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Preferential attachment hypergraph with vertex deactivation

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Abstract

In the area of complex networks so far hypergraph models have received significantly less attention than the graphs. However, many real-life networks feature multiary relations (co-authorship, protein reactions) thus may be modeled way better by hypergraphs. Also, recent study by Broido and Clauset suggests that a power-law degree distribution is not as ubiquitous in the natural systems as it was thought so far. They experimentally confirm that majority of networks (56% of around 1000 social, biological, technological, transportation, and information networks that undergone the test) favor a power-law with an exponential cutoff over other distributions. We address two above observations by introducing a preferential attachment hypergraph model which allows for a vertex deactivation. The phenomenon of a vertex deactivation is rare in existing theoretical models and omnipresent in real-life scenarios (think of social network accounts which are not maintained forever, collaboration networks in which people eventually retire or technological networks in which devices may break down). We prove that the degree distribution of a proposed model follows a power-law with an exponential cutoff. We also check experimentally that a Scopus collaboration network has the same characteristic. We believe that our model will predict well the behavior of the systems from variety of domains.

1 Introduction

The notion of *complex networks* relates to the mathematical structures modeling large real-life systems. Their omnipresence across different life domains is remarkable. Complex networks model biological networks (e.g., protein or gene interactions schemes, maps of neural connections in the brain), social networks (Facebook, Twitter, Snapchat, collaboration networks), technological networks (power grids, transportation networks), the World Wide Web, etc. They also allow to predict the behavior of the systems, serve as the benchmarks for testing algorithms that are used later in real networks, and, in general, allow to better understand the underlying mechanisms that create those systems in nature. Roughly since 1999 there has been dynamical growth in experimental and theoretical research on complex networks in the areas of computer science, mathematics, and physics. It was the year when Barabási and Albert introduced a seminal model of a *preferential attachment* random graph [4]. The model was based on two mechanisms: growth (the graph was growing over time, gaining a new vertex and some edges at each time step) and preferential attachment (arriving vertex was more likely to attach to other vertices with high degree rather than with low degree). It captured the *small world* (small diameter) and the *rich get richer* (leading to a heavy tailed degree distribution) phenomena commonly observed in nature.

Since then several theoretical models were presented, e.g., [8, 9, 13, 23, 27]. These were mostly graph models concentrated on reflecting three phenomena: a small diameter, a high clustering coefficient and a *power-law* degree distribution. It was thought for a long time that a power-law degree distribution is the most commonly present in nature [5]. However, recent study by Broido and Clauset [7] questions this statement. They performed statistical tests on almost 1000 social, biological, technological, transportation, and information networks and observed that “majority of networks (56%) favor the power-law with cutoff model over other distributions”. The cutoff observed in the tail of a degree distribution may be caused by a finite-size character of the dynamic network, i.e., when the elements deactivate after some time [7]. Even though some theoretical models which allow for deletion or deactivation of vertices were introduced ([12, 24]) just a few of them lead to a degree distribution following a power-law with an exponential cutoff. One of the widely cited is a balls and bins scheme with ball deactivation introduced by Fenner et al. [15, 16]. In this model information about the degree of each element of the network is kept but information about who is connected with whom is lost. Working with graphs one keeps this information, however graphs reflect only binary relations. In practice we encounter many k -ary relations (co-authorship, groups of interests or protein reactions) but nowadays they are often modeled in graphs by cliques which may lead to a profound information loss.

Multiary relations can be captured using hypergraphs. So far hypergraph models have received significantly less attention than the graphs in the area of complex networks. A preferential attachment hypergraph model was first introduced by Wang et al. in [26]. However, it was restricted just to a specific subfamily of uniform acyclic hypergraphs (the analogue of trees within graphs). The first rigorously studied non-uniform hypergraph preferential attachment model was proposed only in 2019 by Avin et al. [3] and featured a power-law degree distribution. Another dynamic hypergraph model with a clear community structure was presented in [17]. The algorithms and software tools for working with hypergraph networks, and even the definitions of some features and measures started appearing only recently [2, 18, 19].

Results. In this paper we propose a preferential attachment hypergraph model in which vertices may become inactive after some time (e.g., a neuron may die in a brain, a researcher from the collaboration network may retire). The hyperedges model multiary and not necessarily uniform relations, we allow for different cardinalities of hyperedges (e.g., articles may have different numbers of co-authors). We prove that the degree distribution of our model follows a power-law with an exponential cutoff and give another real-life example, a Scopus research collaboration network, with this distribution. We believe that our model will be the next step towards developing the hypergraph chapter in the complex networks area and that will serve as an interesting tool in many research domains, being able to reflect better phenomena observed in nature.

Paper organisation. Section 2 contains basic definitions and notation. In Section 3 we introduce the hypergraph model with vertex deactivation and prove that its degree distribution follows a power-law with an exponential cutoff. Section 4 is devoted to estimating one of the parameters that appears in the formula for the degree distribution of our model. The experimental results on real data and the simulations of the model are presented in Section 5. Further works are discussed in Section 6. The Appendix contains several technical proofs.

2 Basic definitions and notation

We define a *hypergraph* H as a pair $H = (V, E)$, where V is a set of vertices and E is a multiset of hyperedges, i.e., non-empty, unordered multisets of V . We allow for a multiple appearance of a vertex in a hyperedge (self-loops) as well as a multiple appearance of a hyperedge in E .

The degree of a vertex v in a hyperedge e , denoted by $d(v, e)$, is the number of times v appears in e . The cardinality of a hyperedge e is $|e| = \sum_{v \in e} d(v, e)$. The degree of a vertex $v \in V$ in H is understood as the number of times it appears in all hyperedges, i.e., $\deg(v) = \sum_{e \in E} d(v, e)$. If $|e| = k$ for all $e \in E$, H is said to be k -uniform.

We consider hypergraphs that grow by adding vertices and/or hyperedges at discrete time steps $t = 0, 1, 2, \dots$ according to some rules involving randomness. The random hypergraph obtained at time t will be denoted by $H_t = (V_t, E_t)$ and the degree of $u \in V_t$ in H_t by $\deg_t(u)$. During this building process some of the vertices may become deactivated. Therefore the set V_t splits into \mathcal{A}_t , the set of vertices active at time t (denote its cardinality by A_t), and \mathcal{I}_t , the set of vertices that are not active at time t (denote its cardinality by I_t); thus $|V_t| = A_t + I_t$. By D_t we denote the sum of degrees of vertices active at time t , i.e., $D_t = \sum_{u \in \mathcal{A}_t} \deg_t(u)$. Moreover, we write Θ_t for the degree of a vertex chosen for deactivation at time t (the exact description of a deactivation procedure is given within the formal description of the model in the next section).

$N_{k,t}$ stands for the number of vertices in H_t of degree k . Thus $\sum_{k \geq 1} N_{k,t} = |V_t|$. Similarly, $A_{k,t}$ is the number of active vertices of degree k at time t and $I_{k,t}$ the number of inactive vertices of degree k at time t (denote the corresponding sets by $\mathcal{A}_{k,t}$ and $\mathcal{I}_{k,t}$, respectively); $\sum_{k \geq 1} A_{k,t} = A_t$, $\sum_{k \geq 1} I_{k,t} = I_t$ and $N_{k,t} = A_{k,t} + I_{k,t}$. We write $f(k) \sim g(k)$ if $f(k)/g(k) \xrightarrow{k \rightarrow \infty} 1$. We say that the degree distribution of a random hypergraph follows a *power-law* if the expected fraction of vertices of degree k is proportional to $k^{-\beta}$ for some exponent $\beta > 1$. Formally, we will interpret it as $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta}$ for some positive constants c and $\beta > 1$. Similarly, we say that the degree distribution of H_t follows a *power-law with an exponential cutoff* if $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k$, where $\gamma \in (0, 1)$.

We say that an event A occurs *with high probability* (whp) if the probability $\mathbb{P}[A]$ depends on a certain number t and tends to 1 as t tends to infinity.

3 Preferential attachment hypergraph with vertex deactivation

The model introduced in this section may be seen as a generalization of a hypergraph model presented by Avin et al. in [3]. The model from [3] allows for two different actions at a single time step - attaching a new vertex by a hyperedge to the existing structure or creating a new hyperedge on already existing vertices. We add another possibility - deactivation of a vertex. Once a vertex is chosen for deactivation, it stays deactivated forever, i.e., it remains in the hypergraph but it can not be chosen to the new hyperedges - its degree freezes and the edges incident with it remain in the hypergraph.

3.1 Model $\mathbf{H}(\mathbf{H}_0, \mathbf{p}_v, \mathbf{p}_e, \mathbf{Y})$

The hypergraph model H is characterized by the following parameters:

1. H_0 - the initial hypergraph, seen at $t = 0$;
2. $p_v, p_e, p_d = 1 - p_e - p_v$ - the probabilities indicating, what are the chances that a particular type of event occurs at a single time step;
3. $Y = (Y_0, Y_1, \dots, Y_t, \dots)$ - independent random variables, giving the cardinalities of the hyperedges that are added at a single time step.

