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# Moduli of Lie $p$ -algebras

Alice Bouillet

**Abstract.** In this paper, we study moduli spaces of finite-dimensional Lie algebras with flat center, proving that the forgetful map from Lie  $p$ -algebras to Lie algebras is an affine fibration, and we point out a new case of existence of a  $p$ -mapping. Then we illustrate these results for the special case of Lie algebras of rank 3, whose moduli space we build and study over  $\mathbb{Z}$ . We extend the classical equivalence of categories between locally free Lie  $p$ -algebras of finite rank with finite locally free group schemes of height 1, showing that the centers of these objects correspond to each other. We finish by analysing the smoothness of the moduli of  $p$ -Lie algebras of rank 3, in particular identifying some smooth components.

## 1 Introduction

When does a finite-dimensional Lie algebra  $\mathfrak{g}$  occur as the Lie algebra of an algebraic group  $G$ , i.e. when is  $\mathfrak{g}$  *algebraic*? A necessary condition for this is that whenever  $\mathfrak{g}$  is embedded as a Lie sub-algebra of some  $\mathfrak{gl}_V$ , the semi-simple and nilpotent parts of the elements  $\text{ad}_x : V \rightarrow V$ ,  $x \in \mathfrak{g}$  belong to  $\mathfrak{g}$ . In this case,  $\mathfrak{g}$  is called *semi-algebraic*. Another necessary condition found by N. Jacobson which is typical to the characteristic  $p$  is that there should exist on  $\mathfrak{g}$  a  $p$ -mapping. In this case,  $\mathfrak{g}$  is called *restrictable*. None of these conditions are sufficient, but at least the datum of a  $p$ -mapping allows us, should  $G$  exist, to reconstruct its Frobenius kernel, which is a finite infinitesimal group scheme (and the converse holds). For the study of general finite group schemes and (smooth) algebraic groups, such group schemes are of fundamental importance.

In this article we study the modular aspects of  $p$ -mappings on finite-dimensional Lie algebras, or equivalently, of finite locally free group schemes killed by Frobenius. Our work can be divided in two parts : in the first half of the paper we study the theoretical aspects, and in the second half we study in detail the three-dimensional case.

In the first part we study Lie algebra vector bundles over a scheme  $S$ , and when  $\text{char}(S) = p > 0$ , we study the  $p$ -mappings on it. The difference of two  $p$ -mappings on  $L$  takes its values in the center  $Z(L)$ , which for this reason plays a key role. Our first main result proved in Theorem 4.1.4 and Theorem 4.1.8 is obtained after restriction to the flattening stratification  $S^* \rightarrow S$  of the center, and is stated as follows.

**Theorem A.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle. Let us define the functor  $X = X(L)$  of the  $p$ -mappings on  $L$ , i.e.  $X(T) = \{p\text{-mappings on } L \times_S T\}$  for all  $S$ -schemes  $T$ . Let  $\text{Frob} : S \rightarrow S$  be the Frobenius morphism. Then,  $X$  is representable by an affine scheme, and is a formally homogeneous space under  $E := \text{Hom}(\text{Frob}_S^* L, Z(L))$ .*

*Now let us define the restrictable locus of  $L$  as follows:*

$$S^{\text{res}} = S^{\text{res}}(L) : \{S\text{-schemes}\} \longrightarrow \text{Set}$$
$$T \longmapsto \begin{cases} \{\emptyset\} & \text{if } L_T \text{ is restrictable over } T \\ \emptyset & \text{otherwise.} \end{cases}$$

*Then if we suppose  $Z(L) \rightarrow S$  flat, the following two conditions are verified:*

1.  $S^{\text{res}}$  is representable by a closed subscheme of  $S$ .
2.  $X \rightarrow S$  factors through  $S^{\text{res}}$  and  $X \rightarrow S^{\text{res}}$  is an affine space under the vector bundle  $E \times_S S^{\text{res}}$ .

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It follows in particular that if  $Z(L)$  is flat over  $S$ , then  $X \rightarrow S^{\text{res}}$  is smooth. An interesting question is to know whether this affine space fibration has global sections. In general this is not the case. We provide a positive result under a condition on the derived Lie algebra  $L' \subset L$ . Namely, we move to the next dimension after the case  $L' = 0$  (where the zero map is a  $p$ -mapping). See Theorem 4.2.1.

**Theorem B.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle, such that  $L'$  is a locally free subbundle of rank 1. We define a map of vector bundles as follows:*

$$\begin{aligned} \alpha : L &\rightarrow \text{End}(L') \simeq \mathbb{G}_a \\ x &\mapsto (\text{ad}(x)|_{L'}) \mapsto \alpha(x). \end{aligned}$$

*Then, the map  $L \rightarrow L$ ,  $x \mapsto \alpha(x)^{p-1}x$  is a  $p$ -mapping on  $L$ . Moreover if  $L$  is free of rank 2, this  $p$ -mapping is unique.*

In the rest of the article, we will put our interest on the moduli stack  $\text{Lie}_n$  of  $n$ -dimensional Lie algebras, and especially on the case  $n = 3$ . For this, we will introduce the moduli space  $L_n$  of *based* Lie algebras locally free of rank  $n$ , with the natural action of  $\text{GL}_n$  on it, by change of basis. We can see that we have the quotient stack presentation  $\text{Lie}_n = [L_n / \text{GL}_n]$ , so we are led to study the orbits of the action of  $\text{GL}_n$ . You can find the classification on those isomorphism classes in Fulton and Harris' book [FH91], but in order to apply our theoretical results and to allow varying primes  $p$ , we reformulate in Subsection 5.1 the classification of 3-dimensional Lie algebras over algebraically closed fields in a characteristic-free way, giving representatives of the isomorphism classes defined over  $\mathbb{Z}$  and  $\mathbb{Z}[T]$ . So let  $k$  be an algebraically closed field of characteristic  $p > 0$ , then any Lie algebras of dimension 3 over  $k$  is isomorphic to exactly one in the following table.

Name	Structure	Orbit dimension	Center dimension	Restrictable	
$\mathfrak{ab}_3$	abelian	0	3	yes	
$\mathfrak{h}_3$	nilpotent	3	1	yes	
$\mathfrak{r}$	solvable	5	0	no	
$\mathfrak{s}$	simple	6	0	$p \neq 2$ yes   $p = 2$ no	
$\mathfrak{g}_t$	$t \notin \mathbb{F}_p$	solvable	5	0	no
	$t \in \mathbb{F}_p \setminus \{0, 1\}$	solvable	5	0	yes
	$t = 0$	solvable	5	1	yes
	$t = 1$	solvable	3	0	yes

Afterward in Subsection 5.2 we supplement the known results by giving more precise information on the scheme structure of the moduli space  $L_3$ , that we define over  $\mathbb{Z}$ . For this, we use liaison theory, as developed by Peskine and Szpiro in [PS74]; in fact  $L_3$  turns out to be a typical case of a reducible scheme whose components are linked.

**Theorem C.** 1) *The functor  $L_3$  is representable by an affine flat  $\mathbb{Z}$ -scheme of finite type.*  
 2) *The scheme  $L_3$  has two relative irreducible components  $L_3^{(1)}$  and  $L_3^{(2)}$  which are both flat with Cohen-Macaulay integral geometric fibers of dimension 6.*

For the end, we will come back to our equivalence between height one group schemes and restricted Lie algebras. In Proposition 6.1.2, we extend the classical equivalence of categories between locally free Lie  $p$ -algebras of finite rank with finite locally free group schemes of height 1, showing that the centers of those objects correspond to each other. That is, let us denote by  $\text{Lie}_n^p(S)$  the category of  $n$ -dimensional restricted  $\mathcal{O}_S$ -Lie algebras, and for  $r \leq n$ , let us denote by  $\text{Lie}_{n,r}^p(S)$  those algebras whose center is locally free of rank  $r$ . With the same idea, let us denote by  $G_n^p(S)$  the category of finite locally free  $S$ -group schemes of order  $p^n$ , of height 1, and for  $r \leq n$  let us denote by  $G_{n,r}^p(S)$  the category of those group schemes whose center is locally free of rank  $p^r$ .

**Theorem D.** *Let  $S$  be a scheme of characteristic  $p > 0$  and let  $G \rightarrow S$  be a finite locally free group scheme of height 1. Let  $Z(G)$  denote its center. Then*

$$Z(\mathrm{Lie}(G)) = \mathrm{Lie}(Z(G)).$$

*Then the classical equivalence of categories*

$$\mathrm{Lie} : G_n^p(S) \xrightarrow{\sim} \mathrm{Lie}_n^p(S)$$

*can be extended to this one*

$$\mathrm{Lie} : G_{n,r}^p(S) \xrightarrow{\sim} \mathrm{Lie}_{n,r}^p(S).$$

So using this, we can focus on the object  $\mathrm{Lie}_{n,r}^p(S)$ , and because we have the quotient stack presentation  $\mathrm{Lie}_n = [\mathbb{L}_n/\mathrm{GL}_n]$ , we can focus on  $\mathbb{L}_n$ , and especially on  $\mathbb{L}_n^{\mathrm{res}}$  the locally closed subscheme of  $\mathbb{L}_n$  where the universal Lie algebra  $\mathbb{L}_n \rightarrow \mathbb{L}_n$  is restrictable. In particular, if  $k$  is an algebraically closed field of characteristic  $p > 0$ , Theorem D and the previous results allow us to count the centerless finite locally free  $k$ -group schemes of order  $p^3$ , of height 1. This number is finite, equal to 1 if  $p = 2$  and  $(p+3)/2$  if  $p \neq 2$  (See Proposition 6.1.3).

For the end, in the subsections 6.2, 6.3 and 6.4, we study the smoothness of the restrictable locus  $\mathbb{L}_3^{\mathrm{res}} \subset \mathbb{L}_3$  of  $\mathbb{L}_3$  in the different flattening strata of the center. For a better understanding of the following theorem, the reader can look at the pictures of Subsection 5.3.

**Theorem E.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $\mathbb{L}_{3,r}^{\mathrm{res}} \rightarrow \mathrm{Spec}(k)$  be the locally closed subscheme of  $\mathbb{L}_3$  where the center  $Z(\mathbb{L}_3)$  is locally free of rank  $r$ , and  $\mathbb{L}_3$  is restrictable.*

1. (i) *If  $p \neq 2$ , the singular locus of  $\mathbb{L}_{3,0}^{\mathrm{res}}$  is the orbit of  $\mathfrak{g}_{-1}$ . The singularity remains after intersection with  $\mathbb{L}_3^{(1)}$  but  $\mathbb{L}_{3,0}^{\mathrm{res}} \cap \mathbb{L}_3^{(2)}$  is smooth.*  
(ii) *If  $p = 2$ , the scheme  $\mathbb{L}_{3,0}^{\mathrm{res}}$  is smooth and remains smooth after intersection with any irreducible component.*
2. *The singular locus of  $\mathbb{L}_{3,1}^{\mathrm{res}}$  is the orbit of  $\mathfrak{h}_3$ . The singularity remains after intersection with  $\mathbb{L}_3^{(2)}$  but  $\mathbb{L}_{3,1}^{\mathrm{res}} \cap \mathbb{L}_3^{(1)}$  is smooth.*
3. *The scheme  $\mathbb{L}_{3,2}^{\mathrm{res}}$  is empty.*
4. *The scheme  $\mathbb{L}_{3,3}^{\mathrm{res}}$  is smooth and remains smooth after intersection with any irreducible component.*

It is well known that in Lie algebra theory, the characteristics  $p = 2$  and  $p = 3$  are special. In the previous result we see that the characteristic  $p = 2$  appears as a special case, and the reader can see that the case  $p = 3$  needs special care e.g. in the proof of Theorem 6.3.2.

Thanks to Theorem D, all the assertions of Theorem E hold also for  $G_{3,r}^p(k)$ , i.e.  $G_{3,r}^p(k)$  splits in two irreducible components that we denote by  $G_{3,r}^{p,(1)}$  and  $G_{3,r}^{p,(2)}$ ; and we can say that if  $p \neq 2$ ,  $G_{3,0}^p(k)$  is singular, but becomes smooth if we intersect with  $G_{3,0}^{p,(2)}$ , if  $p \neq 2$  it is smooth. Moreover  $G_{3,1}^p(k)$  is singular but becomes smooth when we intersect it with  $G_{3,1}^{p,(1)}$ ,  $G_{3,2}^p(k)$  is empty and  $G_{3,3}^p(k)$  is smooth. We refer to Corollary 6.4.1 for more details.

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## 2 Definition and theory of Lie $p$ -algebras over a ring

In this section, we recall basic notations and facts on Lie algebras and Lie  $p$ -algebras. We also recall Jacobson's theorems on existence and uniqueness of  $p$ -mappings for some Lie algebras over a commutative ring. The reader can find the proofs for Lie algebras over a field in Strade and Farnsteiner's book on Modular Lie algebras [SF88], and we verify easily that these proofs do not use the fact that the base ring is a field.

The base rings are commutative with unit.

### 2.1 First definitions and notations on Lie algebras

In the following,  $R$  is a base ring and we say that  $l$  is a Lie algebra over  $R$  if  $l$  is an  $R$ -module together with a bilinear alternating map written  $[\cdot, \cdot]_l : l \otimes_R l \rightarrow l$  or simply  $[\cdot, \cdot]$  if it is clear from context, which respects the Jacobi condition. We also denote  $\text{End}(l)$  the module of  $R$ -endomorphisms of  $l$ .

