrow{\text{Id}}, \llbracket [s]_{\mathcal{N}, P}^+ \rrbracket) \\
 \llbracket Q[\pi, s/\zeta^-] \rrbracket_{\mathcal{N}, Q[\pi, N/\zeta^-]}^+ &= \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^-}(\llbracket \mathcal{N}^\perp \rrbracket)(\llbracket [s]_{\mathcal{N}, P}^+ \rrbracket^\perp, \overrightarrow{\text{Id}}, \overrightarrow{\text{Id}})
 \end{aligned}$$

As a direct conclusion of Lemma 4.3 and Lemma 4.12, we can have the following corollary:

**Corollary 4.13** *Assume that  $\vdash \mathcal{N}, P \mid s : N$ , let  $Q$  be a positive formula and let  $\zeta$  be a variable. Let also  $\pi$  be a type valuation such that  $\zeta^+, \zeta^- \notin \text{dom}(\pi)$ . Then*

$$\begin{aligned}
 \llbracket Q[\pi, s/\zeta^+] \rrbracket_{\mathcal{N}, Q[\pi, P/\zeta^+]}^+ &\in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}(\overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}, \llbracket N^\perp \rrbracket}), \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}(\overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}, \llbracket P \rrbracket)) \\
 \llbracket Q[\pi, s/\zeta^-] \rrbracket_{\mathcal{N}, Q[\pi, N/\zeta^-]}^+ &\in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}(\overrightarrow{\llbracket P \rrbracket}^\perp, \overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}), \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}(\overrightarrow{\llbracket N^\perp \rrbracket}^\perp, \overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}))
 \end{aligned}$$

**Lemma 4.14** *If  $c \rightarrow d$ , then either  $\llbracket c \rrbracket_{\mathcal{N}} = \llbracket d \rrbracket_{\mathcal{N}}$  or  $\llbracket c \rrbracket_{\mathcal{N}, P}^+ = \llbracket d \rrbracket_{\mathcal{N}, P}^+$  depending on the typing derivation of  $c$  and  $d$ .*

**Theorem 4.15** *If  $c \rightarrow^* d$ , then either  $\llbracket c \rrbracket_{\mathcal{N}} = \llbracket d \rrbracket_{\mathcal{N}}$  or  $\llbracket c \rrbracket_{\mathcal{N}, P}^+ = \llbracket d \rrbracket_{\mathcal{N}, P}^+$  depending on the typing derivation of  $c$  and  $d$ .*

**Proof.** By induction on the length of reductions from  $c$  to  $d$ , and using Lemma 4.14.  $\square$

## 5 Two concrete models of $\kappa\mu\text{LLP}$

We recall briefly the main definitions concerning the models **Rel** and **Nuts**, more details can be found in [18].

### 5.1 Sets and relations as a model of $\kappa\mu\text{LLP}$

The category **Rel** has sets as objects, and given sets  $E$  and  $F$ ,  $\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$ . Identity is the diagonal relation and composition is the usual composition of relations, denoted by simple juxtaposition. If  $t \in \mathbf{Rel}(E, F)$  and  $u \subseteq E$  then  $t \cdot u = \{b \in F \mid \exists a \in u (a, b) \in t\}$ .

This category is a well-known model of **LL** in which  $1 = \perp = \{*\}$ ,  $E \otimes F = (E \multimap F) = E \wp F = E \times F$  so that  $(E)^\perp = E$ . As to the additives,  $0 = \top = \emptyset$  and  $\&_{i \in I} E_i = \oplus_{i \in I} E_i = \cup_{i \in I} \{i\} \times E_i$ . The exponentials are given by  $!E = ?E = \mathcal{M}_{\text{fin}}(E)$  (finite multisets of elements of  $E$ ). For the additives and multiplicatives, the operations on morphisms are defined in the obvious way. Let us be more specific about the exponentials. Given  $s \in \mathbf{Rel}(E, F)$ ,  $!s \in \mathbf{Rel}(!E, !F)$  is  $!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid \forall i (a_i, b_i) \in s\}$ ,  $\text{der}(E) \in \mathbf{Rel}(!E, E)$  is given by  $\text{der}(E) = \{([a], a) \mid a \in E\}$  and  $\text{dig}_E \in \mathbf{Rel}(!E, !E)$  is given by  $\text{dig}_E = \{(m_1 + \dots + m_n, [m_1, \dots, m_n]) \mid \forall i m_i \in \mathcal{M}_{\text{fin}}(E)\}$ . Last  $\text{m}^0 \in \mathbf{Rel}(1, !\top)$  is  $\text{m}^0 = \{(*, [])\}$  and

$m_{E,F}^2 \in \mathbf{Rel}(!E \otimes !F, !(E \& F))$  is given by

$$m_{E,F}^2 = \{([a_1, \dots, a_k], [b_1, \dots, b_l]), [(1, a_1), \dots, (1, a_k), (2, b_1), \dots, (2, b_l)] \mid a_1, \dots, a_k \in E \text{ and } b_1, \dots, b_l \in F\}.$$

Weakening  $w_E \in \mathbf{Rel}(!E, 1)$  and  $\text{contr}_E \in \mathbf{Rel}(!E, !E \otimes !E)$  are given by  $w_E = \{([\ ], *)\}$  and  $\text{contr}_E = \{(m_1 + m_2, (m_1, m_2)) \mid m_i \in \mathcal{M}_{\text{fin}}(E) \text{ for } i = 1, 2\}$ .

### 5.1.1 Positive functors on $\mathbf{Rel}$

A functor  $\mathcal{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$  is locally continuous if, for all  $\vec{E}, \vec{F} \in \mathbf{Rel}^n$  and all directed  $D \subseteq \mathbf{Rel}^n(\vec{E}, \vec{F})$ ,  $\mathcal{F}(\cup D) = \cup \{\mathcal{F}(\vec{s}) \mid \vec{s} \in D\}$ .

**Definition 5.1** An  $n$ -ary variable set is a strong functor  $\mathcal{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$  such that  $\overline{\mathcal{F}}$  is locally continuous and maps inclusions to inclusions.

**Definition 5.2** A positive  $n + p$ -ary variable set is a  $(n, p)$ -positive functor  $\mathbb{P} : \mathbf{Rel}^{n+p} \rightarrow \mathbf{Rel}$  such that  $\underline{\mathbb{P}}$  is an  $n + p$ -ary variable set.

## 5.2 Non-uniform totality spaces as a model of $\kappa\mu\text{LLP}$

### 5.2.1 Basic definitions.

Let  $E$  be a set and let  $\mathcal{T} \subseteq \mathcal{P}(E)$ . We define  $\mathcal{T}^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$ . If  $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{P}(E)$  then  $(\mathcal{T})^\perp \subseteq \mathcal{S}^\perp$ . We also have  $\mathcal{T} \subseteq \mathcal{T}^{\perp\perp}$  and therefore  $\mathcal{T}^{\perp\perp\perp} = \mathcal{T}^\perp$ . One nice feature of this duality is:

**Lemma 5.3** Let  $\mathcal{T} \subseteq \mathcal{P}(E)$ , then  $\mathcal{T}^{\perp\perp} = \uparrow\mathcal{T} = \{v \subseteq E \mid \exists u \in \mathcal{T} \ u \subseteq v\}$ .

A non-uniform totality space (NUTS) is a pair  $X = (|X|, \mathcal{T}(X))$  where  $|X|$  is a set and  $\mathcal{T}(X) \subseteq \mathcal{P}(|X|)$  satisfies  $\mathcal{T}(X) = \mathcal{T}(X)^{\perp\perp}$ , that is  $\mathcal{T}(X) = \uparrow\mathcal{T}(X)$ . Of course we set  $X^\perp = (|X|, \mathcal{T}(X)^\perp)$ . We define four basic NUTS:  $0 = (\emptyset, \emptyset)$ ,  $\top = (\emptyset, \{\emptyset\})$  and  $1 = \perp = (\{*\}, \{\{*\}\})$ . Given NUTS  $X_1$  and  $X_2$  we define a NUTS  $X_1 \otimes X_2$  by  $|X_1 \otimes X_2| = |X_1| \times |X_2|$  and  $\mathcal{T}(X_1 \otimes X_2) = \uparrow\{u_1 \times u_2 \mid u_i \in \mathcal{T}(X_i) \text{ for } i = 1, 2\}$ . And then we define  $X \multimap Y = (X \otimes Y^\perp)^\perp$ .

**Lemma 5.4**  $t \in \mathcal{T}(X \multimap Y) \Leftrightarrow \forall u \in \mathcal{T}(X) \ t \cdot u \in \mathcal{T}(Y)$ .

We define the category  $\mathbf{Nuts}$  whose objects are the NUTS and  $\mathbf{Nuts}(X, Y) = \mathcal{T}(X \multimap Y)$ , composition being defined as the usual composition in  $\mathbf{Rel}$  (relational composition) and identities as the diagonal relations. Lemma 5.4 shows that we have indeed defined a category.

### 5.2.2 Multiplicative structure

**Lemma 5.5** Let  $t \subseteq |X| \times |Y|$ . One has  $t \in \mathbf{Nuts}(X, Y)$  iff  $(t)^\perp = \{(b, a) \mid (a, b) \in t\} \in \mathbf{Nuts}(Y^\perp, X^\perp)$ .

**Lemma 5.6** Let  $t \subseteq |X_1 \otimes X_2 \multimap Y|$ . One has  $t \in \mathbf{Nuts}(X_1 \otimes X_2, Y)$  iff for all  $u_1 \in \mathcal{T}(X_1)$  and  $u_2 \in \mathcal{T}(X_2)$  one has  $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$ .

**Lemma 5.7** The bijection  $\alpha_{|X_1|, |X_2|, |Y|}$  is an isomorphism from  $(X_1 \otimes X_2) \multimap Y$  to  $X_1 \multimap (X_2 \multimap Y)$ .

We turn now  $\otimes$  into a functor, its action on morphisms being defined as in  $\mathbf{Rel}$ . Let  $t_i \in \mathbf{Nuts}(X_i, Y_i)$  for  $i = 1, 2$ , we have  $t_1 \otimes t_2 \in \mathbf{Nuts}(X_1 \otimes X_2, Y_1 \otimes Y_2)$  by Lemma 5.6 and by the equation  $(t_1 \otimes t_2) \cdot (u_1 \otimes u_2) = (t_1 \cdot u_1) \otimes (t_2 \cdot u_2)$ . This functor is monoidal, with unit 1 and symmetric monoidality isomorphisms  $\lambda, \rho, \gamma$  and  $\alpha$  defined as in  $\mathbf{Rel}$ .

The SMC category  $\mathbf{Nuts}$  is closed, with  $X \multimap Y$  as internal hom object from  $X$  to  $Y$ , and evaluation morphism  $\text{ev} = \{((a, b), a), b \mid a \in |X| \text{ and } b \in |Y|\}$  which indeed belongs to  $\mathbf{Nuts}((X \multimap Y) \otimes X, Y)$  by Lemma 5.6. This category  $\mathbf{Nuts}$  is also  $*$ -autonomous with dualizing object  $\perp = 1$ .

### 5.2.3 Additive structure.

Let  $(X_i)_{i \in I}$  be an at most countable family of objects of **Nuts**. We define  $X = \&_{i \in I} X_i$  by  $|X| = \cup_{i \in I} \{i\} \times |X_i|$  and  $\mathcal{T}(X) = \{u \subseteq |X| \mid \forall i \in I \pi_i \cdot u \in \mathcal{T}(X_i)\}$ .

It is clear that  $\mathcal{T}(X) = \uparrow \mathcal{T}(X)$  and hence  $X$  is an object of **Nuts**. By definition of  $X$  and by Lemma 5.4 we have  $\forall i \in I \pi_i \in \mathbf{Nuts}(X, X_i)$ . Given  $\vec{t} = (t_i)_{i \in I}$  with  $\forall i \in I t_i \in \mathbf{Nuts}(Y, X_i)$ , we have  $\langle \vec{t} \rangle \in \mathbf{Nuts}(Y, X)$  as easily checked (using Lemma 5.4 again). It follows that  $(\&_{i \in I} X_i, (\pi_i)_{i \in I})$  is the cartesian product of the  $X_i$ 's in **Nuts**. This shows that the category **Nuts** has all countable products and hence is cartesian. Since it is  $*$ -autonomous, the category **Nuts** is also cocartesian. Notice that the final object is  $\top = (\emptyset, \{\emptyset\})$  and that  $0 = \top^\perp = (\emptyset, \emptyset)$ .

### 5.2.4 Exponential.

It is an extension of the multiset exponential of **Rel** with totality. We set  $!|X| = \mathcal{M}_{\text{fin}}(|X|)$  and  $\mathcal{T}(!X) = \{\mathcal{M}_{\text{fin}}(u) \mid u \in \mathcal{T}(X)\}^{\perp\perp} = \uparrow \{\mathcal{M}_{\text{fin}}(u) \mid u \in \mathcal{T}(X)\}$ .

**Remark 5.8** A formula is interpreted as exactly the same set in **Rel** and **Nuts** (with the additional totality structure in **Nuts** of course). Similarly a proof is interpreted as the same set in both models; **Nuts** gives us additionally that this set is total.