Here is how the structure of $H = H(H_0, p_v, p_e, Y)$ is being built. We start with some non-empty hypergraph H_0 at $t = 0$. We assume for simplicity that H_0 consists of a hyperedge of cardinality 1 over a single vertex. Nevertheless, all the proofs may be generalized to any

initial H_0 having constant number of vertices and constant number of hyperedges with constant cardinalities. ‘Vertices chosen from \mathcal{A}_t in proportion to degrees’ means that active vertices are chosen independently (possibly with repetitions) and the probability that any u from \mathcal{A}_t is chosen is

$$\mathbb{P}[u \text{ is chosen}] = \frac{\deg_t(u)}{\sum_{v \in \mathcal{A}_t} \deg_t(v)} = \frac{\deg_t(u)}{D_t}$$

($\deg_t(u)$ and $\deg_t(v)$ refer to the degrees of u and v in the whole H_t). For $t \geq 0$ we form H_{t+1} from H_t choosing only one of the following events according to p_v, p_e, p_d .

- With probability p_v : Add one vertex v . Draw a value y being a realization of Y_t . Then select $y - 1$ vertices from \mathcal{A}_t in proportion to degrees; add a new hyperedge consisting of v and the $y - 1$ selected vertices.
- With probability p_e : Draw a value y being a realization of Y_t . Then select y vertices from \mathcal{A}_t in proportion to degrees; add a new hyperedge consisting of the y selected vertices.
- With probability p_d : Choose one vertex from \mathcal{A}_t in proportion to degrees. Deactivate it, i.e., $\mathcal{A}_{t+1} = \mathcal{A}_t \setminus \{v\}$ and $\mathcal{I}_{t+1} = \mathcal{I}_t \cup \{v\}$.

Remark. After setting $p_d = 0$ above, the model boils down to the one presented in [3].

Remark. As the hypergraph gets large, the probability of creating a self-loop can be well bounded and is quite small provided that the sizes of hyperedges are reasonably bounded.

Note that if we want a process to continue then it is reasonable to demand that, on average, we add more vertices to the system than we deactivate. Therefore we always assume $p_v > p_d$. Then the probability that the process terminates (i.e., that we arrive at the moment in which all vertices are deactivated) equals $(p_d/p_v)^i$, where i is the number of active vertices at time $t = 0$, in our case $i = 1$ (compare with the probability that the gambler’s fortune won’t increase forever, [14]). Whenever it happens, we restart the simulation.

3.2 Degree distribution of $\mathbf{H}(\mathbf{H}_0, \mathbf{p}_v, \mathbf{p}_e, \mathbf{Y})$

In this section we prove that the degree distribution of $H = H(H_0, p_v, p_e, Y)$ follows a power-law with an exponential cutoff under four assumptions. First two of them address the distributions of the cardinalities of hyperedges (Y_t) added step by step. We assume that their expectation is constant and their variance sublinear in t , which, we feel, is in accordance with many real-life systems (in particular, with the scientific collaboration networks we are working with experimentally in Section 5). The third assumption tells that we will restrict ourselves to only such distributions of Y_t for which the distribution of D_t (the sum of degrees of active vertices at time t) remains concentrated. Similar assumption one finds in other papers on complex network models, e.g. in [3] by Avin et al. (where a model of a preferential attachment hypergraph with the degree distribution following a power-law is presented) or in [21, 20] by Krapivsky et al. (where the models in which the arriving vertex attaches to the existing node w with probability proportional to $(\deg w)^r$ with $r < 1$ is studied). The fourth assumption was also already present in the literature on models with degree distribution following a power-law with an exponential cutoff (consult [15, 16]). It assumes the existence of the limit $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$. Throughout the paper we prove that the average sum of degrees of vertices chosen for deactivation ($\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$) is of order $\Theta(t)$, however we are not able to (just as the authors of [15] or [16]) theoretically justify the existence of the stated limit. We leave it as the assumption strongly supported by simulations in Section 5. We also explain in Section 4 how the limiting value may be obtained, assuming that the limit exists. **Assumptions**

1. $\mathbb{E}[Y_t] = \mu \in \mathbb{R}_{>0}$ for all $t > 0$.
2. $\text{Var}[Y_t] = o(t)$.
3. $\mathbb{P}[D_t \notin \mathbb{E}[D_t] + o(t)] = o(1/t)$.
4. $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$.

Before we formally state and prove the main theorem we introduce several technical lemmas and theorems that will be helpful later on.

Theorem 1 (Stolz-Cesàro theorem) *Let $(a_t)_{t \geq 1}$ and $(b_t)_{t \geq 1}$ be the sequences of real numbers. Assume that $(b_t)_{t \geq 1}$ is strictly monotone and divergent. If $\lim_{t \rightarrow \infty} \frac{a_{t+1} - a_t}{b_{t+1} - b_t} = g$ then $\lim_{t \rightarrow \infty} \frac{a_t}{b_t} = g$.*

Lemma 1 ([10], Chapter 3.3) *Let $(a_t)_{t \geq 1}$, $(b_t)_{t \geq 1}$ and $(c_t)_{t \geq 1}$ be the sequences of real numbers, where $b_t \xrightarrow{t \rightarrow \infty} b > 0$, $c_t \xrightarrow{t \rightarrow \infty} c$ and a_t satisfies the recursive relation $a_{t+1} = (1 - \frac{b_t}{t}) a_t + c_t$. Then $\lim_{t \rightarrow \infty} \frac{a_t}{t} = \frac{c}{1+b}$.*

The proofs of Lemmas 2, 3, and 4 can be found in the Appendix.

Lemma 2 *If $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim c \cdot k^{-\beta} \gamma^k (\frac{1}{k} + \delta)$ for some positive constants c, β, γ, δ then $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim \frac{c}{p_v} k^{-\beta} \gamma^k (\frac{1}{k} + \delta)$. (Here “ \sim ” refers to the limit by $k \rightarrow \infty$.)*

Lemma 3 *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$ and $\text{Var}[Y_t] = o(t)$. Then $\mathbb{E} \left[\frac{Y_t^2}{D_{t-1}^2} \right] = o(\frac{1}{t})$.*

Lemma 4 *Let $D_t = \mathbb{E}[D_t] + o(t)$ whp. Then $\mathbb{E} \left[\frac{A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[A_{k,t}]}{\mathbb{E}[D_t]} + o(1)$ for each $k \geq 1$.*

Fact 1 *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$. Assume that $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$. Then $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[D_t]}{t} = (p_v + p_e)\mu - p_d\theta$.*

Proof. The initial hypergraph H_0 consists of a single vertex of degree 1. Since at time $t \geq 1$ we add a hyperedge of cardinality Y_t with probability $p_v + p_e$ and we deactivate a vertex of degree Θ_t with probability p_d we get

$$\mathbb{E}[D_t] = 1 + (p_v + p_e) \sum_{\tau=1}^t \mathbb{E}[Y_\tau] - p_d \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]. \quad (1)$$

The conclusion follows. ■

Theorem 2 *Consider a hypergraph $H = H(H_0, p_e, p_v, Y)$ for any $t > 0$. By Assumptions (1-4) the degree distribution of H follows a power-law with an exponential cutoff, i.e.,*

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right)$$

$$\text{for } \beta = \frac{\mu(p_v + p_e) - p_d\theta}{p_v(\mu - 1) + p_e\mu + p_d}, \quad \gamma = \frac{p_v(\mu - 1) + p_e\mu}{p_v(\mu - 1) + p_e\mu + p_d},$$

$$\delta = \frac{p_d}{\mu(p_v + p_e) - p_d\theta}, \quad c = \frac{\beta \cdot \Gamma(1 + \beta)}{\gamma},$$

where $\Gamma(x)$ stands for the gamma function ($\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$).

Remark. The theorem and its proof presented below remain true if we relax Assumption (3) just to $D_t = \mathbb{E}[D_t] + o(t)$ whp. Nevertheless, we leave the stronger version of (3) on the list of assumptions as it will be needed in Section 4 for estimating θ .

Remark. Setting $p_d = 0$ in the above theorem (i.e., considering the process without deactivation) results in the power-law degree distribution, namely $\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-(\beta+1)}$, where $\beta = \frac{\mu}{\mu - p_v}$ and $c = \beta \cdot \Gamma(1 + \beta)$. This is in accordance with the result obtained in [3].

The proof below contains shortcuts in calculations. See the Appendix for the full proof.

Proof. We take a standard master equation approach that can be found e.g. in Chung and Lu book [10] about complex networks. However, we apply it separately to the number of active vertices and the number of deactivated vertices.