**2.1.1. Definition.** For any  $R$ -algebra  $R'$ , the bilinear map  $[\cdot, \cdot]_l \otimes_R \text{id}_{R'}$  defines a  $R'$ -Lie algebra structure on the  $R'$ -module  $l_{R'} := l \otimes_R R'$ .

**2.1.2. Remark.** If we suppose  $l$  to be locally free of finite rank, we have the usual isomorphism

$$\text{End}(l) \xleftarrow{\sim} l^\vee \otimes l.$$

Moreover, let  $R'$  be an  $R$ -algebra. We can write

$$\text{End}(l \otimes_R R') \simeq (l \otimes_R R')^\vee \otimes_{R'} (l \otimes_R R') \simeq l^\vee \otimes_R l \otimes_R R' \simeq \text{End}(l) \otimes_R R'.$$

Thus in this case, the formation of  $\text{End}$  commutes with base change.

**2.1.3. Definition.** We denote by  $\text{ad}$  the following morphism:

$$\begin{aligned} \text{ad} : l &\longrightarrow \text{End}(l) \\ x &\longmapsto (y \mapsto [x, y]). \end{aligned}$$

Thus the formation of  $\text{ad}$  commutes with base change.

**2.1.4. Definition.** We define the *center* of  $l$  as  $Z(l) := \ker(\text{ad} : l \rightarrow \text{End}(l))$ . The center is an ideal of  $l$ .

The formation of the center does not commute with base change in general, even if  $l$  is free. For example let us consider the Lie algebra  $\mathfrak{sl}_2$  over the ring of the integers. Then  $Z(\mathfrak{sl}_2) = \{0\}$  but  $Z(\mathfrak{sl}_2 \otimes \mathbb{F}_2) = \text{Span}(h)$ .

## 2.2 Some properties of $p$ -mappings and restrictable Lie algebras

In this subsection, we fix a prime  $p > 0$  and all base rings are  $\mathbb{F}_p$ -algebras. Let  $R$  be such a ring. We write  $\text{Frob}$  for its Frobenius endomorphism. Let  $l$  be a Lie algebra on  $R$ .

**2.2.1. Definition.** We say that a mapping  $(\cdot)^{[p]} : l \rightarrow l$  is a  *$p$ -mapping* if:

$$\text{(AL1) for all } x \in l, \text{ad}_{x^{[p]}} = (\text{ad}_x)^p$$

$$\text{(AL2) for all } \lambda \in R \text{ and } x \in l, (\lambda x)^{[p]} = \lambda^p x^{[p]}$$

$$\text{(AL3) for all } x, y \in l, (x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$$

$$\text{where for all } i, s_i(x, y) := -\frac{1}{i} \sum_u \text{ad}_{u(1)} \text{ad}_{u(2)} \dots \text{ad}_{u(p-1)}(y),$$

and  $u$  ranges through the maps from  $\{1, \dots, p-1\}$  to  $\{x, y\}$  taking  $i$  times the value  $x$ .

These three conditions are called *Jacobson's identities*.

For instance, we have

$$s_1(x, y) = -[y, \underbrace{[y, \dots, [x, y]]}_{p-1}] \text{ and } s_{p-1}(x, y) = [x, \underbrace{[x, \dots, [x, y]]}_{p-1}].$$

**2.2.2. Definition.** A Lie algebra equipped with a  $p$ -mapping is called *Lie  $p$ -algebra* or we say that it is *restricted*. If a Lie algebra can be equipped with a  $p$ -mapping, we say that it is *restrictable*.

**2.2.3. Examples.**

1. Let  $A$  be an associative  $R$ -algebra, equipped with the bracket  $[x, y] \mapsto xy - yx$ . Then,  $x \mapsto x^p$  is a  $p$ -mapping.
2. Let  $G \rightarrow \text{Spec}(R)$  be a group scheme. Then  $\text{Lie}(G)$  can be equipped with a natural  $p$ -mapping. You can look at Demazure and Gabriel's book [DG70] Chapitre II, §7, n°2 for more details. For example, if we take  $G = \text{GL}_n$ , the corresponding  $p$ -mapping on the matrix space  $\text{Lie}(\text{GL}_n) = \text{M}_n(R)$  is given by  $x \mapsto x^p$ , and for  $G = \mathbb{G}_a$ , it is given by  $x \mapsto 0$  on  $\text{Lie}(G) = R$ . See [DG70] Chapitre II. §7, n°2, example 2.2.

**2.2.4. Definition.**

1. Let  $(l_1, \gamma_1)$  and  $(l_2, \gamma_2)$  be two Lie  $p$ -algebras and let  $f : l_1 \rightarrow l_2$  be a Lie algebra homomorphism. We say that  $f$  is a  *$p$ -morphism* if for all  $x \in l_1$ ,  $f(\gamma_1(x)) = \gamma_2(f(x))$ .
2. Let  $l$  be a Lie  $p$ -algebra over  $R$  and  $m$  a Lie subalgebra of  $l$ . We say that  $m$  is a  *$p$ -subalgebra* if it is stable by the  $p$ -mapping. If, moreover,  $m$  is an ideal in  $l$ , we say that  $m$  is a  *$p$ -ideal* of  $l$ .

### 2.2.5. Examples.

1. Let  $A$  be an associative  $R$ -algebra. A linear map  $D : A \rightarrow A$  is called a *derivation* if for all  $f, g \in A$ ,  $D(fg) = D(f)g + fD(g)$ . If  $D$  is a derivation of  $A$ , then a simple computation shows that  $D^p$  is again a derivation of  $A$ . Thus, the set of derivations on  $A$ , denoted by  $\text{Der}_R(A)$  is a  $p$ -subalgebra of  $\text{End}_R(A)$  (which is a Lie  $p$ -algebra from Example 2.2.3 1.).
2. Let  $l$  be a Lie  $p$ -algebra over  $R$  with a  $p$ -mapping denoted by  $(\cdot)^{[p]}$ . Then the center  $Z(l)$  is a  $p$ -ideal. Indeed let  $x \in Z(l)$ , and  $y \in l$ , then, using (AL 1), we have  $[x^{[p]}, y] = \text{ad}_x^p(y) = 0$ . Moreover  $Z(l)$  is stable by bracket thanks to the Jacobi condition.

The next proposition shows that we can endow the image (under a Lie algebra morphism) of a Lie  $p$ -algebra with a natural  $p$ -mapping.

**2.2.6. Proposition.** *Let  $(l_1, (\cdot)^{[p]})$  be a Lie  $p$ -algebra over  $R$ . Suppose that  $f : l_1 \rightarrow l_2$  is a Lie algebra morphism such that  $\ker(f)$  is a  $p$ -ideal of  $l_1$ . Then there exists exactly one  $p$ -mapping on  $f(l_1)$  such that  $f : l_1 \rightarrow f(l_1)$  is a  $p$ -morphism.*

*Proof.* See [SF88], Chapter 2, Section 2.1, Proposition 1.4. □

**2.2.7. Theorem.** *Let  $l$  be a Lie algebra over  $R$ .*

1. *Let  $\gamma_1$  and  $\gamma_2$  be two  $p$ -mappings on  $l$ . Then  $\gamma_2 - \gamma_1 : l \rightarrow Z(l)$  is Frobenius-semi-linear.*
2. *Conversely, let  $\phi : l \rightarrow Z(l)$  be a Frobenius-semi-linear map, and  $\gamma_1$  a  $p$ -mapping on  $l$ . Then,  $\gamma_1 + \phi : l \rightarrow l$  is also a  $p$ -mapping.*

*Proof.* See [SF88], Chapter 2, Section 2.2, Proposition 2.1. □

The following corollary is a rewording of the previous theorem. It will be useful for the following sections where we will present results on Lie  $p$ -algebras but in a geometric way.

**2.2.8. Corollary.** *Let  $l$  be a Lie algebra over  $R$ . We define*

$$E := \text{Hom}_{\text{Frob}}(l, Z(l)) = \text{Hom}_R(l \otimes_{R, \text{Frob}} R, Z(l))$$

*the set of Frobenius semi-linear maps from  $l$  to  $Z(l)$  and let  $X$  denote the set of  $p$ -mappings on  $l$ . Then the map:*

$$\begin{aligned} E \times X &\rightarrow X \times X \\ (\phi, \gamma) &\mapsto (\phi + \gamma, \gamma) \end{aligned}$$

*is bijective.* □

In particular, the theorem says that if there exists a  $p$ -mapping on  $l$ , it is unique if and only if  $E = \{0\}$ , i.e. if  $l$  is locally free of finite rank, the  $p$ -mapping is unique if and only if  $Z(l) = \{0\}$ .

The next proposition shows that the hypothesis (AL 1) is essential in the definition of a  $p$ -mapping, and gives an equivalent condition for a Lie algebra to be restrictable.

**2.2.9. Theorem. (Jacobson)** *Let  $l$  be a Lie algebra, free over  $R$  with basis  $\{x_i\}_{i \in I}$ . Let us assume that for all  $i \in I$ , there exists  $y_i \in l$  such that  $\text{ad}_{x_i}^p = \text{ad}_{y_i}$ . Then, there exists a unique  $p$ -mapping  $(\cdot)^{[p]} : l \rightarrow l$  such that for all  $i \in I$ ,  $x_i^{[p]} = y_i$ .*

*Proof.* You can find the proof in [SF88], Chapter 2, Section 2.2, Theorem 2.3, but the initial version is due to Jacobson, in [J62], Chapter 5, Section 7, Theorem 11. □

**2.2.10. Example.** (Zassenhaus). You can have a look at [SF88], Chapter 1, Section 2.7, Theorem 7.9, or at Zassenhaus's article: [Z39] for more details. Let  $l$  be a free Lie algebra over  $R$ , with Killing form denoted by  $B$ . We suppose that  $B$  is *non-degenerate*, that is we suppose that the following map

$$\begin{aligned} l &\longrightarrow \text{Hom}_R(l, R) \\ x &\longmapsto B(x, \cdot) \end{aligned}$$

is an isomorphism. Then there exists a unique  $p$ -mapping on  $l$ .

# 3 Basics on vector bundles and Lie algebra vector bundles

## 3.1 Vector bundles, quotient and image

In this section,  $S$  is a base scheme. We will study vector bundles equipped with a bracket, in order to study Lie algebras in families. We start by giving standard definitions and notations about vector bundles. We use the notation  $\mathbb{O}_S$  for the ring scheme  $\mathrm{Spec}(\mathcal{O}_S[X])$ .

**3.1.1. Definition.** Throughout all this paper, we call a *generalized vector bundle* any scheme which is an  $\mathbb{O}_S$ -module, isomorphic to an  $\mathbb{O}_S$ -module of the form  $\mathbb{V}(\mathcal{F}) := \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}))$  with  $\mathcal{F}$  any quasi-coherent  $\mathcal{O}_S$ -module.

We also call *vector bundles* those for which  $\mathcal{F}$  is locally free of finite rank. In this case, we use the usual covariant equivalence for which the sheaf of sections of our scheme is  $\mathcal{F}^\vee$ .

**3.1.2. Remark.** Let  $F = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}^\vee))$  be a vector bundle. Then  $\mathcal{F}$  is the restriction of the functor of points of  $F$  to the small Zariski site of  $S$ , that is, to the open subschemes  $U \hookrightarrow S$ .

**3.1.3. Definition.** Let  $f : E \rightarrow F$  be a morphism of generalized vector bundles over  $S$ . We define the *kernel* and the *image* of  $f$  as the fppf kernel sheaf of  $f$  and the fppf image sheaf of  $f$ , i.e. for all fppf covers  $T \rightarrow S$ , we have

$$\mathrm{im}(f)(T) = \{y \in F(T), \exists T' \rightarrow T \text{ fppf covering and } x' \in E(T') \text{ such that } f(x') = y|_{T'}\}.$$

**3.1.4. Remark.**

1. It is known that the kernel is representable, but the image is not representable in general. In the following we will give a necessary and sufficient condition to have the representability of our image.
2. In the category of fppf sheaves, the formation of the image commutes with base change. Indeed, let  $f : E \rightarrow F$  be a morphism of fppf sheaves on  $S$  and  $S' \rightarrow S$  a morphism. Let us denote  $I$  for the fppf image sheaf of  $f$ ,  $f' : E' := E \times_S S' \rightarrow F' := F \times_S S'$ . By universal property of the fppf image, we have an injective map  $i : I' \rightarrow I \times_S S'$ .  
Let us show that  $i$  is surjective. Let  $y' \in (I \times_S S')(T)$  for some  $T \rightarrow S$ . Then  $y' = (y, s') \in I(T) \times S'(T)$ . Then by definition of  $I$ , one can find  $T' \rightarrow T$  an fppf cover and  $x' \in E(T')$  such that  $f(x') = y|_{T'}$ . Then  $i(f(x'), s'|_{T'}) = y'|_{T'}$  and the map is a surjective fppf sheaf map.
3. In the following, every exact sequence of generalized vector bundles will be understood as an exact sequence of fppf sheaves.