### 5.2.5 Variable non-uniform totality spaces (VNUTS)

Let  $E$  be a set, we use  $\text{Tot}(E)$  for the set of all *totality candidates* on  $E$ , that is, of all subsets  $\mathcal{T}$  of  $\mathcal{P}(E)$  such that  $\mathcal{T} = \mathcal{T}^{\perp\perp}$  (remember that  $\mathcal{T}^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} u \cap u' \neq \emptyset\}$ ). In other words  $\mathcal{T} \in \text{Tot}(E)$  means that  $\uparrow \mathcal{T}$  by Lemma 5.3. Ordered by  $\subseteq$ , this set  $\text{Tot}(E)$  is a complete lattice.

**Definition 5.9** Let  $n \in \mathbb{N}$ , an  $n$ -ary VNUTS is a pair  $\mathcal{X} = (|\mathcal{X}|, \mathcal{T}(\mathcal{X}))$  where  $|\mathcal{X}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$  is a variable set  $|\mathcal{X}| = (|\overline{\mathcal{X}}|, |\widehat{\mathcal{X}}|)$  and  $\mathcal{T}(\mathcal{X})$  is an operation which with each  $n$ -tuple  $\vec{A}$  of objects of **Nuts** associates  $\mathcal{T}(\mathcal{X})(\vec{A}) \in \text{Tot}(|\overline{\mathcal{X}}|(|\vec{A}|))$  in such a way that

- 1 for any  $\vec{t} \in \mathbf{Nuts}^n(\vec{A}, \vec{B})$ , the element  $|\overline{\mathcal{X}}|(\vec{t})$  of  $\mathbf{Rel}(|\overline{\mathcal{X}}|(|\vec{A}|), |\overline{\mathcal{X}}|(|\vec{B}|))$  belongs to  $\mathbf{Nuts}(\overline{\mathcal{X}}(\vec{A}), \overline{\mathcal{X}}(\vec{B}))$  (where  $\overline{\mathcal{X}}(\vec{A})$  denotes the NUTS  $(|\overline{\mathcal{X}}|(|\vec{A}|), \mathcal{T}(\mathcal{X})(\vec{A}))$ )
- 2 and for any  $\vec{B} \in \text{Obj}(\mathbf{Nuts}^n)$  and any  $A \in \text{Obj}(\mathbf{Nuts})$  one has  $|\widehat{\mathcal{X}}|_{|A|, |\vec{B}|} \in \mathbf{Nuts}(!A \otimes \overline{\mathcal{X}}(\vec{B}), \overline{\mathcal{X}}(!A \otimes \vec{B}))$ .

**Definition 5.10** Let  $n, p \in \mathbb{N}$ , an  $n + p$ -ary positive VNUTS is a pair  $\mathcal{P} = (|\mathcal{P}|, \mathcal{T}(\mathcal{P}))$  such that

- 1  $|\mathcal{P}|$  is an  $n + p$ -ary positive variable set,
- 2  $\underline{\mathcal{X}} = (|\mathcal{P}|, \mathcal{T}(\mathcal{P}))$  is an  $n + p$ -ary VNUTS, and
- 3 for any  $\vec{A}, \vec{B} \in \text{Obj}(\mathbf{Nuts}^n)$  one has  $|\widehat{\mathcal{P}}|_{|\vec{A}|, |\vec{B}|} \in \mathbf{Nuts}(|\underline{\mathcal{P}}|(\vec{A}, !\vec{B}), !(|\underline{\mathcal{P}}|(\vec{A}, \vec{B})))$ .

**Lemma 5.11** Any  $n + p$ -ary positive VNUTS  $\mathcal{P} : \mathbf{Nuts}^{n+p} \rightarrow \mathbf{Nuts}$  induces a  $(n, p)$ -positive functor  $\mathbb{P} : \mathbf{Nuts}^{n+p} \rightarrow \mathbf{Nuts}$  which satisfies

- $|\underline{\mathbb{P}}(\vec{A}, \vec{B})| = |\underline{\mathcal{P}}|(|\vec{A}|, |\vec{B}|)$ ,
- $\mathcal{T}(\underline{\mathbb{P}}(\vec{A}, \vec{B})) = \mathcal{T}(\mathcal{P})(\vec{A}, \vec{B})$ ,
- $\underline{\mathbb{P}}(\vec{t}) = |\underline{\mathcal{P}}|(\vec{t}) \in \mathbf{Nuts}(\underline{\mathcal{P}}(\vec{A}_1, \vec{B}_1), \underline{\mathcal{P}}(A_2, \vec{B}_2))$   
for  $\vec{t} \in \mathbf{Nuts}^{n+p}((\vec{A}_1, \vec{B}_1), (A_2, \vec{B}_2))$ ,
- $\widehat{\underline{\mathbb{P}}}_{A, \vec{B}} = |\widehat{\underline{\mathcal{P}}}|_{|A|, |\vec{B}|}$
- and  $\widetilde{\underline{\mathbb{P}}}_{\vec{A}, \vec{B}} = |\widetilde{\underline{\mathcal{P}}}|_{|\vec{A}|, |\vec{B}|}$

and  $\mathcal{P}$  can be retrieved from  $\mathbb{P}$ .

### 5.3 Rel and the point typing system

In Section 3 we have introduced a point typing system. The points of this typing system are essentially the same thing as points of the relational model, the only difference being that a point  $a : P$  contains the  $\sigma b$  construct at places corresponding to occurrences of  $\mu$  or  $\nu$  type constructs in  $P$ . It is easy to see that there is a bijective correspondence between the  $a : P$  and the  $a' \in \llbracket P \rrbracket$  in **Rel**. To simplify notations we consider this correspondence as the identity function.

**Lemma 5.12** *Let  $P$  be a closed positive formula. Then  $a \tilde{P} [a_1, \dots, a_n]$  iff  $(a, [a_1, \dots, a_n]) \in h_{\llbracket P \rrbracket}$ .*

**Theorem 5.13** *Let  $\Phi = (\alpha_1 : a_1 : N_1, \dots, \alpha_k : a_k : N_k)$  be a negative point typing context and let  $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_n : N_n)$ . Then*

- $(a_1, \dots, a_n, b) \in \llbracket t \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$  iff  $\vdash \Phi \mid t : b : N$
- $(a_1, \dots, a_n, c, b) \in \llbracket t \rrbracket_{\mathcal{N}, P}^{\mathbf{Rel}}$  iff  $\vdash \Phi, c : P \mid t : b : N$
- $(a_1, \dots, a_n, b) \in \llbracket q \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$  iff  $\vdash \Phi \mid q : b : P$
- $(a_1, \dots, a_n) \in \llbracket c \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$  iff  $c \vdash \Phi$
- $(a_1, \dots, a_n, b) \in \llbracket c \rrbracket_{\mathcal{N}, P}^{\mathbf{Rel}}$  iff  $c \vdash \Phi, b : P$

The proof of Theorem 5.13 is a simple verification, and it uses Lemma 5.12.

### 5.4 Examples of nat

As we can see in [18], the interpretation of **nat** in **Nuts** is a totality space  $(\mathbb{N}, \{u \subseteq \mathbb{N} \mid u \neq \emptyset\})$ . So,  $\llbracket \mathbf{nat}^\perp \rrbracket = (\mathbb{N}, \{\mathbb{N}\})$ . Hence  $\llbracket !(\mathbf{nat}^\perp) \rrbracket = (\mathcal{M}_{\text{fin}}(\mathbb{N}), \{\mathcal{M}_{\text{fin}}(\mathbb{N})\})$ , since  $\{\mathcal{M}_{\text{fin}}(\mathbb{N})\} = \uparrow \{\mathcal{M}_{\text{fin}}(u) \mid u \in \{\mathbb{N}\}\}$ . The inductive definition of  $h_{\text{nat}}$  means this set is the least one satisfying

- $(\bar{0}, k[\bar{0}]) \in h_{\text{nat}}$  for any  $k \in \mathbb{N}$ , and
- If  $(\bar{n}, [\bar{n}_1, \dots, \bar{n}_k]) \in h_{\text{nat}}$ ,  
then  $(\bar{n} + \bar{1}, [n_1 + \bar{1}, \dots, n_k + \bar{1}]) \in h_{\text{nat}}$ .

Hence we have  $h_{\text{nat}} = \{(\bar{n}, k[\bar{n}]) \mid k, n \in \mathbb{N}\}$ .

## Conclusion

We have decomposed the proof of termination results for  $\kappa\mu\text{LLP}$  (such as in Section 3.6) in two steps: first a normalization result for a non-idempotent intersection typing system in the style of [12, 13, 4] and then a purely semantical result: non-uniform totality spaces are a denotational model of  $\kappa\mu\text{LLP}$  where proofs have the same interpretation as in **Rel**.

This approach was particularly useful here since, due to the presence of the  $\tilde{\kappa}$  construct and the alternation of polarities in  $\kappa\mu\text{LLP}$ , we have not (yet) been able to define a direct realizability interpretation of  $\kappa\mu\text{LLP}$  formulas as sets of terms. In some sense we have handled denotationally the logically complex part of this intended realizability semantics. One can also see our construction as a realizability interpretation of formulas graded by the points of the relational model.

This method does not seem to say anything interesting about formulas which have an empty interpretation in **Rel**. This drawback is not so dramatic since, as explained in the conclusion of [18], no finite information (of type **nat** typically) can be extracted *internally to  $\kappa\mu\text{LLP}$*  from proofs of such formulas. Nevertheless the problem of proving some form of normalization for them is an interesting challenge.

Another goal for further work will be to analyze the expressive power of  $\kappa\mu\text{LLP}$  as a programming language. It seems clear that we can embed faithfully a version of Gödel's T in  $\kappa\mu\text{LLP}$  but we can certainly do much more.

A further direction of work will consist in investigating a related polarized calculus for the circular setting: while the construction  $\bar{\kappa}_{N, \zeta} \alpha.c$  would be much simplified, the challenge will be to account for the global validity condition.

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## 6 Appendix

### 6.1 Definition of substitution of terms in formulas

- ▷ If  $Q = \zeta^+$  then  $Q[\pi, s/\zeta^+] = s$ .
- ▷ If  $Q = \zeta_i^+ \in \text{dom}(\pi)$  then  $Q[\pi, s/\zeta^+] = \bullet$  with  $\vdash \mathcal{N}, P_i \mid \bullet : P_i^\perp$ .
- ▷ If  $Q = 1$  we set  $Q[\pi, s/\zeta^+] = \bullet$  with  $\vdash \mathcal{N}, 1 \mid \bullet : \perp$ .
- ▷ Assume  $Q = Q_1 \otimes Q_2$ . Let  $s_i = Q_i[\pi, s/\zeta^+]$ . By inductive hypothesis we have  $\vdash \mathcal{N}, Q_i[\pi, P/\zeta^+] \mid s_i : Q_i^\perp[\pi^\perp, N/\zeta^-]$  so that  $\vdash \mathcal{N}, \alpha_i : Q_i^\perp[\pi^\perp, N/\zeta^-] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i[\pi, P/\zeta^+]$  for  $i = 1, 2$ . Hence  $\vdash \mathcal{N}, \alpha_1 : Q_1^\perp[\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp[\pi^\perp, N/\zeta^-] \mid (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)) : (Q_1 \otimes Q_2)[\pi, P/\zeta^+]$ . So we have  $\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)) \vdash \mathcal{N}, \alpha_1 : Q_1^\perp[\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp[\pi^\perp, N/\zeta^-], (Q_1 \otimes Q_2)[\pi, P/\zeta^+]$  and  $\vdash \mathcal{N}, Q[\pi, P/\zeta^+] \mid \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2))) : Q^\perp[\pi^\perp, N/\zeta^-]$ . Hence we set  $(Q_1 \otimes Q_2)[\pi, s/\zeta^+] = \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)))$ .
- ▷ Assume  $Q = 0$ . We have  $\vdash \mathcal{N}, 0 \mid \langle \rangle : \top$  and we set  $\top[\pi, s/\zeta^+] = \langle \rangle$ .
- ▷ If  $Q = Q_1 \oplus Q_2$ , we use the same notations as for  $\otimes$ . We have  $\vdash \mathcal{N}, \alpha_i : Q_i^\perp[\pi^\perp, N/\zeta^-] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i[\pi, P/\zeta^+]$  hence  $\vdash \mathcal{N}, \alpha_i : Q_i^\perp[\pi^\perp, N/\zeta^-] \mid \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i)) : (Q_1 \oplus Q_2)[\pi, P/\zeta^+]$ . So  $\vdash \mathcal{N}, (Q_1 \oplus Q_2)[\pi, P/\zeta^+] \mid \kappa\alpha_i.(\bullet \star \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i))) : Q_i^\perp[\pi^\perp, N/\zeta^-]$  so we set  $(Q_1 \oplus Q_2)[\pi, s/\zeta^+] = \langle \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.(s_1 \star \alpha_1))), \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.(s_2 \star \alpha_2))) \rangle$ .
- ▷ Assume that  $Q = \mu\xi.R$  and let us set  $R_P = R[P/\zeta^+]$  and  $R_N = R[N^\perp/\zeta^+]$ . Let  $\rho = \pi \cdot (\mu\xi.R_P[\pi]/\xi^+)$ . By inductive hypothesis we have defined  $t = R[\rho, s/\zeta^+]$  with  $\vdash \mathcal{N}, R_P[\rho] \mid t : R_N^\perp[\rho^\perp]$ . We have  $\vdash \mathcal{N}, \alpha : R_N^\perp[\rho^\perp] \mid \tilde{\kappa}.(t \star \alpha) : R_P[\rho]$ . Notice that, due to the definition of  $\rho$ , we have  $R_S[\rho] = R_S[\pi][\mu\xi.R_P[\pi]/\xi^+]$  for  $S = P, N$ . It follows that  $\vdash \mathcal{N}, \alpha : R_N^\perp[\rho^\perp] \mid \text{fd}(\tilde{\kappa}.(t \star \alpha)) : \mu\xi.R_P[\pi]$ . Therefore  $(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha))) \vdash \mathcal{N}, \mu\xi.R_P[\pi], \alpha : R_N^\perp[\rho^\perp] = (R_N[\pi][\mu\xi.R_P[\pi]/\xi^+])^\perp$  so we define  $(\mu\xi.R)[\pi, s/\zeta^+]$  as  $\bar{\kappa}_{R_N^\perp[\pi^\perp], \zeta} \alpha.(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha)))$  by  $(\mathbf{t}\nu^1)$ .