Recall that $N_{k,t}$ denotes the number of vertices of degree k at time t . We need to show that $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right)$ for the proper constants c, β, γ and δ . However, by Lemma 2 we know that it suffices to show that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right).$$

Recall that $N_{k,t} = A_{k,t} + I_{k,t}$. First, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t}$ using the mathematical induction on k . In this part we follow closely the lines of the proof that can be found in [3]. Consider the case $k = 1$. Since H_0 consists of a single hyperedge of cardinality 1 over a single vertex, we have $A_{1,0} = 1$. To formulate a master equation, let us make the following observation for $t \geq 1$. An active vertex remains in $\mathcal{A}_{1,t}$ if it had degree 1 at step $t - 1$ and was neither selected to a hyperedge, nor deactivated. Recall that a vertex from $\mathcal{A}_{1,t-1}$ is chosen at step t in a single trial to the new hyperedge with probability $1/D_{t-1}$ thus the chance that it won't be selected to the hyperedge of cardinality y equals $(1 - 1/D_{t-1})^y$. Also, in each step, with probability p_v , a single new active vertex of degree 1 is added to the hypergraph. Let \mathcal{F}_t denote a σ -algebra associated with the probability space at step t . For $t \geq 1$ we have

$$\begin{aligned} \mathbb{E}[A_{1,t} | \mathcal{F}_{t-1}] &= p_v A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right)^{Y_{t-1}} + p_e A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right)^{Y_t} \\ &\quad + p_d A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) + p_v. \end{aligned} \tag{2}$$

After taking the expectation on both sides of (2) we derive upper and lower bounds on $\mathbb{E}[A_{1,t}]$. By Bernoulli's inequality ($(1 + x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x \geq -1$), Lemma 4 (thus by Assumption (3) necessary for it) and the independence of Y_t from $A_{1,t-1}$ and D_{t-1}

$$\begin{aligned} \mathbb{E}[A_{1,t}] &\geq p_v \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_{t-1}}{D_{t-1}} \right) \right] + p_e \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t}{D_{t-1}} \right) \right] \\ &\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\ &= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e \mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v - o(1). \end{aligned} \tag{3}$$

On the other hand, since $(1 - x)^n \leq 1/(1 + nx)$ for $x \in [0, 1]$ and $n \in \mathbb{N}$, and $A_{1,t-1} \leq t$, by Lemma 3 (thus by Assumptions (1) and (2) necessary for it) and Lemma 4 (thus by Assumption

(3)) we have

$$\begin{aligned}
\mathbb{E}[A_{1,t}] &\leq p_v \mathbb{E} \left[\frac{A_{1,t-1}}{1 + (Y_t - 1)/D_{t-1}} \right] + p_e \mathbb{E} \left[\frac{A_{1,t-1}}{1 + Y_t/D_{t-1}} \right] \\
&\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\
&= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1).
\end{aligned} \tag{4}$$

From (3) and (4) we get

$$\mathbb{E}[A_{1,t}] = \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1).$$

Now, we apply Lemma 1 to the above equation choosing

$$a_t = \mathbb{E}[A_{1,t}], \quad b_t = \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]/t}, \quad c_t = p_v + o(1).$$

We have $\lim_{t \rightarrow \infty} c_t = p_v$ and, by Fact 1 (thus by Assumptions (1) and (4) implying it), $\lim_{t \rightarrow \infty} b_t = \frac{p_v(\mu-1)+p_e\mu+p_d}{\mu(p_v+p_e)-p_d\theta} =: 1/\beta$ thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{1,t}]}{t} = \frac{p_v}{1 + 1/\beta} =: \bar{A}_1.$$

Now, we assume that the limit $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k-1,t}]}{t}$ exists and equals \bar{A}_{k-1} and we will show by induction on k that the analogous limit for $\mathbb{E}[A_{k,t}]$ exists. Let us again formulate a master equation, this time for $k > 1$. We have $A_{k,0} = 0$ and for $t \geq 1$ an active vertex appears in $A_{k,t}$ if it was active at step $t-1$, had degree $k-l$ and was chosen exactly l times to a hyperedge, or it had degree k and was not selected for deactivation. Let $B(l, n, p) = \binom{n}{l} p^l (1-p)^{n-l}$. We have

$$\begin{aligned}
\mathbb{E}[A_{k,t} | \mathcal{F}_{t-1}] &= p_v \sum_{l=0}^{\min\{Y_t-1, k-1\}} A_{k-l, t-1} B \left(l, Y_t - 1, \frac{k-l}{D_{t-1}} \right) + p_e \sum_{l=0}^{\min\{Y_t, k-1\}} A_{k-l, t-1} B \left(l, Y_t, \frac{k-l}{D_{t-1}} \right) \\
&\quad + p_d A_{k, t-1} \left(1 - \frac{k}{D_{t-1}} \right).
\end{aligned}$$

Taking the expectation on both sides we get

$$\mathbb{E}[A_{k,t}] = \mathbb{E}[\psi] + p_v \mathbb{E}[\varphi(Y_t - 1)] + p_e \mathbb{E}[\varphi(Y_t)], \quad \text{where}$$

$$\begin{aligned}
\psi &= p_v \sum_{l=0}^1 A_{k-l, t-1} B \left(l, Y_t - 1, \frac{k-l}{D_{t-1}} \right) + p_e \sum_{l=0}^1 A_{k-l, t-1} B \left(l, Y_t, \frac{k-l}{D_{t-1}} \right) \\
&\quad + p_d A_{k, t-1} \left(1 - \frac{k}{D_{t-1}} \right)
\end{aligned}$$

$$\text{and } \varphi(n) = \sum_{l=2}^{\min\{n, k-1\}} A_{k-l, t-1} B \left(l, n, \frac{k-l}{D_{t-1}} \right).$$

We will show that only the term $\mathbb{E}[\psi]$ is significant and that the terms $\mathbb{E}[\varphi(Y_t - 1)]$ and $\mathbb{E}[\varphi(Y_t)]$ converge to 0 as $t \rightarrow \infty$. We have

$$\varphi(Y_t) \leq \sum_{l=2}^{k-1} A_{k-l, t-1} \binom{Y_t}{l} \left(\frac{k-l}{D_{t-1}} \right)^l \left(1 - \frac{k-l}{D_{t-1}} \right)^{Y_t-l} = O(t) \frac{Y_t^2}{D_{t-1}^2}.$$

Hence by Lemma 3 (thus by Assumptions (1) and (2)) we get $\mathbb{E}[\varphi(Y_t)] = o(1)$ and, similarly, $\mathbb{E}[\varphi(Y_t - 1)] = o(1)$. The bounds for $\mathbb{E}[\psi]$ can be derived analogously to the ones for $\mathbb{E}[A_{1,t}]$ and they give

$$\begin{aligned} \mathbb{E}[A_{k,t}] &= \mathbb{E}[A_{k,t-1}] \left(1 - \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]} \right) \\ &\quad + \mathbb{E}[A_{k-1,t-1}] \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]} + o(1). \end{aligned} \quad (5)$$

Recall that by the induction assumption $\lim_{t \rightarrow \infty} \mathbb{E}[A_{k-1,t}]/t = \bar{A}_{k-1}$. Now, we apply again Lemma 1 to the above equation choosing

$$a_t = \mathbb{E}[A_{k,t}], b_t = \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]/t}, c_t = \frac{\mathbb{E}[A_{k-1,t-1}]}{t} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]/t} + o(1).$$

By Fact 1 (thus by Assumptions (1) and (4)) we have $\lim_{t \rightarrow \infty} b_t = k/\beta$ and $\lim_{t \rightarrow \infty} c_t = \bar{A}_{k-1} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mu(p_v + p_e) - p_d\theta}$ thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} = \bar{A}_k = \bar{A}_{k-1} \frac{(k-1)\gamma}{k + \beta}, \quad (6)$$

where $\gamma = \frac{p_v(\mu-1) + p_e\mu}{p_v(\mu-1) + p_e\mu + p_d}$. Thus we got

$$\bar{A}_1 = p_v\beta \frac{1}{(1 + \beta)}, \quad \bar{A}_2 = p_v\beta \frac{\gamma}{(1 + \beta)(2 + \beta)}, \dots, \quad \bar{A}_k = p_v\beta \frac{\gamma^{k-1}(k-1)!}{(1 + \beta)(2 + \beta) \dots (k + \beta)}.$$

Since $\lim_{k \rightarrow \infty} \frac{\Gamma(k)k^\alpha}{\Gamma(k+\alpha)} = 1$ for constant $\alpha \in \mathbb{R}$ we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} = \bar{A}_k = \frac{p_v\beta}{\gamma} \frac{\gamma^k \Gamma(1 + \beta) \Gamma(k)}{\Gamma(k + \beta + 1)} \sim p_v \cdot c \cdot \gamma^k k^{-(\beta+1)} \quad (7)$$

with $c = \frac{\beta \cdot \Gamma(1 + \beta)}{\gamma}$.

Now, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t}$. We have $I_{k,0} = 0$ for all $k \geq 1$. For $t \geq 1$ the expected number of inactive vertices of degree $k \geq 1$ at step t , given \mathcal{F}_{t-1} , can be expressed as

$$\mathbb{E}[I_{k,t} | \mathcal{F}_{t-1}] = I_{k,t-1} + p_d A_{k,t-1} \frac{k}{D_{t-1}},$$

since inactive vertices of degree k remain in $I_{k,t}$ forever and a vertex of degree k becomes inactive if it was selected in step $t-1$ for deactivation. Taking the expectation on both sides, by Lemma 4 (thus by Assumption (3)), we obtain

$$\mathbb{E}[I_{k,t}] = \mathbb{E}[I_{k,t-1}] + p_d \mathbb{E}[A_{k,t-1}] \frac{k}{\mathbb{E}[D_{t-1}]} + o(1).$$

Then, by Fact 1 (thus by Assumptions (1) and (4)),

$$\lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k,t-1}]) = \lim_{t \rightarrow \infty} p_d k \frac{\mathbb{E}[A_{k,t-1}]}{t} \frac{t}{\mathbb{E}[D_{t-1}]} + o(1) = \bar{A}_k k \delta,$$

where $\delta = \frac{p_d}{(p_v + p_e)\mu - p_d\theta}$. And, by Stolz–Cesàro theorem (Theorem 1), we obtain

$$\bar{I}_k := \lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t} = \lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k,t-1}]) = \bar{A}_k k \delta. \quad (8)$$

Finally, by (7) and (8)

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}] + \mathbb{E}[I_{k,t}]}{t} = \bar{A}_k + \bar{I}_k = \bar{A}_k(1 + k\delta) \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right).$$

■

4 Estimating the limiting value θ

This section is devoted to estimating θ which appears as one of the parameters in the degree distribution of our hypergraph model H (consult Theorem 2). Recall that Θ_t stands for the degree of a vertex chosen for deactivation at time t and the last of our four assumptions needed to prove Theorem 2 reads as **4.** $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$.