**3.1.5. Definition.** Let  $X \rightarrow S$  be a vector bundle and  $Y \hookrightarrow X$  an  $\mathbb{O}_S$ -submodule of  $X$ . We say that  $Y$  is a *subbundle* of  $X$  if  $Y$  is a vector bundle and  $X/Y$  is also a vector bundle.

**3.1.6. Remark.** It is equivalent to be a subbundle of  $X$  and to be a locally direct factor of  $X$ .

**3.1.7. Proposition.** *Let  $F \rightarrow S$  be a generalized vector bundle. Let us write  $F = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}))$  for a given quasi-coherent  $\mathcal{O}_S$ -module  $\mathcal{F}$ . Then:*

1.  $F \rightarrow S$  is of finite presentation if and only if  $\mathcal{F}$  is of finite presentation.
2. If  $\mathcal{F}$  is of finite presentation, then

$$F \rightarrow S \text{ is flat} \Leftrightarrow F \rightarrow S \text{ is smooth} \Leftrightarrow \mathcal{F} \text{ is locally free of finite rank.}$$

*Proof.* See Görtz and Wedhorn's book [GW20], Chapter 7, Proposition 7.41. □

For the following, it will be useful to characterize when an  $\mathbb{O}_S$ -submodule of a vector bundle is in fact a subbundle. In order to do this, we establish these two preliminary lemmas.



### 3.1.8. Lemma.

1. Let  $R$  be a Noetherian ring. Then any surjective endomorphism  $\alpha : R \rightarrow R$  is an automorphism.
2. Let  $R$  be a ring and  $\alpha : R' \rightarrow R'$  a surjective  $R$ -algebra morphism. Then if  $R \rightarrow R'$  is of finite presentation,  $\alpha$  is an automorphism.

*Proof.* 1. For a contradiction, let us assume that  $\alpha$  is not injective: let  $x \in \ker(\alpha), x \neq 0$  and  $n \in \mathbb{N}$ . Then  $\alpha^n$  is surjective, so there exists  $y \in R$  such that  $x = \alpha^n(y)$ . Thus,  $\alpha^{n+1}(y) = 0$ . Then  $y \in \ker(\alpha^{n+1}) \setminus \ker(\alpha^n)$ . Thus, the sequence  $(\ker(\alpha^n))_{n \geq 0}$  is not stationary, then we get a contradiction.

2. Now we suppose that  $R$  is any ring and  $R \rightarrow R'$  is of finite presentation. Then by standard arguments, there exists a subring  $R_0 \subset R$  of finite type over  $\mathbb{Z}$  and an  $R_0$ -algebra  $R_0 \rightarrow R'_0$  of finite presentation such that  $R' \simeq R'_0 \otimes_{R_0} R$ . Then if  $\alpha : R' \rightarrow R'$  is a surjective  $R$ -algebra morphism, we can write  $\alpha = \alpha_0 \otimes_{R_0} \text{id}_R : R'_0 \otimes R \rightarrow R'_0 \otimes R$  where  $\alpha_0 : R'_0 \rightarrow R'_0$  is surjective. Then thanks to the previous point,  $\alpha_0$  is an automorphism, then so is  $\alpha$ , as we wanted.  $\square$

**3.1.9. Lemma.** Let  $X \rightarrow S$  be a scheme and  $G \rightarrow S$  a flat group scheme of finite presentation, acting on  $X \rightarrow S$ . Let  $\pi : X \rightarrow Y$  be a faithfully flat  $S$ -morphism of finite presentation and  $G$ -invariant. Let us assume that the morphism

$$\begin{aligned} G \times_S X &\rightarrow X \times_Y X \\ (g, x) &\mapsto (x, g \cdot x) \end{aligned}$$

is an isomorphism. Then,  $Y$  is the quotient of  $X$  by  $G$  in the category of fppf sheaves on  $S$ .

*Proof.* Let  $F$  be an fppf sheaf on  $S$  and  $f : X \rightarrow F$  a  $G$ -invariant morphism. As  $X \rightarrow Y$  is an fppf morphism and  $F$  is an fppf sheaf, the following sequence is exact:

$$F(Y) \xrightarrow{\pi^*} F(X) \rightrightarrows F(X \times_Y X)$$

and this sequence is isomorphic to this one:

$$F(Y) \xrightarrow{\pi^*} F(X) \begin{array}{c} \xrightarrow{\text{act.}} \\ \xrightarrow{\text{proj.}} \end{array} F(G \times_S X).$$

And this proves the lemma.  $\square$

**3.1.10. Proposition.** Let  $E \rightarrow S$  be a vector bundle and  $F \hookrightarrow E$  an  $\mathcal{O}_S$ -submodule of finite presentation. Then  $F$  is a subbundle of  $E$  if and only if  $F \rightarrow S$  is flat.

*Proof.* Let us assume  $F$  is a subbundle of  $E$ . Then by definition,  $F \rightarrow S$  is flat. Conversely, let us suppose  $F \rightarrow S$  is flat. Then, thanks to Proposition 3.1.7, we know that its sheaf of sections is locally free of finite rank. Then  $F$  is a vector bundle. We only need to show that  $E/F$  is also a vector bundle. Let us denote by  $\mathcal{E}$  and by  $\mathcal{F}$  the sheaves of sections of  $E$  and  $F$ . Then  $E$  and  $\mathcal{E}$ , and  $F$  and  $\mathcal{F}$  determine each other. Moreover, for any  $f : S' \rightarrow S$  base change and for any vector bundle  $V \rightarrow S$ , we have  $(V \times_S S')|_{Z_{\text{ar}}} = f^*(V|_{Z_{\text{ar}}})$ , and because a monomorphism of schemes remains a monomorphism after any base change, we know that the injection  $\mathcal{F} \hookrightarrow \mathcal{E}$  remains injective after any base change. Then the cokernel  $\mathcal{Y}$  of this injection is  $\mathcal{O}_S$ -flat. Because it is also of finite presentation, it is locally free of finite rank thanks to Proposition 3.1.7. Let us show now that  $Y := \mathbf{Spec}(\text{Sym}(\mathcal{Y}^\vee))$  is actually the quotient  $E/F$ . We have the following exact sequence:

$$0 \rightarrow \mathcal{F} \hookrightarrow \mathcal{E} \rightarrow \mathcal{Y} \rightarrow 0.$$

Dualizing this sequence, we obtain:

$$0 \rightarrow \mathcal{Y}^\vee \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{F}^\vee \rightarrow 0.$$

As  $F$  is a subgroup of  $E$ , it acts on  $E$  by left translation. We then have the action morphism

$$\begin{aligned} F \times_S E &\rightarrow E \times_S E \\ (f, e) &\mapsto (f + e, e) \end{aligned}$$

given on the rings by:

$$\begin{aligned} \Phi : \mathrm{Sym}(\mathcal{E}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee) &\rightarrow \mathrm{Sym}(\mathcal{F}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee) \\ 1 \otimes X &\mapsto 1 \otimes X \\ X \otimes 1 &\mapsto \bar{X} \otimes 1 + 1 \otimes X \end{aligned}$$

for all  $X \in \mathcal{E}^\vee = \mathrm{Sym}^1(\mathcal{E}^\vee)$ . Using the definition we see that the elements of the form  $X \otimes 1 - 1 \otimes X$  with  $X \in \mathcal{Y}^\vee$  are in the kernel of  $\Phi$ , then we obtain a factorized map

$$\tilde{\Phi} : \mathrm{Sym}(\mathcal{E}^\vee) \otimes_{\mathrm{Sym}(\mathcal{Y}^\vee)} \mathrm{Sym}(\mathcal{E}^\vee) \rightarrow \mathrm{Sym}(\mathcal{F}^\vee) \otimes_{\mathrm{Sym}(\mathcal{O}_S^\vee)} \mathrm{Sym}(\mathcal{E}^\vee).$$

Let us show that  $\tilde{\Phi}$  is an isomorphism. First, one can see that the source and the target of  $\tilde{\Phi}$  are sheaves of polynomial algebras, with the same number of variables, equal to  $\mathrm{rk}(\mathcal{F}) + \mathrm{rk}(\mathcal{E})$ . Moreover,  $\tilde{\Phi}$  is surjective because

$$\begin{aligned} \tilde{\Phi}(1 \otimes X) &= 1 \otimes X \text{ and} \\ \tilde{\Phi}(X \otimes 1 - 1 \otimes X) &= \bar{X} \otimes 1. \end{aligned}$$

Thus Lemma 3.1.8 2. shows that  $\tilde{\Phi}$  is an isomorphism. Then we have an isomorphism

$$F \times_S E \xrightarrow{\sim} E \times_Y E.$$

Hence, using Lemma 3.1.9, we see that  $Y$  is the quotient of  $E$  by  $F$  in the category of fppf sheaves on  $S$ , so  $E/F = Y = \mathbf{Spec}(\mathrm{Sym}(\mathcal{Y}^\vee))$  is a vector bundle and  $F$  is a subbundle of  $E$ .  $\square$

**3.1.11. Proposition.** *Let  $E_1$  and  $E_2$  be two generalized vector bundles. Let  $f : E_1 \rightarrow E_2$  be a morphism of generalized vector bundles. If  $E_1$  is of finite presentation and if  $E_2$  is of finite type, then  $\ker(f)$  is of finite presentation.*

*Proof.* By definition, we have:

$$\begin{aligned} \ker(f) &= \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}_1) \otimes_{\mathrm{Sym}(\mathcal{F}_2)} \mathcal{O}_S) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{F}_1) \otimes_{\mathrm{Sym}(\mathcal{F}_2)} \mathrm{Sym}(\mathcal{F}_2)/(\mathcal{F}_2)) \\ &= \mathbf{Spec}(\mathrm{Sym}_{\mathrm{Sym}(\mathcal{F}_2)}(\mathcal{F}_1)/f^\#(\mathcal{F}_2)). \end{aligned}$$

Then because  $\mathcal{F}_1$  is of finite presentation and  $\mathcal{F}_2$  is of finite type,  $\ker(f)$  is of finite presentation.  $\square$

We recall the following result, known as the ‘‘Critère de platitude par fibres’’:

**3.1.12. Theorem.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of schemes over  $S$ . Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Let  $x \in X$ . Set  $y := f(x)$  and  $s \in S$  the image of  $x$  in  $S$ . Let us suppose one of these two hypotheses is true:*

1.  $S$ ,  $X$  and  $Y$  are locally noetherian, and  $\mathcal{F}$  is coherent.
2.  $X$  and  $Y$  are locally of finite presentation and  $\mathcal{F}$  is of finite presentation.

*Then, if  $\mathcal{F}_x \neq 0$ , these two conditions are equivalent:*

1.  $\mathcal{F}$  is flat over  $S$  at  $x$ , and  $\mathcal{F}_s$  is flat over  $Y_s$  at  $x$ .
2.  $Y$  is flat over  $S$  at  $y$  and  $\mathcal{F}$  is flat over  $Y$  at  $x$ .

*Proof.* See [EGA4], troisième partie, théorème 11.3.10.  $\square$

The next statement is a general result about images and kernels of morphisms of vector bundles, for which we could not find a proof in the literature. It gives conditions for the kernel and the image of a vector bundle morphism to be subbundles. For this result, we first recall that, for any morphism of schemes  $f : X \rightarrow Y$ , there exists a smallest closed subscheme of  $Y$  that factorizes  $f$ . We denote it by  $\text{imsc}(f)$  and it is called *the schematic image of  $f$* . See [GW20] Definition and Lemma 10.29.

**3.1.13. Theorem.** *Let  $f : E := \mathbf{Spec}(\text{Sym}(\mathcal{E}^\vee)) \rightarrow F := \mathbf{Spec}(\text{Sym}(\mathcal{F}^\vee))$  be a morphism of  $S$ -vector bundles with kernel  $K$  and with image  $I$ . Then the following are equivalent:*

- 1)  $K \rightarrow S$  is flat.
- 2)  $I \rightarrow S$  is representable by an  $S$ -scheme of finite presentation.

Moreover, when these conditions are satisfied, we have:

- (i)  $K$  is a subbundle of  $E$  and  $I$  is a subbundle of  $F$ . Moreover, the induced morphism  $E/K \rightarrow I$  is an isomorphism.
- (ii)  $I = \text{imsc}(f)$ .
- (iii) The sheaf of sections of  $K$  is  $\mathcal{K} := \ker(\mathcal{E} \rightarrow \mathcal{F})$ , that of  $I$  is  $\mathcal{I} := \text{im}(\mathcal{E} \rightarrow \mathcal{F})$ , and  $\mathcal{E}/\mathcal{K} \simeq \mathcal{I}$ . Moreover, the formation of  $K, I$ , and  $\mathcal{K}, \mathcal{I}$  commute with base change.

*Proof.* 1)  $\implies$  2)

We denote by  $\mathcal{K}$  the sheaf of sections of  $K$ . Because  $E$  and  $F$  are both of finite presentation, Proposition 3.1.11 tells us that  $K$  is of finite presentation, then we can apply Proposition 3.1.10 to say that  $K$  is a subbundle of  $E$ . Let us write the following exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

and let us denote  $Y := \mathbf{Spec}(\text{Sym}(\mathcal{Q}^\vee))$ . Doing the same proof as in Proposition 3.1.10, we see that  $\mathcal{Q}$  is locally free of finite rank, and  $Y$  is the quotient of  $E$  by  $K$  in the category of fppf sheaves on  $S$ . Then  $I = Y = E/K$  is representable by an  $S$ -scheme of finite type, given by  $I = \mathbf{Spec}(\text{Sym}(\mathcal{Q}^\vee))$ . Because  $\mathcal{Q}$  is the cokernel of the injection  $\mathcal{K} \hookrightarrow \mathcal{E}$ , we can write  $\mathcal{Q} \hookrightarrow \mathcal{F}$  so we get a surjection  $\mathcal{F}^\vee \rightarrow \mathcal{Q}^\vee$  hence  $I \hookrightarrow F$  is a closed immersion. But because  $I$  factorises  $f$ , by definition of the schematic image, we have  $I \simeq \text{imsc}(f)$ .

