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Hierarchical heterogeneous one-dimensional problem in linear viscoelastic media

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ABSTRACT

We address the homogenisation of a linear viscoelastic and hierarchical composite material in a one-dimensional (1D) framework via a three-scale asymptotic homogenisation method. We consider a family of heterogeneous problems with periodic, rapidly-oscillating and piece-wise coefficients that model a structure with two hierarchical levels of organisation. Here, we assume continuity contact conditions at the interfaces among the constituents and set a straightforward geometrical configuration in order to gain a better insight of the multiscale problem. The main goal is to provide a general overview of the procedure and validate the approach by means of a comparison between the solution of the original heterogeneous problem, the homogenised problem and the formal asymptotic solution. In addition, we show that the three-scale approach presents a clear improvement over the recursive two-scale one, and we illustrate the convergence of the solutions towards the solution of the homogenised problem when the asymptotic parameters approach.

1. Introduction

There are numerous works that study the modelling of multiscale hierarchical heterogeneous media using homogenisation techniques. In particular, the asymptotic homogenisation method (AHM) is a powerful tool, which has been widely considered in the study of composite materials, see, e.g., Bensoussan et al. (1978), Sanchez-Palencia (1980) and Bakhvalov and Panasenko (1989). In general, the multiscale AHM takes advantage of the information available at the smaller scales of a given heterogeneous medium to predict the effective properties at its larger scales, which dramatically reduces the computational complexity of the original mathematical problems (see, e.g., Lukkassen and Milton (2000), Penta and Gerisch (2017a), Ramírez-Torres et al. (2018b, 2019b), Dong et al. (2019) and Yang et al. (2019a)). This multiscale homogenisation procedure requires the solution of cell problems with input data corresponding to the homogenised material properties obtained from the previous steps.

From the theoretical point of view, Trucu et al. (2012) proposed a three-scale convergence analysis where the asymptotic parameters independently approach zero. Before that Allaire and Briane (1996) illustrated the advantages of the method of reiterated homogenisation, which can be found in Sanchez-Palencia (1980), to a heat problem

in composites. Recently, a three-scale asymptotic approximation technique has been proposed in Ramírez-Torres et al. (2018b, 2019a) with particular interest in hierarchical laminated and fibre-reinforced elastic composites. Additionally, in Dong et al. (2019) and Yang et al. (2019a,b), the authors study the properties of thermo-mechanical, non-ageing and ageing viscoelastic composites with multiple spatial scales by using a three-scale asymptotic expansion and a periodic layout of the heterogeneities in the structures. They also provide a finite element algorithm based on inverse Laplace transform and the three-scale asymptotic homogenisation to obtain the numerical results. Moreover, in Nasirov et al. (2020), the authors employed a three-scale formulation of asymptotic homogenisation to predict the mechanical properties of short fibre-reinforced composites manufactured using the fused filament fabrication process. In Chen et al. (2021), the homogenised and local response of unidirectional fuzzy fibre nanocomposites undergoing inelastic deformations is simulated by proposing a new hybrid hierarchical homogenisation approach. In addition, the Authors in Yang et al. (2021) determine the nonlinear mechanical properties of 3D braided composites by means of a higher-order three-scale reduced homogenisation approach.

The aim of this work is to apply the three-scale asymptotic homogenisation approach introduced in Ramírez-Torres et al. (2018b) to a

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viscoelastic and hierarchical structure in a one-dimensional framework. This simplified setting allows to fully solve the original heterogeneous problem and to compare the solution with the one obtained by the homogenisation approach. With this purpose, it is intended to establish a methodology using a three-scale framework, which is more general than the one studied, e.g., in Rodríguez-Ramos et al. (2020) for two-scale viscoelastic composites.

The novelty of this work relies on the solution of the hierarchical one-dimensional problem. To the best of our knowledge, there are no studies addressing the one-dimensional problem for viscoelastic and hierarchical heterogeneous media by means of the three-scale asymptotic homogenisation approach in its entirety. We notice that, in Cruz-González et al. (2020a) a three-dimensional viscoelastic problem is addressed via the three-scale asymptotic approach given in Ramírez-Torres et al. (2018b), and the effective properties are determined from the solutions of the local problems. However, in this work, we additionally solve the heterogeneous and homogenised problem, and construct the formal asymptotic solution, where the smallness parameters play a fundamental role. In addition, here, we formalise the fact that the initial condition for the homogenised viscoelastic model at $t = 0$ s is given as the homogenised elastic limit case. Furthermore, we offer evidence of the advantages of applying the hierarchical approach with respect to the recursive homogenisation for the geometric configuration under study. This proposal provides a methodology that could serve as a guide for further studies of viscoelastic, multidimensional problems representing real-life scenarios.

The manuscript is organised as follows. In Section 2, we present the geometrical description of the composite material, we introduce the fundamental notions related to the separation of scales and periodicity, and we formulate the linear viscoelastic heterogeneous problem. In Section 3, we consider the correspondence principle and the Laplace–Carson transform, and we rewrite the problem in the Laplace–Carson domain. Then, in Section 4, we describe the three-scale asymptotic homogenisation approach and obtain the main results concerning to the different hierarchical levels of organisation. Additionally, in Section 5, we solve a heterogeneous problem, the corresponding homogenised problem, and determine the asymptotic solution. Finally, in Section 6, we present the numerical results and compare different approaches, highlighting the potential of the three-scale method and showing the convergence of the solutions.

2. Formulation of the problem

In this section, we introduce the geometrical description of the hierarchical structure at the different levels of organisation. Besides, we consider the separation of scales and provide a direct link between the physical variables and the characteristic length scales of the composites. Furthermore, we discuss the concept of periodicity in terms of the local variables. Finally, we present the heterogeneous problem for linear viscoelastic materials in a one-dimensional framework.

2.1. Topology of the macrostructure

We identify a hierarchical composite material with the open and bounded set in the one-dimensional Euclidean space $\mathcal{B} =]0, L[$, where $L > 0$ (see Fig. 1(a)).

Extending the picture reported in Ramírez-Torres et al. (2021), we suppose that \mathcal{B} presents a periodic configuration at two different levels of organisation. In particular, at the first level of organisation, the ε_1 -structural level, we assume that \mathcal{B} comprises a two-phase composite with constituents specified by the open subsets

$$\mathcal{B}_1^{(\varepsilon_1)} := \bigcup_{i=0}^N]x_{2i}^{(1)}, x_{2i+1}^{(1)}[\subset \mathcal{B} \quad \text{and} \quad \mathcal{B}_2^{(\varepsilon_1)} := \bigcup_{i=0}^N]x_{2i+1}^{(1)}, x_{2i+2}^{(1)}[\subset \mathcal{B}, \quad (1)$$

so that $\overline{\mathcal{B}} = \overline{\mathcal{B}_1^{(\varepsilon_1)}} \cup \overline{\mathcal{B}_2^{(\varepsilon_1)}}$ and $\overline{\mathcal{B}_1^{(\varepsilon_1)}} \cap \overline{\mathcal{B}_2^{(\varepsilon_1)}} = \mathcal{B}_1^{(\varepsilon_1)} \cap \mathcal{B}_2^{(\varepsilon_1)} = \emptyset$. Here, the bar symbol indicates the closure of the set. We notice that, for

every $i = 0, \dots, N$, the intervals $]x_{2i}^{(1)}, x_{2i+1}^{(1)}[$ and $]x_{2i+1}^{(1)}, x_{2i+2}^{(1)}[$ represent different constituents (see Fig. 1(b)). Additionally, we denote with $\mathcal{I}^{(\varepsilon_1)}$ the interface between $\mathcal{B}_1^{(\varepsilon_1)}$ and $\mathcal{B}_2^{(\varepsilon_1)}$, namely, $\mathcal{I}^{(\varepsilon_1)} = \overline{\mathcal{B}_1^{(\varepsilon_1)}} \cap \overline{\mathcal{B}_2^{(\varepsilon_1)}} = \bigcup_{i=0}^N \{x_{2i+1}^{(1)}\}$.

Moreover, in this particular setting, we consider that $\mathcal{B}_1^{(\varepsilon_1)}$ is at same time a two-phase composite material. This assumption introduces the ε_2 -structural level. In this situation, the open subsets

$$\mathcal{B}_1^{(\varepsilon_2)} := \bigcup_{i=0}^M]x_{2i}^{(2)}, x_{2i+1}^{(2)}[\subset \mathcal{B}_1^{(\varepsilon_1)} \quad \text{and} \quad \mathcal{B}_2^{(\varepsilon_2)} := \bigcup_{i=0}^M]x_{2i+1}^{(2)}, x_{2i+2}^{(2)}[\subset \mathcal{B}_1^{(\varepsilon_1)} \quad (2)$$

form the constituent $\mathcal{B}_1^{(\varepsilon_1)}$. So, for every $i = 0, \dots, M$, the intervals $]x_{2i}^{(2)}, x_{2i+1}^{(2)}[$ and $]x_{2i+1}^{(2)}, x_{2i+2}^{(2)}[$ represent different materials (see Fig. 1(c)). Furthermore, we have that $\overline{\mathcal{B}_1^{(\varepsilon_1)}} = \overline{\mathcal{B}_1^{(\varepsilon_2)}} \cup \overline{\mathcal{B}_2^{(\varepsilon_2)}}$ and $\overline{\mathcal{B}_1^{(\varepsilon_2)}} \cap \overline{\mathcal{B}_2^{(\varepsilon_2)}} = \mathcal{B}_1^{(\varepsilon_2)} \cap \mathcal{B}_2^{(\varepsilon_2)} = \emptyset$. In particular, the symbol $\mathcal{I}^{(\varepsilon_2)}$ is used to denote the interface between $\mathcal{B}_1^{(\varepsilon_2)}$ and $\mathcal{B}_2^{(\varepsilon_2)}$, that is, $\mathcal{I}^{(\varepsilon_2)} = \overline{\mathcal{B}_1^{(\varepsilon_2)}} \cap \overline{\mathcal{B}_2^{(\varepsilon_2)}} = \bigcup_{i=0}^M \{x_{2i+1}^{(2)}\}$.

2.2. Separation of scales

The use of the asymptotic homogenisation technique requires the sharp separation of the length scales that characterise the hierarchical levels. In this framework, we assume that there exist three distinct length scales ℓ_2 , ℓ_1 and L_c which are related to the characteristic sizes of the periodic micro-structure, meso-structure and the whole composite, respectively. This is formalised by assuming that the dimensionless smallness parameters ε_1 and ε_2 are defined as (Ramírez-Torres et al., 2019a)

$$\varepsilon_1 := \frac{\ell_1}{L_c} \ll 1 \quad \text{and} \quad \varepsilon_2 := \frac{\ell_2}{L_c} \ll \varepsilon_1. \quad (3)$$

The parameters ε_1 and ε_2 are assumed to be constant, as done in several works on both classical two-scale homogenisation (see Bensoussan et al. (1978), Bakhvalov and Panasenko (1989) and Cioranescu and Donato (1999)) and three-scale homogenisation (see Ramírez-Torres et al. (2018b, 2019a)). In addition, it is worth mentioning that in a more general scenario than the one studied in this work, such as the case of a composite material subjected to deformation and change of the internal structure, the characteristic lengths and the scaling parameters may also depend on the spatial variable x and on time t (see Ramírez-Torres et al. (2018a)).

Based on the above considerations, we can explicitly specify the three-scale nature of a given field $\Phi(x, t)$. Specifically, by adapting the considerations discussed in Di Stefano et al. (2020), Penta and Gerisch (2017b) and Ramírez-Torres et al. (2021) to our case, we introduce the notation $\Phi(x, t) = \check{\Phi}(x, t; \ell_1, \ell_2, L_c)$ to consider the dependency of the field on the characteristic length scales. In particular, by introducing the dimensionless variables

$$\bar{x} := \frac{x}{L_c}, \quad \bar{y} := \frac{x}{\ell_1} \quad \text{and} \quad \bar{z} := \frac{x}{\ell_2}, \quad (4)$$

where x is said to be the physical spatial variable, whereas \bar{x} , \bar{y} and \bar{z} represent the macroscopic, mesoscopic and the microscopic dimensionless spatial variables, respectively, we can write

$$\Phi^{(\varepsilon)}(x, t) = \check{\Phi}(x, t; \ell_1, \ell_2, L_c) = \bar{\Phi}(x/L_c, x/\ell_1, x/\ell_2, t) = \phi(\bar{x}, \bar{y}, \bar{z}, t), \quad (5)$$

where by means of Eq. (3), \bar{x} , \bar{y} and \bar{z} can be related through the expressions,

$$\bar{y} = \frac{x}{\ell_1} = \frac{x/L_c}{\ell_1/L_c} = \frac{\bar{x}}{\varepsilon_1}, \quad (6a)$$

$$\bar{z} = \frac{x}{\ell_2} = \frac{x/L_c}{\ell_2/L_c} = \frac{\bar{x}}{\varepsilon_2}. \quad (6b)$$

That is, $\Phi^{(\varepsilon)}$ is reformulated in terms of three formally independent variables \bar{x} , \bar{y} and \bar{z} , which are representative of the geometrical configuration. Within this dimensionless setting, \mathcal{B} becomes $\mathcal{X} :=]0, L/L_c[$.