Let us start with showing that $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ is of order $\Theta(t)$.

Lemma 5 *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$. Then $p_d \leq \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \leq 1 + \frac{p_v(\mu-1) + p_e\mu}{p_d}$.*

Proof. By equation (1) we get $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \frac{1}{p_d} (1 + (p_v + p_e)\mu t - \mathbb{E}[D_t])$. Note that $\mathbb{E}[D_t] \geq \mathbb{E}[A_t] = 1 + (p_v - p_d)t$ (we assume $p_v > p_d$) thus on one hand $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \leq t \left(1 + \frac{p_v(\mu-1) + p_e\mu}{p_d} \right)$ and on the other $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \geq \mathbb{E}[I_t] = p_d t$. ■

Unfortunately, we were not able to prove that the limit $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ exists. However, we support this assumption by simulations in Section 5. Whereas in this section we show, assuming that the limit exists, how to estimate it.

Throughout this section $F(a, b; c; z)$ denotes the Gaussian hypergeometric function, i.e.,

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \text{for } a, b, c, z \in \mathbb{C}, \quad |z| < 1 \quad \text{and } c \notin \mathbb{Z}_{\leq 0},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol.

Lemma 6 *Assume that $\mathbb{P}[D_t \notin \mathbb{E}[D_t] + o(t)] = o(1/t)$. Then*

$$\mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t}]}{\mathbb{E}[D_t]} + o(1).$$

The proof can be found in the Appendix.

Theorem 3 *Assume that the conditions (1–4) from Section 3 hold. Then θ is a fixed point of the function $R(x) := \frac{F(2, 2, \rho(x), \gamma)}{F(1, 2, \rho(x), \gamma)}$, where $\gamma = \frac{p_v(\mu-1) + p_e\mu}{p_v(\mu-1) + p_e\mu + p_d}$ and $\rho(x) = 2 + \frac{\mu(p_v + p_e) - p_d x}{p_v(\mu-1) + p_e\mu + p_d}$.*

Proof. Recall that $\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ and Θ_t is the degree of a vertex chosen for deactivation at time t . Let \mathcal{F}_t denote a σ -algebra associated with the probability space at step t . We have $\mathbb{E}[\Theta_t | \mathcal{F}_{t-1}] = \sum_{k \geq 1} k \frac{k A_{k,t-1}}{D_{t-1}}$, hence taking expectation on both sides, applying Lemma 6 (thus by Assumption (3)) and noting that $D_t = \sum_{k \geq 1} k A_{k,t}$ we get

$$\mathbb{E}[\Theta_t] = \mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t-1}}{D_{t-1}} \right] = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t-1}]}{\mathbb{E}[D_{t-1}]} + o(1) = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t-1}]}{\mathbb{E}[\sum_{k \geq 1} k A_{k,t-1}]} + o(1).$$

Now, by equation (7) (thus by Assumptions (1-4) needed to prove Theorem 2) we write

$$\lim_{t \rightarrow \infty} \mathbb{E}[\Theta_t] = \frac{\sum_{k \geq 1} k^2 \bar{A}_k}{\sum_{k \geq 1} k \bar{A}_k} = \frac{F(2, 2, \rho(\theta), \gamma)}{F(1, 2, \rho(\theta), \gamma)}.$$

Finally, setting $a_t = \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ and $b_t = t$ in Stolz-Cesàro theorem (Theorem 1) we obtain

$$\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \frac{F(2, 2, \rho(\theta), \gamma)}{F(1, 2, \rho(\theta), \gamma)}.$$

■

From now on we consider the behavior of $R(x)$ only in the interval $[0, \hat{\theta}]$, where $\hat{\theta} = \frac{(p_v + p_e)\mu}{p_d}$ since we know that the limiting value θ we are looking for belongs there. Indeed, by Lemma 5 we know that it is at least p_d and at most $\frac{(p_v + p_e)\mu - p_v + p_d}{p_d}$ and we work by $p_v > p_d$ to ensure that, on average, we add more vertices to the network than we deactivate. Remind that the function $F(a, b; c; z)$ is defined for $|z| < 1$ and $c \notin \mathbb{Z}_{\leq 0}$. Therefore, since $0 < \gamma < 1$ and $\rho(x)$ is positive on $[0, \hat{\theta}]$, both $F(1, 2, \rho(x), \gamma)$ and $F(2, 2, \rho(x), \gamma)$ are always defined, continuous and positive on $[0, \hat{\theta}]$. This implies that $R(x)$ is continuous on $[0, \hat{\theta}]$. Below we will justify that $R(x)$ has just one fixed point in the interval $[0, \hat{\theta}]$ and that a fixed-point iteration method will converge here. We start with recalling Banach Fixed Point Theorem.

Theorem 4 (Banach Fixed Point Theorem) *Let (S, d) be a non-empty complete metric space with a contraction mapping $R : S \rightarrow S$. Then R admits a unique fixed point s^* in S ($R(s^*) = s^*$). Furthermore, s^* can be found as follows: start with an arbitrary element $s_0 \in S$ and define a sequence $\{s_n\}_{n \geq 1}$ by $s_n = R(s_{n-1})$ for $n \geq 1$. Then $\lim_{n \rightarrow \infty} s_n = s^*$.*

Thus we aim at showing that $R(x)$ is a contraction mapping on $[0, \hat{\theta}]$. From now on let $F_1(x) = F(1, 2, \rho(x), \gamma)$ and $F_2(x) = F(2, 2, \rho(x), \gamma)$ for γ and $\rho(x)$ as in Theorem 3.

The proofs of Lemmas 7, 8, and 9 can be found in the Appendix.

Lemma 7 *The function $R(x) = \frac{F_2(x)}{F_1(x)}$ can be also expressed as $R(x) = x - \frac{p_v}{p_d} + \frac{1}{1-\gamma} \frac{\rho(x)-1}{F_1(x)}$, where γ and $\rho(x)$ are as in Theorem 3.*

Lemma 8 *$R(x)$ strictly increases on $[0, \hat{\theta}]$.*

Lemma 9 *The function $R(x)$ is a contraction mapping on $[0, \hat{\theta}]$.*

Corollary 1 *Assume that the conditions (1 – 4) from Section 3 hold (in particular, $\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$). Then θ is a unique fixed point of $R(x)$ in $[0, \hat{\theta}]$, such that $\lim_{n \rightarrow \infty} \theta_n = \theta$, where $\theta_{n+1} = R(\theta_n)$ and θ_0 can take any value in $[0, \hat{\theta}]$.*

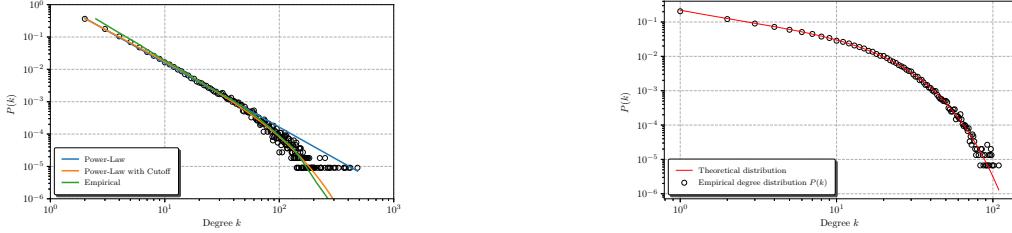
Proof. The proof follows directly from the fact that θ is a fixed point of $R(x)$ (Theorem 3), the fact that $R(x)$ is a contraction mapping, defined on a complete metric space (Lemma 9), and the Banach fixed-point theorem (Theorem 4). ■

Remark. The speed of convergence of the fixed-point iteration method may be described by a Lipschitz constant for R , denoted here by q : $d(\theta, \theta_{n+1}) \leq \frac{q}{1-q} d(\theta_{n+1}, \theta_n)$. If we conjecture that $R(x)$ is convex on $[0, \hat{\theta}]$ then we easily get (remembering that $R(x)$ is increasing) that the best Lipschitz constant for R is $q = \sup_{x \in [0, \hat{\theta}]} R'(x) = R'(\hat{\theta}) = 1 + \frac{1-\gamma}{\gamma} \ln(1-\gamma)$. However, proving the convexity of $R(x)$ seems very demanding.

In the next section we present the results of applying the fixed-point iteration method to estimate θ for the exemplary random hypergraph following our model.