2)  $\implies$  1)

Let us suppose  $I$  is representable by an  $S$ -scheme of finite presentation. In order to prove that  $K \rightarrow S$  is flat, it is sufficient to prove that  $E \rightarrow I$  is flat. Let  $s \in S$ . Then  $I_s$  is the image of  $E_s \rightarrow F_s$  and  $K_s$  is its kernel. Because the formation of the kernel and of the image commutes with base change, we have an isomorphism of fppf sheaves

$$E_s/K_s \xrightarrow{\sim} I_s$$

then  $E_s \rightarrow I_s$  is flat. Then using Theorem 3.1.12, we obtain that  $E \rightarrow I$  is flat. Moreover, the morphism  $E \rightarrow I$  is surjective in the topological sense because it is surjective as a morphism of fppf sheaves, then  $I \rightarrow S$  is flat.

Let us suppose now that these conditions are satisfied. Then looking at the proof of 1)  $\implies$  2), we see that  $K$  and  $E/K$  are vector bundles on  $S$ . Using this same proof, we see that  $I$  is also a subbundle of  $F$ , and that  $E/K \simeq I \simeq \text{imsc}(f)$ . The first part of (iii) is true because  $K$  is a subbundle of  $E$  and  $I$  is a subbundle of  $F$ . Then for the last assertion, we have to say that the formation of  $K$  commutes with base change because it is a kernel, then because  $K$  and  $\mathcal{K}$  determine each other, we see that  $\mathcal{K}$  commutes with base change. Finally,  $I$  commutes with base change because it is a quotient, and then  $\mathcal{I}$  commutes with base change because it is determined by  $I$ .  $\square$

## 3.2 Definitions and notations on Lie algebra vector bundles

**Notations:** In the following,  $\mathcal{L}$  is a Lie  $\mathcal{O}_S$ -algebra locally free of finite rank, whose bracket is denoted by

$$[\cdot, \cdot] : \mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{L} \rightarrow \mathcal{L}.$$

We denote by  $\mathcal{E}nd(\mathcal{L})$  the  $\mathcal{O}_S$ -module of  $\mathcal{O}_S$ -endomorphisms of  $\mathcal{L}$ . We denote by

$$L := \mathbf{Spec}(\mathrm{Sym}(\mathcal{L}^\vee))$$

the associated vector bundle, and  $[\cdot, \cdot] : L \times L \rightarrow L$  the morphism of schemes we deduce from the bracket of  $\mathcal{L}$ , inducing a Lie  $S$ -algebra structure on  $L$ . We call these kinds of objects *Lie algebra vector bundles*. We denote by

$$\begin{aligned} ad : \mathcal{L} &\longrightarrow \mathcal{E}nd(\mathcal{L}) \\ y &\longmapsto (x \mapsto [x, y]) \end{aligned}$$

and  $ad : L \rightarrow \mathrm{End}(L) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{E}nd(\mathcal{L})^\vee))$  the corresponding morphism of schemes.

**3.2.1. Remark.** By definition, the formation of  $ad$  and  $\mathrm{End}(L)$  commutes with base change, and because  $\mathcal{L}$  is locally free of finite rank, the formation of  $\mathcal{E}nd(\mathcal{L})$  does too.

**3.2.2. Definition.** Let  $L$  be a Lie algebra vector bundle over  $S$ . We define

$$Z(L) := \ker(ad : L \rightarrow \mathrm{End}(L)).$$

This kernel is called the *center* of  $L$ .

**3.2.3. Remark.** The center  $Z(L)$  is a closed subscheme of  $L$  because  $\mathrm{End}(L)$  is affine, then separated, hence the neutral section is closed.

**3.2.4. Proposition.** *The center  $Z(L)$  of a Lie algebra vector bundle is of finite presentation.*

*Proof.* This is straightforward from Proposition 3.1.11. □

Then we see that we are in good conditions for using Proposition 3.1.7 with the center of a Lie algebra vector bundle.

Moreover, using Theorem 3.1.13 (iii), we know that if the center  $Z(L) \rightarrow S$  is flat, then it is determined by its sheaf of Zariski sections, which is given by

$$\mathcal{Z}(\mathcal{L}) := \ker(ad : \mathcal{L} \rightarrow \mathcal{E}nd(\mathcal{L})),$$

and so we have

$$Z(L) = \mathbf{Spec}(\mathrm{Sym}(\mathcal{Z}(\mathcal{L})^\vee)).$$

**3.2.5. Counter-example.** The hypothesis " $Z(L) \rightarrow S$  flat" is essential. Here is a counter-example: let  $R$  be a ring and  $L$  be the Lie  $R$ -algebra with basis  $\{x, y\}$  and bracket defined by  $[x, y] = ax$  for some  $a \in R$  such that  $a \nmid 0$ , that is, geometrically  $L = \mathrm{Spec}(\mathrm{Sym}(Rx^* \oplus Ry^*))$ . Let  $\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$  be an open immersion. Hence  $R'$  is a flat  $R$ -algebra. Hence  $a \nmid 0$  in  $R'$ . So using the previous notation, we have  $\mathcal{Z}(\mathcal{L}) = 0$ . But whenever  $R'$  is a  $R/(a)$ -algebra, we have  $Z(L)(R') = L(R')$ . Hence the  $Z(L)$  and its sheaf of Zariski sections do not determine each other.

In the following, we say a little bit more about the derived Lie algebra of a Lie algebra generalised vector bundle.

**3.2.6. Definition.** Let  $\mathcal{L}$  be an  $\mathcal{O}_S$ -module in Lie algebras. We define its *derived Lie algebra* as the image sheaf of  $[\cdot, \cdot] : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L}$ .

Let  $L \rightarrow S$  be a Lie algebra generalised vector bundle. We define its *derived Lie algebra*  $L'$  as the fppf image sheaf of  $[\cdot, \cdot] : L \otimes L \rightarrow L$ .

A priori, the derived Lie algebra of a vector bundle can be only seen as an fppf sheaf but it is not necessarily representable. But thanks to Theorem 3.1.13, we know that under some conditions, the derived Lie algebra will be easy to manipulate:

**3.2.7. Lemma.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle. Let us assume that  $L'$  is a subbundle of  $L$ . Then the formation of the derived Lie algebra commutes with base change.*

**3.2.8. Remark.** We recall that thanks to Theorem 3.1.13, we know that the condition " $L'$  is a subbundle of  $L$ " is equivalent to asking " $K = \ker([\cdot, \cdot] : L \otimes L \rightarrow L) \rightarrow S$ " is flat, i.e. "the kernel  $K$  is a subbundle of  $L \otimes L$ ", and Theorem 3.1.13 (i) implies that we have an isomorphism  $(L \otimes L)/K \xrightarrow{\sim} L'$ .

*Proof.* This comes from Theorem 3.1.13 (iii).  $\square$

**3.2.9. Remark.** The derived Lie algebra, as an fppf sheaf, has the following universal property: all morphisms of fppf sheaves in Lie algebras  $f : L \rightarrow \mathcal{A}$  where  $\mathcal{A}$  is a sheaf in abelian Lie algebras, can be uniquely factorized by  $L/L'$ :

$$\begin{array}{ccc} L & \xrightarrow{f} & \mathcal{A} \\ \downarrow & \nearrow \exists! \bar{f} & \\ L/L' & & \end{array} .$$

**3.2.10. Corollary.** *With the conditions of Lemma 3.2.7,  $L' = \mathbf{Spec}(\mathrm{Sym}(\mathcal{L}'^\vee))$ .*

*Proof.* In these conditions,  $L'$  is a vector bundle so its sheaf of sections and itself determine each other. Then, thanks to the previous remark, we see that its sheaf of sections satisfy the universal property of the sub-derived module, then we have the result.  $\square$

## 4 The scheme of Lie $p$ -algebra structures

### 4.1 Main results on the $p$ -mapping functor and the restrictable locus

From now on,  $S$  is a scheme of characteristic  $p > 0$ , and we globalize the definition of a  $p$ -mapping from Lie algebras to a definition on Lie algebra vector bundles as follows.

**4.1.1. Definition.** Let  $L$  be a Lie algebra generalised vector bundle. We say that a morphism of schemes  $(\cdot)^{[p]} : L \rightarrow L$  is a  $p$ -mapping on  $L$  if for all  $S$ -schemes  $T$ , it is a  $p$ -mapping on  $L(T)$ .

**4.1.2. Definition.** Let  $X \rightarrow S$  be an  $S$ -scheme, and let  $E$  be a generalised vector bundle over  $S$ . We say that  $X$  is a *formally homogeneous space under  $E$*  if  $E$  acts on  $X$  such that the action map

$$\begin{aligned} E \times_S X &\rightarrow X \times_S X \\ (e, x) &\mapsto (e \cdot x, x) \end{aligned}$$

is a scheme isomorphism.

Moreover, if  $E$  is a vector bundle, we say that  $X$  is a *formally affine space under  $E$* . Moreover, if  $X \rightarrow S$  has local sections, i.e. if  $X \rightarrow S$  is a sheaf epimorphism for the fppf topology, we say that  $X$  is an *affine space under  $E$* .

**4.1.3. Remark.** One can show that the second condition is equivalent to have local sections for the étale or for the Zariski topology. This is because  $H_{\mathrm{fppf}}^1(S, E) \simeq H_{\mathrm{Zar}}^1(S, E)$ . See Milnes's book on étale cohomology [M80], Chapter III, §3, Proposition 3.7.

**Notations:** Let  $X$  be a scheme of characteristic  $p > 0$ . We denote by  $\mathrm{Frob}_X : X \rightarrow X$  or simply  $\mathrm{Frob}$  the absolute Frobenius morphism of the scheme  $X$ .

Let  $L \rightarrow S$  be a Lie algebra vector bundle. Let us denote by  $E$  the generalised vector bundle of Frobenius-semilinear morphisms between  $L$  and  $Z(L)$ :

$$E := \mathrm{Hom}_{\mathrm{Frob}}(L, Z(L)) = \mathrm{Hom}(\mathrm{Frob}_S^* L, Z(L)) = (\mathrm{Frob}_S^* L)^\vee \otimes Z(L)$$

where the tensor product is taken in the category of vector bundles over  $S$ . If  $Z(L)$  is a vector bundle, then so is  $E$ .

**4.1.4. Theorem.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle. Let us define a set-valued functor as follows:*

$$\begin{aligned} X : \{S\text{-schemes}\} &\longrightarrow \text{Set} \\ T &\longmapsto \{p\text{-mappings on } L \times_S T\}. \end{aligned}$$

*Then,  $X$  is representable by an affine scheme, and is a formally homogeneous space under  $E$ .*

*Proof.* Let  $\mathcal{L}$  be the Zariski sheaf of sections of  $L \rightarrow S$ . Let us show that  $X$  is representable. Because the claim is local on the target, we can suppose  $S = \text{Spec}(R)$  affine, small enough so that  $\mathcal{L}$  is free with basis  $x_1, \dots, x_n$  on  $S$ , i.e.

$$\mathcal{L} = \mathcal{O}_S x_1 \oplus \dots \oplus \mathcal{O}_S x_n, \quad x_i \in \mathcal{L}(S) = L(S) \text{ and } L = \text{Spec}(\mathcal{O}_S[x_1^*, \dots, x_n^*]).$$

Let us define for all  $i$  the following morphism:

$$f_i : S \xrightarrow{x_i} L \xrightarrow{\text{ad}} \text{End}(L) \xrightarrow{p} \text{End}(L)$$

where  $p : \text{End}(L) \rightarrow \text{End}(L)$  maps an endomorphism to its  $p$ -power. Then, by definition of the fiber product, for all  $T \rightarrow S$  and  $i \in \{1, \dots, n\}$ , we have

$$(L \times_{\text{End}(L), (\text{ad}, f_i)} S)(T) = \{y \in L(T), \text{ad}_y = (\text{ad}_{x_i})|_T^p\}.$$

Then, by Jacobson's Theorem 2.2.9, the map

$$\begin{aligned} X &\rightarrow (L \times_{\text{End}(L); (\text{ad}, f_1)} S) \times (L \times_{\text{End}(L); (\text{ad}, f_2)} S) \times \dots \times (L \times_{\text{End}(L); (\text{ad}, f_n)} S) \\ \gamma &\mapsto (\gamma(x_1), \dots, \gamma(x_n)) \end{aligned}$$

is an isomorphism. This shows that  $X$  is representable. Let us show now that  $X$  is a formally homogeneous space under  $E$ . Let  $T \rightarrow S$  be an  $S$ -scheme. We can write

$$E(T) = \text{Hom}_{\mathbb{G}_a, T\text{-mod}}(\text{Frob}^* L \times T, Z(L) \times_S T) = \text{Hom}_{\mathbb{G}_a, T\text{-mod}}(\text{Frob}^* L \times_S T, Z(L \times_S T))$$

$$\text{and } X(T) = \{p\text{-structures on } L \times_S T\}.$$

Then the morphism

$$\begin{aligned} E \times_S X &\rightarrow X \times_S X \\ (\phi, \gamma) &\mapsto (\phi + \gamma, \gamma) \end{aligned}$$

is well-defined and is an isomorphism thanks to Corollary 2.2.8.  $\square$

**4.1.5. Remark.** If we suppose moreover that  $Z(L) \rightarrow S$  flat, then  $E$  is a vector bundle, so  $X$  is a formally affine space under  $E$ .