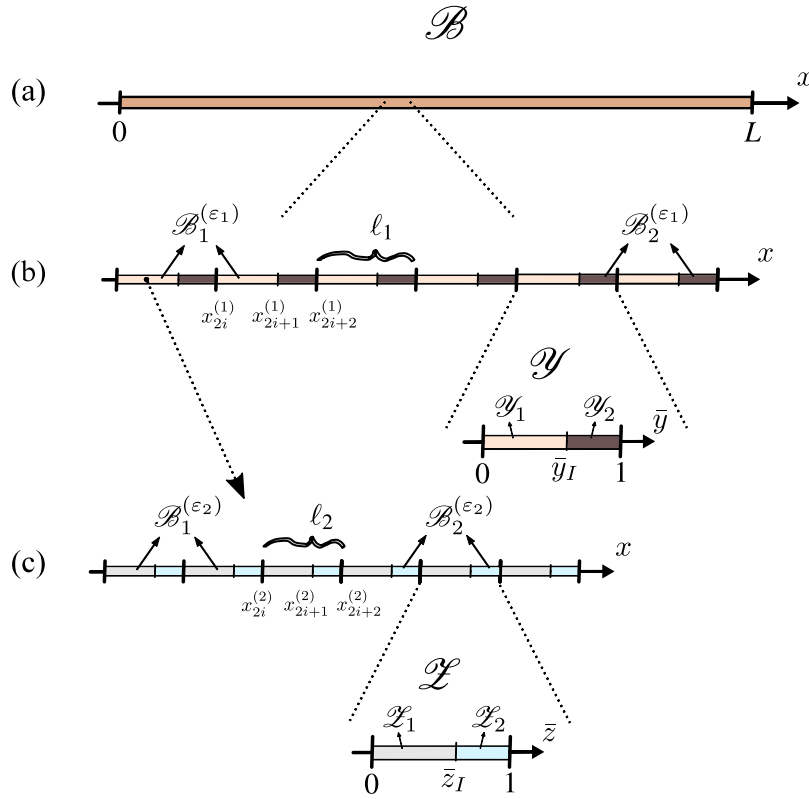


Fig. 1. (a) Viscoelastic heterogeneous bar with two hierarchical levels of organisation ((b) and (c)). This figure presents a two-phase configuration at both structural levels.

Accordingly, the variables \bar{x} , \bar{y} and \bar{z} vary in \mathcal{X} , $\mathcal{X}/\varepsilon_1 =]0, L/\ell_1[=]0, \frac{1}{\varepsilon_1}L/L_c[$ and $\mathcal{X}/\varepsilon_2 =]0, L/\ell_2[=]0, \frac{1}{\varepsilon_2}L/L_c[$, respectively. Therefore, for each time t , $\Phi^{(\varepsilon)}$ is defined, in general, as $\phi(\bar{x}, \bar{y}, \bar{z}, \cdot) : \mathcal{D}_{\bar{x}} \times \mathcal{D}_{\bar{y}} \times \mathcal{D}_{\bar{z}} \rightarrow \mathcal{S}$, where $\mathcal{D}_{\bar{x}} \subseteq \mathcal{X}$, $\mathcal{D}_{\bar{y}} \subseteq \mathcal{X}/\varepsilon_1$ and $\mathcal{D}_{\bar{z}} \subseteq \mathcal{X}/\varepsilon_2$ and \mathcal{S} denotes the one-dimensional Euclidean space.

Now, by considering the expressions in (3)–(6b) and the chain rule, the spatial derivative of $\Phi^{(\varepsilon)}$ can be computed as follows,

$$\begin{aligned} \frac{\partial \Phi^{(\varepsilon)}(x, t)}{\partial x} &= \frac{1}{L_c} \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{x}} + \frac{1}{\ell_1} \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{y}} + \frac{1}{\ell_2} \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{z}} \\ &= \frac{1}{L_c} \left(\frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{x}} + \frac{1}{\varepsilon_1} \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{y}} + \frac{1}{\varepsilon_2} \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{z}} \right). \end{aligned} \quad (7)$$

2.3. Topology of the micro- and meso-structure

At the micro-scale, the representative periodic cell is represented as the open interval $]0, \ell_2[$, which in a dimensionless formalism is identified as $\mathcal{X} =]0, 1[\subset \mathcal{X}/\varepsilon_2$. Additionally, the two constituents in \mathcal{X} are denoted by the non-empty, open subsets $\mathcal{X}_1 =]0, \bar{z}_I[$ and $\mathcal{X}_2 =]\bar{z}_I, 1[$, where the interface between them is the set with a single element $\mathcal{I}_{\mathcal{X}} = \overline{\mathcal{X}_1} \cap \overline{\mathcal{X}_2} = \{\bar{z}_I\}$, with $\bar{z}_I \in]0, 1[$ (see Fig. 1(c)). Furthermore, the constraints $\overline{\mathcal{X}} = \overline{\mathcal{X}_1} \cup \overline{\mathcal{X}_2}$ and $\overline{\mathcal{X}_1} \cap \overline{\mathcal{X}_2} = \mathcal{X}_1 \cap \overline{\mathcal{X}_2} = \emptyset$ are satisfied.

Similarly, the representative periodic cell at the meso-scale is the open interval $]0, \ell_1[$, which in a dimensionless setting is given by $\mathcal{Y} =]0, 1[\subset \mathcal{X}/\varepsilon_1$. In this case, the two constituents in \mathcal{Y} are denoted by the non-empty, open subsets $\mathcal{Y}_1 =]0, \bar{y}_I[$ and $\mathcal{Y}_2 =]\bar{y}_I, 1[$, with interface $\mathcal{I}_{\mathcal{Y}} = \overline{\mathcal{Y}_1} \cap \overline{\mathcal{Y}_2} = \{\bar{y}_I\}$, with $\bar{y}_I \in]0, 1[$ (see Fig. 1(b)). Analogously, $\overline{\mathcal{Y}} = \overline{\mathcal{Y}_1} \cup \overline{\mathcal{Y}_2}$ and $\overline{\mathcal{Y}_1} \cap \overline{\mathcal{Y}_2} = \mathcal{Y}_1 \cap \overline{\mathcal{Y}_2} = \emptyset$. It is worth mentioning that every field ϕ used in the model is assumed to be defined in $\mathcal{Y} \setminus \{\bar{y}_I\} \subset \mathcal{D}_{\bar{y}}$ and $\mathcal{X} \setminus \{\bar{z}_I\} \subset \mathcal{D}_{\bar{z}}$, and for the interface points we consider interface conditions.

For the sake of simplicity, we restrict our analysis to the particular case in which the elementary periodic cells \mathcal{Y} and \mathcal{X} , at each level of

organisation, can be uniquely chosen independently of \bar{x} and \bar{y} in the case of the ε_2 -structural level, and independently of the macroscopic variable \bar{x} in the case of the ε_1 -structural level. These assumptions are referred to as *macroscopic uniformity* (see, e.g., Holmes (2012), Penta et al. (2014, 2015), Di Stefano et al. (2020) and Ramírez-Torres et al. (2021)). One of the implications of this assumption is that, if the field ϕ satisfies the required properties, the operations of differentiation and integration commute. Thus, if we define the average operators on the elementary cells \mathcal{X} and \mathcal{Y} as

$$\langle \phi(\bar{x}, \bar{y}, \bar{z}, t) \rangle_{\bar{z}} := \frac{1}{|\mathcal{X}|} \int_{\mathcal{X}} \phi(\bar{x}, \bar{y}, \bar{z}, t) d\bar{z}, \quad (8a)$$

and

$$\langle \phi(\bar{x}, \bar{y}, \bar{z}, t) \rangle_{\bar{y}} := \frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}} \phi(\bar{x}, \bar{y}, \bar{z}, t) d\bar{y}, \quad (8b)$$

we can write

$$\frac{\partial}{\partial \bar{y}} \langle \phi(\bar{x}, \bar{y}, \bar{z}, t) \rangle_{\bar{z}} = \left\langle \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{y}} \right\rangle_{\bar{z}}, \quad (9a)$$

$$\frac{\partial}{\partial \bar{x}} \langle \phi(\bar{x}, \bar{y}, \bar{z}, t) \rangle_{\bar{z}} = \left\langle \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{x}} \right\rangle_{\bar{z}}, \quad (9b)$$

$$\frac{\partial}{\partial \bar{x}} \langle \phi(\bar{x}, \bar{y}, \bar{z}, t) \rangle_{\bar{y}} = \left\langle \frac{\partial \phi(\bar{x}, \bar{y}, \bar{z}, t)}{\partial \bar{x}} \right\rangle_{\bar{y}}. \quad (9c)$$

The above relationships are useful when obtaining the homogenised problems at the ε_1 -structural level and at the macro-scale.

We notice that, in the case of non-macroscopically uniform structures, Reynolds' transport theorem should be considered (see for instance Holmes, 2012; Penta and Gerisch, 2017b; Burrige and Keller, 1981; Ramírez-Torres et al., 2021 in the context of poroelasticity and diffusion).

2.4. Periodicity

Further considerations are required for the boundary points of the periodic cells and their relation with the assumption of periodicity. To

this end, we follow the ideas given in [Ramírez-Torres et al. \(2021\)](#) and adapted to the present three-scale framework. For this purpose, and in view of the definitions provided in [Cioranescu and Donato \(1999\)](#), we say that a function Φ is ℓ_i -periodic, with $i = 1, 2$, if $\Phi(x, t) = \Phi(x + s\ell_i, t)$ for all integers s . Now, in our three-scale homogenisation approach, the periodicity of Φ is reinterpreted in terms of ϕ . However, as pointed out in [Ramírez-Torres et al. \(2021\)](#), since ϕ can be undefined for some values of the dimensionless microscopic, \bar{y} , and mesoscopic, \bar{z} , variables, it is needed to understand the periodicity in the sense

$$\phi(\bar{x}, \bar{y}, \zeta_{\bar{z}}^{\pm}, t) = \phi(\bar{x}, \bar{y}, (\zeta_{\bar{z}} + 1)^{\pm}, t), \quad (10a)$$

$$\phi(\bar{x}, \zeta_{\bar{y}}^{\pm}, \bar{z}, t) = \phi(\bar{x}, (\zeta_{\bar{y}} + 1)^{\pm}, \bar{z}, t), \quad (10b)$$

where, for all $\zeta_{\bar{z}}$ and $\zeta_{\bar{y}}$ for which both lateral limits exist,

$$\phi(\bar{x}, \bar{y}, \zeta_{\alpha}^{\pm}, t) := \lim_{\alpha \rightarrow \zeta_{\alpha}^{\pm}} \phi(\bar{x}, \bar{y}, \bar{z}, t), \quad (11)$$

with $\alpha = \bar{y}, \bar{z}$.

Now, based on the macroscopic uniformity assumption and the 1-periodicity of ϕ in relation to \bar{y} and \bar{z} , we are able to reduce the analysis to the case of a representative periodic cell. For this purpose, let us consider the restriction of ϕ to the unit cells $\mathcal{Y} =]0, 1[$ and $\mathcal{X} =]0, 1[$ as piecewise functions. That is

$$\phi(\bar{x}, \bar{y}, \bar{z}, t)|_{\mathcal{X}} = \begin{cases} \phi_1(\bar{x}, \bar{y}, \bar{z}, t), & \bar{z} \in \mathcal{X}_1, \\ \phi_2(\bar{x}, \bar{y}, \bar{z}, t), & \bar{z} \in \mathcal{X}_2, \end{cases} \quad (12a)$$

and

$$\phi(\bar{x}, \bar{y}, \bar{z}, t)|_{\mathcal{Y}} = \begin{cases} \phi_3(\bar{x}, \bar{y}, \bar{z}, t), & \bar{y} \in \mathcal{Y}_1, \\ \phi_4(\bar{x}, \bar{y}, \bar{z}, t), & \bar{y} \in \mathcal{Y}_2. \end{cases} \quad (12b)$$

Thus, using Eqs. (10a) and (10b), we can describe the periodicity at $\zeta_{\bar{z}} = 0$ and $\zeta_{\bar{y}} = 0$ as follows

$$\phi(\bar{x}, \bar{y}, 0^+, t) = \phi(\bar{x}, \bar{y}, 1^+, t), \quad (13a)$$

$$\phi(\bar{x}, 0^+, \bar{z}, t) = \phi(\bar{x}, 1^+, \bar{z}, t). \quad (13b)$$

In particular, to satisfy possible continuity conditions for ϕ at the boundary of the periodic cells (see Eqs. (12a) and (12b)), we have that

$$\phi_2(\bar{x}, \bar{y}, 1^-, t) = \phi_1(\bar{x}, \bar{y}, 1^+, t), \quad (14a)$$

$$\phi_4(\bar{x}, 1^-, \bar{z}, t) = \phi_3(\bar{x}, 1^+, \bar{z}, t). \quad (14b)$$

So, combining the results in Eqs. (13a)–(14b) and because of the periodicity assumption,

$$\phi_2(\bar{x}, \bar{y}, 1^-, t) = \phi_1(\bar{x}, \bar{y}, 0^+, t), \quad (15a)$$

$$\phi_4(\bar{x}, 1^-, \bar{z}, t) = \phi_3(\bar{x}, 0^+, \bar{z}, t). \quad (15b)$$

2.5. Further assumptions

By following the formalism adopted so far, all equations should be written in dimensionless form. However, for simplicity of notation, we drop the ‘bar’ notation in the subsequent discussions. Therefore, with a slight abuse of notation, the expression in (5) takes the form

$$\Phi^{(\varepsilon)}(x, t) = \phi(x, y, z, t), \quad (16)$$

with

$$y = \frac{x}{\varepsilon_1} \quad \text{and} \quad z = \frac{x}{\varepsilon_2}. \quad (17)$$

2.6. Statement of the problem

We consider a one-dimensional physical model that describes a heterogeneous bar of length L with linear viscoelastic behaviour and two hierarchical levels of organisation. For the sake of simplicity, we

neglect inertial terms and assume that the body force $f(x)$ is independent of ε and in particular depends on the macroscale only. We refer to [Penta et al. \(2021\)](#) for the case in which an inhomogeneous body force admits a classical Helmholtz decomposition. In this situation, the body force $f(x)$ will be given by the gradient and the curl of scalar and vector potentials. Therefore, it will present microscale variations of the type x/ε .

So, for all $t \in]-\infty, +\infty[$, the one-dimensional, balance of linear momentum together with the interface, boundary and initial conditions reads

$$-\frac{\partial \sigma^{(\varepsilon)}(x, t)}{\partial x} = f(x), \quad x \in \mathcal{B} \setminus (\mathcal{S}^{(\varepsilon_1)} \cup \mathcal{S}^{(\varepsilon_2)}), \quad (18a)$$

$$\llbracket u^{(\varepsilon)}(x, t) \rrbracket = 0, \quad \llbracket \sigma^{(\varepsilon)}(x, t) \rrbracket = 0, \quad x \in \mathcal{S}^{(\varepsilon_1)} \cup \mathcal{S}^{(\varepsilon_2)}, \quad (18b)$$

Boundary conditions

$$u^{(\varepsilon)}(0, t) = u_0(t), \quad u^{(\varepsilon)}(L, t) = u_L(t), \quad (18c)$$

Initial condition

$$u^{(\varepsilon)}(x, 0) = u_e^{(\varepsilon)}(x), \quad x \in \mathcal{B}, \quad (18d)$$

where $\sigma^{(\varepsilon)}$ is the one-dimensional representation of stress and $u^{(\varepsilon)}$ is the global displacement. Besides, the double bracket in (18b), also called contrast or jump operator, is defined as

$$\llbracket \Phi^{(\varepsilon)}(x_j, t) \rrbracket := \lim_{x \rightarrow x_j^+} \Phi^{(\varepsilon)}(x, t) - \lim_{x \rightarrow x_j^-} \Phi^{(\varepsilon)}(x, t) = \Phi^{(\varepsilon)}(x_j^+, t) - \Phi^{(\varepsilon)}(x_j^-, t). \quad (19)$$

As observed in Eq. (18b), we impose continuity conditions at the interfaces between the different constituents.