▷ Assume last that  $Q = !R^\perp$ . By inductive hypothesis we have defined  $t = R[\pi^\perp, s/\zeta^-]$  which satisfies  $\vdash \mathcal{N}, R[\pi^\perp, N/\zeta^-] \mid t : R^\perp[\pi, P/\zeta^+]$ . It follows that  $\vdash \mathcal{N}, \alpha : R^\perp[\pi, P/\zeta^+] \mid \tilde{\kappa}.(t \star \alpha) : R[\pi^\perp, N/\zeta^-]$  and hence  $\vdash \mathcal{N}, \alpha : R^\perp[\pi, P/\zeta^+] \mid \text{der}(\tilde{\kappa}.(t \star \alpha)) : ?R[\pi^\perp, N/\zeta^-]$  so that  $\vdash \mathcal{N}, \beta : ?R[\pi^\perp, N/\zeta^-] \mid \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta) : R^\perp[\pi, P/\zeta^+]$ . Next we obtain  $\vdash \mathcal{N}, \beta : ?R[\pi^\perp, N/\zeta^-] \mid \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^\dagger : !R^\perp[\pi, P/\zeta^+]$  so that  $\vdash \mathcal{N}, !R^\perp[\pi, P/\zeta^+] \mid \kappa\beta.(\bullet \star \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^\dagger) : ?R[\pi^\perp, N/\zeta^-]$  and we define  $!R^\perp[\pi, s/\zeta^+]$  as the term  $\kappa\beta.(\bullet \star \kappa\alpha.(\text{der}(\tilde{\kappa}.(t \star \alpha)) \star \beta)^\dagger)$ .

In the same induction we define the negative substitution  $Q[\pi, s/\zeta^-]$ ; this is done as follows.

▷ If  $Q$  is a variable, it cannot be  $\zeta^-$  since  $Q$  is positive and we must have  $Q = \zeta_i^+$ . It follows that  $Q[\pi, N/\zeta^-] = P_i$  and  $Q^\perp[\pi^\perp, P/\zeta^+] = P_i^\perp$  so that we set  $Q[\pi, s/\zeta^-] = \bullet$ . The cases  $Q = 1$  and  $Q = 0$  are similar.

▷ Assume  $Q = Q_1 \otimes Q_2$ . Let  $s_i = Q_i[\pi, s/\zeta^-]$  so that  $\vdash \mathcal{N}, Q_i[\pi, N/\zeta^-] \mid s_i : Q_i^\perp[\pi^\perp, P/\zeta^+]$ . It follows that  $\vdash \mathcal{N}, \alpha_i : Q_i^\perp[\pi^\perp, P/\zeta^+] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i[\pi, N/\zeta^-]$  for  $i = 1, 2$ . Hence  $\vdash \mathcal{N}, \alpha_1 : Q_1^\perp[\pi^\perp, P/\zeta^+], \alpha_2 : Q_2^\perp[\pi^\perp, P/\zeta^+] \mid (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)) : (Q_1 \otimes Q_2)[\pi, N/\zeta^-]$ . So we set  $(Q_1 \otimes Q_2)[\pi, s/\zeta^+] = \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)))$ . The other cases are similarly analogous to the definition of the positive substitution.

## 6.2 Proof of Lemma 3.1

**Proof.** By induction on  $a$  assuming that we have a derivation of  $a : R[P/\zeta^+]$ . We consider several cases, according to the shape of  $R$ .

▷ If  $R = \zeta^+$  we choose  $j \in J$  and set  $a^0 = (j, \zeta)$ ,  $b_j = a$ .

▷ If  $R = \xi^+ \neq \zeta^+$  we must have  $a = (j, \xi)$  and we set  $a^0 = a$ . In that case we have  $\text{rg}_\zeta a^0 = \emptyset$  and so we have no  $b_k$ 's to define.

▷ If  $R = 1$  we must have  $a = *$  and we take  $a^0 = a$ . As before  $\text{rg}_\zeta a^0 = \emptyset$  and so we have no  $b_k$ 's to define.

▷ If  $R$  is  $R_1 \otimes R_2$  so that  $a = (a_1, a_2)$  with  $a_i : R_i[P/\zeta^+]$  for  $i = 1, 2$ . Let  $J_1, J_2 \subseteq J$  be disjoint and infinite. By inductive hypothesis, for  $i = 1, 2$ , we can find  $a_i^0 : R$  with  $\text{rg}_\zeta a_i^0 \subseteq J_i$  as well as families  $\vec{b}(i) = (b(i)_j)_{j \in \text{rg}_\zeta a_i^0}$  such that  $\forall j b(i)_j : P$ ,  $(a_i^0, \vec{b}(i))$  is adapted and  $a_i = a_i^0 \{b(i)_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a_i^0}$ . For  $j \in \text{rg}_\zeta a^0 = \text{rg}_\zeta a_1^0 \uplus \text{rg}_\zeta a_2^0$  we set  $b_j = b(i)_j$  where  $i \in \{1, 2\}$  is uniquely determined by  $j \in \text{rg}_\zeta a_i^0$ , defining  $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$ . Let  $\xi$  be a literal,  $j, j' \in \text{rg}_\zeta a^0$  with  $j \neq j'$  and let  $K = \text{rg}_\xi b_j \cap \text{rg}_\xi b_{j'}$ . If  $j, j' \in \text{rg}_\xi a_i^0$  for  $i = 1$  or  $i = 2$  then  $K = \emptyset$  since  $(a_i^0, \vec{b}(i))$  is adapted. If  $j \in \text{rg}_\xi a_i^0$  and  $j' \in \text{rg}_\xi a_{3-i}^0$  then  $K = \emptyset$  because we know that  $\text{rg}_\xi b_j \subseteq \text{rg}_\xi a_i$ : this is due to the fact that, by inductive hypothesis  $a_i = a_i^0 \{b_k / (k, \zeta)\}_{k \in \text{rg}_\zeta a_i^0}$ . Similarly  $\text{rg}_\xi b_{j'} \subseteq \text{rg}_\xi a_{3-i}$  and moreover  $\text{rg}_\xi a_i \cap \text{rg}_\xi a_{3-i} = \emptyset$  because  $a : R_1[P/\zeta^+] \otimes R_2[P/\zeta^+]$ . For the same reason, if  $j \in \text{rg}_\zeta a_i^0$  we have  $\text{rg}_\xi b(i)_j \cap \text{rg}_\xi a_{3-i}^0 = \emptyset$ . It follows that the pair  $(a^0, \vec{b})$  is adapted. The fact that  $a = a^0 \{b_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a^0}$  is an immediate consequence of the inductive hypothesis. The case  $R = !R_0$  (and hence  $a = [a_1, \dots, a_n]$ ) is dealt with similarly (applying the inductive hypothesis to the  $a_i$ 's) and the case  $R = R_1 \oplus R_2$  is straightforward: we have  $a = (i, a_0)$  with  $a_0 : R_i[P/\zeta^+]$  for  $i = 1$  or  $i = 2$  and the inductive hypothesis directly applies to  $a_0$ .

▷ Last if that  $R = \mu\xi.Q$  with  $\xi \neq \zeta$  and  $\xi$  does not occur in  $P$ , so that  $a = \sigma(a_0)$  and that we have a derivation of  $a_0 : R'$  where  $R' = Q[P/\zeta^+][\mu\xi.Q[P/\zeta^+]/\xi^+] = Q[R/\xi^+][P/\zeta^+]$ . By inductive hypothesis applied to  $a_0$  there is  $a_0^0 : Q[R/\xi^+]$  and a family  $(c_j)_{j \in \text{rg}_\xi(a_0^0)}$  satisfying the required properties wrt.  $a^0$ . We take  $a^0 = \sigma(a_0^0)$  so that  $\text{rg}_\zeta a^0 = \text{rg}_\zeta a_0^0$  and we set  $b_j = c_j$  for each  $j$  in that set.  $\square$

## 6.3 Proof of Lemma 3.2

**Proof.** Let  $p = \tilde{\kappa}.c \in |b|_Q$ . This means that  $c \rightarrow^* \bullet \star q$  with  $q \in |b|_Q \setminus \tilde{\kappa} = ||b||_Q$ . It follows that  $s \star p \rightarrow c[s/\bullet] \rightarrow^* (\bullet \star q)[s/\bullet] = s \star q \in \perp(a : P)$  (notice that  $\bullet$  cannot occur free in  $q$  since  $q$  is a typed

positive term). It follows that  $s \star p \in \perp(a : P)$ .  $\square$

#### 6.4 Proof of Lemma 3.3

**Proof.** By induction on the derivation of  $a \widetilde{P} [a_1, \dots, a_n]$  we prove that  $\|a\|_P \subseteq \|a_i\|_P$  which implies the announced inclusions. If the derivation consists of (s-1) then we have  $P = 1$  and  $\forall i \ a = a_i = *$  so that the statement obviously holds. If the derivation ends with (s- $\otimes$ ) then  $P = P_1 \otimes P_2$ ,  $a = (a^1, a^2)$ ,  $a_i = (a_i^1, a_i^2)$  and  $a^j \widetilde{P}_j [a_1^j, \dots, a_n^j]$  for  $j = 1, 2$ . The inductive hypothesis gives  $\|a^j\|_{P_j} \subseteq \|a_i^j\|_{P_j}$  for  $j = 1, 2$  and for each  $i$  whence the announced inclusion by definition of  $\|(a^1, a^2)\|_{P_1 \otimes P_2}$ . The case where the last rule is (s- $\oplus$ ) is similar. If the last rule is (s-!) then  $P = !N$ ,  $a = [b_1, \dots, b_k]$ ,  $a_i = [b_j \mid j \in J_i]$  with  $\underline{k} = \bigsqcup_{i=1}^n J_i$ . Let  $p \in \|a\|_P$  so that  $p = s^!$  with  $\forall j \in \underline{k} \forall q \in |b_j|_{N^\perp} \ s \star q \in \perp$ . So *a fortiori* for each  $i \in \underline{n}$  one has  $\forall j \in J_i \forall q \in |b_j|_{N^\perp} \ s \star q \in \perp$ , that is  $p \in \|a_i\|_P$ . Assume that the last rule is (s- $\mu$ ) so that  $P = \mu\zeta.Q$ ,  $a = \sigma(b)$ ,  $a_i = \sigma(b_i)$  for  $i \in \underline{n}$  and we have  $b \ Q [P/\zeta^+] [b_1, \dots, b_n]$ . Let  $p \in \|a\|_P$ , which means that  $p = \text{fd}(q)$  with  $q \in |b|_{Q[P/\zeta^+]}$ . For  $i \in \underline{n}$  we have  $q \in \|b_i\|_{Q[P/\zeta^+]}$  by inductive hypothesis and hence  $p \in \|a_i\|_P$ .  $\square$

#### 6.5 Proof of Lemma 3.5

**Proof.** By induction on the pairs  $(Q, d)$ , ordered lexicographically, following the definition of substitution of terms in formulas in Section 2.2. It is important to notice that the universal quantification on  $\pi$  is part of the statement we prove by induction. We set  $J = \text{rg}_\eta d$ .

▷ If  $Q = 1$  then  $d = *$ ,  $J = \emptyset$ ,  $Q [\pi, s/\eta^+] = \bullet$ ,  $d\{b_j/(j, \eta)\}_{j \in J} = d\{c_j/(j, \eta)\}_{j \in J} = *$  and  $(a_i \widetilde{P}_i [])_{i=1}^n$ . We have  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, * : 1 \mid \bullet : * : \perp$  as required. The case  $Q = \zeta^+ \neq \eta^+$  is similar.

▷ If  $Q = \eta^+$ ,  $d = (j, \eta)$  for some  $j \in I$  then  $J = \{j\}$  and hence  $d\{b_k/(k, \eta)\}_{k \in J} = b_j$  and  $d\{c_k/(k, \eta)\}_{k \in J} = c_j$ . Moreover we have  $(a_i^j \widetilde{P}_i [a_i])_{i=1}^n$  so that  $a_i^j = a_i$  for each  $i \in \underline{n}$  by Lemma 3.4. We have  $Q [\pi, s/\eta^+] = s$  and the first conclusion is identical to the typing assumption on  $s$ . Since  $Q [\pi, s/\eta^-] = \bullet$  the second conclusion is obtained as in the previous cases.