For the next theorem, we recall that a scheme  $X \rightarrow S$  is said to be *essentially free* if we can find a cover of  $S$  by affine opens  $S_i$ , and for all  $i$  an affine and faithfully flat  $S_i$ -scheme  $S'_i$ , and a cover  $(X'_{i,j})_j$  of  $X'_i := X \times_S S'_i$  by affine opens  $X'_{i,j}$  such that for all  $(i, j)$ , the ring of functions of  $X'_{i,j}$  is a free module on the ring of  $S'_i$ .

**4.1.6. Theorem.** *Let  $S$  be a scheme. Let  $Z \rightarrow S$  be essentially free and let  $Y \hookrightarrow Z$  be a closed subscheme of  $Z$ . Then, the Weil restriction defined by*

$$\begin{aligned} \Pi_{Z/S}(Y) : \{S\text{-schemes}\} &\longrightarrow \text{Set} \\ T &\longmapsto \begin{cases} \{\emptyset\} & \text{if } Z_T = Y_T \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

*is representable by a closed subscheme of  $S$ .*

*Proof.* See [SGA3] Tome 2, exposé VIII, Théorème 6.4. □

**4.1.7. Lemma.** *Let*

$$0 \rightarrow K \rightarrow E \xrightarrow{\pi} F \rightarrow 0$$

*be an exact sequence of vector bundles (i.e. seen as fppf sheaves) on a scheme  $S$ . Then,  $\pi$  is surjective, Zariski-locally on  $S$ .*

*Proof.* By hypothesis,  $E \rightarrow F$  is a  $K_F$ -torsor for the fppf topology. Let  $f : S \rightarrow F$  a section on  $F$ . Let  $E \times_F S$  be the fiber product made with the section  $f$ . Then by base change,  $E \times_F S$  is a  $K_F \times_F S$ -torsor for the fppf topology. But  $K_F \times_F S = K$  and

$$H_{fppf}^1(S, K) = H_{Zar}^1(S, K)$$

because  $K$  is a vector bundle over  $S$  (see [M80] for more details). Then  $E \times_F S$  is a  $K$ -torsor over  $S$ , for the Zariski topology. Then there exist open immersions  $g : S' \rightarrow S$  and  $h : S' \rightarrow E \times_F S$  such that this diagram commutes:

$$\begin{array}{ccc} & & S' \\ & \swarrow h & \downarrow g \\ E \times_F S & \longrightarrow & S \\ \text{pr}_1 \downarrow & & \downarrow f \\ E & \xrightarrow{\pi} & F \end{array}$$

Then the Zariski section we are looking for is given by  $\text{pr}_1 \circ h : S' \rightarrow E$ . □

**4.1.8. Theorem.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle whose center  $Z(L) \rightarrow S$  is flat. Let us recall the notation  $E := \text{Hom}_{\text{Frob}}(L, Z(L))$ . Let  $X \rightarrow S$  be the functor of  $p$ -mappings on  $L$  defined above, and let  $S^{\text{res}} = S^{\text{res}}(L)$  be defined as:*

$$S^{\text{res}} : \{S\text{-schemes}\} \longrightarrow \text{Set}$$

$$T \longmapsto \begin{cases} \{\emptyset\} & \text{if } L_T \text{ is Zar-loc. restrictable over } T \\ \emptyset & \text{otherwise.} \end{cases}$$

*Then the following two conditions are verified:*

1.  $S^{\text{res}}$  is representable by a closed subscheme of  $S$ .
2.  $X \rightarrow S$  factors through  $S^{\text{res}}$  and  $X \rightarrow S^{\text{res}}$  is an affine space under the vector bundle  $E \times_S S^{\text{res}}$ .

**4.1.9. Remark.**

- The functor  $S^{\text{res}}$  could have been defined as the unique sub-functor of  $S$  such that

$$L_T \text{ is Zar-loc. restrictable} \Leftrightarrow T \rightarrow S \text{ can be factorized by } S^{\text{res}}.$$

Indeed  $T \rightarrow S$  can be factorized by  $S^{\text{res}}$  if and only if  $S^{\text{res}}(T) \neq \emptyset$  if and only if  $L_T$  is Zar-loc. restrictable. Let  $F$  be a subfunctor of  $S$  such that  $L_T$  is Zar-loc. restrictable if and only if  $T \rightarrow S$  can be factorized by  $F$ . But for all  $T$ ,  $F(T) \subset \text{Hom}_S(T, S) = \{*\}$ . Hence by definition,  $S^{\text{res}}$  is the only subfunctor of  $S$  satisfying the property above.

- We could have defined  $S^{\text{res}}$  to be the locus where a Lie algebra is fppf-loc. restrictable, because this is less restrictive, but the following results will show that those conditions are the same.
- By Yoneda, we can see  $X(X) = \text{Hom}_S(X, X) \neq \emptyset$  because  $\text{id} \in X(X)$ . But by definition,  $\text{id} \in X(X)$  corresponds to a  $p$ -mapping on  $L_X$ . Then  $L_X$  is Zar-loc. restrictable and we call this mapping the *universal  $p$ -mapping on  $L_X$* .

*Proof.* 1. Let  $I$  be the image of  $\text{ad}$ . Because  $Z(L)$  is flat,  $I$  is a subbundle of  $\text{End}(L)$  by Theorem 3.1.13 (i). Let  $\rho : I \rightarrow \text{End}(L)$  be the  $p$ -th power map, restricted to  $I$ . Let  $W = W(L)$  be the subfunctor of  $S$  defined by:

$$W : \{S\text{-schemes}\} \longrightarrow \text{Set}$$

$$T \longmapsto \begin{cases} \{\emptyset\} & \text{if } I_T \text{ is stable by } \rho \\ \emptyset & \text{otherwise.} \end{cases}$$

Let us show that  $W$  is representable by a closed subscheme of  $S$ . Let  $T \rightarrow S$  be an  $S$ -scheme. Then,  $I_T$  is  $\rho$ -stable if and only if  $\rho^{-1}(I_T) \xrightarrow{\sim} I_T$ . But  $I \hookrightarrow \text{End}(L)$  is closed thanks to Theorem 3.1.13 (ii), and closed immersion are stable by base change. Then  $\rho^{-1}(I)$  is a closed subscheme of  $I$ . We know that  $I \rightarrow S$  is essentially free because it is a vector bundle, then using Theorem 4.1.6 with  $\rho^{-1}(I) \hookrightarrow I$ , we see that  $W$  is a closed subscheme of  $S$ .

Let us now show that  $W = S^{\text{res}}$ . First, let us show  $S^{\text{res}} \subset W$ . Let  $T \rightarrow S$ . If  $S^{\text{res}}(T) = \emptyset$ , there is nothing to prove. Let us suppose  $S^{\text{res}}(T) \neq \emptyset$ . Then by definition  $L_T$  is Zar-loc. restrictable over  $T$ , hence there exists a  $p$ -mapping on  $L_T$ , locally on  $T$  for the Zariski topology. We denote this  $p$ -mapping by  $\gamma$ . We want to show that  $I_T$  is stable by  $\rho$ . That means we want to show that there exists a map  $\sigma : I_T \rightarrow I_T$  such that the following diagram commutes:

$$\begin{array}{ccc} I_T & \xrightarrow{\rho_T} & \text{End}(L_T) \\ & \searrow \sigma & \uparrow i \\ & & I_T. \end{array}$$

Thanks to Theorem 3.1.13, we know that  $I_T = L_T/Z(L_T)$ . But  $Z(L_T) \subset L_T$  is an ideal of  $L_T$ , and thanks to (AL 1), it is stable by any  $p$ -mapping, so it is stable by  $\gamma$ . Then  $\gamma$  induces a  $p$ -mapping that we can write  $\sigma : L_T/Z(L_T) \rightarrow L_T/Z(L_T)$  by Proposition 2.2.6. If we denote by  $\pi : L_T \rightarrow L_T/Z(L_T)$  the quotient morphism, we have a commutative diagram, where  $p : \text{End}(L_T) \rightarrow \text{End}(L_T)$  is the  $p$ -power:

$$\begin{array}{ccccc} L_T & \xrightarrow{\pi} & L_T/Z(L_T) & = & I_T \xrightarrow{i} \text{End}(L_T) \\ \downarrow \gamma & & \downarrow \sigma & & \downarrow p \\ L_T & \xrightarrow{\pi} & L_T/Z(L_T) & = & I_T \xrightarrow{i} \text{End}(L_T) \end{array}$$

The right-hand square of this diagram is commutative thanks to axiom (AL 1). As  $\rho = p \circ i$ , we can calculate

$$\rho \circ \pi = p \circ i \circ \pi = i \circ \sigma \circ \pi.$$

But  $\pi$  is an epimorphism in the category of schemes, then we obtain  $\rho = i \circ \sigma$ , i.e  $\rho$  factors via  $i$  as we wanted. Then,  $S^{\text{res}} \subset W$ .

Conversely, let  $T \rightarrow S$  be such that  $I_T$  is stable by  $\rho$ . Let us show that  $X(T)$  is nonempty, locally for the Zariski topology on  $T$ . As everything is local on  $S$ , and  $Z_T$  is locally a direct factor in  $L_T$ , we can assume  $T$  is affine, small enough such that  $L = \mathbf{Spec}(\mathcal{O}_T[x_1^*, \dots, x_n^*])$ . Then we have the exact sequence

$$0 \rightarrow Z(L) \rightarrow L \xrightarrow{\text{ad}} I \rightarrow 0$$

and Lemma 4.1.7 says that  $\text{ad}$  is surjective, Zariski locally on  $T$ . Then, thanks to Jacobson's Theorem 2.2.9, we know that we have existence of a  $p$ -mapping, Zariski locally on  $T$ . Then  $S^{\text{res}} = W$ , so  $S^{\text{res}}$  is representable by a closed subscheme of  $S$ .

2. Let  $T \rightarrow S$  be an  $S$ -scheme. Let  $\gamma \in X(T)$ . Then by definition,  $L_T$  is restrictable so  $S^{\text{res}}(T) = \{\emptyset\}$ . Then we define this map:

$$\begin{array}{ccc} X(T) & \rightarrow & S^{\text{res}}(T) \\ \gamma & \mapsto & \{\emptyset\} \end{array}$$



that factorizes  $X \rightarrow S$ . Thanks to Theorem 4.1.4, and because  $X \times_S X = X \times_{S^{\text{res}}} X$ , we know that

$$E_{S^{\text{res}}} \times_{S^{\text{res}}} X \simeq X \times_{S^{\text{res}}} X.$$

We need to show that  $X \rightarrow S^{\text{res}}$  is a sheaf epimorphism for the fppf topology. It suffices to show that for all  $T \rightarrow S$  such that  $L_T$  is Zar-loc. restrictable, we can find an fppf morphism  $T' \rightarrow T$  such that there exists a  $p$ -mapping on  $L_{T'}$ . We just have to take for  $T'$  the Zariski covering on which  $L_T$  possesses a  $p$ -mapping.  $\square$

**4.1.10. Corollary.** *With the same hypothesis, if  $Z(L) = \{0\}$ , then  $X \simeq S^{\text{res}}$ , so  $X \rightarrow S$  is a closed immersion.*

## 4.2 A case of existence of a $p$ -mapping

In general it is not easy to decide if a given finite-dimensional Lie algebra or a Lie algebra vector bundle admits a  $p$ -mapping. Here is a brief review of the easiest cases we have already seen, where such existence is known to hold :

1. Associative Lie algebras, with the Frobenius map.
2. Lie algebras of group schemes.
3. Lie algebras whose Killing form is nondegenerate (Zassenhaus).
4. Somewhat opposite to 3. is the abelian case, where  $\gamma = 0$  is a  $p$ -mapping.

The last case corresponds to the situation where the derived Lie algebra has rank 0. In the rest of the section, we will extend that case to the mildly non-abelian case where the derived Lie algebra has rank 1.

Let  $E$  be a vector bundle of rank 1. We remind to the reader that in this case we have a canonical isomorphism, given on the functor of points by:

$$\begin{aligned} \mathbb{G}_a &\xrightarrow{\sim} \text{End}(E) \\ f &\mapsto m_f \end{aligned}$$

where  $m_f$  is the multiplication by  $f$ .