5 Experimental results

In order to verify the obtained results and legitimacy of assumptions made we ran numerous simulations of proposed model trying different sets of parameters. In this section we present the results of simulated $\tilde{H} = H(H_0, p_v = 0.3, p_e = 0.49, p_d = 0.21, Y_t = 3)$ (this choice is arbitrary, we observed similar results by other parameters). We also compare them with the behavior of a real collaboration network G . G was built upon data extracted from Scopus [25], these were 239414 computer science articles published in between 1990 and 2018 by 258145 different authors. Each author was treated as a node and every publication corresponded to a hyperedge between its co-authors.

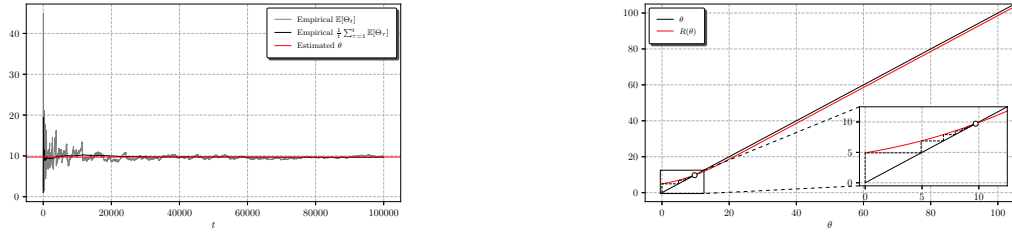


(a) The degree distribution of the Scopus collaboration network G .

(b) The comparison of a theoretical and simulated degree distribution of the hypergraph \tilde{H} .

Figure 1: The degree distributions of a real-life and simulated hypergraphs.

We used statistical tools from [11] to fit and compare theoretical distributions with the real degree distribution of G . One finds the result in Figure 1a which shows that a power-law with an exponential cutoff is a good fit here (this is just one of many examples of real-life networks that follow this distribution [7]). Figure 1b shows the degree distribution of \tilde{H} - the simulated one closely corresponds to the theoretical one that we obtain by Theorem 2.



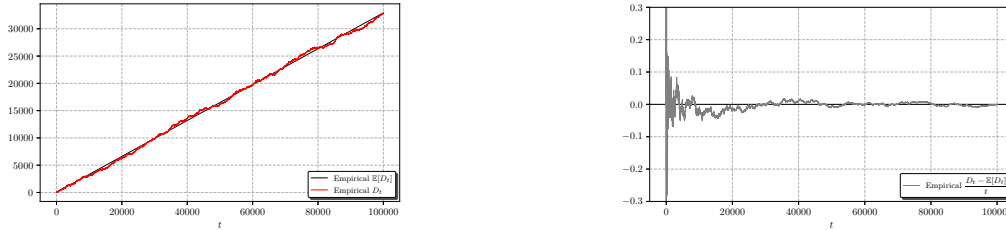
(a) The empirical average deactivated degree (grey and black curves) as compared to the estimated θ (red line) in the hypergraph \tilde{H} .

(b) The visualization of the fixed-point iteration method applied to the model \tilde{H} , starting from $\theta_0 = 0$.

Figure 2: Experimental results on the parameter θ and the fixed-point iteration method.

Next, we investigated the evolution of the average degree of a vertex selected for deactivation in \tilde{H} and compared it with the value of θ calculated using the fixed-point iteration method (Theorem 1). Figure 2a shows the convergence of the empirical average degree of a deactivated vertex to the estimated value of θ which supports both, our assumption (4) as well as the method for evaluating θ (see Figure 2b for its visualization).

Furthermore, we checked empirically the value of $\mathbb{E}[D_t]$ in the simulated \tilde{H} (we ran 10000 simulations up to 100000 steps). The empirical $\mathbb{E}[D_t]$ appeared to be linear with the slope $\hat{\alpha} = 0.32900254$ (Figure 3a). We then calculated the slope of the actual $\mathbb{E}[D_t]$ using the fixed-point iteration method to compute θ and then plugging it into equation from Fact 1. It yielded



(a) The empirical $\mathbb{E}[D_t]$ and the trajectory of D_t for the hypergraph \tilde{H} .

(b) Check of the concentration of D_t for the hypergraph \tilde{H} .

Figure 3: Experimental results on the sum of degrees of active vertices D_t in the simulated hypergraph \tilde{H} .

$\alpha = 0.32904168$ which closely corresponds to $\hat{\alpha}$. Finally, the result seen in Figure 3b supports our assumption (3) about the concentration of D_t ($D_t \in \mathbb{E}[D_t] + o(t)$ whp).

6 Conclusions

To the best of our knowledge, we have presented the first complex network model which allows for multiary relations and deactivation of elements. Both those eventualities occur naturally in real-life systems thus we believe that the model will find a wide application in many research domains. We have also proved that its degree distribution follows a power-law with an exponential cutoff, which, according to the broad study of Broido and Clauset [7], is the distribution most often observed in nature.

In further research we would like to investigate deeper some natural networks and observe how well our model reflects them. Maybe the need of generalizing the model will occur, e.g., by modifying the form of the attachment function or introducing the possibility of adding isolated vertices. The other interesting direction of study is to make the attachment rule dependent not only on the degrees of vertices but also on their additional own characteristic (called *fitness* in the literature [6]).

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A Preferential attachment hypergraph with vertex deactivation

Lemma 10 (Chernoff Bounds, [22], Chapter 4.2) *Let Z_1, Z_2, \dots, Z_t be independent indicator random variables with $\mathbb{P}[Z_i = 1] = p_i$ and $\mathbb{P}[Z_i = 0] = 1 - p_i$. Let $Z = \sum_{i=1}^t Z_i$ and $m = \mathbb{E}[Z] = \sum_{i=1}^t p_i$. Then*

$$\mathbb{P}[|Z - m| \geq \delta m] \leq 2e^{-m\delta^2/3}$$

for all delta $\delta \in (0, 1)$.

Corollary 2 *Since $|V_t|$ follows a binomial distribution with parameters t and p_v , setting $\delta = \sqrt{\frac{9 \ln t}{p_v t}}$ in Chernoff bounds (Lemma 10) we get*

$$\mathbb{P}[||V_t| - \mathbb{E}[|V_t|]| \geq \sqrt{9p_v t \ln t}] \leq 2/t^3.$$

Below we restate and prove Lemmas 2, 3, 4, 6, and 7.

Lemma 3 *If $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right)$ for some positive constants c, β, γ, δ then*

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim \frac{c}{p_v} k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right).$$

(Here “ \sim ” refers to the limit by $k \rightarrow \infty$.)

Proof. Let B denote the event $[||V_t| - \mathbb{E}[|V_t|]| < \sqrt{9p_v t \ln t}]$ and B^C its complement. We have

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] = \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} | B \right] \mathbb{P}[B] + \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} | B^C \right] \mathbb{P}[B^C].$$

Since $N_{k,t} \leq |V_t|$ and $\mathbb{E}[|V_t|] = p_v t$, by Corollary 2 we get

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \leq \frac{\mathbb{E}[N_{k,t}]}{\mathbb{E}[|V_t|] - \sqrt{9p_v t \ln t}} \cdot 1 + 1 \cdot \frac{2}{t^3} \sim \frac{\mathbb{E}[N_{k,t}]}{p_v t},$$

and on the other hand

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \geq \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} | B \right] \mathbb{P}[B] \geq \frac{\mathbb{E}[N_{k,t}]}{\mathbb{E}[|V_t|] + \sqrt{9p_v t \ln t}} \left(1 - \frac{2}{t^3} \right) \sim \frac{\mathbb{E}[N_{k,t}]}{p_v t}.$$

■

Lemma 4 *Assume that $\mathbb{E}[Y_t] = \mu$ for all $t > 0$ and $\text{Var}[Y_t] = o(t)$. Then*

$$\mathbb{E} \left[\frac{Y_t^2}{D_{t-1}^2} \right] = o\left(\frac{1}{t}\right).$$

Proof. By the fact that $D_{t-1} \geq A_{t-1}$ and Y_t is independent of A_{t-1} we have

$$\mathbb{E} \left[\frac{Y_t^2}{D_{t-1}^2} \right] \leq \mathbb{E} \left[\frac{1}{A_{t-1}^2} \right] \mathbb{E}[Y_t^2] = \mathbb{E} \left[\frac{1}{A_{t-1}^2} \right] (\text{Var}[Y_t] + \mu^2). \quad (9)$$

Note that A_{t-1} follows a binomial distribution with parameters $t - 1$ and $p = (p_v - p_d)$ (recall that we assume $p_v > p_d$ throughout the paper) thus setting $\delta = \sqrt{\frac{6 \ln t}{p(t-1)}}$ in Chernoff bounds (Lemma 10) we may write

$$\mathbb{P}[|A_{t-1} - (t-1)p| \geq \sqrt{6p \cdot (t-1) \ln t}] \leq 2/t^2.$$

Let B denote the event $[|A_{t-1} - (t-1)p| < \sqrt{p \cdot t \ln t^2}]$ and B^C its complement. Note that $A_{t-1} \geq 1$. We have

$$\begin{aligned} \mathbb{E} \left[\frac{1}{A_{t-1}^2} \right] &= \mathbb{E} \left[\frac{1}{A_{t-1}^2} | B \right] \mathbb{P}[B] + \mathbb{E} \left[\frac{1}{A_{t-1}^2} | B^C \right] \mathbb{P}[B^C] \\ &\leq \frac{1}{\left(p(t-1) - \sqrt{p \cdot t \ln t^2} \right)^2} \cdot 1 + 1 \cdot \frac{2}{t^2} = \Theta \left(\frac{1}{t^2} \right). \end{aligned} \quad (10)$$