In the present framework, the composite behaves as a non-ageing linear viscoelastic material so that the constitutive relation for $\sigma^{(\varepsilon)}$ can be written as (see [Christensen \(1982\)](#))

$$\sigma^{(\varepsilon)}(x, t) = \int_0^t \mathbb{R}^{(\varepsilon)}(x, t - \tau) \frac{\partial}{\partial \tau} \left(\frac{\partial u^{(\varepsilon)}(x, \tau)}{\partial x} \right) d\tau, \quad (20)$$

where $\mathbb{R}^{(\varepsilon)}$ is the relaxation modulus. Notice that the properties of the phases are encoded in $\mathbb{R}^{(\varepsilon)}$, which is assumed to be a smooth real function of x in $(\mathcal{B} \setminus \mathcal{S}) \times]-\infty, +\infty[$, but discontinuous on $\mathcal{S} \times]-\infty, +\infty[$, with $\mathcal{S} = \mathcal{S}^{(\varepsilon_1)} \cup \mathcal{S}^{(\varepsilon_2)}$.

An important remark is related to the initial condition given in Eq. (18d). In this regard, we consider that a suitable initial condition for the non-ageing linear viscoelastic heterogeneous problem (18a)–(18d) corresponds to the solution of the equivalent problem in the elastic case, denoted in Eq. (18d) by $u_e^{(\varepsilon)}(x)$. This result is reached at $t = 0$ s because for this value of time the constitutive law in Eq. (20), and hence the whole problem, become the equivalent elastic one. In [Appendix A](#), we provide, for the particular case of a two-scale configuration, details on the implications of this assumption on the homogenisation process. In the remainder of this work, the notation $\Phi_e^{(\varepsilon)}(x)$ stands for the elastic counterpart (instant elastic response) of $\Phi^{(\varepsilon)}(x, t)$.

Finally, mimicking the two scale case ([Bakhtvalov and Panasenkov, 1989; Persson et al., 1993](#)), we consider that $f \in C^\infty(\mathcal{B})$ and that there exist two real constants α, β such that $0 < \alpha \leq \mathbb{R}^{(\varepsilon)}(x, t) \leq \beta < +\infty$ for all $x \in \mathcal{B}$ and $t \in]-\infty, +\infty[$ as $\{\varepsilon_1, \varepsilon_2\} \rightarrow 0$.

3. Reformulation in Laplace–Carson space

The integral equation (20) refers to the constitutive law for non-ageing, linear viscoelastic materials, and can be manipulated by means of integral transforms (see, e.g., [Christensen \(1982\)](#)). In particular, the Laplace–Carson transform, which is given by

$$\Phi^{(\varepsilon)}(x, p) = p \int_0^\infty e^{-pt} \Phi^{(\varepsilon)}(x, t) dt, \quad \forall t \geq 0, \quad (21)$$

where p is the variable in the Laplace–Carson space, reduces (20) to an expression representing the constitutive relation in classical elasticity theory (see, for instance, Lakes (2009)). This methodology originally proposed by Hashin (1965) is known as the elastic–viscoelastic correspondence principle.

Hence, based on the above considerations, the original system (18a)–(18d), together with (20), can be written in the Laplace–Carson domain as follows

$$-\frac{\partial}{\partial x} \left(\mathbb{R}^{(\epsilon)}(x, p) \frac{\partial u^{(\epsilon)}(x, p)}{\partial x} \right) = f(x), \quad x \in \mathcal{B} \setminus \mathcal{J}, \quad (22a)$$

$$\llbracket u^{(\epsilon)}(x, p) \rrbracket = 0, \quad \left[\left[\mathbb{R}^{(\epsilon)}(x, p) \frac{\partial u^{(\epsilon)}(x, p)}{\partial x} \right] \right] = 0, \quad x \in \mathcal{J}, \quad (22b)$$

Boundary conditions

$$u^{(\epsilon)}(0, p) = u_0(p), \quad u^{(\epsilon)}(L, p) = u_L(p), \quad (22c)$$

Initial condition

$$u^{(\epsilon)}(x, 0) = u_e^{(\epsilon)}(x), \quad x \in \mathcal{B}. \quad (22d)$$

where $p \in [0, +\infty]$. We recall that the dependency on the variable p means that the quantity is defined in the Laplace–Carson space.

4. Solution via a three-scale asymptotic homogenisation method

In this section, we adapt the three-scale asymptotic homogenisation method introduced in Ramírez-Torres et al. (2018b) to the one-dimensional viscoelastic problem stated above. It is worth noticing that when both asymptotic parameters are approaching zero, we are in presence of a strongly heterogeneous structure. Therefore, we apply the three-scale asymptotic technique to reduce the complexity of the model specified in (22a)–(22d) by finding an “equivalent” homogenised problem through the construction of a formal asymptotic solution.

The three-scale AHM proposes the solution of the heterogeneous problem (22a)–(22d) as a formal series expansion in powers of ϵ_1 and ϵ_2 . In the Laplace–Carson domain, it reads

$$u^{(\epsilon)}(x, p) = \tilde{u}^{(0)}(x, y, z, p) + \sum_{i=1}^{+\infty} \tilde{u}^{(i)}(x, y, z, p) \epsilon_2^i, \quad (23)$$

where $\tilde{u}^{(0)}$ is defined as

$$\tilde{u}^{(0)}(x, y, z, p) = u^{(0)}(x, y, z, p) + \sum_{i=1}^{+\infty} u^{(i)}(x, y, z, p) \epsilon_1^i, \quad (24)$$

and $u^{(i)}$ and $\tilde{u}^{(i)}$ are assumed to be 1-periodic functions with respect to the variables y and z , for all natural numbers i , for all $x \in \mathcal{B}$, and for all $p \in [0, +\infty]$.

The homogenisation procedure starts by collecting information at the smallest scale, i.e., the ϵ_2 -structure level and then taking the study up to the larger scales, i.e., the ϵ_1 -structure level and the macro-scale. The solution (23)–(24) is built according to the definition of the *formal asymptotic solution* (see Bakhvalov and Panasenko (1989) for more details). In particular, we consider the same order of the approximation at the different levels of organisation, namely $\mathbf{O}(\epsilon_2)$ and $\mathbf{O}(\epsilon_1)$. In what follows (unless necessary), the variable dependence is dropped out for convenience.

4.1. Homogenisation at the ϵ_2 -structure level

After replacing (23) into (22a) and using (7), we have

$$\begin{aligned} & -\mathfrak{L}_{xx} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) - \epsilon_1^{-1} \mathfrak{L}_{xy} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) \\ & - \epsilon_2^{-1} \mathfrak{L}_{xz} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) - \epsilon_1^{-1} \mathfrak{L}_{yx} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) \\ & - \epsilon_1^{-2} \mathfrak{L}_{yy} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) - \epsilon_1^{-1} \epsilon_2^{-1} \mathfrak{L}_{yz} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) \end{aligned}$$

$$\begin{aligned} & - \epsilon_2^{-1} \mathfrak{L}_{zx} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) - \epsilon_1^{-1} \epsilon_2^{-1} \mathfrak{L}_{zy} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) \\ & - \epsilon_2^{-2} \mathfrak{L}_{zz} \left(\tilde{u}^{(0)} + \sum_{i=1}^{+\infty} \tilde{u}^{(i)} \epsilon_2^i \right) = f(x). \end{aligned} \quad (25)$$

where the following operator is defined,

$$\mathfrak{L}_{\alpha\beta}(\phi(x, y, z, p)) := \frac{1}{L_c^2} \frac{\partial}{\partial \alpha} \left(\mathbb{R}(x, y, z, p) \frac{\partial \phi(x, y, z, p)}{\partial \beta} \right), \quad (26)$$

for $\alpha, \beta = x, y, z$.

The homogenisation procedure follows by grouping the terms of Eq. (25) in the same powers of ϵ_2 , keeping ϵ_1 fixed. Then, in order to have the order of approximation $\mathbf{O}(\epsilon_2)$ in the construction of the formal asymptotic solution, all the coefficients of such power having non-positive exponents are set equal to zero. Then, the following sequence of problems are obtained,

$$\epsilon_2^{-2} : \quad \mathfrak{L}_{zz}(\tilde{u}^{(0)}) = 0, \quad (27a)$$

$$\epsilon_2^{-1} : \quad \mathfrak{L}_{zz}(\tilde{u}^{(1)}) = -\mathfrak{L}_{xz}(\tilde{u}^{(0)}) - \epsilon_1^{-1} \mathfrak{L}_{yz}(\tilde{u}^{(0)}) - \mathfrak{L}_{zx}(\tilde{u}^{(0)}) - \epsilon_1^{-1} \mathfrak{L}_{zy}(\tilde{u}^{(0)}), \quad (27b)$$

$$\begin{aligned} \epsilon_2^0 : \quad & \mathfrak{L}_{zz}(\tilde{u}^{(2)}) = -\mathfrak{L}_{xx}(\tilde{u}^{(0)}) - \epsilon_1^{-1} \mathfrak{L}_{xy}(\tilde{u}^{(0)}) - \mathfrak{L}_{xz}(\tilde{u}^{(1)}) - \epsilon_1^{-1} \mathfrak{L}_{yx}(\tilde{u}^{(0)}) \\ & - \epsilon_1^{-2} \mathfrak{L}_{yy}(\tilde{u}^{(0)}) - \epsilon_1^{-1} \mathfrak{L}_{yz}(\tilde{u}^{(1)}) - \mathfrak{L}_{zx}(\tilde{u}^{(1)}) - \epsilon_1^{-1} \mathfrak{L}_{zy}(\tilde{u}^{(1)}) - f. \end{aligned} \quad (27c)$$

Analogously, by replacing Eq. (23) into (22b) and multiplying the resulting expressions by ϵ_2^{-2} and ϵ_2^{-1} , respectively, the interface conditions are rewritten as follows

$$\llbracket \epsilon_2^{-2} \tilde{u}^{(0)} + \epsilon_2^{-1} \tilde{u}^{(1)} + \epsilon_2^0 \tilde{u}^{(2)} + \dots \rrbracket = 0, \quad (28a)$$

$$\begin{aligned} & \left[\left[\frac{\mathbb{R}}{L_c} \left[\epsilon_2^{-2} \frac{\partial \tilde{u}^{(0)}}{\partial z} + \epsilon_2^{-1} \left(\frac{\partial \tilde{u}^{(0)}}{\partial x} + \epsilon_1^{-1} \frac{\partial \tilde{u}^{(0)}}{\partial y} + \frac{\partial \tilde{u}^{(1)}}{\partial z} \right) \right. \right. \right. \\ & \left. \left. \left. + \epsilon_2^0 \left(\frac{\partial \tilde{u}^{(1)}}{\partial x} + \epsilon_1^{-1} \frac{\partial \tilde{u}^{(1)}}{\partial y} + \frac{\partial \tilde{u}^{(2)}}{\partial z} \right) \right] \right] = 0. \end{aligned} \quad (28b)$$

Before going further, it is worth mentioning that in order to solve the problems (27a)–(27c) equipped with the corresponding interface conditions that emanates from (28a)–(28b), the following lemma is useful.

Lemma 1. *Let $a(\zeta)$, $F_0(\zeta)$, and $F_1(\zeta)$ be 1-periodic piecewise-differentiable functions with finite jump discontinuities in $\zeta \in \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ ($0 < \zeta < 1$) and positive $a(\zeta)$ bounded over $[0, 1]$. Then, a necessary and sufficient condition for the existence of a 1-periodic solution $N(\zeta)$ of the problem*

$$\frac{d}{d\zeta} \left(a(\zeta) \frac{dN(\zeta)}{d\zeta} \right) = F_0(\zeta) + \frac{dF_1(\zeta)}{d\zeta}, \quad (29a)$$

$$\llbracket N(\zeta) \rrbracket_{\zeta_j} = 0, \quad (29b)$$

$$\left[\left[a(\zeta) \frac{dN(\zeta)}{d\zeta} - F_1(\zeta) \right] \right]_{\zeta_j} = 0, \quad (29c)$$

for all $\zeta \notin \{\zeta_1, \zeta_2, \dots, \zeta_n\}$, is that

$$\langle F_0(\zeta) \rangle_{\zeta} \equiv \int_0^1 F_0(\zeta) d\zeta = 0. \quad (30)$$

So, if $N(\zeta)$ is a 1-periodic solution of the problem (29a)–(29c), there exists a family of 1-periodic solutions given by $N_1(\zeta) = N(\zeta) + C$, where C is an arbitrary real constant. That is, the solution is unique up to a ζ -constant function. The proof of this Lemma, in the context of elastic composites, is given in Section 2 of Álvarez-Borges et al. (2014) and we refer to Álvarez-Borges et al. (2018) for a version of the lemma in the case of imperfect contact conditions.