**4.2.1. Theorem.** *Let  $L \rightarrow S$  be a Lie algebra vector bundle, such that  $L'$  is a locally free subbundle of rank 1. We define a map of vector bundles as follows:*

$$\begin{aligned} \alpha : L &\rightarrow \text{End}(L') \simeq \mathbb{G}_a \\ x &\mapsto (\text{ad}(x)|_{L'}) \mapsto \alpha(x). \end{aligned}$$

*Then, the map*

$$\begin{aligned} L &\rightarrow L \\ x &\mapsto \alpha(x)^{p-1}x \end{aligned}$$

*is a  $p$ -mapping on  $L$ . Moreover if  $L$  is locally free of rank 2, this  $p$ -mapping is unique.*

*Proof.* In order to prove this, we can suppose  $L$  is free, such that  $L'$  is free, given by  $L' = \mathbb{G}_a \cdot v$ . Then the bracket is given on the functor of points by:

$$\begin{aligned} [\cdot, \cdot] : L \times L &\rightarrow L \\ (x, y) &\mapsto f(x, y)v \end{aligned}$$

where  $f$  is a bilinear alternating form. Let us start by showing that for all  $T \rightarrow S$ , for all  $x \in L(T)$ , we have this equality:  $f(x, v) = \alpha(x)$ . Let us denote by  $\phi : \mathbb{G}_a \simeq \text{End}(L')$ . So we have to show that

$f(x, v) = \phi^{-1}(\text{ad}(x)|_{L'})$ , i.e. we have to show that  $\phi(f(x, v)) = \text{ad}(x)|_{L'}$ . Then let  $y \in L'(T)$ . We can write  $y = \lambda v$  with some  $\lambda \in \mathbb{G}_a(T)$ . Then  $\phi(f(x, v))(y) = \lambda f(x, v)v$  and  $(\text{ad}(x)|_{L'})(y) = \lambda f(x, v)v$  so we have the equality we wanted.

Let us show now that the map

$$\begin{aligned} (\cdot)^{[p]} : L &\rightarrow L \\ x &\mapsto \alpha(x)^{p-1}x = f(x, v)^{p-1}v \end{aligned}$$

verifies (AL 1). Let  $T$  be an  $S$ -scheme and let  $x, y \in L(T)$ . Then

$$\text{ad}_{x^{[p]}}(y) = f(f(x, v)^{p-1}x, y)v = f(x, v)^{p-1}f(x, y)v.$$

Moreover, we can write

$$\text{ad}_x(\text{ad}_x(y)) = \text{ad}_x(f(x, y)v) = f(x, y)\text{ad}_x(v) = f(x, y)f(x, v)v.$$

Then by induction, we found

$$\text{ad}_x^p(y) = f(x, y)f(x, v)^{p-1}v.$$

Thus, (AL 1) is checked.

The condition (AL 2) is directly checked by definition. Let us show that our map respects (AL 3). Let  $x, y \in L(T)$ . Then,

$$\begin{aligned} (x + y)^{[p]} &= f(x + y, v)^{p-1}(x + y) \\ &= \sum_{k=0}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} (x + y) \\ &= \sum_{k=0}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} x + \sum_{k=0}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} y \\ &= \sum_{k=0}^{p-2} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} x + x^{[p]} + \sum_{k=1}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} y + y^{[p]} \\ &= x^{[p]} + y^{[p]} + \sum_{k=1}^{p-1} \binom{p-1}{k-1} f(x, v)^{k-1} f(y, v)^{p-k} x + \sum_{k=1}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} y. \end{aligned}$$

Let  $k \in [1, p-1]$ . Let us compute  $s_k$ . First, for all  $x, y \in l$ ,

$$\text{ad}_x(\text{ad}_y(v)) = \text{ad}_y(\text{ad}_x(v)) = f(x, v)f(y, v)v.$$

Let us recall

$$s_k(x, y) = -\frac{1}{k} \sum_u \text{ad}_{u(1)} \text{ad}_{u(2)} \dots \text{ad}_{u(p-1)}(y)$$

where  $u$  ranges through the maps  $\{1, \dots, p-1\} \rightarrow \{x, y\}$  taking  $k$  times the value  $x$ . In this sum, all terms corresponding to a map  $u$  such that  $u(p-1) = y$  are zero because  $\text{ad}_y(y) = 0$  and  $\text{ad}_z$  is linear for all  $z \in L(T)$ . Then,

$$\begin{aligned} s_k(x, y) &= -\frac{1}{k} \sum_u \text{ad}_{u(1)} \text{ad}_{u(2)} \dots \text{ad}_{u(p-2)}(f(x, y)v) \\ &= -\frac{f(x, y)}{k} \sum_u \text{ad}_{u(1)} \text{ad}_{u(2)} \dots \text{ad}_{u(p-2)}(v) \end{aligned}$$

where  $u$  ranges through the maps  $u : \{1, \dots, p-2\} \rightarrow \{x, y\}$  taking  $k-1$  times the value  $x$ .

Then there are  $\binom{p-2}{k-1}$  such maps and, thanks to the previous remark, they all give the same term in the sum, which is  $\text{ad}_x^{k-1}((\text{ad}_y)^{p-2-(k-1)}(v))$ . Then,

$$s_k = -\frac{f(x, y)}{k} \binom{p-2}{k-1} f(x, v)^{k-1} f(y, v)^{p-k-1} v = -\frac{1}{k} \binom{p-2}{k-1} f(x, v)^{k-1} f(y, v)^{p-k-1} f(x, y)v.$$

But,

$$\frac{p-1}{k} \binom{p-2}{k-1} = \binom{p-1}{k}.$$

Then,

$$\begin{aligned} s_k &= \binom{p-1}{k} f(x, v)^{k-1} f(y, v)^{p-k-1} (f(x, v)y - f(y, v)x) \\ &= \binom{p-1}{k} f(x, v)^k f(y, v)^{p-k-1} y - \binom{p-1}{k} f(x, v)^{k-1} f(y, v)^{p-k} x. \end{aligned}$$

Moreover,

$$\binom{p-1}{k-1} + \binom{p-1}{k} = \binom{p}{k} = 0 \text{ for all } k \in [1, p-1].$$

Hence,

$$\begin{aligned} \sum_{k=1}^{p-1} s_k &= \sum_{k=1}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} y - \sum_{k=1}^{p-1} \binom{p-1}{k} f(x, v)^{k-1} f(y, v)^{p-k} x \\ &= \sum_{k=1}^{p-1} \binom{p-1}{k} f(x, v)^k f(y, v)^{p-1-k} y + \sum_{k=1}^{p-1} \binom{p-1}{k-1} f(x, v)^{k-1} f(y, v)^{p-k} x. \end{aligned}$$

Then the map we set verifies (AL1), (AL2) and (AL3). Finally, this map is a  $p$ -mapping on our Lie algebra  $L$ .

Let us now suppose that  $L$  is locally free of rank 2. Let us show that the  $p$ -mapping we have defined above is unique. As the functor  $X$  of the  $p$ -mappings is a sheaf for the Zariski topology, we can suppose  $S = \text{Spec}(R)$  is affine, small enough so that  $L$  is free. As we suppose  $\text{rk}(L') = 1$ ,  $\text{rk}(L) = 2$  and because the bracket is an alternating bilinear map, we can write for all  $x, y \in L(S)$ ,

$$[x, y] = \det(x, y)v$$

for a certain  $v \in L(S)$ .

Let us write  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L(S) \simeq R^2$  and  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in Z(L(S))$ . Then for all  $y \in L(S)$ ,  $[x, y] = 0$ .

By taking  $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , we can write:

$$\begin{cases} x_1 v_1 = x_1 v_2 = 0 \\ x_2 v_1 = x_2 v_2 = 0 \end{cases}.$$

But  $(v_1, v_2) = R$ . Indeed let  $q \in \text{Spec}(R)$ . Let us suppose  $(v_1, v_2) \subset q$ . Then the bracket on  $L(T) \otimes R_q/qR_q$  is the zero morphism. This is impossible because the derived Lie algebra commutes with base change and so has rank 1 on  $R_q/qR_q$ . Then  $(v_1, v_2)$  is not contained in any maximal ideal so  $(v_1, v_2) = R$ . Then there exist  $a, b \in R$  such that  $av_1 + bv_2 = 1$  and  $x_1 = x_1(av_1 + bv_2) = 0$ . With the same arguments, we obtain  $x_2 = 0$ . Thus  $Z(L(T)) = \{0\}$  and the  $p$ -mapping is unique thanks to Corollary 2.2.8.  $\square$

## 5 The moduli space of Lie $p$ -algebras of rank 3

In the remaining sections, we illustrate the previous results in the case of three-dimensional Lie algebras. As stated in the introduction, let us denote by  $\text{Lie}_n$  the moduli stack of  $n$ -dimensional Lie algebras, and  $\text{L}_n$  the moduli space of *based* Lie algebras. Then we have the quotient stack presentation  $\text{Lie}_n = [\text{L}_n/\text{GL}_n]$  where  $\text{GL}_n$  acts by change of basis, by this action for any  $S$ -scheme  $T$ :

$$\begin{aligned} \text{GL}_n(T) \times \text{L}_n(T) &\rightarrow \text{L}_n(T) \\ (M, [\cdot, \cdot]_T) &\mapsto [\cdot, \cdot]'_T := (v \otimes w \mapsto M^{-1}[Mv, Mw]_T). \end{aligned}$$

Hence we are led to studying the  $\mathrm{GL}_n$ -equivariant geometry of  $L_n$ . In the following we will focus on the case  $n = 3$ . For a fixed prime  $p$ , we are interested in the moduli stack  $\mathrm{Lie}_3^p$  of restricted Lie algebras. For this, we use the morphism  $\pi : \mathrm{Lie}_3^p \rightarrow \mathrm{Lie}_3$  to the moduli stack of three-dimensional Lie algebras. Thanks to Theorem 4.1.8, after passing to the flattening stratification of the center of the universal Lie algebra, the map  $\pi$  is an affine bundle. Before studying  $\mathrm{Lie}_3^p$ , we will focus on  $\mathrm{Lie}_3$ .

For our purposes, it is important to obtain a description available in all characteristics. Even better, by defining  $L_3$  as a functor over  $\mathbb{Z}$  and proving its representability we gain insight into its fine scheme structure and the way the fibers vary. For all this section, we denote by  $L_{3,k}$  the base change of  $L_3$  with a field  $k$ . Here is a summary of our main results:

### 5.0.1. Theorem.

- 1) *The functor  $L_3$  is representable by an affine flat  $\mathbb{Z}$ -scheme of finite type.*
- 2) *The scheme  $L_3$  has two relative irreducible components  $L_3^{(1)}$  and  $L_3^{(2)}$  which are both flat with Cohen-Macaulay integral geometric fibers of dimension 6.*

In 2) it is noteworthy that the component we call  $L_3^{(1)}$  is very simple: it is isomorphic to 6-dimensional affine space  $\mathbb{A}_{\mathbb{Z}}^6$ . This is crucial because it turns out that the other component  $L_3^{(2)}$  is *linked* to it in the sense of liaison theory as developed by Peskine and Szpiro in [PS74], which provides powerful tools to deduce its properties.

Here we use the terminology "relative irreducible components" in the sense that  $L_3^{(1)}$  and  $L_3^{(2)}$  are flat of finite presentation over  $\mathbb{Z}$ , and that for all algebraically closed fields  $k$ ,  $L_{3,k}^{(1)}$  and  $L_{3,k}^{(2)}$  are the irreducible components of  $L_{3,k}$ . We use this terminology because we are over the ring of integers  $\mathbb{Z}$  then it makes more sense. For more details the reader can have a look at [R11], where the definition is given in 2.1.1, with a small (but not important for us) difference. Following the notation of *loc. cit.*, we will show in the following that  $\mathrm{Irr}(L_3/\mathbb{Z}) = \mathrm{Spec}(\mathbb{Z}) \amalg \mathrm{Spec}(\mathbb{Z})$ .

## 5.1 Classification over an algebraically closed field

To begin with, we recall the classification of isomorphism classes of three-dimensional Lie algebras over any algebraically closed field  $k$ . That is, the description of the ( $\mathrm{GL}_3$ -orbits of) geometric points of the moduli space  $L_{3,k}$ . Historically, the isomorphism classes of complex and real three-dimensional Lie algebras were classified as early as 1898 in Bianchi's paper [B98]. After the development of the algebraic theory of Lie algebras, the topic appeared in the lecture notes of Jacobson's course [J62]. From this moment the focus shifted to the algebraic variety structure of the set  $L_n$  of  $n$ -dimensional Lie algebras in work of Vergne [V66], Carles [C79], Carles and Diakit  [CD84], Kirillov and Neretin [KN84] and others. There, emphasis was put on low dimensions. Note that this bibliographic selection is by no means complete. Here, in order to allow varying primes  $p$ , we need to reformulate the classification of 3-dimensional Lie algebras over algebraically closed fields in a characteristic-free way.

### 5.1.1. Some Lie algebras: four discrete ones, and a family.

We introduce the five Lie algebras involved in the classification in a way that allows a characteristic-free statement. Notationally speaking, if  $l$  is a Lie algebra over a ring  $R$ , free of rank 3 with basis  $\{x, y, z\}$  and bracket defined by  $[x, y] = ax + by + cz$ ,  $[x, z] = dx + ey + fz$ ,  $[y, z] = gx + hy + iz$  for some coefficients  $a, \dots, i \in R$ , then we say that

$$\text{“the Lie algebra structure of } l \text{ is given by the matrix } \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \text{.”}$$

Moreover, for any Lie algebra  $l$  and any  $v \in l$ , the map  $\mathrm{ad}_v$  is linear, so we will always represent this linear map by its matrix in the base  $\{x, y, z\}$ .