▷ If  $Q = Q_1 \otimes Q_2$  then  $d = (d_1, d_2)$  with  $(d_k : Q_k [\pi])_{k=1,2}$ . Let  $K = \{1, 2\}$  and  $J_k = \text{rg}_\eta d_k$  for  $k \in K$  so that  $J = J_1 \uplus J_2$ . By Lemma 3.4, for each  $i \in \underline{n}$  and  $k \in K$  there is  $a(k)_i : N_i^\perp$  such that  $(a(k)_i \widetilde{N}_i^\perp [a_i^j \mid j \in J_k])_{i=1}^n$  for  $k \in K$  and  $(a_i \widetilde{N}_i^\perp [a(1)_i, a(2)_i])_{i=1}^n$ . By inductive hypothesis we have  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{b_j/(j, \eta)\}_{j \in J_k} : Q [\pi, P/\eta^+] \mid Q_k [\pi, s/\eta^+] : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q^\perp [\pi^\perp, N/\eta^-]$  for  $k \in K$ . Pick some conneutral  $(e_i : N_i)_{i=1}^n$  and  $f_k : Q_k [\pi, N^\perp/\eta^+]$ , we have  $\vdash (\alpha_i : e_i : N_i)_{i=1}^n, \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp [\pi^\perp, N/\eta^-], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp [\pi^\perp, N/\eta^-] \mid \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k [\pi, N^\perp/\eta^+]$ . Since  $(a(k)_i \widetilde{N}_i^\perp [e_i, a(k)_i])_{i=1}^n$  we get  $s_k \star \beta_k \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{b_j/(j, \eta)\}_{j \in J_k} : Q_k [\pi, P/\eta^+], \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp [\pi^\perp, N/\eta^-], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp [\pi^\perp, N/\eta^-]$  where  $s_k = Q_k [\pi, s/\eta^+]$  and consequently  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp [\pi^\perp, N/\eta^-], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp [\pi^\perp, N/\eta^-] \mid \tilde{\kappa}.(s_k \star \beta_k) : d_k\{b_j/(j, \eta)\}_{j \in J_k} : Q [\pi, P/\eta^+]$ . Using the fact that  $(a_i \widetilde{N}_i^\perp [a(1)_i, a(2)_i])_{i=1}^n$  and the conneutrality of  $f_1, f_2$  we get  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, \beta_1 : d_1\{c_j/(j, \eta)\}_{j \in J_1} : Q_1^\perp [\pi^\perp, N/\eta^-], \beta_2 : d_2\{c_j/(j, \eta)\}_{j \in J_2} : Q_2^\perp [\pi^\perp, N/\eta^-] \mid (\tilde{\kappa}.(s_1 \star \beta_1), \tilde{\kappa}.(s_2 \star \beta_2)) : (d_1\{b_j/(j, \eta)\}_{j \in J_1}, d_2\{b_j/(j, \eta)\}_{j \in J_2}) : Q [\pi, P/\eta^+]$ . Since we have  $d\{b_j/(j, \eta)\}_{j \in J} = (d_1\{b_j/(j, \eta)\}_{j \in J_1}, d_2\{b_j/(j, \eta)\}_{j \in J_2})$  and similarly for  $d\{c_j/(j, \eta)\}_{j \in J}$ , we finally get  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, d\{b_j/(j, \eta)\}_{j \in J} : Q [\pi, P/\eta^+] \mid \tilde{\kappa}(\beta_1, \beta_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \beta_1), \tilde{\kappa}.(s_2 \star \beta_2))) : d\{c_j/(j, \eta)\}_{j \in J} : Q^\perp [\pi^\perp, N/\eta^-]$  as contended. The case  $Q = Q_1 \oplus Q_2$  is similar. The second conclusion is obtained in the same way. The case  $Q = Q_1 \oplus Q_2$  is similar.

▷ Assume that  $Q = !R^\perp$ , so that  $d = [d_k \mid k \in K]$  with  $d_k : R [\pi^\perp]$ . For  $k \in K$  let  $J_k = \text{rg}_\eta d_k$  so that  $J = \bigsqcup J_k$ . By Lemma 3.4, for each  $i \in \underline{n}$  and  $k \in K$  there is  $a(k)_i : N_i^\perp$  such that  $(a(k)_i \widetilde{N}_i^\perp [a_i^j \mid j \in J_k])_{i=1}^n$  for  $k \in K$  and  $(a_i \widetilde{N}_i^\perp [a(k)_i \mid k \in K])_{i=1}^n$ . By inductive hypothesis, for each  $k \in K$  we have  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{c_j/(j, \eta)\}_{j \in J_k} : R [\pi, N/\eta^-] \mid t : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp [\pi^\perp, P/\eta^+]$ , where  $t = R [\pi, s/\eta^-]$ , so  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp [\pi^\perp, P/\eta^+] \mid \tilde{\kappa}.(t \star \beta) : d_k\{c_j/(j, \eta)\}_{j \in J_k} : R [\pi, N/\eta^-]$ .

So  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp[\pi^\perp, P/\eta^+] \mid \mathbf{der}(\tilde{\kappa}.(t \star \beta)) : [d_k\{c_j/(j, \eta)\}_{j \in J_k}] : ?R[\pi, N/\eta^-]$  and hence  $\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \gamma : [d_k\{b_j/(j, \eta)\}_{j \in J_k}] : ?R[\pi, N/\eta^-] \mid \kappa\beta.(\mathbf{der}(\tilde{\kappa}.(t \star \beta)) \star \gamma) : d_k\{c_j/(j, \eta)\}_{j \in J_k} : R^\perp[\pi^\perp, P/\eta^+]$ . Since  $(a_i \widetilde{N_i^\perp} [a(k)_i \mid k \in K])_{i=1}^n$  and  $[d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] !R^\perp[\pi^\perp, N^\perp/\eta^+] [[d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] \mid k \in K]$  we have  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, \gamma : [d_k\{b_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : ?R[\pi, N/\eta^-] \mid \kappa\beta.(\mathbf{der}(\tilde{\kappa}.(t \star \beta)) \star \gamma)^\dagger : [d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : !R^\perp[\pi^\perp, P/\eta^+]$  and therefore we have  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, [d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : !R^\perp[\pi^\perp, P/\eta^+] \mid \kappa\gamma.(\bullet \star \kappa\beta.(\mathbf{der}(\tilde{\kappa}.(t \star \beta)) \star \gamma)^\dagger) : [d_k\{b_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : ?R[\pi, N/\eta^-]$  as required. The second conclusion is dealt with similarly.

▷ Assume that  $Q = \mu\zeta.R$ . We set  $R_P = R[P/\eta^+]$  and  $R_N = R[N^\perp/\eta^+]$ . We have  $d = \sigma(d^0)$  with  $d^0 : R[\pi][Q[\pi]/\zeta^+]$ . By Lemma 3.1 we can find  $f : R[\pi]$  as well as a family  $(d_l : Q[\pi])_{l \in L}$  where  $L = \mathbf{rg}_\zeta f$  such that  $d^0 = f\{d_l/(l, \zeta)\}_{l \in L}$ . Let  $J = \mathbf{rg}_\eta d$ , we have  $J = J' \uplus \bigsqcup_{l \in L} \widetilde{J_l}$  where  $J' = \mathbf{rg}_\eta f$  and  $J_l = \mathbf{rg}_\eta d_l$  for each  $l \in L$ . By Lemma 3.4 we can find  $a'_i, a_i^l : N_i^\perp$  such that  $a_i \widetilde{N_i^\perp} [a'_i] + [a_i^l \mid l \in L]$  and  $a_i \widetilde{N_i^\perp} [a_i^j \mid j \in J']$  and  $a_i^l \widetilde{N_i^\perp} [a_i^j \mid j \in J_l]$ , for all  $i = 1, \dots, n$ . Let  $l \in L$ . We set  $f_l = d_l\{b_j/(j, \eta)\}_{j \in J_l}$  and  $g_l = d_l\{c_j/(j, \eta)\}_{j \in J_l}$ . By inductive hypothesis (since  $\mathbf{sz} d_l \leq \mathbf{sz} d^0 < \mathbf{sz} d$ ) we have, for all  $l \in L$ ,

$$\vdash (\alpha_i : a_i^l : N_i)_{i=1}^n, f_l : \mu\zeta.R_P[\pi] \mid Q[\pi, s/\eta^+] : g_l : \nu\zeta.R_N^\perp[\pi^\perp] \quad (2)$$

Let  $\rho = \pi \cdot (\mu\zeta.R_P[\pi]/\zeta^+)$ . Notice that all the free variables of  $R$ , but possibly  $\eta^+, \eta^-$  are in  $\mathbf{dom}(\rho)$ . Let  $d^1 = f\{f_l/(l, \zeta)\}_{l \in L}$  so that  $d^1 : R[\rho]$  since  $f : R[\pi]$  and  $f_l : \mu\zeta.R_P[\pi]$ . Notice that  $\mathbf{rg}_\eta d^1 = J'$ . We apply the inductive hypothesis to  $(R, d^1)$  and get  $\vdash (\alpha_i : a'_i : N_i)_{i=1}^n, d^1\{b_j/(j, \eta)\}_{j \in J'} : R[\rho, P/\eta^+] \mid R[\rho, s/\eta^+] : d^1\{c_j/(j, \eta)\}_{j \in J'} : R^\perp[\rho^\perp, N/\eta^-]$ . Notice that  $R[\rho, P/\zeta^+] = R_P[\pi][\mu\zeta.R_P[\pi]/\zeta^+]$  and  $R[\rho, N^\perp/\zeta^+] = R_{N^\perp}[\pi][\mu\zeta.R_P[\pi]/\zeta^+]$  so that

$$\begin{aligned} \bullet \star \mathbf{fd}(\tilde{\kappa}.(R[\rho, s/\eta^+] \star \alpha)) \vdash (\alpha_i : a'_i : N_i)_{i=1}^n, \\ \sigma(d^1\{b_j/(j, \eta)\}_{j \in J'}) : \mu\zeta.R_P[\pi], \\ \alpha : d^1\{c_j/(j, \eta)\}_{j \in J'} : R_N[\pi]^\perp \left[ (\mu\zeta.R_P[\pi])^\perp / \zeta^- \right] \quad (3) \end{aligned}$$

We have

$$\begin{aligned} \sigma(d^1\{b_j/(j, \eta)\}_{j \in J'}) &= d\{b_j/(j, \eta)\}_{j \in J} \\ d^1\{c_j/(j, \eta)\}_{j \in J'} & \\ &= f\{c_j/(j, \eta)\}_{j \in J'} \{f_l/(l, \zeta)\}_{l \in L} \end{aligned}$$

hence by (2) and (3) applying rule  $(\mathbf{i}\text{-}\nu^\perp)$  and using also the fact that  $a_i \widetilde{N_i^\perp} [a'_i] + [a_i^l \mid l \in L]$ , we get  $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, d\{b_j/(j, \eta)\}_{j \in J} : \mu\zeta.R_P[\pi] \mid \mu\zeta.R[\pi, s/\eta^+] : \sigma(f)\{c_j/(j, \eta)\}_{j \in J'} \{g_l/(l, \zeta)\}_{l \in L} : \nu\zeta.(R_N[\pi])^\perp$  and notice that  $\sigma(f)\{c_j/(j, \eta)\}_{j \in J'} \{g_l/(l, \zeta)\}_{l \in L} = d\{c_j/(j, \eta)\}_{j \in \mathbf{rg}_\eta d}$  since  $g_l = d_l\{c_j/(j, \eta)\}_{j \in J_l}$  and  $d = \sigma(f)\{d_l/(l, \zeta)\}_{l \in L}$ ; the announced statement is proven. For the second conclusion we proceed similarly.  $\square$

### 6.6 Proof of Lemma 3.6

**Proof.** By induction on the pairs  $(Q, d)$ , ordered lexicographically, following the definition of substitution of terms in formulas in Section 2.2.

▷ The cases  $Q = 1$  and  $Q = \eta^+$  are trivial.

▷ If  $Q = Q_1 \otimes Q_2$  then  $d = (d_1, d_2)$  with  $(d_k : Q_k[\pi])_{k=1,2}$  and  $J = J_1 \uplus J_2$  where  $J_k = \mathbf{rg}_\eta d_k$  and by inductive hypothesis we have  $s_k = Q_k[\pi, s/\eta^+] \in |g_k|_{N_k}(f_k : P_k)$  where  $f_k = d_k(\vec{b})$ ,  $N_k = Q_k^\perp[\pi^\perp, N/\eta^-]$ ,  $g_k = d_k(\vec{c})$  and  $P_k = Q_k[\pi, P/\eta^+]$  for  $k = 1, 2$ . It suffices to prove that  $t \in |(g_1, g_2)|_{N_1 \wp N_2}((f_1, f_2) :$

$P_1 \otimes P_2$ ) where  $t = \kappa(\beta_1, \beta_2) \cdot (\bullet \star (\tilde{\kappa} \cdot (s_1 \star \beta_1), \tilde{\kappa} \cdot (s_2 \star \beta_2)))$ . We use Lemma 3.2 so let  $q_k \in |g_k|_{Q_k^\perp}$  for  $k = 1, 2$ , we have  $t \star (q_1, q_2) \rightarrow (\bullet \star (\tilde{\kappa} \cdot (s_1 \star q_1), \tilde{\kappa} \cdot (s_2 \star q_2)))$ . We have  $s_k \star q_k \in \perp(f_k : P_k)$  and hence  $(\tilde{\kappa} \cdot (s_1 \star q_1), \tilde{\kappa} \cdot (s_2 \star q_2)) \in \|(f_1, f_2)\|_{P_1 \otimes P_2}$  so that  $\bullet \star (\tilde{\kappa} \cdot (s_1 \star q_1), \tilde{\kappa} \cdot (s_2 \star q_2)) \in \perp((f_1, f_2) : P_1 \otimes P_2)$  and hence  $t \star (q_1, q_2) \in \perp((f_1, f_2) : P_1 \otimes P_2)$ . The case  $Q = Q_1 \oplus Q_2$  is similar.