The problems concerning the ϵ_2 -structural level are presented below

Problem for ϵ_2^{-2}

For all $p \in [0, +\infty]$,

$$\mathfrak{L}_{zz}(\tilde{u}^{(0)}(x, y, z, p)) = 0, \quad z \in \mathcal{Z} \setminus \mathcal{J}_{\mathcal{Z}}, \quad (31a)$$

$$\llbracket \tilde{u}^{(0)}(x, y, z, p) \rrbracket = 0, \quad z \in \mathcal{J}_{\mathcal{Z}}, \quad (31b)$$

$$\left[\left[\frac{\mathbb{R}(y, z, p)}{L_c} \frac{\partial \tilde{u}^{(0)}(x, y, z, p)}{\partial z} \right] \right] = 0, \quad z \in \mathcal{I}_{\mathcal{Z}}, \quad (31c)$$

Initial condition

$$\tilde{u}^{(0)}(x, y, z, 0) = \tilde{u}_e^{(0)}(x, y, z), \quad z \in \mathcal{Z}. \quad (31d)$$

In Eq. (31a), the problem has the trivial solution $\tilde{u}^{(0)}(x, y, z, p) \equiv 0$. Then, Lemma 1 ensures that $\tilde{u}^{(0)}(x, y, z, p)$ is a solution of (31a)–(31d) if and only if it is independent of z , namely

$$\tilde{u}^{(0)} = \tilde{u}^{(0)}(x, y, p). \quad (32)$$

Thus, from Eqs. (24) and (32), it follows that

$$u^{(0)} = u^{(0)}(x, y, p), \quad (33a)$$

$$u^{(i)} = u^{(i)}(x, y, p). \quad (33b)$$

Problem for ε_2^{-1}

In view of the above result, $\mathfrak{L}_{xz}(\tilde{u}^{(0)}) = 0$ and $\mathfrak{L}_{yz}(\tilde{u}^{(0)}) = 0$ in (27b). Therefore, the problem for $\tilde{u}^{(1)}$ can be reformulated as

$$\mathfrak{L}_{zz}(\tilde{u}^{(1)}) = -\mathfrak{L}_{zx}(\tilde{u}^{(0)}) - \varepsilon_1^{-1} \mathfrak{L}_{zy}(\tilde{u}^{(0)}), \quad z \in \mathcal{Z} \setminus \mathcal{I}_{\mathcal{Z}}, \quad (34a)$$

$$\left[\tilde{u}^{(1)} \right] = 0, \quad z \in \mathcal{I}_{\mathcal{Z}}, \quad (34b)$$

$$\left[\left[\frac{\mathbb{R}}{L_c} \frac{\partial \tilde{u}^{(1)}}{\partial z} \right] \right] = - \left[\left[\frac{\mathbb{R}}{L_c} \left(\frac{\partial \tilde{u}^{(0)}}{\partial x} + \varepsilon_1^{-1} \frac{\partial \tilde{u}^{(0)}}{\partial y} \right) \right] \right], \quad z \in \mathcal{I}_{\mathcal{Z}}, \quad (34c)$$

Initial condition

$$\tilde{u}^{(1)}(x, y, z, 0) = \tilde{u}_e^{(1)}(x, y, z), \quad z \in \mathcal{Z}. \quad (34d)$$

for all $p \in [0, +\infty[$.

Applying Lemma 1 on (34a) and considering (32) and the 1-periodicity of $\mathbb{R}(x, y, z, t)$ with respect to z , the following result holds true

$$\langle \mathfrak{L}_{zx}(\tilde{u}^{(0)}) + \varepsilon_1^{-1} \mathfrak{L}_{zy}(\tilde{u}^{(0)}) \rangle_z = 0. \quad (35)$$

Consequently, the existence and uniqueness, up to a z -constant function, of a solution for the problem (34a)–(34d) is guaranteed. In particular, a solution for (34a)–(34d) is proposed as follows

$$\tilde{u}^{(1)}(x, y, z, p) = \tilde{\chi}(x, y, z, p) \tilde{U}^{(0)}(x, y, p), \quad (36a)$$

$$\tilde{U}^{(0)}(x, y, p) = \frac{\partial \tilde{u}^{(0)}(x, y, p)}{\partial x} + \varepsilon_1^{-1} \frac{\partial \tilde{u}^{(0)}(x, y, p)}{\partial y}, \quad (36b)$$

where the local function $\tilde{\chi}(x, y, z, p)$ is 1-periodic with respect to z .

The substitution of (36a)–(36b) into (34a)–(34d) leads to the ε_2 -local problem, which, for all $p \in [0, +\infty[$, is given by

$$\frac{1}{L_c^2} \frac{\partial}{\partial z} \left(\mathbb{R}(x, y, z, p) \frac{\partial \tilde{\chi}(x, y, z, p)}{\partial z} + \mathbb{R}(x, y, z, p) \right) = 0, \quad z \in \mathcal{Z} \setminus \mathcal{I}_{\mathcal{Z}}, \quad (37a)$$

$$\left[\tilde{\chi}(x, y, z, p) \right] = 0, \quad z \in \mathcal{I}_{\mathcal{Z}}, \quad (37b)$$

$$\left[\left[\frac{1}{L_c} \left(\mathbb{R}(x, y, z, p) \frac{\partial \tilde{\chi}(x, y, z, p)}{\partial z} + \mathbb{R}(x, y, z, p) \right) \right] \right] = 0, \quad z \in \mathcal{I}_{\mathcal{Z}}, \quad (37c)$$

Initial condition

$$\tilde{\chi}(x, y, z, 0) = \tilde{\chi}_e(x, y, z), \quad z \in \mathcal{Z}. \quad (37d)$$

According to the problem (37a)–(37d), a further condition is required to obtain the uniqueness of the solution, for instance, one can enforce that $\langle \tilde{\chi}(x, y, z, p) \rangle_z = 0$, or alternatively, fix the value of $\tilde{\chi}(x, y, z, p)$ at one point of the reference cell \mathcal{Z} (see, e.g., Penta and Gerisch (2017a) and Penta and Gerisch (2015)). Following previous works (Cruz-González et al., 2020b, 2021), we assume the following Dirichlet condition

$$\tilde{\chi}(x, y, 0, p) = 0, \quad \forall p \in [0, +\infty[. \quad (38)$$

Problem for ε_2^0

Because of Lemma 1, the existence and uniqueness of a 1-periodic solution $\tilde{u}^{(2)}$ for the problem (27c) is guaranteed if and only if

$$\langle \mathfrak{L}_{xx}(\tilde{u}^{(0)}) + \varepsilon_1^{-1} \mathfrak{L}_{xy}(\tilde{u}^{(0)}) + \mathfrak{L}_{xz}(\tilde{u}^{(1)}) + \varepsilon_1^{-1} \mathfrak{L}_{yx}(\tilde{u}^{(0)}) + \varepsilon_1^{-2} \mathfrak{L}_{yy}(\tilde{u}^{(0)}) + \varepsilon_1^{-1} \mathfrak{L}_{yz}(\tilde{u}^{(1)}) + \mathfrak{L}_{zx}(\tilde{u}^{(1)}) + \varepsilon_1^{-1} L_{zy}(\tilde{u}^{(1)}) + f \rangle_z = 0. \quad (39)$$

Thus, working with Eq. (39) we find the “intermediate” homogenised problem, after taking into account Eqs. (32), (36a), the periodicity of $\mathbb{R}(x, y, z, p)$ and $\tilde{\chi}(x, y, z, p)$ with respect to z , and the assumption of macroscopic uniformity (see, e.g., Cruz-González et al. (2017)). Thus, the intermediate homogenised problem in the Laplace–Carson space, is given by

$$\frac{1}{L_c^2} \left(\frac{\partial}{\partial x} + \varepsilon_1^{-1} \frac{\partial}{\partial y} \right) \left[\mathbb{R}(x, y, p) \tilde{U}^{(0)}(x, y, p) \right] + f(x) = 0, \quad (40)$$

where $p \in [0, +\infty[$ and $x \in \mathcal{B}_1^{(h_{\varepsilon_1})}$ with $\mathcal{B}_1^{(h_{\varepsilon_1})}$ denoting the homogenised counterpart of $\mathcal{B}_1^{(\varepsilon_1)}$. In particular, \mathbb{R} is the effective coefficient at the ε_1 -structural level, which is given by the expression

$$\mathbb{R}(x, y, p) = \left\langle \mathbb{R}(x, y, z, p) + \mathbb{R}(x, y, z, p) \frac{\partial \tilde{\chi}(x, y, z, p)}{\partial z} \right\rangle_z. \quad (41)$$

4.2. Homogenisation at the ε_1 -structure level

Here, we use the information obtained in Section 4.1 as input values in order to find the macroscopic effective behaviour of the hierarchical composite material. For this purpose, we proceed by replacing (36b) and (24) into (40) so that

$$\begin{aligned} & \check{\mathfrak{L}}_{xx} \left(u^{(0)} + \sum_{i=1}^{+\infty} u^{(i)} \varepsilon_1^i \right) + \varepsilon_1^{-1} \check{\mathfrak{L}}_{xy} \left(u^{(0)} + \sum_{i=1}^{+\infty} u^{(i)} \varepsilon_1^i \right) \\ & + \varepsilon_1^{-1} \check{\mathfrak{L}}_{yx} \left(u^{(0)} + \sum_{i=1}^{+\infty} u^{(i)} \varepsilon_1^i \right) + \varepsilon_1^{-2} \check{\mathfrak{L}}_{yy} \left(u^{(0)} + \sum_{i=1}^{+\infty} u^{(i)} \varepsilon_1^i \right) + f = 0, \end{aligned} \quad (42)$$

where

$$\check{\mathfrak{L}}_{\alpha\beta}(\phi(x, y, p)) := \frac{1}{L_c^2} \frac{\partial}{\partial \alpha} \left(\mathbb{R}(x, y, p) \frac{\partial \phi(x, y, p)}{\partial \beta} \right), \quad (43)$$

for $\alpha, \beta = x, y$.

We proceed similarly to the previous section, i.e. grouping the terms of the resulting Eq. (42) in the same powers of ε_1 . So then, in order to have the order of approximation $\mathbf{O}(\varepsilon_1)$, the following conditions must be satisfied,

$$\varepsilon_1^{-2} : \quad \check{\mathfrak{L}}_{yy}(u^{(0)}) = 0, \quad (44a)$$

$$\varepsilon_1^{-1} : \quad \check{\mathfrak{L}}_{yy}(u^{(1)}) = -\check{\mathfrak{L}}_{xy}(u^{(0)}) - \check{\mathfrak{L}}_{yx}(u^{(0)}), \quad (44b)$$

$$\varepsilon_1^0 : \quad \check{\mathfrak{L}}_{yy}(u^{(2)}) = -\check{\mathfrak{L}}_{xx}(u^{(0)}) - \check{\mathfrak{L}}_{xy}(u^{(1)}) - \check{\mathfrak{L}}_{yx}(u^{(1)}) - f. \quad (44c)$$

In addition, from the interface condition (31b), the expression for $\tilde{u}^{(0)}$ given in (24), and multiplying by ε_1^{-2} , we can deduce that

$$\left[\left[\varepsilon_1^{-2} u^{(0)} + \varepsilon_1^{-1} u^{(1)} + \varepsilon_1^0 u^{(2)} + \dots \right] \right] = 0. \quad (45)$$

Moreover, by considering the relations in (24), (34c), (36a) and (36b), using the cell average operator over \mathcal{Z} and multiplying the result by ε_1^{-1} , we obtain (granted that the operations of limit and integration can be interchanged)

$$\left[\left[\frac{\mathbb{R}}{L_c} \left(\varepsilon_1^{-2} \frac{\partial u^{(0)}}{\partial y} + \varepsilon_1^{-1} \frac{\partial u^{(0)}}{\partial x} + \varepsilon_1^{-1} \frac{\partial u^{(1)}}{\partial y} + \varepsilon_1^0 \frac{\partial u^{(1)}}{\partial x} \right) + \dots \right] \right] = 0. \quad (46)$$

It is worth mentioning that the interface condition (46) holds in the physical domain, whereas we are now addressing the homogenisation process at the ε_1 -structural level, whose mechanical response is given, in effect, by the effective elastic function $\mathbb{R}(x, y, p)$ provided in (41).

Then, the sequence of problems for the ε_1 -structural level are

Problem for ε_1^{-2}

For all $p \in [0, +\infty[$

$$\check{\Delta}_{yy} (u^{(0)}(x, y, p)) = 0, \quad y \in \mathcal{Y} \setminus \mathcal{I}_{\mathcal{Y}}, \quad (47a)$$

$$\llbracket u^{(0)}(x, y, p) \rrbracket = 0, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (47b)$$

$$\left\llbracket \frac{\check{\mathbb{R}}(y, p)}{L_c} \frac{\partial u^{(0)}(x, y, p)}{\partial y} \right\rrbracket = 0, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (47c)$$

Initial Condition

$$u^{(0)}(x, y, 0) = u_e^{(0)}(x, y), \quad y \in \mathcal{Y}. \quad (47d)$$

Similarly to what we discussed in the previous section, Lemma 1 ensures that $u^{(0)}(x, y, p)$ is a solution of (47a)–(47d) if and only if it is y -constant, i.e.,

$$u^{(0)} = u^{(0)}(x, p). \quad (48)$$

This result leads to a simplification in the expression for $\check{u}^{(1)}$ given in (36a) and (36b). Specifically, we have that

$$\check{u}^{(1)}(x, y, z, p) = \check{\chi}(x, y, z, p) \left(\frac{\partial u^{(0)}(x, p)}{\partial x} + \frac{\partial u^{(1)}(x, y, p)}{\partial y} \right). \quad (49)$$

Problem for ε_1^{-1}

Using the result in (48), Eq. (44b), together with the corresponding interface and initial conditions, can be rewritten as

$$\check{\Delta}_{yy} (u^{(1)}) = -\check{\Delta}_{yx} (u^{(0)}), \quad p \in \mathcal{Y} \setminus \mathcal{I}_{\mathcal{Y}}, \quad (50a)$$

$$\llbracket u^{(1)} \rrbracket = 0, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (50b)$$

$$\left\llbracket \frac{\check{\mathbb{R}}}{L_c} \frac{\partial u^{(1)}}{\partial y} \right\rrbracket = - \left\llbracket \frac{\check{\mathbb{R}}}{L_c} \left(\frac{\partial u^{(0)}}{\partial x} \right) \right\rrbracket, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (50c)$$

Initial Condition

$$u^{(1)}(x, y, 0) = u_e^{(1)}(x, y), \quad y \in \mathcal{Y}, \quad (50d)$$

where $p \in [0, +\infty[$.