The first four Lie algebras are defined over the ring of integers  $R = \mathbb{Z}$ :

- (1) the *abelian Lie algebra*  $\mathfrak{ab}_3$  with structure given by the zero matrix,
- (2) the *Heisenberg Lie algebra*  $\mathfrak{h}_3$  with structure matrix  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
- (3) the *Lie algebra*  $\mathfrak{r}$ , with structure matrix  $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ ,
- (4) the *simple Lie algebra*  $\mathfrak{s}$  with structure matrix  $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ .

The fifth Lie algebra is a family defined over the polynomial ring  $R = \mathbb{Z}[T]$ :

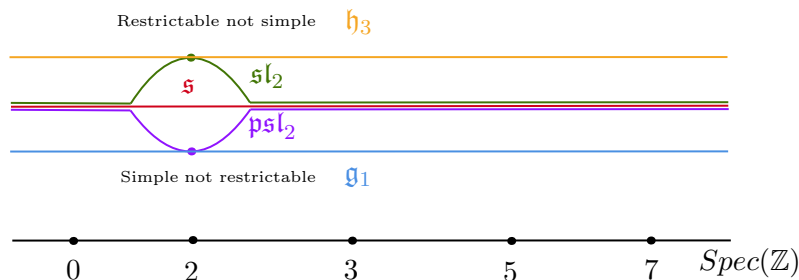
- (5) the Lie algebra  $\mathfrak{g}_T$  is defined by the structure matrix  $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & T & 0 \end{pmatrix}$ .

### 5.1.2. More about the simple Lie algebra.

The reader wondering about the place of  $\mathfrak{sl}_2$  and  $\mathfrak{psl}_2$  in the picture will find the following explanations useful. Let us write  $\{X, Y, H\}$  and  $\{X', Y', Z'\}$  for the classical bases of  $\mathfrak{sl}_2$  and  $\mathfrak{psl}_2$ , and  $\{x, y, z\}$  for that of  $\mathfrak{s}$ . We can write a sequence of morphisms of  $\mathbb{Z}$ -Lie algebras:

$$\mathfrak{sl}_2 \xrightarrow{\pi} \mathfrak{psl}_2 \xrightarrow{f} \mathfrak{s} \xrightarrow{\text{ad}} \mathfrak{gl}_3$$

with  $\pi$  and  $f$  given by  $X \mapsto X' \mapsto 2x$ ,  $Y \mapsto Y' \mapsto y$ ,  $H \mapsto 2Z' \mapsto 2z$ . The morphism  $f$  is an isomorphism over  $\mathbb{Z}[1/2]$ , but a contraction onto the subalgebra generated by  $y$  in the fiber at the prime  $p = 2$ . For any algebraically closed field  $k$ , the Lie algebra  $\mathfrak{s} \otimes k$  is the only simple Lie three-dimensional  $k$ -algebra because in characteristic  $p \neq 2$  we have  $\mathfrak{s} \otimes k \simeq \mathfrak{sl}_2 \otimes k$ , while if  $p = 2$  the algebra  $\mathfrak{s} \otimes k$  is known as  $W(1, \underline{2})'$ , the derived algebra of the *Jacobson-Witt algebra*. See [SF88], § 4.2 for more on  $W(n, \underline{m})$ , and especially Strade's paper [S07], Theorem 3.2 for the case of characteristic 2. In characteristic  $p = 2$ , the algebra  $\mathfrak{sl}_2$  happens to be isomorphic with the Heisenberg algebra  $\mathfrak{h}_3$ . Moreover, again when 2 is invertible the morphism  $\pi$  is an isomorphism so  $\mathfrak{psl}_2$  is isomorphic to  $\mathfrak{sl}_2$  i.e. to  $\mathfrak{s}$ . But in characteristic 2, using the adjoint representation  $\mathfrak{psl}_2$  is  $\mathfrak{gl}_3$  we see the bracket is given by the one denoted by  $\mathfrak{g}_1$  is the above classification. The following picture gives a summary of the situation.



It can be surprising to find that the group  $U_3$  of upper-triangular unipotent matrices of size 3 and the reductive group  $SL_2$  have the same Lie algebra in characteristic 2. But those Lie algebras are not isomorphic as Lie  $p$ -algebras. Indeed, let  $k$  be a field of characteristic 2. Seeing  $U_3$  as a subgroup of  $GL_3$ , we can see  $\mathfrak{h}_3$  as a 2-subalgebra of  $M_3(k)$ , where the 2-mapping on  $M_3(k)$  is the square map. Then we obtain that the 2-mapping on  $\mathfrak{h}_3$  is  $\gamma \equiv 0$ . Doing the same for  $SL_2$ , i.e. seeing the group  $SL_2 \subset GL_2$ , we obtain the 2-mapping:  $x, y \mapsto 0$  and  $z \mapsto z$ .

**5.1.3. More about the family  $\mathfrak{g}_T$ .** The Lie algebra  $\mathfrak{g}_0$  has center of dimension 1 and a 1-dimensional derived Lie algebra  $\mathfrak{g}'_0 = \text{Span}(y)$ . Now let us suppose  $t \in k$  for some field  $k$  and  $t \neq 0$ . Then the Lie algebra  $\mathfrak{g}_t$  has a trivial center and 2-dimensional derived Lie algebra  $\mathfrak{g}'_t = \text{Span}(y, z)$ . The adjoint action  $\text{ad} : \mathfrak{g}_t \rightarrow \text{End}(\mathfrak{g}'_t)$  factors through  $\mathfrak{g}_t^{\text{ab}} := \mathfrak{g}_t / \mathfrak{g}'_t$  which is free of rank 1. Any generator of  $\mathfrak{g}_t^{\text{ab}}$  is of the form  $ux$  for some unit  $u \in k^\times$  and acts on  $\mathfrak{g}'_t$  with eigenvalues  $\{u, ut\}$ . We see that the ratio of eigenvalues is well-defined up to inversion: that is, the class of  $t$  modulo the equivalence relation  $t \sim t^{-1}$  is independent of  $u$  and thus intrinsic to  $\mathfrak{g}_t$ . In this way we see that for every field  $k$  and elements  $t, t' \in k^\times$  we have:  $\mathfrak{g}_t \simeq \mathfrak{g}_{t'}$  if and only if  $t' \in \{t, t^{-1}\}$ .

Here is the main theorem of this subsection:

**5.1.4. Theorem.** *Let  $k$  be an algebraically closed field and denote by  $p$  its characteristic. Then any Lie algebra of dimension 3 over  $k$  is isomorphic to exactly one in the following table.*

Name		Structure	Orbit dimension	Center dimension	Restrictable
$\mathfrak{ab}_3$		<i>abelian</i>	0	3	<i>yes</i>
$\mathfrak{h}_3$		<i>nilpotent</i>	3	1	<i>yes</i>
$\mathfrak{r}$		<i>solvable</i>	5	0	<i>no</i>
$\mathfrak{s}$		<i>simple</i>	6	0	$p \neq 2$ <i>yes</i>   $p = 2$ <i>no</i>
$\mathfrak{g}_t$	$t \notin \mathbb{F}_p$	<i>solvable</i>	5	0	<i>no</i>
	$t \in \mathbb{F}_p \setminus \{0, 1\}$	<i>solvable</i>	5	0	<i>yes</i>
	$t = 0$	<i>solvable</i>	5	1	<i>yes</i>
	$t = 1$	<i>solvable</i>	3	0	<i>yes</i>

We split the proof in three parts: first of all we list the isomorphism classes (5.1.5), then we compute the dimensions of the orbits (5.1.7) and finally we determine the restrictable Lie algebras (5.1.8).

**5.1.5. Proof of the statement on isomorphism classes.** In order to have the list of the different orbits, we are following the proof in Fulton and Harris's book [FH91], Chapter 10. In this chapter the proof is divided in three parts, depending on the dimension of the derived Lie algebra. In this book though, the classification is done over the ring of complex numbers. The reader can verify that the proof can be generalised to any field of characteristic  $\neq 2$ , and up to a change of basis for the Lie algebra whose Lie structure is given by the matrix

$$\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

we find the classification we claim in the theorem (indeed changing  $X$  into  $2X$  and  $H$  into  $2H$ , we find the Lie algebra  $\mathfrak{s}$ ).

Now let us suppose  $\text{char}(k) = 2$ . The reader can verify that the proof done in [FH91] can still be generalised until the *loc. cit.* §10.4, where the authors consider Lie algebras with derived Lie algebra of rank 3. Indeed, in this part, they use an argument that is no longer true in characteristic 2: a certain endomorphism denoted by  $\text{ad}_H$  has three eigenvalues:  $0, \alpha$  and  $-\alpha$ , and because  $\alpha \neq 0$ , these three eigenvalues are different, then this endomorphism is diagonalizable. So now let us transform this argument in our case. So let  $\mathfrak{g}$  be a Lie algebra over  $k$ , with derived Lie algebra of rank 3. Let us do the same proof as done in *loc. cit.* §10.4 until this argument. Then, changing the eigenvector  $X$  of  $\text{ad}_H$  for the eigenvalue  $\alpha$  into  $\alpha X$ , and changing  $H$  in  $\alpha^{-1}H$ , we found that  $\text{ad}_H$  has 0 and 1 as eigenvalues. If  $\text{ad}_H$  is diagonalizable, we can apply the proof of [FH91]. Otherwise, we can apply the Jordan–Chevalley decomposition to  $\text{ad}_H$  and so we can suppose there is a basis  $\{X, Y, H\}$  of  $\mathfrak{g}$  such that  $[H, X] = X$  and  $[H, Y] = X + Y$ . Then thanks to the Jacobi condition, we know that

$$[H, [X, Y]] = [X, [H, Y]] + [Y, [H, X]] = [X, Y] + [X, Y] = 0.$$

Then  $[X, Y] = \beta H$  with  $\beta \neq 0$  because the derived Lie algebra of  $\mathfrak{g}$  is of dimension 3. Changing  $X$  into  $aX$  and  $Y$  into  $aY$  where  $a^2 = \beta^{-1}$ , we can suppose  $\beta = 1$  and using the matrix notation, we can suppose the bracket of  $\mathfrak{g}$  is given by

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

in the basis  $\{X, Y, H\}$ . Using the basis  $x = X, y = X + Y + H$  and  $z = X + H$ , we obtain

$$\begin{cases} [x, y] = [X, Y] + [X, H] = H + X = z \\ [x, z] = [X, H] = X = x \\ [y, z] = [X, H] + [Y, X] + [Y, H] + [H, X] = [X, Y] + [Y, H] = H + X + Y = y. \end{cases}$$

Hence we finally find the Lie algebra structure of  $\mathfrak{g}$ , so we find our classification.  $\square$

**5.1.6. Remark.** Here we use the terminology of [FH91] for the Lie algebras  $\mathfrak{g}_t$ , in particular for the Lie algebra  $\mathfrak{g}_{-1}$ . Actually you can find in the literature (for example in [KN84]) the terminology  $\mathfrak{m}(2)$  for this one. This name is due to the fact it is the Lie algebra of the group  $M(2)$  of euclidean motions of the plane.

### 5.1.7. Proof of the statement on the dimension of the orbits.

From now on, we use the notation  $o(l)$  for the orbit of a Lie algebra  $l$  under the group  $GL_3$ . In order to find the dimension of the orbits, we can calculate the dimension of the stabilizer, and use the orbit-stabilizer relation. Let  $l$  be a Lie algebra over  $k$ , i.e.  $l \in L_{3,k}(k)$ . Then the orbit of  $l$  is the image of this  $k$ -morphism:

$$\begin{aligned} GL_3(k) &\longrightarrow L_{3,k}(k) \\ A &\longmapsto A \cdot l. \end{aligned}$$

Let  $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \in GL_3(k)$  be a matrix in the stabilizer of  $o(l)$ . Then, we write

$$[Av, Aw] = A[v, w]$$

for the elements of the basis and we can find the equations for the stabilizer. For example let us fix a  $t$  in some field  $k$  and let us do it for  $\mathfrak{g}_t$ . We obtain these conditions:

$$\begin{cases} (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})y + t(a_{1,1}a_{3,2} - a_{3,1}a_{1,2})z = a_{1,2}x + a_{2,2}y + a_{3,2}z, \\ (a_{1,1}a_{2,3} - a_{1,3}a_{2,1})y + t(a_{1,1}a_{3,3} - a_{3,1}a_{1,3})z = ta_{1,3}x + ta_{2,3}y + ta_{3,3}z \\ (a_{1,2}a_{2,3} - a_{2,2}a_{1,3})y + t(a_{1,2}a_{3,3} - a_{1,3}a_{3,2})z = 0 \end{cases}$$

Then for instance if  $t = 0$ , the conditions of the stabilizer are now:

$$\begin{cases} a_{1,2} = a_{3,2} = 0, a_{1,1}a_{2,2} = a_{2,2} \\ a_{1,1}a_{2,3} - a_{1,3}a_{2,1} = 0 \\ a_{2,2}a_{1,3} = 0 \end{cases}.$$

But  $\det(A) = a_{2,2}a_{3,3} \neq 0$  then  $a_{1,3} = 0$  and  $a_{1,1} = 1$ , so  $a_{2,3} = 0$ . Hence

$$\text{Stab}(\mathfrak{g}_0) = \left\{ A \in GL_3(k), A = \begin{pmatrix} 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & 0 & a_{3,3} \end{pmatrix} \right\}.$$

Then  $\dim(\text{Stab}(\mathfrak{g}_0)) = 4$ , so  $\dim(o(\mathfrak{g}_0)) = 5$ . Now let us suppose  $t \neq 0$  and  $t \neq 1$ . Doing the same type of calculation, we obtain again:

$$\text{Stab}(\mathfrak{g}_t) = \left\{ A \in GL_3(k), A = \begin{pmatrix} 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & 0 & a_{3,3} \end{pmatrix} \right\}.$$

Then  $\dim(\text{Stab}(\mathfrak{g}_t)) = 4$  and  $\dim(o(\mathfrak{g}_t)) = 5$ .