Thus by (9) and (10), since $\text{Var}[Y_t] = o(t)$, we obtain

$$\mathbb{E} \left[\frac{Y_t^2}{D_{t-1}^2} \right] \leq \frac{1}{t^2} (\text{Var}[Y_t] + \mu^2) = o \left(\frac{1}{t} \right).$$

■

Lemma 5 *Assume that $D_t = \mathbb{E}[D_t] + o(t)$ whp. Then for each $k \geq 1$*

$$\mathbb{E} \left[\frac{A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[A_{k,t}]}{\mathbb{E}[D_t]} + o(1).$$

Proof. Denote the event $[D_t \in \mathbb{E}[D_t] + o(t)]$ by B and its complement by B^C . Since $\mathbb{P}[B^C] = o(1)$, $\frac{A_{k,t}}{D_t} \leq 1$ and $\mathbb{E}[D_t] = \Omega(t)$ (note that $\mathbb{E}[D_t] \geq \mathbb{E}[A_t] = 1 + (p_v - p_d)t$ and we assume $p_v > p_d$), we have

$$\begin{aligned} \mathbb{E} \left[\frac{A_{k,t}}{D_t} \right] &= \mathbb{E} \left[\frac{A_{k,t}}{D_t} | B \right] \mathbb{P}[B] + \mathbb{E} \left[\frac{A_{k,t}}{D_t} | B^C \right] \mathbb{P}[B^C] \\ &= \frac{\mathbb{E}[A_{k,t}]}{\mathbb{E}[D_t] + o(t)} + o(1) = \frac{\mathbb{E}[A_{k,t}]}{\mathbb{E}[D_t]} + o(1). \end{aligned}$$

■

Lemma 6 *Consider a hypergraph $H = H(H_0, p_e, p_v, Y)$ for any $t > 0$. By Assumptions (1-4) the degree distribution of H follows a power-law with an exponential cutoff, i.e.,*

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right)$$

$$\begin{aligned} \text{for } \beta &= \frac{\mu(p_v + p_e) - p_d \theta}{p_v(\mu - 1) + p_e \mu + p_d}, & \gamma &= \frac{p_v(\mu - 1) + p_e \mu}{p_v(\mu - 1) + p_e \mu + p_d}, \\ \delta &= \frac{p_d}{\mu(p_v + p_e) - p_d \theta}, & c &= \frac{\beta \cdot \Gamma(1 + \beta)}{\gamma}, \end{aligned}$$

where $\Gamma(x)$ stands for the gamma function ($\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$).

Proof. We take a standard master equation approach that can be found e.g. in Chung and Lu book [10] about complex networks. However, we apply it separately to the number of active vertices and the number of deactivated vertices.

Recall that $N_{k,t}$ denotes the number of vertices of degree k at time t . We need to show that $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right)$ for the proper constants c, β, γ and δ . However, by Lemma 2 we know that it suffices to show that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right).$$

Recall that $N_{k,t} = A_{k,t} + I_{k,t}$. First, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t}$ using the mathematical induction on k . In this part we follow closely the lines of the proof that can be found in [3]. Consider the case $k = 1$. Since H_0 consists of a single hyperedge of cardinality 1 over a single vertex, we have $A_{1,0} = 1$. To formulate a master equation, let us make the following observation for $t \geq 1$. An active vertex remains in $\mathcal{A}_{1,t}$ if it had degree 1 at step $t - 1$ and was neither selected to a hyperedge, nor deactivated. Recall that a vertex from \mathcal{A}_{t-1} is chosen at step t in a single trial to the new hyperedge with probability $1/D_{t-1}$ thus the chance that it won't be selected to the hyperedge of cardinality y equals $(1 - 1/D_{t-1})^y$. Also, in each step, with probability p_v , a single new active vertex of degree 1 is added to the hypergraph. Let \mathcal{F}_t denote a σ -algebra associated with the probability space at step t . For $t \geq 1$ we have

$$\begin{aligned} \mathbb{E}[A_{1,t} | \mathcal{F}_{t-1}] &= p_v A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right)^{Y_{t-1}} + p_e A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right)^{Y_t} \\ &\quad + p_d A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right) + p_v. \end{aligned} \tag{11}$$

After taking the expectation on both sides of (11) we derive upper and lower bounds on $\mathbb{E}[A_{1,t}]$. By Bernoulli's inequality ($(1 + x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x \geq -1$), Lemma 4 (thus by Assumption (3) necessary for it) and the independence of Y_t from $A_{1,t-1}$ and D_{t-1} we obtain

$$\begin{aligned} \mathbb{E}[A_{1,t}] &\geq p_v \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t - 1}{D_{t-1}}\right) \right] + p_e \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t}{D_{t-1}}\right) \right] \\ &\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right) \right] + p_v \\ &= p_v \mathbb{E}[A_{1,t-1}] \left(1 - \frac{\mathbb{E}[Y_t] - 1}{\mathbb{E}[D_{t-1}]}\right) + p_e \mathbb{E}[A_{1,t-1}] \left(1 - \frac{\mathbb{E}[Y_t]}{\mathbb{E}[D_{t-1}]}\right) \\ &\quad + p_d \mathbb{E}[A_{1,t-1}] \left(1 - \frac{1}{\mathbb{E}[D_{t-1}]}\right) + p_v - o(1) \\ &= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]}\right) + p_v - o(1). \end{aligned} \tag{12}$$

On the other hand, since $(1 - x)^n \leq 1/(1 + nx)$ for $x \in [0, 1]$ and $n \in \mathbb{N}$, and $A_{1,t-1} \leq t$, by Lemma 3 (thus by Assumptions (1) and (2) necessary for it) and Lemma 4 (thus by Assumption

(3)) we have

$$\begin{aligned}
\mathbb{E}[A_{1,t}] &\leq p_v \mathbb{E} \left[\frac{A_{1,t-1}}{1 + (Y_t - 1)/D_{t-1}} \right] + p_e \mathbb{E} \left[\frac{A_{1,t-1}}{1 + Y_t/D_{t-1}} \right] \\
&\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\
&= p_v \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t - 1}{D_{t-1} + Y_t - 1} \right) \right] + p_e \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t}{D_{t-1} + Y_t} \right) \right] \\
&\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\
&\leq p_v \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t - 1}{D_{t-1}} + \frac{(Y_t - 1)^2}{D_{t-1}^2} \right) \right] + p_e \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t}{D_{t-1}} + \frac{Y_t^2}{D_{t-1}^2} \right) \right] \\
&\quad + p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\
&= p_v \mathbb{E}[A_{1,t-1}] \left(1 - \frac{\mathbb{E}[Y_t] - 1}{\mathbb{E}[D_{t-1}]} \right) + p_e \mathbb{E}[A_{1,t-1}] \left(1 - \frac{\mathbb{E}[Y_t]}{\mathbb{E}[D_{t-1}]} \right) \\
&\quad + p_d \mathbb{E}[A_{1,t-1}] \left(1 - \frac{1}{\mathbb{E}[D_{t-1}]} \right) + p_v + p_v \mathbb{E} \left[O(t) \frac{(Y_t - 1)^2}{D_{t-1}^2} \right] + p_e \mathbb{E} \left[O(t) \frac{Y_t^2}{D_{t-1}^2} \right] + o(1) \\
&= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1).
\end{aligned} \tag{13}$$

From (12) and (13) we get

$$\mathbb{E}[A_{1,t}] = \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1).$$

Now, we apply Lemma 1 to the above equation choosing

$$a_t = \mathbb{E}[A_{1,t}], \quad b_t = \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]/t}, \quad c_t = p_v + o(1).$$

We have $\lim_{t \rightarrow \infty} c_t = p_v$ and, by Fact 1 (thus by Assumptions (1) and (4) implying it), $\lim_{t \rightarrow \infty} b_t = \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mu(p_v + p_e) - p_d\theta} =: 1/\beta$ thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{1,t}]}{t} = \frac{p_v}{1 + 1/\beta} =: \bar{A}_1.$$

Now, we assume that the limit $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k-1,t}]}{t}$ exists and equals \bar{A}_{k-1} and we will show by induction on k that the analogous limit for $\mathbb{E}[A_{k,t}]$ exists. Let us again formulate a master equation, this time for $k > 1$. We have $A_{k,0} = 0$ and for $t \geq 1$ an active vertex appears in $A_{k,t}$ if it was active at step $t - 1$, had degree $k - l$ and was chosen exactly l times to a hyperedge, or it had degree k and was not selected for deactivation. Let $B(l, n, p) = \binom{n}{l} p^l (1 - p)^{n-l}$. We have

$$\begin{aligned}
\mathbb{E}[A_{k,t} | \mathcal{F}_{t-1}] &= p_v \sum_{l=0}^{\min\{Y_t-1, k-1\}} A_{k-l, t-1} B \left(l, Y_t - 1, \frac{k-l}{D_{t-1}} \right) + p_e \sum_{l=0}^{\min\{Y_t, k-1\}} A_{k-l, t-1} B \left(l, Y_t, \frac{k-l}{D_{t-1}} \right) \\
&\quad + p_d A_{k, t-1} \left(1 - \frac{k}{D_{t-1}} \right).
\end{aligned}$$