Now, taking into account Eq. (48) and the 1-periodicity of $\check{\mathbb{R}}(x, y, t)$ with respect to y , we can deduce that

$$\left\langle \check{\Delta}_{yx} (u^{(0)}) \right\rangle_y = 0. \quad (51)$$

Hence, the existence and uniqueness, up to a y -constant function, of the solution for the problem (50a)–(50d) is guaranteed by means of Lemma 1, and a solution is proposed as follows

$$u^{(1)}(x, y, p) = \chi(x, y, p) \frac{\partial u^{(0)}(x, p)}{\partial x}, \quad (52)$$

where the local function $\chi(x, y, p)$ is 1-periodic with respect to y and, for all $p \in [0, \infty[$, solution of the ε_1 -local problem

$$\frac{1}{L_c^2} \frac{\partial}{\partial y} \left(\check{\mathbb{R}}(x, y, p) \frac{\partial \chi(x, y, p)}{\partial y} + \check{\mathbb{R}}(x, y, p) \right) = 0, \quad y \in \mathcal{Y} \setminus \mathcal{I}_{\mathcal{Y}}, \quad (53a)$$

$$\llbracket \chi(x, y, p) \rrbracket = 0, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (53b)$$

$$\left\llbracket \frac{1}{L_c} \left(\check{\mathbb{R}}(x, y, p) \frac{\partial \chi(x, y, p)}{\partial y} + \check{\mathbb{R}}(x, y, p) \right) \right\rrbracket = 0, \quad y \in \mathcal{I}_{\mathcal{Y}}, \quad (53c)$$

Initial Condition

$$\chi(x, y, 0) = \chi_e(x, y), \quad y \in \mathcal{Y}. \quad (53d)$$

Additionally, we include the following condition to guarantee the uniqueness,

$$\chi(x, 0, p) = 0, \quad \forall p \in [0, +\infty[. \quad (54)$$

Problem for ε_1^0

According to Lemma 1, the existence of a solution for Eq. (44c) requires that the following condition must be satisfied

$$\left\langle \check{\Delta}_{xx}(u^{(0)}) + \check{\Delta}_{xy}(u^{(1)}) + \check{\Delta}_{yx}(u^{(1)}) + f \right\rangle_y = 0. \quad (55)$$

So, from (55) and taking into account Eqs. (48) and (52), the periodicity of $\check{\mathbb{R}}(x, y, p)$ and the assumption of macroscopic uniformity, we reach the fully homogenised problem at the macroscale, which in Laplace–Carson space, reads

$$-\frac{1}{L_c^2} \frac{\partial}{\partial x} \left(\check{\mathbb{R}}(x, p) \frac{\partial u^{(0)}(x, p)}{\partial x} \right) = f(x), \quad (56)$$

where $p \in [0, +\infty[$ and $x \in \mathcal{B}^{(h)}$, with $\mathcal{B}^{(h)}$ being the idealised homogeneous structure. In addition, $\check{\mathbb{R}}$ represents the effective coefficient of the hierarchical composite and is given by the expression

$$\check{\mathbb{R}}(x, p) = \left\langle \check{\mathbb{R}}(x, y, p) + \check{\mathbb{R}}(x, y, p) \frac{\partial \chi(x, y, p)}{\partial y} \right\rangle_y. \quad (57)$$

Finally, from the boundary and initial conditions (22c)–(22d), we can conclude that the leading order term in the asymptotic expansion must satisfy the boundary and initial conditions

$$u^{(0)}(0, p) = u_0(p), \quad u^{(0)}(L, p) = u_L(p), \quad p \in [0, +\infty[, \quad (58a)$$

$$u^{(0)}(x, 0) = u_e^{(0)}(x), \quad x \in \mathcal{B}^{(h)}. \quad (58b)$$

4.3. Relation between the effective relaxation modulus and the effective creep compliance

The one-dimensional mathematical relationship between the effective relaxation modulus $\check{\mathbb{R}}$ and the effective creep compliance $\check{\mathbb{J}}(t)$, is given in the Laplace–Carson space as follows (refer to Hashin (1972) for further details)

$$\check{\mathbb{R}}(p)\check{\mathbb{J}}(p) = 1. \quad (59)$$

Furthermore, the application of the inverse of the Laplace–Carson transform in (59) leads to the Stieltjes convolution integral, (see Hanyga and Seređnyńska (2007))

$$\int_0^t \check{\mathbb{R}}(\tau)\check{\mathbb{J}}(t - \tau)d\tau = t, \quad t > 0. \quad (60)$$

This last integral equation can be solved numerically by using, for instance, one of the numerical integration schemes outlined in Park and Kim (1999). So, if the effective relaxation modulus is known in the time domain, then the effective creep compliance can be derived. These relations will be useful for computational purposes.

5. Numerical calculations

The scope of this section is twofold. First, we solve the heterogeneous problem by adapting the semi-analytical technique used in Álvarez-Borges et al. (2014). That is, in this case, we consider the solution of the original problem (18a)–(18d) without going through the homogenisation process. In the second part of this section, we use the results derived in this work to solve the homogenised and local problems. In doing this, our aim is to compare the solutions given by these two approaches.

5.1. The heterogeneous problem

In the present framework, we neglect the body force $f(x)$ in the heterogeneous problem (18a)–(18d). Moreover, we set $L = L_c = 1$ cm and assume the boundary conditions

$$u^{(\varepsilon)}(0, t) = 0 \text{ cm}, \quad u^{(\varepsilon)}(1, t) = 1 \text{ cm}. \quad (61)$$

for all $t \in] - \infty, +\infty[$.

Furthermore, along with the assumption of macroscopic uniformity, we consider that the material properties of each individual constituent do not depend of the global variable x . Specifically, we chose the relaxation modulus (see Eq. (20)) to be given by the expression,

$$\mathbb{R}^{(\varepsilon)}(x, t) = \begin{cases} \mathbb{R}_1(t), & \text{if } x \in \mathcal{B}_1^{(\varepsilon_2)}, \\ \mathbb{R}_2, & \text{if } x \in \mathcal{B}_2^{(\varepsilon_2)}, \\ \mathbb{R}_3, & \text{if } x \in \mathcal{B}_2^{(\varepsilon_1)}. \end{cases} \quad (62)$$

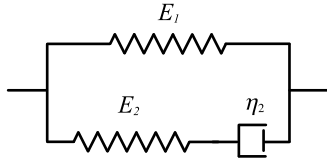


Fig. 2. Zener model.

Table 1

Material properties.

E_0 (kPa)	g	τ (s)	\mathbb{R}_2 (kPa)	\mathbb{R}_3 (kPa)
75	0.8	15	90	105

As observed in Eq. (62), the relaxation modulus for the first constituent is assumed to depend on time, corresponding to a viscoelastic material, whereas the other two constituents have constants values and describe elastic materials. In particular, the viscoelastic constituent is modelled by means of the Maxwell-like relaxation representation of the Zener model, also denominated Standard Linear Solid (S.L.S.) (Mainardi and Spada, 2011),

$$\mathbb{R}_1(t) = E_1 + E_2 \exp(-t/\tau), \quad (63)$$

where E_i ($i = 1, 2$) represents the elastic modulus of the spring i in Fig. 2 and $\tau = \eta_2/E_2$ is the relaxation time. In particular, Eq. (63) can be equivalently rewritten as

$$\mathbb{R}_1(t) = E_0 \{1 - g[1 - \exp(-t/\tau)]\}, \quad (64)$$

where $E_0 = E_1 + E_2$ is the instantaneous elastic relaxation modulus and $g := E_1/E_0$ (see Che-Yu (2020)). Here, we fix the parameters appearing in Eqs. (62) and (64) as shown in Table 1.

Now, to solve the heterogeneous problem described in (18a)–(18d), we continue by setting the values of the scaling parameters that describe the two levels of organisation. Here, we choose $\varepsilon_1 = 1/3$ and $\varepsilon_2 = 1/9$. These values of ε_1 and ε_2 lead to the set of 14 discontinuity points \mathcal{J}_d of the relaxation function $\mathbb{R}^{(\varepsilon)}(x, t)$, namely

$$\mathcal{J}_d = \left\{ \frac{4}{45}, \frac{1}{9}, \frac{1}{5}, \frac{2}{9}, \frac{1}{3}, \frac{19}{45}, \frac{4}{9}, \frac{8}{15}, \frac{5}{9}, \frac{2}{3}, \frac{34}{45}, \frac{7}{9}, \frac{13}{15}, \frac{8}{9} \right\}. \quad (65)$$

In addition, we assume that the volumetric fractions of \mathcal{X}_1 and \mathcal{X}_2 are $V_f^{(\mathcal{X}_1)} = 0.8$ and $V_f^{(\mathcal{X}_2)} = 1 - V_f^{(\mathcal{X}_1)} = 0.2$, respectively. In the case of the volumetric fractions of \mathcal{Y}_1 and \mathcal{Y}_2 , we consider that

$$V_f^{(\mathcal{Y}_1)} = (n_2/n_1 - N_c)\varepsilon_2/\varepsilon_1 \quad \text{and} \quad V_f^{(\mathcal{Y}_2)} = 1 - V_f^{(\mathcal{Y}_1)}, \quad (66)$$

where $n_1 = 1/\varepsilon_1 = 3$, $n_2 = 1/\varepsilon_2 = 9$ and N_c is a parameter that we introduce and serves as an acronym for *number of cells*.

The purpose of the first expression appearing in Eq. (66) is to avoid overlapping and unexpected cuts of the periodicity when the constituents $\mathcal{B}_1^{(\varepsilon_2)}$ and $\mathcal{B}_2^{(\varepsilon_2)}$ at the ε_2 -structural level are coupled with the constituent $\mathcal{B}_2^{(\varepsilon_1)}$ at the ε_1 -structural level. Additionally, the number N_c provides the quantity of \mathcal{X} -periodic cells that are equal in length to \mathcal{Y}_2 in the x -axis. Here, we consider $N_c = 1$, so that $V_f^{(\mathcal{Y}_1)} = 0.6667$. Fig. 3 shows the geometrical configuration of the viscoelastic heterogeneous material \mathcal{B} for the set of fixed parameters.

It is worth noting that, to solve the original heterogeneous problem (18a)–(18d) with boundary conditions specified in (61), we also need to consider the boundary points within the set of discontinuity points. This is an important difference with respect to the homogenised problem for which the relevant information of the internal structure is captured through the representative periodic cell and, then, extended according to periodicity conditions. So, in the homogenised case, the interface points represent the only discontinuity points of interest of $\mathbb{R}^{(\varepsilon)}(x, t)$ (see Eqs. (18a)–(18b)).

In Fig. 4, the graph of the function $\mathbb{R}^{(\varepsilon)}(x, t)$ is plotted for $t = 0$ s (elastic limit case).

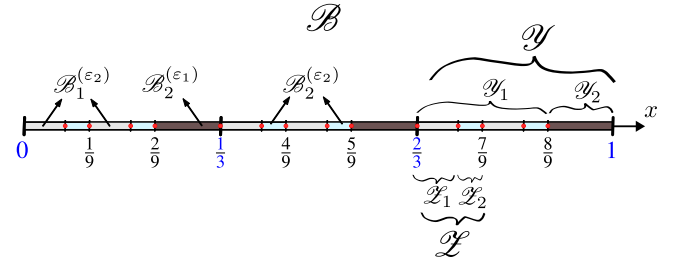


Fig. 3. Schematic of the heterogeneous material. The red dots are used to visualise the discontinuities. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

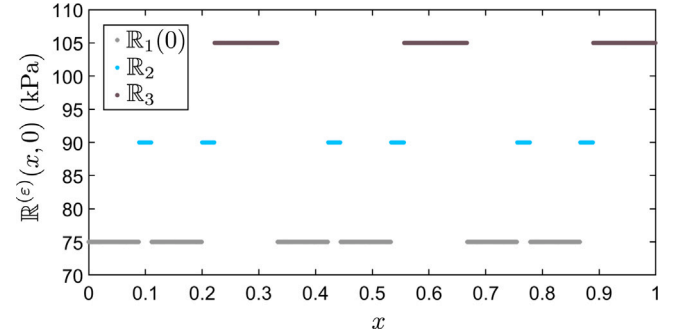


Fig. 4. Plot of the relaxation function at $t = 0$ s.

Similar to what was done in Section 3, the solution of the heterogeneous problem is rewritten in the Laplace–Carson domain, as follows

$$-\frac{1}{L_c^2} \frac{\partial}{\partial x} \left(\mathbb{R}^{(\varepsilon)}(x, p) \frac{\partial u^{(\varepsilon)}(x, p)}{\partial x} \right) = 0, \quad x \in \mathcal{B} \setminus \mathcal{J}_d, \quad (67a)$$

$$\left[[u^{(\varepsilon)}(x, p)] \right] = 0, \quad \left[\left[\frac{\mathbb{R}^{(\varepsilon)}(x, p)}{L_c} \frac{\partial u^{(\varepsilon)}(x, p)}{\partial x} \right] \right] = 0, \quad x \in \mathcal{J}_d, \quad (67b)$$

$$u^{(\varepsilon)}(0, p) = 0, \quad u^{(\varepsilon)}(1, p) = 1, \quad (67c)$$

$$u^{(\varepsilon)}(x, 0) = u_c^{(\varepsilon)}(x), \quad x \in \mathcal{B}, \quad (67d)$$

for all $p \in [0, +\infty[$.

In particular, for each discrete value of p , denoted by p_j , it is possible to solve the elastic-type heterogeneous problem (67a)–(67d) following the ideas put forward in Álvarez-Borges et al. (2014). Specifically, we propose an analytical solution of the problem (67a)–(67d) as follows

$$u_{\text{het}}^{(1/3, 1/9)}(x, p_j) = \sum_{i=0}^{14} w_{ij}(x) \mathbb{1}_{|\alpha_i, \alpha_{i+1}[}, \quad (68)$$

where $w_{ij}(x)$ are linear functions defined as $w_{ij}(x) = m_i x + n_i$, and $\mathbb{1}_{|\alpha_i, \alpha_{i+1}[}$ denotes the characteristic function of the interval $|\alpha_i, \alpha_{i+1}[$, being these intervals the ones defined by the discontinuity points in (65) with $\alpha_0 = 0$ and $\alpha_{15} = 1$. We determine the unknowns m_i, n_i with $i = 0, \dots, 14$, by solving the linear system of the form (see Appendix B for more details)

$$\mathcal{M} \cdot \mathcal{X}_{\text{incog}} = \mathcal{B}, \quad (69)$$

where \mathcal{M} , $\mathcal{X}_{\text{incog}}$ and \mathcal{B} are defined in (B.5). It is worth noticing that $n_0 = 0$ due to the first equation of (67c), therefore it is not considered as an unknown quantity, i.e. $n_0 \notin \mathcal{X}_{\text{incog}}$.