▷ If  $Q = !R^\perp$  then  $d = [d_1, \dots, d_k]$  with  $J = \biguplus_{l=1}^k J_l$  where  $J_l = \mathbf{rg}_\eta d_l$  for each  $l \in \underline{k}$ . By inductive hypothesis  $s' = R[\pi^\perp, s/\eta^-] \in |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}(d_l(\vec{c}) : R[\pi^\perp, N/\eta^-])$  for each  $l \in \underline{k}$  and we must prove that  $t \in \|[d_l(\vec{c}) \mid l \in L]\|_{R[\pi^\perp, N/\eta^-]}([d_l(\vec{b}) \mid l \in L] : !R^\perp[\pi, P/\eta^+])$  where  $t = !R^\perp[\pi, s/\eta^+] = \kappa\beta \cdot (\bullet \star (\kappa\alpha \cdot (\mathbf{der}(\tilde{\kappa} \cdot (s' \star \alpha)) \star \beta)))$  where  $s' = R[\pi^\perp, s/\eta^-]$  and for this we apply Lemma 3.2. Let  $s_1 \in \bigcap_{l \in L} |d_l(\vec{c})|_{R^\perp[\pi, N^\perp/\eta^+]}$ , it suffices to prove that  $t \star s_1 \in \perp([d_l(\vec{b}) \mid l \in L] : !R^\perp[\pi, P/\eta^+])$ . We have  $t \star s_1 \rightarrow \bullet \star (\kappa\alpha \cdot (\mathbf{der}(\tilde{\kappa} \cdot (s' \star \alpha)) \star s_1))$  so it is sufficient to prove  $(\kappa\alpha \cdot (\mathbf{der}(\tilde{\kappa} \cdot (s' \star \alpha)) \star s_1)) \in \|[d_l(\vec{b}) \mid l \in L]\|_{!R^\perp[\pi, P/\eta^+]}$ , and hence it is enough to prove that  $\kappa\alpha \cdot (\mathbf{der}(\tilde{\kappa} \cdot (s' \star \alpha)) \star s_1) \in \bigcap_{l \in L} |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}$ . So let  $q \in \bigcup_{l \in L} |d_l(\vec{b})|_{R[\pi^\perp, P^\perp/\eta^-]}$ , it suffices to prove that  $(\mathbf{der}(\tilde{\kappa} \cdot (s' \star q)) \star s_1) \in \perp$ . Let  $l \in L$ , since  $s' \in |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}(d_l(\vec{c}) : R[\pi^\perp, N/\eta^-])$  we have  $s' \star q \in \perp(d_l(\vec{c}) : R[\pi^\perp, N/\eta^-])$  and hence  $\tilde{\kappa} \cdot (s' \star q) \in |d_l(\vec{c})|_{R[\pi^\perp, N/\eta^-]}$  and this holds for all  $l \in L$ , that is  $\tilde{\kappa} \cdot (s' \star q) \in \bigcup_{l \in L} |d_l(\vec{c})|_{R[\pi^\perp, N/\eta^-]}$ . Since  $s_1 \in \bigcap_{l \in L} |d_l(\vec{c})|_{R^\perp[\pi, N^\perp/\eta^+]}$  it follows that  $s_1 \star \tilde{\kappa} \cdot (s' \star q) \in \perp$ , hence  $\mathbf{der}(\tilde{\kappa} \cdot (s' \star q)) \star s_1 \in \perp$  as expected since  $\mathbf{der}(\tilde{\kappa} \cdot (s' \star q)) \star s_1 \rightarrow s_1 \star \tilde{\kappa} \cdot (s' \star q)$ .

▷ Assume that  $Q = \mu\zeta.R$ . We set  $R_P = R[P/\eta^+]$  and  $R_N = R[N^\perp/\eta^+]$ ,  $Q_P = Q[P/\eta^+]$  and  $Q_N = Q[N^\perp/\eta^+]$ , and also  $t = Q[\pi, s/\eta^+]$ . We also use  $\rho = \pi \cdot (Q_P[\pi]/\zeta^+)$ . We have  $d = \sigma(d^0)$  with  $d^0 : R[\pi][Q[\pi]/\zeta^+]$ . By Lemma 3.1 we can find  $f : R[\pi]$  as well as a family  $(d_l : Q[\pi])_{l \in L}$  where  $L = \mathbf{rg}_\zeta f$  such that  $d^0 = f\{d_l/(l, \zeta)\}_{l \in L}$ . Let  $J = \mathbf{rg}_\eta d$ , we have  $J = J' \uplus \biguplus_{l \in L} J_l$  where  $J' = \mathbf{rg}_\eta f$  and  $J_l = \mathbf{rg}_\eta d_l$  for each  $l \in L$ . By inductive hypothesis applied to  $(Q, d_l)$  (since  $\mathbf{sz} d_l \leq \mathbf{sz} d^0 < \mathbf{sz} d$ ) we have  $t \in |g_l|_{Q_N^\perp[\pi^\perp]}(f_l : Q_P[\pi])$  where  $f_l = d_l(\vec{b})$  and  $g_l = d_l(\vec{c})$  for each  $l \in L$ , since  $\mathbf{sz} d_l \leq \mathbf{sz} d^0 < \mathbf{sz} d$ . Notice that  $f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L} = d^0(\vec{c})$  by definition of the  $g_l$ 's. We must prove that  $t \in |d(\vec{c})|_{Q_N^\perp[\pi^\perp]}(d(\vec{b}) : Q_P[\pi])$  so let  $p \in \|[d(\vec{c})]\|_{Q_N[\pi]}$ , it suffices to prove that  $t \star p \in \perp(d(\vec{b}) : Q_P[\pi])$ . We have  $p = \mathbf{fd}(q)$  with  $q \in |d^0(\vec{c})|_{R_N[\pi, Q_N[\pi]/\zeta^+]}$ . We have (see Section 2.2)  $t = Q[\pi, s/\eta^+] = \bar{\kappa}_{Q_N^\perp[\pi^\perp], \zeta^+} \alpha \cdot c$  where  $c = \bullet \star \mathbf{fd}(\tilde{\kappa} \cdot (s' \star \alpha))$  where  $s' = R[\rho, s/\eta^+]$ . So  $t \star p \rightarrow (R_N[\pi, t/\zeta^+] \star q) [\kappa\alpha \cdot c/\bullet]$ .

Notice that  $f(\vec{c}) = f\{c_j/(j, \eta)\}_{j \in J'}$  satisfies  $f(\vec{c}) : R_N[\pi]$  and hence by inductive hypothesis applied to  $(R, f(\vec{c}))$  we have that  $R_N[\pi, t/\zeta^+]$  belongs to the set

$$|f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L}|_{R_N^\perp[\pi^\perp, Q_N^\perp[\pi^\perp]/\zeta^-]}(f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L} :$$

$R_N[\pi, Q_P[\pi]/\zeta^+])$  since we have seen that  $t \in |g_l|_{Q_N^\perp[\pi^\perp]}(f_l : Q_P[\pi])$  for each  $l \in L$ . Since  $q \in |d^0(\vec{c})|_{R_N[\pi, Q_N[\pi]/\zeta^+]}$  and  $d^0(\vec{c}) = f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L}$  it follows that  $c' = R_N[\pi, t/\zeta^+] \star q \in \perp(f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L} : R_N[\pi, Q_P[\pi]/\zeta^+])$  which means  $c' \rightarrow^* \bullet \star r$  for some  $r \in \|[f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L}]\|_{R_N[\pi, Q_P[\pi]/\zeta^+]}$  and therefore  $c' [\kappa\alpha \cdot c/\bullet] \rightarrow^* \kappa\alpha \cdot c \star r = \kappa\alpha \cdot (\bullet \star \mathbf{fd}(\tilde{\kappa} \cdot (s' \star \alpha))) \star r \rightarrow \bullet \star \mathbf{fd}(\tilde{\kappa} \cdot (s' \star r))$ .

Notice next that  $f' = f\{f_l/(l, \zeta)\}_{l \in L} : R[\pi, Q_P[\pi]/\zeta^+] = R[\rho]$ . Hence by ind. hyp. applied to  $(R, f')$  and since  $f'(\vec{c}) = f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L}$  we have  $s' \in |f'(\vec{c})|_{R^\perp[\rho^\perp, N/\eta^-]}(f'(\vec{b}) : R[\rho, P/\eta^+])$  and hence  $s' \star r \in \perp(f'(\vec{b}) : R[\rho, P/\eta^+])$ . Notice that  $f'(\vec{b}) = d^0(\vec{b})$  and hence we have  $\tilde{\kappa} \cdot (s' \star r) \in |d^0(\vec{b})|_{R[\rho, P/\eta^+]}$  and therefore, by definition of  $\rho$ ,  $\mathbf{fd}(\tilde{\kappa} \cdot (s' \star r)) \in \|[d(\vec{b})]\|_{Q_P[\pi]}$ . So we have  $\bullet \star \mathbf{fd}(\tilde{\kappa} \cdot (s' \star r)) \in \perp(d(\vec{b}) : Q_P[\pi])$  and hence  $t \star p \in \perp(d(\vec{b}) : Q_P[\pi])$  as contended.

▷ The second statement of the lemma is proved similarly in the same induction of course since the case  $Q = !R^\perp$  (change of polarity) for a given statement uses the other one as an inductive hypothesis.  $\square$

## 6.7 Proof of Theorem 3.7

**Proof.** By induction on the point derivation  $\delta$  for  $c$ ,  $t$  and  $p$ . To increase readability we use  $c'$  for  $c[p_i/\alpha_i]_{i=1}^n$  and similarly for  $s$  and  $p$ . In the induction, we use the notations introduced in the statement of the theorem to avoid boring sentences introducing new symbols. But one has to keep in mind that the statement proven by induction contains the universal quantification on the  $p_i$ 's.

$\triangleright$   $\delta$  consists of **(i-n)** so that we are in case **(v)** with  $p = \alpha_j$  for some  $j \in \underline{n}$ , and  $a = a_j$ . In that case we have  $p' = p_j$  and the expected conclusion follows from  $\|a_j\|_{P_j} \subseteq |a_j|_{P_j}$ .

$\triangleright$   $\delta$  consists of **(i-1)** so that we are in case **(v)** with  $p = ()$  and  $a = *$ . We have  $p' = ()$  so that  $p' \in \|*\|_1 \subseteq |*|_1$ .

$\triangleright$   $\delta$  ends with a **(i- $\otimes$ )** so that we are in case **(v)** with  $p = (q_1, q_2)$ ,  $a = (b_1, b_2)$  and we have subderivations  $\delta_j$  of  $\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid q_j : b_j : Q_j$  for  $j \in \underline{2}$  and moreover  $a_i \tilde{P}_i [a_i^1, a_i^2]$  for all  $i \in \underline{n}$ . For each  $i \in \underline{n}$  we know that  $p_i \in \|a_i\|_{P_i}$  and hence by Lemma 3.3 we have  $p_i \in \|a_i^j\|_{P_i}$  for each  $i \in \underline{n}$  and  $j \in \underline{2}$ . Hence by inductive hypothesis  $q_j' \in |b_j|_{Q_j}$  so that  $p' = (q_1', q_2') \in \|(b_1, b_2)\|_{Q_1 \otimes Q_2} \subseteq |(b_1, b_2)|_{Q_1 \otimes Q_2}$ .

$\triangleright$   $\delta$  ends with a left **(i- $\oplus$ )** the case of a right **(i- $\oplus$ )** being of course completely similar. We are in case **(v)** and  $P = Q_1 \oplus Q_2$ ,  $p = \text{in}_1(q)$  and  $a = (1, b)$  and we have a subderivation  $\delta_1$  of  $\vdash \Phi \mid q : b : Q_1$  so that by inductive hypothesis  $q' \in |b|_{Q_1}$  and hence  $p' = \text{in}_1(q') \in \|(1, b)\|_{Q_1 \oplus Q_2} \subseteq |(1, b)|_{Q_1 \oplus Q_2}$ .

$\triangleright$   $\delta$  ends with a **(i- $\tilde{\kappa}$ )**. We are in case **(v)** with  $p = \tilde{\kappa}.c$  and we have a subderivation  $\delta_1$  of  $\delta$  which has  $c \vdash \Phi, a : P$  as conclusion. By inductive hypothesis we have  $c' \in \perp(a : P) = |a|_{P^\bullet} = \|a\|_{P^\bullet}$  and hence  $p' = \tilde{\kappa}.c' \in |a|_P$  as required.