Taking the expectation on both sides we get

$$\mathbb{E}[A_{k,t}] = \mathbb{E}[\psi] + p_v \mathbb{E}[\varphi(Y_t - 1)] + p_e \mathbb{E}[\varphi(Y_t)], \quad \text{where}$$

$$\begin{aligned} \psi &= p_v \sum_{l=0}^1 A_{k-l,t-1} B\left(l, Y_t - 1, \frac{k-l}{D_{t-1}}\right) + p_e \sum_{l=0}^1 A_{k-l,t-1} B\left(l, Y_t, \frac{k-l}{D_{t-1}}\right) \\ &\quad + p_d A_{k,t-1} \left(1 - \frac{k}{D_{t-1}}\right) \\ &= A_{k,t-1} \left(p_v \left(1 - \frac{k}{D_{t-1}}\right)^{Y_t-1} + p_e \left(1 - \frac{k}{D_{t-1}}\right)^{Y_t} + p_d \left(1 - \frac{k}{D_{t-1}}\right) \right) \\ &\quad + A_{k-1,t-1} \frac{k-1}{D_{t-1}} \left(p_v (Y_t - 1) \left(1 - \frac{k-1}{D_{t-1}}\right)^{Y_t-2} + p_e Y_t \left(1 - \frac{k-1}{D_{t-1}}\right)^{Y_t-1} \right) \end{aligned}$$

$$\text{and } \varphi(n) = \sum_{l=2}^{\min\{n, k-1\}} A_{k-l,t-1} B\left(l, n, \frac{k-l}{D_{t-1}}\right).$$

We will show that only the term $\mathbb{E}[\psi]$ is significant and that the terms $\mathbb{E}[\varphi(Y_t - 1)]$ and $\mathbb{E}[\varphi(Y_t)]$ converge to 0 as $t \rightarrow \infty$. We have

$$\begin{aligned} \varphi(Y_t) &\leq \sum_{l=2}^{k-1} A_{k-l,t-1} \binom{Y_t}{l} \left(\frac{k-l}{D_{t-1}}\right)^l \left(1 - \frac{k-l}{D_{t-1}}\right)^{Y_t-l} \\ &\leq O(t) \sum_{l=2}^{k-1} \binom{Y_t}{l} \left(\frac{k-l}{D_{t-1}}\right)^l \left(1 - \frac{k-l}{D_{t-1}}\right)^{Y_t-l} \\ &\leq O(t) \sum_{l=2}^{k-1} Y_t^l \left(\frac{k}{D_{t-1}}\right)^l \left(1 - \frac{1}{D_{t-1}}\right)^{Y_t-k+1} \\ &\leq O(t) \frac{Y_t^2 k^2}{D_{t-1}^2} e^{-Y_t/D_{t-1}} e^{k-1} \sum_{l=2}^{k-1} \left(\frac{Y_t k}{D_{t-1}}\right)^{l-2} \\ &= O(t) \frac{Y_t^2}{D_{t-1}^2} e^{-Y_t/D_{t-1}} \sum_{l=2}^{k-1} \left(\frac{Y_t k}{D_{t-1}}\right)^{l-2}. \end{aligned}$$

Then, if $Y_t \leq D_{t-1}$, we have

$$\varphi(Y_t) \leq O(t) \frac{Y_t^2}{D_{t-1}^2} k^{k-2} = O(t) \frac{Y_t^2}{D_{t-1}^2}.$$

Otherwise,

$$\begin{aligned} \varphi(Y_t) &\leq O(t) \frac{Y_t^2}{D_{t-1}^2} e^{-Y_t/D_{t-1}} \frac{(Y_t k/D_{t-1})^{k-2} - 1}{(Y_t k/D_{t-1}) - 1} \\ &\leq O(t) \frac{Y_t^2}{D_{t-1}^2} e^{-Y_t/D_{t-1}} \frac{(Y_t/D_{t-1})^{k-2}}{k-1} k^{k-2} \\ &\leq O(t) \frac{Y_t^2}{D_{t-1}^2} e^{-(k-2)} \frac{(k-2)^{k-2}}{k-1} k^{k-2} = O(t) \frac{Y_t^2}{D_{t-1}^2}, \end{aligned}$$

where the last inequality follows from the fact that $e^{-x} x^\alpha$ is maximized at $x = \alpha$. Hence by Lemma 3 (thus by Assumptions (1) and (2)) in both above cases we get $\mathbb{E}[\varphi(Y_t)] = o(1)$ and,

similarly, $\mathbb{E}[\varphi(Y_t - 1)] = o(1)$. Now, we derive the bounds for $\mathbb{E}[\psi]$ analogous to the ones derived for $\mathbb{E}[A_{1,t}]$.

$$\begin{aligned}
\mathbb{E}[\psi] &= \mathbb{E} \left[A_{k,t-1} \left(p_v \left(1 - \frac{k}{D_{t-1}} \right)^{Y_t-1} + p_e \left(1 - \frac{k}{D_{t-1}} \right)^{Y_t} + p_d \left(1 - \frac{k}{D_{t-1}} \right) \right) \right] \\
&\quad + \mathbb{E} \left[A_{k-1,t-1} \frac{k-1}{D_{t-1}} \left(p_v (Y_t - 1) \left(1 - \frac{k-1}{D_{t-1}} \right)^{Y_t-2} + p_e Y_t \left(1 - \frac{k-1}{D_{t-1}} \right)^{Y_t-1} \right) \right] \\
&\geq \mathbb{E} \left[A_{k,t-1} \left(p_v \left(1 - \frac{(Y_t-1)k}{D_{t-1}} \right) + p_e \left(1 - \frac{Y_t k}{D_{t-1}} \right) + p_d \left(1 - \frac{k}{D_{t-1}} \right) \right) \right] \\
&\quad + \mathbb{E} \left[A_{k-1,t-1} \frac{k-1}{D_{t-1}} \left(1 - \frac{(k-1)(Y_t-1)}{D_{t-1}} \right) (p_v (Y_t - 1) + p_e Y_t) \right] \\
&= \mathbb{E}[A_{k,t-1}] \left(1 - \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]} \right) \\
&\quad + \mathbb{E}[A_{k-1,t-1}] \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]} + o(1).
\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}[\psi] &\leq \mathbb{E} \left[A_{k,t-1} \left(1 - \frac{k(p_v(Y_t-1) + p_e Y_t + p_d)}{D_{t-1}} \right) \right] + \mathbb{E} \left[O(t) \frac{Y_t^2}{D_{t-1}^2} \right] \\
&\quad + \mathbb{E} \left[A_{k-1,t-1} \frac{k-1}{D_{t-1}} (p_v(Y_t-1) + p_e Y_t) \right] \\
&= \mathbb{E}[A_{k,t-1}] \left(1 - \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]} \right) \\
&\quad + \mathbb{E}[A_{k-1,t-1}] \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]} + o(1).
\end{aligned} \tag{15}$$

By (14) and (15) we get

$$\begin{aligned}
\mathbb{E}[A_{k,t}] &= \mathbb{E}[A_{k,t-1}] \left(1 - \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]} \right) \\
&\quad + \mathbb{E}[A_{k-1,t-1}] \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]} + o(1).
\end{aligned} \tag{16}$$

Recall that by the induction assumption $\lim_{t \rightarrow \infty} \mathbb{E}[A_{k-1,t}]/t = \bar{A}_{k-1}$. Now, we apply again Lemma 1 to the above equation choosing

$$\begin{aligned}
a_t &= \mathbb{E}[A_{k,t}], \quad b_t = \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]/t}, \\
c_t &= \frac{\mathbb{E}[A_{k-1,t-1}]}{t} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]/t} + o(1).
\end{aligned}$$

By Fact 1 (thus by Assumptions (1) and (4)) we have

$$\lim_{t \rightarrow \infty} b_t = k/\beta \quad \text{and} \quad \lim_{t \rightarrow \infty} c_t = \bar{A}_{k-1} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mu(p_v + p_e) - p_d\theta}$$

thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} = \bar{A}_k = \bar{A}_{k-1} \frac{(k-1)\gamma}{k + \beta}, \tag{17}$$

where $\gamma = \frac{p_v(\mu-1)+p_e\mu}{p_v(\mu-1)+p_e\mu+p_d}$. Thus we got

$$\begin{aligned}\bar{A}_1 &= p_v\beta \frac{1}{(1+\beta)}, \quad \bar{A}_2 = p_v\beta \frac{\gamma}{(1+\beta)(2+\beta)}, \quad \dots, \\ \bar{A}_k &= p_v\beta \frac{\gamma^{k-1}(k-1)!}{(1+\beta)(2+\beta)\dots(k+\beta)}.\end{aligned}$$

Since $\lim_{k \rightarrow \infty} \frac{\Gamma(k)k^\alpha}{\Gamma(k+\alpha)} = 1$ for constant $\alpha \in \mathbb{R}$ we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} = \bar{A}_k = \frac{p_v\beta \gamma^k \Gamma(1+\beta)\Gamma(k)}{\gamma \Gamma(k+\beta+1)} \sim p_v \cdot c \cdot \gamma^k k^{-(\beta+1)} \quad (18)$$

with $c = \frac{\beta\Gamma(1+\beta)}{\gamma}$.