The last step to obtain the solution of the heterogeneous problem in the time domain involves the inversion of the solution Eq. (68). For this purpose, we employ a numerical approach using Matlab's function INV LAP (see Juraj (2020) and Valsa and Brančik (1998)). In particular, each fixed value p_j is defined using the formula $p_j := t_j/t_i$ for $j =$

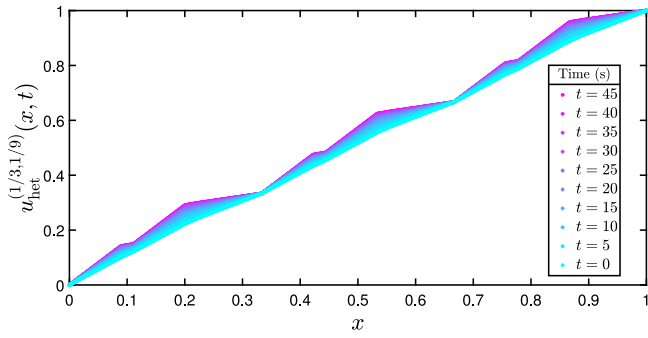


Fig. 5. Solution of the heterogeneous problem (67a)–(67d) in the time domain.

$1, \dots, (ns + nd + 1)$, where t_i is a point of the discretised time interval $t = [t_1, t_2, \dots, t_N]$, ns , nd and t_j are parameters of Valsa's numerical inversion method (see, e.g., Juraj (2020), Valsa and Bránčik (1998) and Cruz-González et al. (2021)). As a drawback, this numerical algorithm presents issues close to $t = 0$ s, however, in our approach the particular case at $t = 0$ s is obtained from the elastic problem (instant elastic response).

We remark that the original version of the INV LAP's script is only conceived for the inversion of the Laplace transform. In this and previous works of ours (Cruz-González et al., 2020a,b, 2021), we adapted the code to perform the inversion of the Laplace–Carson transform in a discrete space of solutions arising from the multiscale analysis (see Figure 2 of Cruz-González et al. (2021)).

So using the semi-analytical approach given above, in Fig. 5, we plot the solution of the heterogeneous problem (67a)–(67d), which we denote by $u_{\text{het}}^{(1/3,1/9)}(x,t)$ (see Eq. (68)), for different values of time. We note how the heterogeneities influence the solution as time increases being more in-homogeneous.

5.2. The homogenised problem and the asymptotic solution

Considering the assumptions given in the above section, the homogenised problem at the macro-scale takes the form

$$\frac{\hat{\mathbb{R}}(p)}{L_c^2} \frac{\partial^2 u^{(0)}(x,p)}{\partial x^2} = 0, \quad x \in \mathcal{B}^h, \quad (70a)$$

$$u^{(0)}(0,p) = 0, \quad u^{(0)}(1,p) = 1, \quad (70b)$$

$$u^{(0)}(x,0) = u_e^{(0)}(x) = x, \quad \forall x \in \mathcal{B}^h. \quad (70c)$$

where $p \in [0, +\infty[$. In contrast with the heterogeneous problem, the solution of the homogenised problem at the macro-scale can be computed in a more simpler way. Particularly, the solution of (70a) with boundary and initial conditions (70b) and (70c) reads

$$u^{(0)}(x,p) = x. \quad (71)$$

We note that the solution of the homogenised problem, which represents the leading order term in the asymptotic expansion (23)–(24), does not feature dependency on time, neither any contribution that makes it “oscillates” as the original solution of the heterogeneous problem. This information, however, is encrypted in the terms of higher powers of ε_1 and ε_2 . With this in mind, we consider the truncation of the expansion for $u^{(\varepsilon)}$ up to the first order of ε_1 and ε_2 , so that, by taking into account the results in Eqs. (48), (49) and (52), we have

$$u_{\text{ThreeScale}}^{(\varepsilon_1, \varepsilon_2)}(x,p) = u^{(0)}(x,p) + \chi^{(\varepsilon_1, \varepsilon_2)}(y,p) \frac{\partial u^{(0)}(x,p)}{\partial x} \varepsilon_1 + \tilde{\chi}^{(\varepsilon_1, \varepsilon_2)}(z,p) \left(1 + \frac{\partial \chi^{(\varepsilon_1, \varepsilon_2)}(y,p)}{\partial y} \right) \frac{\partial u^{(0)}(x,p)}{\partial x} \varepsilon_2, \quad (72)$$

which for $\varepsilon_1 = 1/3$ and $\varepsilon_2 = 1/9$ takes the form

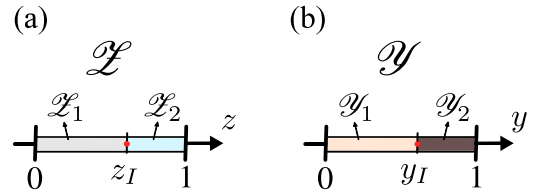


Fig. 6. Images (a)–(b) show the periodic cells for the ε_2 - and ε_1 -structural levels, respectively.

$$u_{\text{ThreeScale}}^{(1/3,1/9)}(x,p) = u^{(0)}(x,p) + \frac{1}{3} \chi^{(1/3,1/9)}(y,p) \frac{\partial u^{(0)}(x,p)}{\partial x} + \frac{1}{9} \tilde{\chi}^{(1/3,1/9)}(z,p) \left(1 + \frac{\partial \chi^{(1/3,1/9)}(y,p)}{\partial y} \right) \frac{\partial u^{(0)}(x,p)}{\partial x}. \quad (73)$$

Therefore, apart from the solution of the homogenised problem (71), we need to compute the solutions of the local problems (79a) and (79b) to fully determine Eq. (72). In this respect, we solve the ε_2 - and ε_1 -local problems by means of a similar approach to the one performed in Section 5.1 concerning the heterogeneous problem. However, for this purpose, we need to take into account the discontinuity points within the periodic cells since we have a two-phase composite material at both structural levels (see Fig. 6).

By considering the expression introduced in (62) for the relaxation modulus and according to the structure of the periodic cells at the different levels of organisation (see Fig. 6), we write

$$\mathbb{R}(z,t) = \begin{cases} \mathbb{R}_1(t), & \text{if } z \in \mathcal{Z}_1 = [0, z_I], \\ \mathbb{R}_2, & \text{if } z \in \mathcal{Z}_2 =]z_I, 1], \end{cases} \quad (74)$$

and

$$\check{\mathbb{R}}(y,t) = \begin{cases} \check{\mathbb{R}}(t), & \text{if } y \in \mathcal{Y}_1 = [0, y_I], \\ \mathbb{R}_3, & \text{if } y \in \mathcal{Y}_2 =]y_I, 1]. \end{cases} \quad (75)$$

We remark that the effective relaxation modulus $\check{\mathbb{R}}$ in Eq. (75) is derived by means of the relaxation moduli \mathbb{R}_1 and \mathbb{R}_2 in the homogenisation process at the ε_2 -structural level.

Because of the assumptions made so far, after integrating the local problems (37a)–(38) and (53a)–(54), we can write

$$\frac{1}{L_c^2} \left[\mathbb{R}(z,p) \frac{\partial \tilde{\chi}(z,p)}{\partial z} + \mathbb{R}(z,p) \right] = \check{\mathbb{R}}(p), \quad z \in \mathcal{Z} \setminus \mathcal{Z}_{\mathcal{Z}}, \quad (76a)$$

$$\llbracket \tilde{\chi}(z,p) \rrbracket = 0, \quad z \in \mathcal{Z}_{\mathcal{Z}}, \quad (76b)$$

Uniqueness condition

$$\tilde{\chi}(0,p) = 0, \quad (76c)$$

Initial condition

$$\tilde{\chi}(z,0) = \tilde{\chi}_e(z), \quad z \in \mathcal{Z}, \quad (76d)$$

and

$$\frac{1}{L_c^2} \left[\check{\mathbb{R}}(y,p) \frac{\partial \chi(y,p)}{\partial y} + \check{\mathbb{R}}(y,p) \right] = \hat{\mathbb{R}}(p), \quad y \in \mathcal{Y} \setminus \mathcal{Y}_{\mathcal{Y}}, \quad (77a)$$

$$\llbracket \chi(y,p) \rrbracket = 0, \quad y \in \mathcal{Y}_{\mathcal{Y}}, \quad (77b)$$

Uniqueness condition

$$\chi(0,p) = 0, \quad (77c)$$

Initial condition

$$\chi(y,0) = \chi_e(y), \quad y \in \mathcal{Y}, \quad (77d)$$

where $p \in [0, +\infty[$ and the effective coefficients $\check{\mathbb{R}}$ and $\hat{\mathbb{R}}$ can be recast in the form

$$\check{\mathbb{R}}(p) = \langle \mathbb{R}^{-1}(z,p) \rangle_z^{-1} = \frac{\mathbb{R}_1(p)\mathbb{R}_2}{V_f^{(\mathcal{Z}_2)}\mathbb{R}_1(p) + V_f^{(\mathcal{Z}_1)}\mathbb{R}_2}, \quad (78a)$$

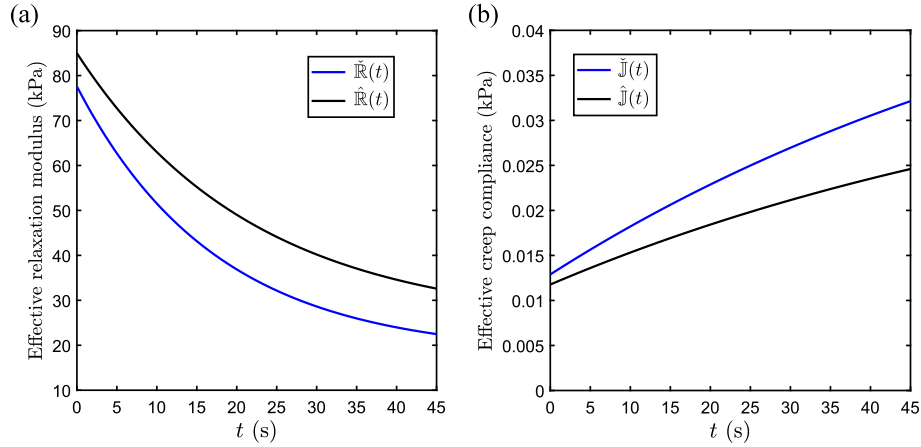


Fig. 7. Comparison of the effective relaxation moduli (a) and the effective creep compliances (b) in the time domain.

$$\hat{\mathbb{R}}(p) = \langle \check{\mathbb{R}}^{-1}(y, p) \rangle_y^{-1} = \frac{\check{\mathbb{R}}(p)\mathbb{R}_3}{V_f^{(\mathcal{Y}2)}\hat{\mathbb{R}}(p) + V_f^{(\mathcal{Y}1)}\mathbb{R}_3}. \quad (78b)$$

We note that expressions (78a) and (78b) have a similar structure, with the only difference being related to the hierarchical level and the corresponding relaxation function we use in each of them. So, it shows the pattern also observed when considering the recursive or double homogenisation performed by several authors when obtaining the macroscopic properties of such kind of composites (see, for instance, Bensoussan et al. (1978), Allaire and Briane (1996), Auriault et al. (2009) and Guinovart-Díaz et al. (2005)). However, as it will be clear later on, the three-scale asymptotic approach adopted in this work presents some advantages with respect to the recursive homogenisation scheme.

Finally, as it was done in the case of the heterogeneous problem (see Section 5.1), the analytical solutions in the Laplace–Carson domain for the local problems (76a)–(76d) and (77a)–(77d) can be proposed as follows

$$\check{\chi}^{(1/3,1/9)}(z, p_j) = w_{0j}^{(z)}(z)\mathbb{1}_{]0,z_r[} + w_{1j}^{(z)}(z)\mathbb{1}_{]z_r,1[}, \quad (79a)$$

$$\chi^{(1/3,1/9)}(y, p_j) = w_{0j}^{(y)}(y)\mathbb{1}_{]0,y_r[} + w_{1j}^{(y)}(y)\mathbb{1}_{]y_r,1[}, \quad (79b)$$

where $w_{ij}^{(\alpha)}$ with $\alpha = y, z$ and $i = 0, 1$ are linear functions in the form $w_{ij}^{(\alpha)}(\alpha) = m_i^{(\alpha)}\alpha + n_i^{(\alpha)}$, similar as in Eq. (68). In particular, the unknowns $m_i^{(\alpha)}, n_i^{(\alpha)}$ with $\alpha = y, z$ and $i = 0, 1$ are found by solving the linear systems of equations (see Appendix C for more details)

$$\mathcal{M}^{(z)} \cdot z_{\text{incog}} = \mathcal{B}^{(z)}, \quad (80)$$

and

$$\mathcal{M}^{(y)} \cdot y_{\text{incog}} = \mathcal{B}^{(y)}, \quad (81)$$

where $\mathcal{M}^{(z)}, \mathcal{M}^{(y)}, z_{\text{incog}}, y_{\text{incog}}, \mathcal{B}^{(z)}, \mathcal{B}^{(y)}$ are defined in (C.3) and (C.6). Once again, $n_0^{(z)} = 0$ and $n_0^{(y)} = 0$ due to (76c) and (77c), respectively. Hence, $n_0^{(z)} \notin z_{\text{incog}}$ and $n_0^{(y)} \notin y_{\text{incog}}$.

6. Numerical results

6.1. Effective coefficients and local solutions

In Fig. 7(a), we show the plots of the effective relaxation modulus at the ε_1 -structural level (see Eq. (78a)) and at the macro-scale (see Eq. (78b)) in the time domain. In addition, in Fig. 7(b), we present the equivalent effective creep compliance at the ε_1 -structural level and at the macro-scale by means of the relations given in Section 4.3. As shown in Fig. 7(a), both effective relaxation moduli decrease with time, with $\hat{\mathbb{R}}$ having higher values. On the contrary, in Fig. 7(b), the effective

creep compliances increase with time and, because of the inverse relation in Eq. (59), the effective creep compliance at the macro-scale exhibits lower values.