$\triangleright$   $\delta$  ends with **(i- $\mu$ )** so that we are in case **(v)** with  $P = \mu\zeta.Q$ ,  $p = \text{fd}(q)$  and  $a = \sigma(b)$  and  $\delta$  has a subderivation  $\delta_1$  whose conclusion is  $\vdash \Phi \mid q : b : Q[P/\zeta^+]$ . By inductive hypothesis we have  $q' \in |b|_{Q[P/\zeta^+]}$  and hence  $p' = \text{fd}(q') \in \|\sigma(b)\|_{\mu\zeta.Q} \subseteq |a|_P$ .

$\triangleright$   $\delta$  ends with **(i- $!$ )** so that we are in case **(v)** with  $P = !N$ ,  $p = s^!$  and  $a = [b_j \mid j \in J]$  and  $\delta$  has a subderivation  $\delta_j$  of  $\vdash (\alpha_i : a_i^j : P_i^\perp) \mid s : b_j : N$  for each  $j \in J$  and moreover we have  $a_i \tilde{P}_i [a_i^j \mid j \in J]$  for each  $i \in \underline{n}$ . Since  $\|a_i\|_{P_i} \subseteq \|a_i^j\|_{P_i}$  for each  $i \in \underline{n}$  and  $j \in J$  we have  $s' \in |b_j|_N$  by inductive hypothesis applied to  $\delta_j$  for each  $j \in J$  and hence  $p' = (s')^! \in \|[b_j \mid j \in J]\|_{!N} \subseteq |a|_P$ .

$\triangleright$   $\delta$  is **(i- $\bullet$ )** so that we are in case **(iv)** with  $N = P^\perp$ ,  $s = \bullet$ ,  $b = a$  and  $a_i \tilde{P}_i []$  for each  $i \in \underline{n}$ . We have  $s' = \bullet$  so that for all  $p \in \|a\|_P$  we have  $s' \star p \in \perp(a : P)$  which means that  $s' \in |a|_{P^\perp(a : P)}$  as required.

$\triangleright$   $\delta$  ends with a left **(i- $\&$ )** (the case of a right **(i- $\&$ )** is of course completely similar) and we are in case **(iii)** with  $N = N_1 \& N_2$ ,  $a = (1, a_1)$ ,  $s = \langle s_1, s_2 \rangle$  and we have a subderivation  $\delta_1$  whose conclusion is  $\vdash \Phi \mid s_1 : a_1 : N_1$ . The other subderivation  $\delta_2$  makes sure that  $s_2$  is well typed in the typing system of Figure 2. Let  $p \in \|(1, a_1)\|_{N_1^\perp \oplus N_2^\perp}$  that is  $p = \text{in}_1(p_1)$  with  $p_1 \in |a_1|_{N_1^\perp}$ . We have  $s' \star p = \langle s_1', s_2' \rangle \star \text{in}_1(p_1) \rightarrow s_1' \star p_1 \in \perp$  by inductive hypothesis and hence  $s' \in |a|_N$ .

$\triangleright$   $\delta$  ends with a left **(i- $\&$ )** (the case of a right **(i- $\&$ )** is of course completely similar) and we are in case **(iv)** with  $N = N_1 \& N_2$ ,  $b = (1, b_1)$ ,  $s = \langle s_1, s_2 \rangle$  and we have a subderivation  $\delta_1$  whose conclusion is  $\vdash \Phi, a : P \mid s_1 : b_1 : N_1$ . The other subderivation  $\delta_2$  makes sure that  $s_2$  is well typed in the typing system of Figure 2. Let  $p \in \|(1, b_1)\|_{N_1^\perp \oplus N_2^\perp}$  that is  $p = \text{in}_1(p_1)$  with  $p_1 \in |b_1|_{N_1^\perp}$ . We have  $s' \star p = \langle s_1', s_2' \rangle \star \text{in}_1(p_1) \rightarrow s_1' \star p_1 \in \perp(a : P)$  by inductive hypothesis and hence  $s' \in |b|_N(a : P)$ .

$\triangleright$   $\delta$  ends with **(i- $\kappa$ )** and we are in case **(iii)** with  $s = \kappa\alpha.c$  and we have a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi, \alpha : a : N$ . Let  $p \in \|a\|_{N^\perp}$  we have  $s' \star p \rightarrow c' [p/\alpha] \in \perp$  by inductive hypothesis. It follows that  $s' \in |a|_N$ .

$\triangleright$   $\delta$  ends with **(i- $\kappa$ )** and we are in case **(iv)** with  $s = \kappa\alpha.c$  and we have a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi, a : P \alpha : b : N$ . Let  $p \in \|b\|_{N^\perp}$  we have  $s' \star p \rightarrow c' [p/\alpha] \in \perp(a : P)$  by inductive hypothesis. It follows that  $s' \in |b|_N(a : P)$  as required.

$\triangleright$   $\delta$  ends with **(i- $\perp$ )** and we are in case **(iii)** with  $N = \perp$ ,  $a = *$ ,  $s = \kappa_\perp.c$  and  $\delta$  has a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi$ . We have  $s' \star () = \kappa_\perp.c' \star () \rightarrow c'$  and since, by inductive hypothesis,  $c' \in \perp$ , it follows that  $s' \in |*|_\perp$ .

$\triangleright$   $\delta$  ends with **(i- $\perp$ )** and we are in case **(iv)** with  $N = \perp$ ,  $b = *$ ,  $s = \kappa_{\perp}.c$  and  $\delta$  has a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi$ . We have  $s' \star () = \kappa_{\perp}.c' \star () \rightarrow c'$  and since, by inductive hypothesis,  $c' \in \perp(a : P)$ , it follows that  $s' \in |*|_{\perp}(a : P)$ .

$\triangleright$   $\delta$  ends with **(i- $\wp$ )** and we are in case **(iii)** with  $N = N_1 \wp N_2$ ,  $a = (a_1, a_2)$ ,  $s = \kappa(\alpha_1, \alpha_2).c$  and we have a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi, \alpha_1 : a_1 : N_1, \alpha_2 : a_2 : N_2$ . Let  $p \in \|(a_1, a_2)\|_{N_1^{\perp} \otimes N_2^{\perp}}$ , that is  $p = (q_1, q_2)$  with  $q_j \in |a_j|_{N_j^{\perp}}$  for  $j \in \underline{2}$ . We have  $s' \star p \rightarrow c'[q_1/\alpha_1, q_2/\alpha_2] \in \perp$  by inductive hypothesis. It follows that  $s' \in |a|_N$ .

$\triangleright$   $\delta$  ends with **(i- $\wp$ )** and we are in case **(iv)** with  $N = N_1 \wp N_2$ ,  $b = (b_1, b_2)$ ,  $s = \kappa(\alpha_1, \alpha_2).c$  and we have a subderivation  $\delta_1$  whose conclusion is  $c \vdash \Phi, \alpha_1 : b_1 : N_1, \alpha_2 : b_2 : N_2, a : P$ . Let  $p \in \|(b_1, b_2)\|_{N_1^{\perp} \otimes N_2^{\perp}}$ , that is  $p = (q_1, q_2)$  with  $q_j \in |b_j|_{N_j^{\perp}}$  for  $j \in \underline{2}$ . We have  $s' \star p \rightarrow c'[q_1/\alpha_1, q_2/\alpha_2] \in \perp(a : P)$  by inductive hypothesis. It follows that  $s' \in |b|_N(a : P)$ .

$\triangleright$   $\delta$  ends with **(i-?)** so that we are in case **(iii)** (notice that there no case **(iv)** for this rule) with  $N = ?Q$ ,  $a = [b]$ ,  $s = \text{der}(p)$  and  $\delta$  has a subderivation  $\delta_1$  whose conclusion is  $\vdash \Phi \mid p : b : Q$ . Let  $p \in \|[b]\|_{!Q^{\perp}}$ , that is  $p = t'$  where  $t \in |b|_{Q^{\perp}}$ . We have  $s' \star p \rightarrow t \star p' \in \perp$  since  $p' \in |b|_Q$  by inductive hypothesis.

$\triangleright$   $\delta$  ends with **(i-cut)** and we are in case **(i)** with  $c = s \star p$  and  $\delta$  has two subderivations  $\delta_1$  and  $\delta_2$  with conclusions  $\vdash (\alpha_i : a_i^1 : P_i^{\perp})_{i=1}^n \mid s : b : Q^{\perp}$  and  $\vdash (\alpha_i : a_i^2 : P_i^{\perp})_{i=1}^n \mid p : b : Q$  and  $a_i \tilde{P}_i [a_i^1, a_i^2]$  for all  $i \in \underline{n}$  so that by Lemma 3.3 we have  $p_i \in |a_i^j|_{P_i}$  for all  $i \in \underline{n}$  and  $j \in \underline{2}$ . By inductive hypothesis we have  $s' \in |b|_{Q^{\perp}}$  and  $p' \in |b|_Q$  so that  $c' = s' \star p' \in \perp$ .

$\triangleright$   $\delta$  ends with **(i-cut)** and we are in case **(ii)** with  $c = s \star p$  and  $\delta$  has two subderivations  $\delta_1$  and  $\delta_2$  with conclusions  $\vdash (\alpha_i : a_i^1 : P_i^{\perp})_{i=1}^n, a : P \mid s : b : Q^{\perp}$  and  $\vdash (\alpha_i : a_i^2 : P_i^{\perp})_{i=1}^n \mid p : b : Q$  and  $a_i \tilde{P}_i [a_i^1, a_i^2]$  for all  $i \in \underline{n}$  so that by Lemma 3.3 we have  $p_i \in |a_i^j|_{P_i}$  for all  $i \in \underline{n}$  and  $j \in \underline{2}$ . By inductive hypothesis we have  $s' \in |b|_{Q^{\perp}}(a : P)$  and  $p' \in |b|_Q$  so that  $c' = s' \star p' \in \perp(a : P)$ .

$\triangleright$   $\delta$  ends with **(i- $\nu$ )**, remember that this rule is given at the beginning of Section 3.2. Assume that we are in case **(iv)** with  $N = \nu\zeta.R^{\perp}$ ,  $s = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.(c; t)$ , we have a point  $d : R$ , a subderivation  $\delta''$  of  $\delta$  whose conclusion is  $\vdash (\alpha_i : a_i'' : P_i^{\perp}), a : P \mid t : h : Q^{\perp}$ , a subderivation  $\delta'$  of  $\delta$  whose conclusion is  $c \vdash (\alpha_i : a_i' : P_i^{\perp}), h : Q, \alpha : d\{h_l/(l, \eta)\}_{l \in L} : R^{\perp} [Q^{\perp}/\zeta^-]$ , and for each  $l \in L = \text{rg}_{\zeta}d$  we have a subderivation  $\delta_l$  of  $\delta$  whose conclusion is  $\vdash (\alpha_i : a_i^l : P_i^{\perp})_{i=1}^n, h_l : Q \mid u : f_l : \nu\zeta.R^{\perp}$  where  $u = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.c = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.(c; \bullet)$ . With these notations we have  $b = \sigma(d)\{f_l/(l, \zeta)\}_{l \in L}$ . Moreover  $a_i \tilde{P}_i [a_i', a_i''] + [a_i^l \mid l \in L]$  so that by Lemma 3.3 we have  $p_i \in |a_i^l|_{P_i}$  for all  $i \in \underline{n}$  and  $l \in L$ , and  $p_i \in |a_i'|_{P_i}$ ,  $p_i \in |a_i''|_{P_i}$  for all  $i \in \underline{n}$ . Let  $p \in \|[b]\|_{N^{\perp}}$  so that  $p = \text{fd}(q)$  with  $q \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N^{\perp}/\zeta^+]}$ . Then we have  $s' \star p \rightarrow (R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet]$ . By inductive hypothesis applied to the  $\delta_l$ 's we have  $u' \in |f_l|_N(h_l : Q)$  for each  $l \in L$  and hence by Lemma 3.6,  $R[u'/\zeta^+] \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N/\zeta^-]}(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$ , therefore:

$R[u'/\zeta^+] \star q \in \perp(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$  which means that  $R[u'/\zeta^+] \star q \rightarrow^* \bullet \star r$  with  $r \in \|[d\{h_l/(l, \zeta)\}_{l \in L}]\|_{R[Q/\zeta^+]}$ . It follows that  $(R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet] \rightarrow^* \kappa\alpha.c' [t'/\bullet] \star r \rightarrow c' [r/\alpha] [t'/\bullet]$  since  $r \notin \tilde{\kappa}$ ,  $r$  is closed and  $t'$  has no free names. By inductive hypothesis applied to  $\delta'$  (with substituting positive terms the  $p_i$ 's and  $r$  which satisfies  $r \in |d\{h_l/(l, \zeta)\}_{l \in L}|_{R[Q/\zeta^+]}$  as required) we have  $c' [r/\alpha] \in \perp(h : Q)$  and hence  $\tilde{\kappa}.(c' [r/\alpha]) \in |h|_Q$ . We also have  $t' \in |h|_{Q^{\perp}}(a : P)$  by inductive hypothesis and hence  $t' \star \tilde{\kappa}.(c' [r/\alpha]) \in \perp(a : P)$  so that  $c' [r/\alpha] [t'/\bullet] \in \perp(a : P)$  and hence  $s' \star p \in \perp(a : P)$ . Since this holds for all  $p \in \|[b]\|_{N^{\perp}}$  we have proven that  $s' \in |b|_N(a : P)$  as required.