Now, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t}$. We have $I_{k,0} = 0$ for all $k \geq 1$. For $t \geq 1$ the expected number of inactive vertices of degree $k \geq 1$ at step t , given \mathcal{F}_{t-1} , can be expressed as

$$\mathbb{E}[I_{k,t} | \mathcal{F}_{t-1}] = I_{k,t-1} + p_d A_{k,t-1} \frac{k}{D_{t-1}},$$

since inactive vertices of degree k remain in $I_{k,t}$ forever and a vertex of degree k becomes inactive if it was selected in step $t-1$ for deactivation. Taking the expectation on both sides, by Lemma 4 (thus by Assumption (3)), we obtain

$$\mathbb{E}[I_{k,t}] = \mathbb{E}[I_{k,t-1}] + p_d \mathbb{E}[A_{k,t-1}] \frac{k}{\mathbb{E}[D_{t-1}]} + o(1).$$

Then, by Fact 1 (thus by Assumptions (1) and (4)),

$$\begin{aligned}\lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k,t-1}]) &= \lim_{t \rightarrow \infty} p_d k \frac{\mathbb{E}[A_{k,t-1}]}{t} \frac{t}{\mathbb{E}[D_{t-1}]} + o(1) \\ &= \bar{A}_k \frac{p_d k}{(p_v + p_e)\mu - p_d \theta} = \bar{A}_k k \delta,\end{aligned}$$

where $\delta = \frac{p_d}{(p_v + p_e)\mu - p_d \theta}$. And, by Stolz–Cesàro theorem (Theorem 1), we obtain

$$\bar{I}_k := \lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t} = \lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k,t-1}]) = \bar{A}_k k \delta. \quad (19)$$

Finally, by (7) and (19)

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}] + \mathbb{E}[I_{k,t}]}{t} = \bar{A}_k + \bar{I}_k = \bar{A}_k (1 + k\delta) \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right).$$

■

B Estimating the limiting value θ

Lemma 8 *Assume that $\mathbb{P}[D_t \notin \mathbb{E}[D_t] + o(t)] = o(1/t)$. Then*

$$\mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t}]}{\mathbb{E}[D_t]} + o(1).$$

Proof. Denote the event $[D_t \in \mathbb{E}[D_t] + o(t)]$ by B and its complement by B^C . Let us work assuming that $k \leq t$ (indeed, in our model it is very unlikely that a vertex achieves degree greater than t after t steps). Since

$$D_t = \sum_{k \geq 1} k A_{k,t}, \quad \frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \leq \frac{t \sum_{k \geq 1} k A_{k,t}}{D_t} = t, \quad \mathbb{P}[B^C] = o(1/t),$$

and $\mathbb{E}[D_t] = \Omega(t)$ (note that $\mathbb{E}[D_t] \geq \mathbb{E}[A_t] = 1 + (p_v - p_d)t$ and we assume $p_v > p_d$), we have

$$\begin{aligned} \mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \right] &= \mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \middle| B \right] \mathbb{P}[B] + \mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \middle| B^C \right] \mathbb{P}[B^C] \\ &\leq \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t}]}{\mathbb{E}[D_t] + o(t)} + t \cdot o(1/t) = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t}]}{\mathbb{E}[D_t]} + o(1). \end{aligned}$$

■

Lemma 11 *The function $R(x) = \frac{F(2,2,\rho(x),\gamma)}{F(1,2,\rho(x),\gamma)}$ can be also expressed as*

$$R(x) = x - \frac{p_v}{p_d} + \frac{1}{1-\gamma} \frac{\rho(x) - 1}{F(1,2,\rho(x),\gamma)},$$

where γ and $\rho(x)$ are as in Theorem 3.

Proof. We will use the Gauss' contiguous relations (consult [1]). Let $a, b, c, z \in \mathbb{C}$ with $|z| < 1$ and $c \notin \mathbb{Z}_{\leq 0}$. Let $F(z) = F(a, b; c; z)$, $F(a+, z) = F(a+1, b; c; z)$ and $F(a-, z) = F(a-1, b; c; z)$. Then

$$a(F(a+, z) - F(z)) = \frac{(c-a)F(a-, z) + (a-c+bz)F(z)}{1-z}$$

which is equivalent to

$$\frac{F(a+, z)}{F(z)} = \frac{2a-c+(b-a)z}{a(1-z)} + \frac{(c-a)F(a-, z)}{a(1-z)F(z)}. \quad (20)$$

Since $F(0, b; c; z) = 1$, plugging $a = 1$, $b = 2$, $c = \rho(x)$ and $z = \gamma$ into (20), we get the result. ■

Lemma 12 *$R(x)$ strictly increases on $[0, \hat{\theta}]$.*

Proof. First, note that the derivative of the hypergeometric function with respect to c is

$$F'(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} (\psi(c) - \psi(c+n)),$$

where $\psi(x)$ denotes the digamma function ($\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$). Since the digamma function increases on $(0, +\infty)$, we can see that $F'(a, b; c; z)$ is negative when parameters of the function are positive.

Now, observe that since $\rho'(x) = \gamma - 1$, we have

$$F'_1(x) = (\gamma - 1)F'(1, 2; \rho(x); \gamma) \quad \text{and} \quad F'_2(x) = (\gamma - 1)F'(2, 2; \rho(x); \gamma),$$

and thus, given that $\gamma < 1$, they are both positive. In order to determine the sign of

$$R'(x) = \frac{F_2'(x)F_1(x) - F_2(x)F_1'(x)}{F_1(x)^2},$$

we only need to determine the sign of its numerator. By considering the Cauchy product of $F_2'(x)$ and $F_1(x)$, we obtain

$$F_2'(x)F_1(x) = (\gamma - 1) \sum_{k=0}^{\infty} \gamma^k \sum_{n=0}^k (n+1) \frac{(2)_n}{(\rho(x))_n} \frac{(2)_{k-n}}{(\rho(x))_{k-n}} (\psi(\rho(x)) - \psi(\rho(x) + n)).$$

Similarly, for $F_2(x)$ and $F_1'(x)$, we have

$$F_2(x)F_1'(x) = (\gamma - 1) \sum_{k=0}^{\infty} \gamma^k \sum_{n=0}^k (k-n+1) \frac{(2)_n}{(\rho(x))_n} \frac{(2)_{k-n}}{(\rho(x))_{k-n}} (\psi(\rho(x)) - \psi(\rho(x) + n)).$$

Finally, we express the difference between these two expressions as

$$\begin{aligned} F_2'(x)F_1(x) - F_2(x)F_1'(x) = \\ (\gamma - 1) \sum_{k=0}^{\infty} \gamma^k \sum_{n=0}^k (2n-k) \frac{(2)_n}{(\rho(x))_n} \frac{(2)_{k-n}}{(\rho(x))_{k-n}} (\psi(\rho(x)) - \psi(\rho(x) + n)). \end{aligned}$$

We now check the sign of the inner sum. Observe that the sum of two elements with indices $n = i$ and $n = k - i$ is

$$(2i-k) \frac{(2)_i}{(\rho(\theta))_i} \frac{(2)_{k-i}}{(\rho(x))_{k-i}} (\psi(\rho(x) + k - i) - \psi(\rho(x) + i)).$$

Since the digamma function increases on $(0, +\infty)$, the inner sum is negative, which, together with $\gamma < 1$, implies $F_2'(x)F_1(x) - F_2(x)F_1'(x) > 0$. We conclude that $R'(x) > 0$. \blacksquare

Lemma 13 *The function $R(x)$ is a contraction mapping on $[0, \hat{\theta}]$.*

Proof. Remind that a function $f : S \mapsto S$, defined on a metric space (S, d) , is called a contraction mapping, if there exists a constant $q \in [0, 1)$, such that for all $s_1, s_2 \in S$, we have $d(f(s_1), f(s_2)) \leq qd(s_1, s_2)$. If $f(s)$ is a differentiable function, such that $\sup |f'(s)| < 1$, then $f(s)$ is a contraction mapping with $q = \sup |f'(s)|$.

Using the form of $R(x)$ presented in Lemma 7, we obtain

$$R'(x) = 1 + \frac{1}{1-\gamma} \frac{\rho'(x)F_1(x) - (\rho(x) - 1)F_1'(x)}{F_1(x)^2}.$$

$F_1(x)$ is positive and increases, and $\rho(x) > 1$ and decreases on $[0, \hat{\theta}]$, which implies that the right term of the expression is negative. Since $R(x)$ also increases on $[0, \hat{\theta}]$ (Lemma 8), we have that $|R'(\theta)| \in [0, 1)$ for any $\theta \in [0, \hat{\theta}]$. Therefore, by the extreme value theorem, we know that $|R'(\theta)|$ achieves some maximum value $q \in (0, 1)$. Then, since $[0, \hat{\theta}]$ is a complete metric space and $R([0, \hat{\theta}]) \subseteq [0, \hat{\theta}]$ (using the formula from Lemma 7 it is easy to check that $R(0) > 0$ and $R(\hat{\theta}) < \hat{\theta}$), we conclude that $R(x)$ is a contraction mapping on $[0, \hat{\theta}]$. \blacksquare