In particular, the solutions of the local problems (76a)–(76d) and (77a)–(77d), that is $\check{\chi}^{(1/3,1/9)}$ and $\chi^{(1/3,1/9)}$, are shown in Fig. 8 for different instants of time. Note that the plots (a) and (c) show the behaviour of the local functions in the corresponding periodic cells, whereas graphs (b) and (d) extend this behaviour periodically to the whole domain. Additionally, we notice that the periodic cells in plots (b) and (d) are repeated according to the values $\varepsilon_2 = 1/9$ and $\varepsilon_1 = 1/3$ and using Eq. (17).

6.2. Comparison between different approaches

In this section, we use our previous results to compare the solutions given by (i) the original heterogeneous problem (see Eq. (68)); (ii) the truncated formal asymptotic expansion (see Eq. (72)); and (iii) the truncated formal asymptotic solution that arises from a two-scale analysis (see for instance Cruz-González et al., 2018) given by

$$u_{\text{TwoScale}}^{(\varepsilon_1)}(x, p) = u^{(0)}(x, p) + \chi^{(\varepsilon_1)}(y, p) \frac{\partial u^{(0)}(x, p)}{\partial x} \varepsilon_1, \quad (82)$$

where $\chi^{\varepsilon_1}(y, p)$ is the solution of the local problem which results from a two scale approach (see, for instance, Bakhvalov and Panasenko (1989)). In our case, this would be equivalent to the local problem (77a)–(77d). In particular, for $\varepsilon_1 = 1/3$ the expression (82) reads

$$u_{\text{TwoScale}}^{(1/3)}(x, p) = u^{(0)}(x, p) + \frac{1}{3} \chi^{(1/3)}(y, p) \frac{\partial u^{(0)}(x, p)}{\partial x}. \quad (83)$$

We remark that, if we would like to use a two-scale approach for the aforementioned hierarchical structure, we would need to know the value of the effective coefficient $\check{\mathbb{R}}(t)$. This is because with a two scale framework the ε_2 -structural level cannot be taken into account explicitly (see (82)). In this situation, we can circumvent this downside by setting $\check{\mathbb{R}}(t)$ as done, for example, in Rodríguez-Ramos et al. (2020). Proceeding in this way, we are able to implicitly account the information of the lower scale of the hierarchical structure. Then, we can find the effective coefficient $\hat{\mathbb{R}}(t)$ given in Eq. (78b).

In Fig. 9, we compare, for the fixed value of time $t = 45$ s, the solution of the heterogeneous problem (68), the solution of the homogenised problem (71), and the truncated formal asymptotic solutions given by Eqs. (72) and (82). In particular, it can be noticed that the truncated formal asymptotic expansions are in good agreement with the solution of the heterogeneous problem, being the truncated three-scale asymptotic expansion the nearest to the solution of the original problem. Moreover, we point out that the solution of the homogenised problem, as is trivial from its nature, does not incorporate the spatial variations present in the other solutions. In order to provide further

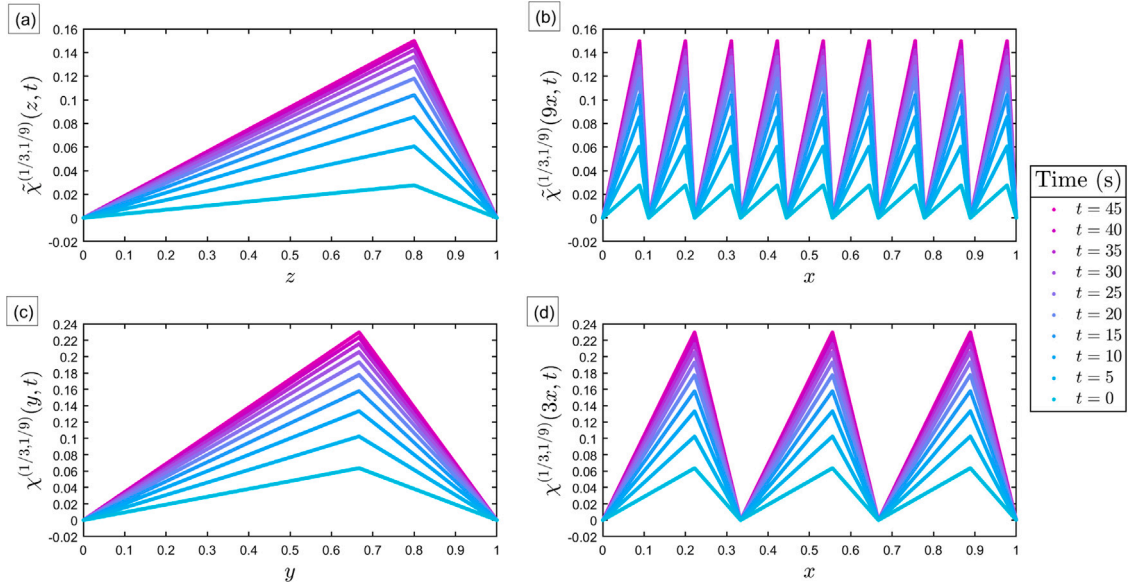


Fig. 8. Graphs (a)–(b) show the spatial distribution of the local function $\tilde{\chi}^{(1/3,1/9)}$ given in (79a), whereas graphs (c)–(d), show the local function $\chi^{(1/3,1/9)}$ as described in (79b). The results are given in the time domain.

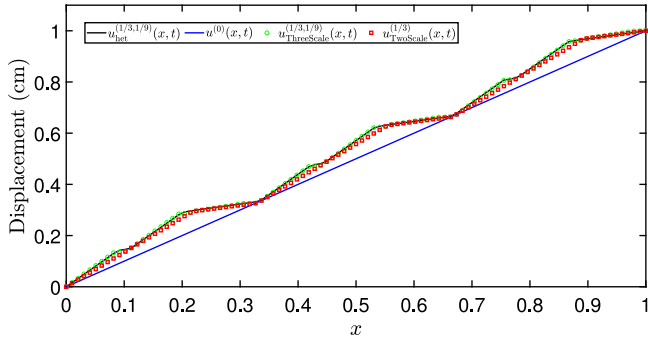


Fig. 9. Comparison of the different approaches studied in this work. The plots corresponds to the fixed value of time $t = 45$ s.

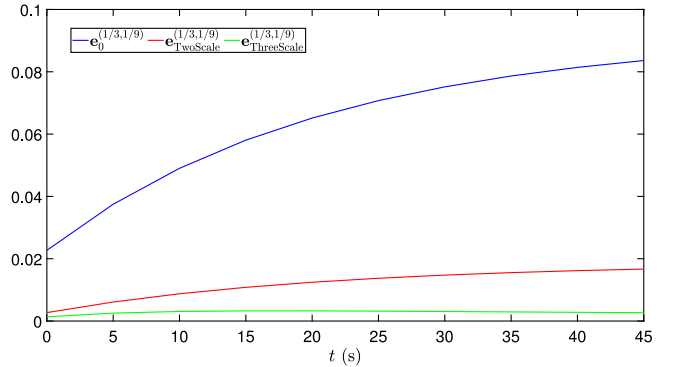


Fig. 10. Comparison of the relative numerical errors given in Eqs. (84a), (84b) and (84c).

evidences on the closeness of the asymptotic solutions to the one from the heterogeneous problem, we introduce the relative numerical errors in L^2 -norm as follows,

$$e_0^{(1/3,1/9)}(t) = \frac{\|u_{het}^{(1/3,1/9)}(x,t) - u^{(0)}(x,t)\|_{L^2}}{\|u_{het}^{(1/3,1/9)}(x,t)\|_{L^2}}, \quad (84a)$$

$$e_{TwoScale}^{(1/3,1/9)}(t) = \frac{\|u_{het}^{(1/3,1/9)}(x,t) - u_{TwoScale}^{(1/3)}(x,t)\|_{L^2}}{\|u_{het}^{(1/3,1/9)}(x,t)\|_{L^2}}, \quad (84b)$$

$$e_{ThreeScale}^{(1/3,1/9)}(t) = \frac{\|u_{het}^{(1/3,1/9)}(x,t) - u_{ThreeScale}^{(1/3,1/9)}(x,t)\|_{L^2}}{\|u_{het}^{(1/3,1/9)}(x,t)\|_{L^2}}. \quad (84c)$$

In Fig. 10, we show the relative numerical errors in L^2 -norm determined by Eqs. (84a), (84b) and (84c). As it can be observed, the relative error concerning the three-scale asymptotic solution with respect to the solution of the original problem is the lowest, which confirms the discussions made so far and the advantage of the three-scale approach with respect to the methods analysed in this work. Therefore, this result illustrates the major accuracy of the three-scale technique to model hierarchical structures over the two-scale one.

Finally, we address the convergence of the solutions. With this aim, we explore the behaviour of the asymptotic formal solutions and the solution of the heterogeneous problem when the scaling parameters ε_1 and ε_2 become smaller. That is, we can extend the procedure in previous sections for different values of ε_1 and ε_2 . Here, we consider three new sets of parameters, namely (i) $\varepsilon_1 = 1/4$, $\varepsilon_2 = 1/16$, (ii) $\varepsilon_1 = 1/6$, $\varepsilon_2 = 1/36$, and (iii) $\varepsilon_1 = 1/9$, $\varepsilon_2 = 1/81$. The results of the comparisons are displayed in Fig. 11, wherein the behaviour of the curves is similar to the one shown in Figs. 9 and 10. However, we have now more information on the convergence behaviour of the results. As observed in Fig. 11 the asymptotic formal solutions and the solution of the heterogeneous problem approach to the solution of the homogenised problem when the parameters become smaller (which is what it is expected).

7. Conclusions

In the present work, we considered the three-scale asymptotic homogenisation technique proposed in Ramírez-Torres et al. (2018a) to model the overall response of a viscoelastic and hierarchical composite material in a one-dimensional framework. In particular, the methodology is fully addressed by deriving the local and homogenised problems, and the effective coefficients for the two levels of organisation examined in this work.

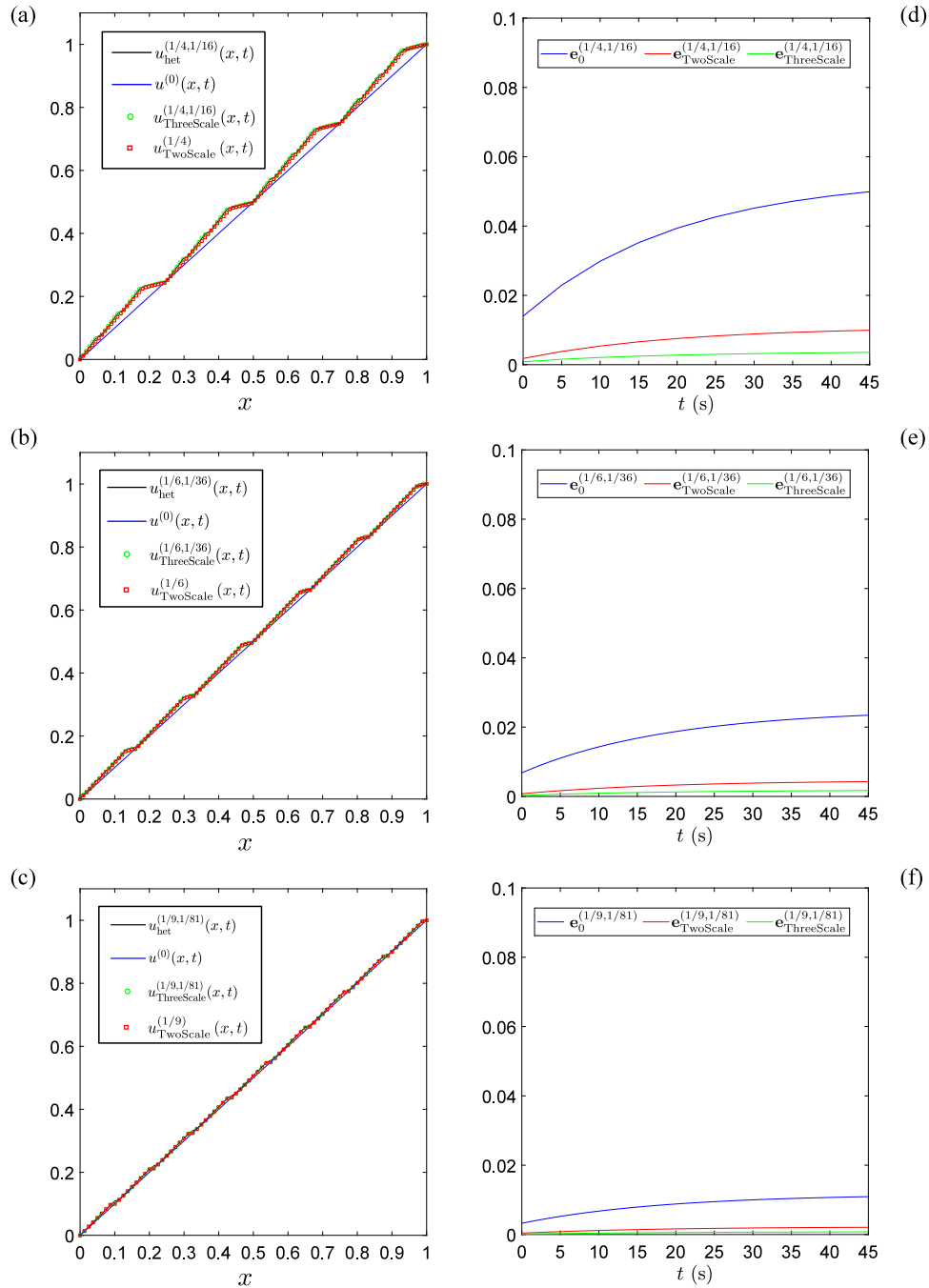


Fig. 11. (a)–(c) Comparison of the different approaches for three new frameworks and the fixed value of time $t = 45$ s. (d)–(f) Relative error in L^2 -norm for the three frameworks.