$\triangleright$   $\delta$  ends with **(i- $\nu$ )** and we are in case **(iii)** with  $N = \nu\zeta.R^{\perp}$ ,  $s = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.(c; t)$ , we have a point  $d : R$ , a subderivation  $\delta''$  of  $\delta$  whose conclusion is  $\vdash (\alpha_i : a_i'' : P_i^{\perp}) \mid t : h : Q^{\perp}$ , a subderivation  $\delta'$  of  $\delta$  whose conclusion is  $c \vdash (\alpha_i : a_i' : P_i^{\perp}), h : Q, \alpha : d\{h_l/(l, \eta)\}_{l \in L} : R^{\perp} [Q^{\perp}/\zeta^-]$ , and for each  $l \in L = \text{rg}_{\zeta}d$  we have a subderivation  $\delta_l$  of  $\delta$  whose conclusion is  $\vdash (\alpha_i : a_i^l : P_i^{\perp})_{i=1}^n, h_l : Q \mid u : f_l : \nu\zeta.R^{\perp}$  where  $u = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.c = \bar{\kappa}_{R^{\perp}, \zeta}\alpha.(c; \bullet)$ . With these notations we have  $b = \sigma(d)\{f_l/(l, \zeta)\}_{l \in L}$ . Moreover  $a_i \tilde{P}_i [a_i', a_i''] + [a_i^l \mid l \in L]$  so that by Lemma 3.3 we have  $p_i \in |a_i^l|_{P_i}$  for all  $i \in \underline{n}$  and  $l \in L$ , and  $p_i \in |a_i'|_{P_i}$ ,  $p_i \in |a_i''|_{P_i}$  for all  $i \in \underline{n}$ . Let  $p \in \|[b]\|_{N^{\perp}}$  so that  $p = \text{fd}(q)$  with  $q \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N^{\perp}/\zeta^+]}$ . Then we have

$s' \star p \rightarrow (R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet]$ . By inductive hypothesis applied to the  $\delta_l$ 's we have  $u' \in |f_l|_N(h_l : Q)$  for each  $l \in L$  and hence, by Lemma 3.6,  $R[u'/\zeta^+] \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N/\zeta^-]}(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$ , therefore  $R[u'/\zeta^+] \star q \in \perp(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$  which means that  $R[u'/\zeta^+] \star q \rightarrow^* \bullet \star r$  with  $r \in |d\{h_l/(l, \zeta)\}_{l \in L}|_{R[Q/\zeta^+]}$ . It follows that  $(R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet] \rightarrow^* \kappa\alpha.c' [t'/\bullet] \star r \rightarrow c' [r/\alpha] [t'/\bullet]$  since  $r \notin \tilde{\kappa}$ ,  $r$  is closed and  $t'$  is closed. By inductive hypothesis applied to  $\delta'$  (with substituting positive terms the  $p_i$ 's and  $r$  which satisfies  $r \in |d\{h_l/(l, \zeta)\}_{l \in L}|_{R[Q/\zeta^+]}$  as required) we have  $c' [r/\alpha] \in \perp(h : Q)$  and hence  $\tilde{\kappa}.(c' [r/\alpha]) \in |h|_Q$ . We also have  $t' \in |h|_{Q^\perp}$  by inductive hypothesis and hence  $t' \star \tilde{\kappa}.(c' [r/\alpha]) \in \perp$  so that  $c' [r/\alpha] [t'/\bullet] \in \perp$  and hence  $s' \star p \in \perp$ . Since this holds for all  $p \in \|b\|_{N^\perp}$  we have proven that  $s' \in |b|_N$  as required.  $\square$

### 6.8 Proof of Lemma 4.2

**Proof.**

Given  $(\vec{A}, \vec{Y}) \in \mathcal{L}^n \times (\mathcal{L}^!)^p$  where  $\vec{Y} = (\vec{Y}, h_{\vec{Y}})$ , we define  $\mathbb{P}^+(\vec{A}, \vec{Y})$  as  $\overline{\mathbb{P}}(\vec{A}, \vec{Y})$  and  $h_{\mathbb{P}^+(\vec{A}, \vec{Y})}$  as the following composition of morphisms in  $\mathcal{L}$ :

$$\overline{\mathbb{P}}(\vec{A}, \vec{Y}) \xrightarrow{\overline{\mathbb{P}}(\vec{A}, h_{\vec{Y}})} \overline{\mathbb{P}}(\vec{A}, \vec{Y}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{Y}}} \mathbb{P}(\vec{A}, \vec{Y})$$

Let  $(f_1, f_2) \in (\mathcal{L}^n \times (\mathcal{L}^!)^p)((\vec{A}_1, \vec{Y}_1), (\vec{A}_2, \vec{Y}_2))$ , we define  $\mathbb{P}^+(f_1, f_2) \in \mathcal{L}^!(\mathbb{P}^+(\vec{A}_1, \vec{Y}_1), \mathbb{P}^+(\vec{A}_2, \vec{Y}_2))$  as  $\overline{\mathbb{P}}(f_1, f_2)$ . And the following diagram commutes which shows that  $\overline{\mathbb{P}}(f_1, f_2)$  is indeed a morphism in  $\mathcal{L}^!$ .

$$\begin{array}{ccc} \overline{\mathbb{P}}(\vec{A}_1, \vec{Y}_1) & \xrightarrow{\overline{\mathbb{P}}(f_1, f_2)} & \overline{\mathbb{P}}(\vec{A}_2, \vec{Y}_2) \\ \overline{\mathbb{P}}(\vec{A}_1, h_{\vec{Y}_1}) \downarrow & & \downarrow \overline{\mathbb{P}}(\vec{A}_2, h_{\vec{Y}_2}) \\ \overline{\mathbb{P}}(\vec{A}_1, \vec{Y}_1) & & \overline{\mathbb{P}}(\vec{A}_2, \vec{Y}_2) \\ \tilde{\mathbb{P}}_{\vec{A}_1, \vec{Y}_1} \downarrow & & \downarrow \tilde{\mathbb{P}}_{\vec{A}_2, \vec{Y}_2} \\ \mathbb{P}(\vec{A}_1, \vec{Y}_1) & \xrightarrow{\mathbb{P}(f_1, f_2)} & \mathbb{P}(\vec{A}_2, \vec{Y}_2) \end{array}$$

$\square$

### 6.9 Proof of Lemma 4.3

**Proof.**

The first item hold because of commutation of the following diagram :

$$\begin{array}{ccc} \mathbb{Z} \otimes \overline{\mathbb{P}}(\vec{A}, \vec{Y}_1, \underline{X}) & \xrightarrow{\mathbb{P}[\mathbb{Z}](\vec{\text{id}}, \vec{\text{id}}, f)} & \overline{\mathbb{P}}(\vec{A}, \vec{Y}_1, \underline{Y}) \\ h_{\mathbb{Z} \otimes \mathbb{P}^+(\vec{A}, \vec{Y}_1, \underline{X})} \downarrow & & \downarrow h_{\mathbb{P}^+(\vec{A}, \vec{Y}_1, \underline{Y})} \\ \mathbb{P}(\mathbb{Z} \otimes \overline{\mathbb{P}}(\vec{A}, \vec{Y}_1, \underline{Y})) & \xrightarrow{\mathbb{P}[\mathbb{Z}](f^\perp, \vec{\text{id}}, \vec{\text{id}})} & \mathbb{P}(\overline{\mathbb{P}}(\vec{A}, \vec{Y}_1, \underline{Y})) \end{array}$$

And the second item is similar to the first one.  $\square$

### 6.10 Proof of Lemma 4.4

**Proof.**

Since  $\mathbb{F} \in \mathcal{L}^{k+1}$ , there is a functor  $\overline{\mu\mathbb{F}} : \mathcal{L}^k \rightarrow \mathcal{L}$  such that  $\overline{\mu\mathbb{F}}(\vec{A})$  is the initial object of the category  $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{A}})$  for any object  $A \in \mathcal{L}$ .

Now, by the construction above, one can extend  $\mu\mathbb{F}$  to a functor  $(\mu\mathbb{F})[Z] : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$ . We take  $(\mu\mathbb{F})[Z]$  as the unique functor  $\mu(\mathbb{F}[Z])$ . Let us take an object  $A \in \mathcal{L}$ . We have  $(\mu\mathbb{F})[Z](\vec{A}) = \overline{\mu\mathbb{F}}(\vec{A})$ . We need to show that  $\mu\mathbb{F}(\vec{A})$  is the initial object of the category  $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$ . First, one can see that  $(\overline{\mu\mathbb{F}}(\vec{A}), h)$  is an object of  $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$  where  $h$  is the following:

$$\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) \xrightarrow{W_Z \otimes \text{Id}} \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) \xrightarrow{\simeq} \overline{\mu\mathbb{F}}(\vec{A})$$

So, let us take an object  $(B, g)$  of  $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$  where  $g \in \mathcal{L}[Z](\overline{\mathbb{F}}_{\vec{A}}(B), B) = \mathcal{L}(\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B), B)$ . We need to provide a morphism  $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})(\overline{\mu\mathbb{F}}(\vec{A}), B)$ . Having a morphism  $\tilde{g} \in \mathcal{L}(\underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}), B)$  is equivalent to have morphism  $\text{cur}'(\tilde{g}) \in \mathcal{L}(\overline{\mu\mathbb{F}}(\vec{A}), \underline{Z} \multimap B)$ . Since  $\overline{\mu\mathbb{F}}(\vec{A})$  is the initial object of  $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{A}})$ , it is enough to have a morphism  $g' \in \mathcal{L}(\overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B), \underline{Z} \multimap B)$  in order to have  $\text{cur}'(\tilde{g})$ . And this is equivalent to provide morphism  $\text{cur}'^{-1}(g') \in \mathcal{L}(\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B), B)$  which is as follows:

$$\begin{array}{ccc} \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) & \xrightarrow{C_Z \otimes \text{Id}} & \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) \\ & & \downarrow \text{Id} \otimes h_Z \otimes \text{Id} \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(!\underline{Z} \otimes (\underline{Z} \multimap B)) & \xleftarrow[\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}]{} & \underline{Z} \otimes !\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) \\ \text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\text{der}_{\underline{Z}} \otimes \text{Id}) \downarrow & & \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \otimes (\underline{Z} \multimap B)) & \xrightarrow[\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\text{ev})]{} & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B) \\ & & \downarrow g \\ & & B \end{array}$$

So,  $\tilde{g}$  is  $\text{cur}'^{-1}(\tilde{g}')$ , and it satisfies the following diagram:

$$\begin{array}{ccc} \underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}) & \xrightarrow{\tilde{g}} & B \\ \simeq \uparrow & & \uparrow g \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & & \\ \downarrow C_Z \otimes \text{Id} & & \\ \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & \xrightarrow{\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}[Z](\tilde{g})} & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B) \end{array} \quad (4)$$

One can see that the following diagram is just unfolding of Diagram 4:

$$\begin{array}{ccc}
 \underline{Z} \otimes \overline{\mu\mathbb{F}}(\overrightarrow{A}) & \xrightarrow{\tilde{g}} & B \\
 \text{Id} \otimes (W_Z \otimes \simeq) \uparrow & & \uparrow g \\
 \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\overline{\mu\mathbb{F}}(\overrightarrow{A})) & & \\
 C_Z \otimes \text{Id} \uparrow & & \\
 \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\overline{\mu\mathbb{F}}(\overrightarrow{A})) & & \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(B) \\
 \downarrow C_Z \otimes \text{Id} & & \uparrow \text{Id} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\tilde{g}) \\
 \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\overline{\mu\mathbb{F}}(\mathbf{A})) & & \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\underline{Z} \otimes \overline{\mu\mathbb{F}}(\overrightarrow{A})) \\
 \downarrow \text{Id} \otimes h_Z \text{ Id} & & \uparrow \text{Id} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\text{der}_{\underline{Z}} \otimes \text{Id}) \\
 \underline{Z} \otimes !\underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(\overline{\mu\mathbb{F}}(\mathbf{A})) & \xrightarrow{\text{Id} \otimes \widehat{\overline{\mathbb{F}}_{\overrightarrow{A}}}} & \underline{Z} \otimes \overline{\mathbb{F}}_{\overrightarrow{A}}(!\underline{Z} \otimes \overline{\mu\mathbb{F}}(\mathbf{A}))
 \end{array}$$

□

### 6.11 Proof of Lemma 4.5

#### Proof.

By Lemma 4.4, we know that there is a unique object  $\mu(\underline{\mathbb{Q}}[Z]) \in \mathcal{L}[Z]$  which is  $(\mu\underline{\mathbb{Q}})[Z]$ . We also know that  $\mu\underline{\mathbb{Q}}$  is the initial object in  $\mathbf{Alg}_{\mathcal{L}}(\underline{\mathbb{Q}})$ , since  $\underline{\mathbb{Q}} \in \mathcal{L}$ . And  $(\mu\underline{\mathbb{Q}})[Z]$  is same as  $\mu\underline{\mathbb{Q}}$  (considering  $\mu\underline{\mathbb{Q}}$ , equivalently, as  $\mu\underline{\mathbb{Q}} : 1 \rightarrow \mathcal{L}$ ). So, to define morphism  $\tilde{g} \in \mathcal{L}[Z](\mu(\underline{\mathbb{Q}}[Z]), \underline{Y})$ , one need to provide a morphism  $g$  in  $\mathcal{L}[Z](\underline{\mathbb{Q}}[Z](Y), Y)$  which is as follows:

$$\underline{Z} \otimes \underline{\mathbb{Q}}[Z](Y) = \underline{Z} \otimes \underline{\mathbb{Q}}(Y) \xrightarrow{g} Y$$

□

### 6.12 Proof of Lemma 4.6

#### Proof.

By definition, we have  $\mu\underline{\mathbb{Q}}^+ = \overline{\mu\underline{\mathbb{Q}}} = \overline{\mu\underline{\mathbb{Q}}} = \mu\underline{\mathbb{Q}}$ . The following diagram commutes which shows that  $\tilde{g}$  is indeed a co-algebra morphism.

$$\begin{array}{ccc}
 !\underline{Z} \otimes \mu\underline{\mathbb{Q}} & \xrightarrow{\tilde{g}} & \underline{Y} \\
 \downarrow h_{!\underline{Z} \otimes \mu\underline{\mathbb{Q}}} & & \downarrow h_Y \\
 !(!\underline{Z} \otimes \mu\underline{\mathbb{Q}}) & \xrightarrow{!\tilde{g}} & !\underline{Y}
 \end{array}$$



## 6.13 Interpretation of proofs

$$\begin{aligned}
 & \llbracket \alpha \rrbracket_{\mathcal{N}, P^\perp}^+ : \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \xrightarrow{W_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \llbracket P \rrbracket}} \llbracket P \rrbracket \\
 & \llbracket () \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{W_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket 1 \rrbracket \\
 & \llbracket (p_1, p_1) \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket p_1 \rrbracket_{\mathcal{N}}^+ \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+}} \llbracket P_1 \otimes P_2 \rrbracket \\
 & \llbracket \text{in}_i(p) \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket p \rrbracket_{\mathcal{N}}^+} \llbracket P_1 \rrbracket \xrightarrow{\bar{\pi}_i} P_1 \oplus P_2 \\
 & \llbracket \tilde{\kappa}.c \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, P}^+} \llbracket P \rrbracket \\
 & \llbracket \text{fd}(p) \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket p \rrbracket_{\mathcal{N}}^+} \llbracket P[\mu\zeta.P/\zeta^+] \rrbracket \simeq \llbracket \mu\zeta.P \rrbracket \\
 & \llbracket t! \rrbracket_{\mathcal{N}}^+ : \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{h_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket !\mathcal{N}^\perp \rrbracket \\
 & \quad \downarrow \llbracket (\text{cur}(t)_{\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket, \perp} \rrbracket) \rrbracket \\
 & \quad \llbracket !N \rrbracket = \llbracket !N \rrbracket \\
 & \llbracket \bullet \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \xrightarrow{W_{\mathcal{N}^\perp} \otimes \llbracket P \rrbracket}} \llbracket P \rrbracket \\
 & \llbracket \langle \rangle \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N} \rrbracket \otimes 0 \simeq 0 \xrightarrow{\text{Init}_{0, P}} \llbracket P \rrbracket \\
 & \llbracket \langle \rangle \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N} \rrbracket \otimes \underline{0} \simeq \underline{0} \xrightarrow{\text{Init}_{0, \perp}} \perp \\
 & \llbracket \langle t_1, t_2 \rangle \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N} \rrbracket \otimes (\llbracket N_1^\perp \rrbracket \oplus \llbracket N_2^\perp \rrbracket) \xrightarrow{\llbracket [t_1]_{\mathcal{N}, P}^+, [t_2]_{\mathcal{N}, P}^+ \rrbracket}} \llbracket P \rrbracket \\
 & \llbracket \langle t_1, t_2 \rangle \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N} \rrbracket \otimes (\llbracket N_1^\perp \rrbracket \oplus \llbracket N_2^\perp \rrbracket) \xrightarrow{\llbracket [t_1]_{\mathcal{N}}, [t_2]_{\mathcal{N}} \rrbracket}} \perp \\
 & \llbracket \kappa\alpha.c \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, N, P}^+} \llbracket P \rrbracket \\
 & \llbracket \kappa\alpha.c \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, N}} \perp \\
 & \llbracket \kappa_\perp.c \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N}^\perp \rrbracket \otimes 1 \simeq \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, P}^+} \llbracket P \rrbracket \\
 & \llbracket \kappa_\perp.c \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N}^\perp \rrbracket \otimes \underline{1} \simeq \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}}} \llbracket P \rrbracket \\
 & \llbracket \kappa(\alpha_1, \alpha_2).c \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N}^\perp \rrbracket \otimes (\llbracket N_1^\perp \rrbracket \otimes \llbracket N_2^\perp \rrbracket) \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, N_1, N_2, P}^+} \llbracket P \rrbracket \\
 & \llbracket \kappa(\alpha_1, \alpha_2).c \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N}^\perp \rrbracket \otimes (\llbracket N_1^\perp \rrbracket \otimes \llbracket N_2^\perp \rrbracket) \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, N_1, N_2}} \perp \\
 & \llbracket \text{der}(p) \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket !P^\perp \rrbracket \xrightarrow{\text{cur}^{-1}((\text{der}_{\llbracket P^\perp \rrbracket})^\perp \circ \llbracket p \rrbracket_{\mathcal{N}}^+)} \perp
 \end{aligned}$$

$$\begin{array}{ccc}
 \llbracket t \star p \rrbracket_{\mathcal{N}, P}^+ : \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 & & \downarrow \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket p \rrbracket_{\mathcal{N}}^+ \\
 \llbracket P \rrbracket & \xleftarrow{\llbracket t \rrbracket_{\mathcal{N}, P}^+} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \\
 \llbracket t \star p \rrbracket_{\mathcal{N}} : \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 & & \downarrow \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket p \rrbracket_{\mathcal{N}}^+ \\
 \perp & \xleftarrow{\llbracket t \rrbracket_{\mathcal{N}}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket
 \end{array}$$

#### 6.14 Proof of Lemma 4.14

**Proof.** We prove it by case analysis of  $c$  and  $d$  based on the reduction system in Figure 3.

▷ If we have  $s \star \tilde{\kappa}.c \rightarrow c[s/\bullet]$ :

$$\begin{aligned}
 \llbracket s \star \tilde{\kappa}.c \rrbracket_{\mathcal{N}} &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket \tilde{\kappa}.c \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket c \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= \llbracket c[s/\bullet] \rrbracket_{\mathcal{N}} \text{ by Lemma 4.11}
 \end{aligned}$$

▷ If we have  $\kappa\alpha.c \star p \rightarrow c[p/\alpha]$ :

$$\begin{aligned}
 \llbracket \kappa\alpha.c \star p \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa\alpha.c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}, P^\perp}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= \llbracket c[p/\alpha] \rrbracket_{\mathcal{N}} \text{ by Lemma 4.9}
 \end{aligned}$$

▷ If we have  $\langle s_1, s_2 \rangle \star \text{in}_i(p) \rightarrow s_i \star p$ :

$$\begin{aligned}
 \llbracket \langle s_1, s_2 \rangle \star \text{in}_i(p) \rrbracket_{\mathcal{N}} &= ((\llbracket \langle s_1, s_2 \rangle \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket \text{in}_i(p) \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket s_1 \rrbracket_{\mathcal{N}}, \llbracket s_2 \rrbracket_{\mathcal{N}}) \circ ( \\
 &\quad \text{Id} \otimes ((\bar{\pi}_i) \circ (\llbracket p \rrbracket_{\mathcal{N}}^+)))) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket s_i \rrbracket_{\mathcal{N}}) \circ ( \\
 &\quad \text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ (C_{\llbracket \mathcal{N}^\perp \rrbracket}) \quad \text{since } \mathcal{L} \text{ is co-cartesian} \\
 &= \llbracket s_i \star p \rrbracket_{\mathcal{N}}
 \end{aligned}$$

▷ If we have  $\kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rightarrow c[p_1/\alpha_1, p_2/\alpha_2]$ :

$$\begin{aligned}
 \llbracket \kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa(\alpha_1, \alpha_2).c \rrbracket_{\mathcal{N}}) \circ ( \\
 &\quad \text{Id} \otimes \llbracket (p_1, p_2) \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}, P_1^\perp, P_2^\perp}) \circ (\text{Id} \otimes (\llbracket p_1 \rrbracket_{\mathcal{N}}^+ \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+)) \circ ( \\
 &\quad \mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \text{ by Diagram (5)} \\
 &= (((\llbracket c \rrbracket_{\mathcal{N}, P_1^\perp, P_2^\perp}) \circ ((\text{Id} \otimes \text{Id} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^+ \otimes \mathbf{W}_{\llbracket P_2 \rrbracket})) \circ ( \\
 &\quad \mathbf{C}_{\llbracket \mathcal{N}^\perp \otimes P_2 \rrbracket})) \circ (\text{Id} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket (c [p_1/\alpha_1]) \rrbracket_{N_2}) \circ (\text{Id} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= \llbracket (c [p_1/\alpha_1]) [p_2/\alpha_2] \rrbracket_{\mathcal{N}} \\
 &= \llbracket c [p_1/\alpha_1, p_2/\alpha_2] \rrbracket_{\mathcal{N}}
 \end{aligned}$$

$$\begin{array}{ccc}
 \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \downarrow \mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket} & & \downarrow \text{Id}_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket} \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket & & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \downarrow \text{Id}_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+ & & \downarrow \text{Id}_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^+ \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^+ \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P_2 \rrbracket & & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P_1 \rrbracket \otimes \llbracket P_2 \rrbracket \\
 \downarrow \mathbf{C}_{\llbracket \mathcal{N} \otimes P_2 \rrbracket} & & \downarrow \llbracket c \rrbracket_{\mathcal{N}, P_1^\perp, P_2^\perp} \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P_2 \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P_2 \rrbracket & & \perp \\
 \downarrow \text{Id} \otimes \text{Id} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^+ \otimes \mathbf{W}_{\llbracket P_2 \rrbracket} & \nearrow \llbracket c \rrbracket_{\mathcal{N}, P_1^\perp, P_2^\perp} & \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P_2 \rrbracket \otimes \llbracket P_1 \rrbracket & & 
 \end{array} \tag{5}$$

▷ If we have  $\kappa_\perp.c \star () \rightarrow c$ :

$$\begin{aligned}
 \llbracket \kappa_\perp.c \star () \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa_\perp.c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket () \rrbracket)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket () \rrbracket)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \mathbf{W}_{\llbracket \mathcal{N}^\perp \rrbracket})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= (\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id}) = \llbracket c \rrbracket_{\mathcal{N}}
 \end{aligned}$$

If we have  $\text{der}(p) \star s^! \rightarrow s \star p$ :

$$\begin{aligned}
 \llbracket \text{der}(p) \star s^! \rrbracket_{\mathcal{N}} &= ((\llbracket \text{der}(p) \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket s^! \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\text{cur}^{-1}((\text{der}_{\llbracket P^\perp \rrbracket})^\perp \circ \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ ((\text{Id} \otimes \\
 &\quad (!(\llbracket \text{cur}(s) \rrbracket_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \llbracket P, \perp \rrbracket_{\mathcal{N}}))) \circ (h_{\llbracket \mathcal{N}^\perp \rrbracket}))) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \\
 &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^\perp \rrbracket}) \text{ by Diagram 6}
 \end{aligned}$$

$$\begin{array}{ccc}
 \llbracket \mathcal{N}^\perp \rrbracket & \xrightarrow{c_{\llbracket \mathcal{N}^\perp \rrbracket}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \\
 \downarrow c_{\llbracket \mathcal{N}^\perp \rrbracket} & & \downarrow \text{Id} \otimes h_{\llbracket \mathcal{N}^\perp \rrbracket} \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket & & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket !\mathcal{N}^\perp \rrbracket \\
 \downarrow \text{Id} \otimes [p]_{\mathcal{N}}^+ & & \downarrow \text{Id} \otimes !(\text{cur}(s)_{\llbracket \mathcal{N}^\perp \rrbracket \otimes [P], \perp} \llbracket \mathcal{N} \rrbracket}) \\
 \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket & & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket !P^\perp \rrbracket \\
 \downarrow [s]_{\mathcal{N}} & \swarrow \text{cur}^{-1}((\text{der}_{\llbracket P^\perp \rrbracket})^\perp \circ [p]_{\mathcal{N}}^+) & \\
 \perp & & 
 \end{array} \tag{6}$$

▷ If we have  $\bar{\kappa}_{Q^\perp, \zeta} \alpha.c \star \text{fd}(p) \rightarrow (Q [\bar{\kappa}_{Q^\perp, \zeta} \alpha.c / \zeta^+] \star p) [\kappa \alpha.c / \bullet]$ : See Figure 8

□