To demonstrate the advantage of this three-scale approach, we computed a semi-analytical solution for the corresponding heterogeneous problem and compare it with the one resulting from the truncation of the formal three-scale asymptotic expansion used here. In doing this, we needed to compute the solutions of the cell problems that can be found using a similar semi-analytical procedure to the one employed to solve the original, heterogeneous problem. Furthermore, we also compared our results with the solution resulting from the homogenised problem. Our numerical simulations confirmed that, under this hierarchical setting, the three-scale approach offers a better agreement with the solution of the heterogeneous problem with respect to the homogenised and two-scale ones. Furthermore, we performed some additional calculations to analyse the convergence of the solutions, and our findings showed a trend towards the solution of the homogenised problem when the scaling parameters approaches to zero.

We note that if we were to compute the errors on the stresses as we did with the displacement, we would have to consider additional terms in the expansion (72). Extending these computations to the three-scale case would involve several changes in our manuscript. For this reason, we preferred not to do it this time and we will consider it in our future research.

We further mention that even though we concentrated on a one-dimensional setting, our theoretical results can be immediately generalised to account for more complex, three dimensional composite materials. In this respect, future developments of the present work are aimed to consider, e.g., the overall behaviour of hierarchical biological tissues such as bones (Ramírez-Torres et al., 2018b). Viscoelastic effects in this context are well-documented (Yamashita et al., 2002) and can

play a crucial role because of their connections with specific physiological conditions (Eberhardsteiner et al., 2014), as well as ageing (Chen et al., 2020).

CRedit authorship contribution statement

O.L. Cruz-González: Drafted, Reviewed and edited the manuscript, Performed investigations, Implementation. **A. Ramírez-Torres:** Reviewed and edited the manuscript, Conceptualisation. **R. Rodríguez-Ramos:** Supervised, Reviewed and edited the manuscript, Conceptualisation. **R. Penta:** Reviewed and edited the manuscript, Conceptualisation. **F. Lebon:** Supervised, Reviewed and edited the manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Initial condition in the homogenisation process of viscoelastic materials

For simplicity in our analysis, we consider the classical two-scale case. However, the ideas discussed here can be adapted with minor modifications to the three-scale approach. Then, let us consider the that the initial condition in (18d), $u_e^{(\epsilon)}$, is solution of the linear elastic problem

$$-\frac{\partial \sigma_e^{(\epsilon)}(x)}{\partial x} = f(x), \quad x \in \mathcal{B} \setminus \mathcal{J}^{(\epsilon_1)}, \quad (A.1a)$$

$$\left[\left[u_e^{(\epsilon)}(x) \right] \right] = 0, \quad \left[\left[\sigma_e^{(\epsilon)}(x) \right] \right] = 0, \quad x \in \mathcal{J}^{(\epsilon_1)}, \quad (A.1b)$$

$$\sigma_e^{(\epsilon)}(x) = \mathbb{R}^{(\epsilon)}(x, 0) \frac{\partial u_e^{(\epsilon)}(x)}{\partial x} = \mathbb{C}^{(\epsilon)}(x) \frac{\partial u_e^{(\epsilon)}(x)}{\partial x}, \quad (A.1c)$$

which has to be supplemented with appropriate boundary conditions.

Considering the classical procedure in two-scale AHM (see, for instance, Bakhvalov and Panasenko (1989), Bensoussan et al. (1978) and Cioranescu and Donato (1999)), the formal asymptotic solution for the elastic problem (A.1a)–(A.1c), truncated up to first order of ϵ_1 , is given by the expression,

$$\begin{aligned} u_e^{(\epsilon)}(x) &= u_e^{(0)}(x, y) + u_e^{(1)}(x, y)\epsilon_1 \\ &= u_e^{(0)}(x) + \chi_e(x, y) \frac{\partial u_e^{(0)}(x)}{\partial x} \epsilon_1. \end{aligned} \quad (A.2)$$

where $u_e^{(0)}$ and χ_e are, respectively, the solutions of the homogenised and local problem. Thus, using a two-scale asymptotic expansion for $u^{(\epsilon)}(x, 0)$ and Eq. (18d), we can deduce that

$$u^{(0)}(x, y, 0) + u^{(1)}(x, y, 0)\epsilon_1 = u_e^{(0)}(x, y) + u_e^{(1)}(x, y)\epsilon_1, \quad (A.3)$$

which leads to the following identifications

$$u^{(0)}(x, y, 0) = u_e^{(0)}(x, y), \quad (A.4a)$$

$$u^{(1)}(x, y, 0) = u_e^{(1)}(x, y), \quad (A.4b)$$

$$\chi(x, y, 0) = \chi_e(x, y). \quad (A.4c)$$

Therefore, from expressions (A.4a)–(A.4c), we can conclude that the initial conditions at $t = 0$ s for all the problems involved in the homogenisation process of the non-ageing linear viscoelastic problem are given by corresponding instantaneous elastic counterpart.

Appendix B. Solution of the heterogeneous problem

Here, we adapt the method shown in Álvarez-Borges et al. (2014) to our case and outline the main ideas leading to the system of Eqs. (69). The solution of the heterogeneous problem (67a)–(67d) can be found, in a first step, by determining the unknowns quantities m_i, n_i with $i = 0, \dots, 14$, involved in the construction of the proposed solution (68). In particular, the first expression appearing in Eq. (67b) leads to

$$\mathcal{J}_d(1)m_0 + n_0 = \mathcal{J}_d(1)m_1 + n_1, \quad (B.1a)$$

$$\mathcal{J}_d(2)m_1 + n_1 = \mathcal{J}_d(2)m_2 + n_2, \quad (B.1b)$$

$$\mathcal{J}_d(3)m_2 + n_2 = \mathcal{J}_d(3)m_3 + n_3, \quad (B.1c)$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathcal{J}_d(14)m_{13} + n_{13} = \mathcal{J}_d(14)m_{14} + n_{14}, \quad (B.1d)$$

and the second expression in (67b) yields

$$\mathbb{R}_1(p_j)m_0 = \mathbb{R}_2m_1, \quad (B.2a)$$

$$\mathbb{R}_2m_1 = \mathbb{R}_1(p_j)m_2, \quad (B.2b)$$

$$\mathbb{R}_1(p_j)m_2 = \mathbb{R}_2m_3, \quad (B.2c)$$

$$\mathbb{R}_2m_3 = \mathbb{R}_3m_4, \quad (B.2d)$$

$$\mathbb{R}_3m_4 = \mathbb{R}_1(p_j)m_5, \quad (B.2e)$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbb{R}_2m_{13} = \mathbb{R}_3m_{14}. \quad (B.2f)$$

Moreover, from Eq. (67c) we have that

$$n_0 = 0, \quad (B.3a)$$

$$m_{14} + n_{14} = 1. \quad (B.3b)$$

Eqs. (B.1a)–(B.3b) can be rewritten as the system of linear equations

$$\mathcal{M} \cdot \mathcal{X}_{\text{incog}} = \mathcal{B}. \quad (B.4)$$

Finally, the structure of each element of (B.4) is summarised below

$$[\mathcal{M}]_{29 \times 29} = \begin{bmatrix} \mathcal{M}_1 & \mathcal{N}_1 \\ \mathcal{M}_2 & \mathcal{N}_2 \\ \mathcal{V}_1 & \mathcal{V}_2 \end{bmatrix}, \quad [\mathcal{X}_{\text{incog}}]_{29 \times 1} = \begin{bmatrix} m_0 \\ \vdots \\ m_{14} \\ n_1 \\ \vdots \\ n_{14} \end{bmatrix}, \quad [\mathcal{B}]_{29 \times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad (B.5)$$

where $[\mathcal{M}_1]_{14 \times 15}, [\mathcal{N}_1]_{14 \times 14}, [\mathcal{M}_2]_{14 \times 15}, [\mathcal{N}_2]_{14 \times 14}, [\mathcal{V}_1]_{1 \times 15}, [\mathcal{V}_2]_{1 \times 14}, \mathcal{M}_z, \mathcal{M}_z^*$ are given in Box I.

Appendix C. Solution of the local problems

The solutions of the local problems follow from a similar analysis as the one presented above. So, we need to determine the unknowns $m_i^{(\alpha)}, n_i^{(\alpha)}$, with $\alpha = y, z$ and $i = 0, 1$, involved in the construction of the proposed solutions (79a) and (79b) for the local problems (76a)–(76d)

$$[\mathcal{M}_1]_{14 \times 15} = \begin{bmatrix} \mathcal{F}_d(1) & -\mathcal{F}_d(1) & & & \mathbf{0} \\ & \ddots & & & \\ & & \ddots & & \\ \mathbf{0} & & & \mathcal{F}_d(14) & -\mathcal{F}_d(14) \end{bmatrix}, \quad [\mathcal{N}_1]_{14 \times 14} = \begin{bmatrix} -1 & & & & \mathbf{0} \\ 1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ \mathbf{0} & & & & 1 & -1 \end{bmatrix}, \quad (\text{B.6})$$

$$[\mathcal{M}_2]_{14 \times 15} = \begin{bmatrix} \mathcal{M}_z(1) & 0 & & & & & & & & & & & & & \mathbf{0} \\ & \ddots & & & & & & & & & & & & & \\ & & \mathcal{M}_z(4) & & & & & & & & & & & & \\ & & & \mathbb{R}_3 & & & & & & & & & & & \\ & & & & \mathcal{M}_z(1) & & & & & & & & & & \\ & & & & & \ddots & & & & & & & & & \\ & & & & & & \mathcal{M}_z(4) & & & & & & & & \\ & & & & & & & \mathbb{R}_3 & & & & & & & \\ & & & & & & & & \mathcal{M}_z(1) & & & & & & \\ & & & & & & & & & \ddots & & & & & \\ & & & & & & & & & & \mathcal{M}_z(4) & & & & \\ \mathbf{0} & & & & & & & & & & & \mathcal{M}_z(4) & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathcal{M}_z^*(1) & & & & & & & & & & & & & \mathbf{0} \\ & \ddots & & & & & & & & & & & & & \\ & & -\mathcal{M}_z^*(3) & & & & & & & & & & & & \\ & & & -\mathbb{R}_3 & & & & & & & & & & & \\ & & & & -\mathcal{M}_z(1) & & & & & & & & & & \\ & & & & & \ddots & & & & & & & & & \\ & & & & & & -\mathcal{M}_z(4) & & & & & & & & \\ & & & & & & & -\mathbb{R}_3 & & & & & & & \\ & & & & & & & & -\mathcal{M}_z(1) & & & & & & \\ & & & & & & & & & \ddots & & & & & \\ & & & & & & & & & & -\mathcal{M}_z(4) & & & & \\ & & & & & & & & & & & 0 & & & -\mathbb{R}_3 \end{bmatrix}, \quad (\text{B.7})$$

$$[\mathcal{N}_2]_{14 \times 14} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}, \quad [\mathcal{V}_1]_{1 \times 15} = [0 \quad \dots \quad 1], \quad [\mathcal{V}_2]_{1 \times 14} = [0 \quad \dots \quad 1], \quad (\text{B.8})$$

$$\mathcal{M}_z = [\mathbb{R}_1(s_j) \quad \mathbb{R}_2 \quad \mathbb{R}_1(s_j) \quad \mathbb{R}_2], \quad \mathcal{M}_z^* = [\mathbb{R}_2 \quad \mathbb{R}_1(s_j) \quad \mathbb{R}_2]. \quad (\text{B.9})$$

Box I.

and (77a)–(77d), respectively. In particular, from Eqs. (76a)–(76c), we obtain the following system of linear equations

$$z_I m_0^{(z)} = z_I m_1^{(z)} + n_1^{(z)}, \quad (\text{C.1a})$$

$$\mathbb{R}_1(p_j) m_0^{(z)} + \mathbb{R}_1(p_j) = \check{\mathbb{R}}(p_j), \quad (\text{C.1b})$$

$$\mathbb{R}_2 m_1^{(z)} + \mathbb{R}_2 = \check{\mathbb{R}}(p_j), \quad (\text{C.1c})$$

which can be rewritten as

$$\mathcal{M}^{(z)} \cdot \tilde{z}_{\text{incog}} = \mathcal{B}^{(z)}, \quad (\text{C.2})$$

where

$$\mathcal{M}^{(z)} = \begin{bmatrix} z_I & -z_I & -1 \\ \mathbb{R}_1(p_j) & 0 & 0 \\ 0 & \mathbb{R}_2 & 0 \end{bmatrix}, \quad \tilde{z}_{\text{incog}} = \begin{bmatrix} m_0^{(z)} \\ m_1^{(z)} \\ n_1^{(z)} \end{bmatrix}, \quad (\text{C.3})$$

$$\mathcal{B}^{(z)} = \begin{bmatrix} 0 \\ \check{\mathbb{R}}(p_j) - \mathbb{R}_1(p_j) \\ \check{\mathbb{R}}(p_j) - \mathbb{R}_2 \end{bmatrix}.$$

Similarly, from Eqs. (77a)–(77c), we obtain

$$y_I m_0^{(y)} = y_I m_1^{(y)} + n_1^{(y)}, \quad (\text{C.4a})$$

$$\check{\mathbb{R}}(p_j) m_0^{(y)} + \check{\mathbb{R}}(p_j) = \hat{\mathbb{R}}(p_j), \quad (\text{C.4b})$$

$$\mathbb{R}_3 m_1^{(y)} + \mathbb{R}_3 = \hat{\mathbb{R}}(p_j). \quad (\text{C.4c})$$

The above system can be rewritten as

$$\mathcal{M}^{(y)} \cdot \mathcal{Y}_{\text{incog}} = \mathcal{B}^{(y)}, \quad (\text{C.5})$$

where

$$\mathcal{M}^{(y)} = \begin{bmatrix} y_I & -y_I & -1 \\ \check{\mathbb{R}}(p_j) & 0 & 0 \\ 0 & \mathbb{R}_3 & 0 \end{bmatrix}, \quad \mathcal{Y}_{\text{incog}} = \begin{bmatrix} m_0^{(y)} \\ m_1^{(y)} \\ n_1^{(y)} \end{bmatrix}, \quad \mathcal{B}^{(y)} = \begin{bmatrix} 0 \\ \hat{\mathbb{R}}(p_j) - \check{\mathbb{R}}(p_j) \\ \hat{\mathbb{R}}(p_j) - \mathbb{R}_3 \end{bmatrix}. \quad (\text{C.6})$$

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