We can do the same calculations for the other orbits in order to find the announced dimensions. The details are left to the reader.  $\square$

**5.1.8. Proof of the statement on the restricted orbits.** Now we can have a look at the restrictable orbits. Let us suppose for this section that  $\text{char}(k) = p > 0$ .

1. On the abelian Lie algebra,  $\gamma \equiv 0$  is a  $p$ -mapping.
2. The Lie algebra  $\mathfrak{h}_3 = \text{Lie}(U_3)$  is algebraic, hence restrictable.
3. The Lie algebra  $\mathfrak{s}$  is restrictable if  $\text{char}(k) \neq 2$ , because then  $\mathfrak{s} \simeq \mathfrak{sl}_2$  so it is algebraic. But if  $\text{char}(k) = 2$ ,  $\mathfrak{s}$  is not restrictable: one can see that  $\text{ad}_x^2$  is not a linear combination of  $\text{ad}_x$ ,  $\text{ad}_y$  and  $\text{ad}_z$ , then the condition (AL 1) can not be verified.
4. Let  $l := \mathfrak{r}$  with basis  $\{x, y, z\}$ . We have

$$\text{ad}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \text{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \text{ad}_z = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then we have

$$(\text{ad}_x)^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence  $(\text{ad}_x)^p$  is not a linear combination of  $\text{ad}_x$ ,  $\text{ad}_y$  and  $\text{ad}_z$ , so we conclude that  $\mathfrak{r}$  is not restrictable.

5. For the end let  $t \in k$  and let us have a look at the Lie algebra  $\mathfrak{g}_t$  with basis  $\{x, y, z\}$ . We have

$$\text{ad}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}; \text{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \text{ad}_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & 0 \end{pmatrix}.$$

Then we have

$$(\text{ad}_x)^p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^p \end{pmatrix} \text{ and } (\text{ad}_y)^p = (\text{ad}_z)^p \equiv 0.$$

Then using Theorem 2.2.9 (Jacobson's theorem), and the definition of a restrictable Lie algebra, we know that  $l$  is restrictable if and only if  $t^p = t$  i.e. if and only if  $t \in \mathbb{F}_p$ .  $\square$

**5.1.9. Example.** Thanks to this classification, we can illustrate Theorem 4.1.8. Indeed, let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $\mathfrak{g}_T := \text{Spec}(k[T]) \rightarrow L_3$ , given on the rings by  $a, c, d, e, g, h, i \mapsto 0$ ,  $b \mapsto 1$  and  $f \mapsto T$ . Let us calculate  $L_3^{\text{res}} \times \mathfrak{g}_T$ . Thanks to what we have done before, we know

$$L_3^{\text{res}} \times \mathfrak{g}_T = \text{Spec } k[T]/(T^p - T).$$

Then we see that  $L_3^{\text{res}} \times \mathfrak{g}_T$  is closed in  $\text{Spec}(k[T])$ .

### 5.1.10. First consequences for the topology of $L_{3,k}$ .

To finish this subsection, we derive the first topological description of the irreducible components of the moduli space that the classification just given affords. Finer information can only be obtained with the more advanced algebraic tools of liaison theory presented in Subsection 5.2. First, note that:

- the points corresponding to the Lie algebras  $\mathfrak{ab}_3$  and  $\mathfrak{h}_3$  are in the closure of the orbit of the simple algebra  $\mathfrak{s}$ ;
- the point corresponding to the Lie algebras  $\mathfrak{r}$  is in the closure of the orbit of the 1-parameter algebra  $\mathfrak{g}_T$  (to see this, let  $k$  be a field and let  $t \in k$ , and consider the Lie algebra defined by the structure matrix  $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & t & 0 \end{pmatrix}$ . For  $t \neq 1$ , the structure constants of this algebra in the basis  $\{x, y, y + (t - 1)z\}$  are those of  $\mathfrak{g}_t$  and when  $t \rightarrow 1$  the limit of this family is  $\mathfrak{r}$ ).



Therefore, in order to single out the irreducible components of  $L_{3,k}$  it is enough to look at  $o(\mathfrak{s})$  and  $o(\mathfrak{g}_T)$ . We consider their orbit morphisms :

$$\text{ev}_{\mathfrak{s}} : \text{GL}_3 \times \text{Spec}(\mathbb{Z}) \longrightarrow L_3 \quad , \quad \text{ev}_{\mathfrak{g}_T} : \text{GL}_3 \times \text{Spec}(\mathbb{Z}[T]) \longrightarrow L_3.$$

We obtain the following result.

**5.1.11. Lemma.** *In each geometric fiber over a point  $\text{Spec}(k) \rightarrow \text{Spec}(\mathbb{Z})$ , the following hold:  $\text{ev}_{\mathfrak{s}}$  and  $\text{ev}_{\mathfrak{g}_T}$  have 6-dimensional image, their sum  $\text{GL}_{3,k} \amalg \text{GL}_{3,k}[T] \rightarrow L_{3,k}$  is dominant, and  $L_{3,k}$  has pure dimension 6 with two irreducible components.*

*Proof.* Everything takes place in the fiber over  $\text{Spec}(k) \rightarrow \text{Spec}(\mathbb{Z})$  so for simplicity we omit  $k$  from the notation. The stabilizer of  $\mathfrak{s}$  has dimension 3, hence its orbit (the image of  $\text{ev}_{\mathfrak{s}}$ ) has dimension 6. For the orbit of  $\mathfrak{g}_T$  we may as well remove the value  $t = 1$  without changing the dimension. Then the stabilizer of  $\mathfrak{g}_T$  is flat, of dimension 2 over  $\text{Spec}(k[T, (T-1)^{-1}])$ , hence it has dimension 3 over  $k$ , and again the orbit (the image of  $\text{ev}_{\mathfrak{g}_T}$ ) has dimension 6. The fact that  $\text{GL}_{3,k} \amalg \text{GL}_{3,k}[T] \rightarrow L_{3,k}$  is dominant follows from the fact that the remaining orbits lie in the closure of those two, as we indicated before the lemma. Finally since both images of  $\text{ev}_{\mathfrak{s}}$  and  $\text{ev}_{\mathfrak{g}_T}$  are distinct, irreducible, of dimension 6, their closures are the irreducible components of  $L_{3,k}$ .  $\square$

## 5.2 Schematic description of the moduli space $L_3$

Let us now focus on the schematic structure of the moduli space of three-dimensional Lie algebras. We first prove the representability of the functor  $L_3$  over the ring of integers.

**5.2.1. Definition.** The *moduli space of based Lie algebras of rank three* is the following functor:

$$\begin{aligned} L_3 : \text{Sch} &\longrightarrow \text{Set} \\ T &\longmapsto \{[\cdot, \cdot] : \mathcal{O}_T^3 \otimes \mathcal{O}_T^3 \rightarrow \mathcal{O}_T^3 ; \text{ where } [\cdot, \cdot] \text{ is a Lie bracket}\}. \end{aligned}$$

**5.2.2. Proposition.** *This functor is representable by a closed subscheme of  $\mathbb{A}_{\mathbb{Z}}^9$ , given by*

$$\text{Spec}(\mathbb{Z}[a, b, c, d, e, f, g, h, i] / \langle ah + di - fg - bg, ie + bd - fh - ae, hc + dc - af - bi \rangle).$$

*Proof.* Let  $T$  be a scheme, and let  $\{x, y, z\}$  be a  $\mathcal{O}_T(T)$ -basis of  $\mathcal{O}_T(T)^3$ .

Let us write  $(a, b, c, d, e, f, g, h, i) \in \mathcal{O}_T(T)^9$  for the coefficients of the Lie bracket  $[\cdot, \cdot]$ , where

$$[x, y] = ax + by + cz, \quad [x, z] = dx + ey + fz \quad \text{and} \quad [y, z] = gx + hy + iz.$$

Then by definition, we have:

$$\begin{aligned} L_3(T) &= \{[\cdot, \cdot] : \mathcal{O}_T^3 \otimes \mathcal{O}_T^3 \rightarrow \mathcal{O}_T^3, \text{ where } [\cdot, \cdot] \text{ is a Lie bracket}\} \\ &\simeq \{(a, \dots, i) \in \mathcal{O}_T(T)^9 ; ah + di - fg - bg = ie + bd - fh - ae = hc + dc - af - bi = 0\}. \end{aligned}$$

One can easily verify that the conditions on the 9-tuple correspond to the Jacobi condition.  $\square$

**Notations:** From now on, we will use the following notations:

-  $Q_1 := ah + di - fg - bg$ ,  $Q_2 := ie + bd - fh - ae$  and  $Q_3 := hc + dc - af - bi$  and  $R_3 := \mathbb{Z}[a, b, c, d, e, f, g, h, i] / (Q_1, Q_2, Q_3)$ , hence  $L_3 = \text{Spec}(R_3)$ . For any ring  $A$ , we write  $R_{3,A}$  for  $R_3 \otimes A$ .

- Let us remark that the Jacobi condition can be written as

$$\begin{cases} Q_1 = ah + di - fg - bg = 0 \\ Q_2 = ie + bd - fh - ae = 0 \\ Q_3 = hc + dc - af - bi = 0 \end{cases} \Leftrightarrow \begin{cases} (a-i)h + (b+f)(-g) + (d+h)i = 0 \\ (a-i)(-e) + (b+f)(-h) + (d+h)b = 0 \\ (a-i)(-f) + (b+f)(-i) + (d+h)c = 0. \end{cases}$$

Then let us denote

$$M := \begin{pmatrix} h & -g & i \\ -e & -h & b \\ -f & -i & c \end{pmatrix} \text{ and } X := \begin{pmatrix} L_1 := a - i \\ L_2 := b + f \\ L_3 := d + h \end{pmatrix}.$$

Then, the Jacobi condition is verified if and only if  $MX = 0$ .

Actually here, we find again the underlying set of the irreducible components of  $L_3$  that we saw in Lemma 5.1.11. Indeed,  $MX = 0$  if and only if  $X = 0$  or  $X \neq 0$  so the matrix  $M$  is not injective, i.e.  $\det(M) = 0$  and there exists a nonzero vector in its kernel.

- For these reasons, we finally set

$$L := (L_1, L_2, L_3), I := (Q_1, Q_2, Q_3) \text{ and } J = (Q_1, Q_2, Q_3, \det(M)) = I + (\det(M)),$$

and we will see that the two irreducible components are given, as schemes, by the ideals  $L$  and  $J$ , and we will give a more precise description of them. When it is clear from the context, we will still write  $I, J$  and  $L$  for those ideals seen in  $R_{3,A}$  for any ring  $A$ .

### 5.2.3. Description of the irreducible components.

**5.2.4. Theorem.** *The affine scheme  $L_3$  can be decomposed in two irreducible components: the first one is*

$$L_3^{(1)} := \text{Spec}(\mathbb{Z}[a, \dots, i]/L) \simeq \mathbb{A}^6$$

and the second one is

$$L_3^{(2)} := \text{Spec}(\mathbb{Z}[a, \dots, i]/J).$$

*These irreducible components are linked to each other, they are both Cohen-Macaulay, flat over  $\mathbb{Z}$  with integral geometric fibers of dimension 6.*

Let  $A$  be any regular ring (for the following we will use  $A = \mathbb{Z}$ ,  $A = \mathbb{Q}$  or  $A = \mathbb{F}_p$ ). We have  $R_{3,A}/(L \otimes A) \simeq A[a, b, c, d, e, g]$  so  $L \otimes A$  is prime in  $L_{3,A}$ . Let us show that the ideal  $L$  describes an irreducible component of  $L_{3,A}$ . Let us denote  $D := A[a, \dots, i]$ .

**5.2.5. Lemma.** *The ideal  $L$  is minimal in  $D$  among the prime ideals containing  $I$ .*

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(D)$  be such that  $I \subset \mathfrak{p} \subset L$ . First of all, because we have

$$M \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix},$$

we obtain

$$\det(M) \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = (\text{com}(M))^t \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}.$$

So we can write

$$(*) \begin{cases} \det(M)L_1 = (-hc + bi)Q_1 + (gc - i^2)Q_2 + (-gb + ih)Q_3 \\ \det(M)L_2 = (ec - bf)Q_1 + (hc + if)Q_2 + (-ie - bh)Q_3 \\ \det(M)L_3 = (ei - hf)Q_1 + (fg + ih)Q_2 + (-h^2 - ge)Q_3 \end{cases}.$$

But  $\det(M) = -ch^2 + gbf + ei^2 - gec + hbi - ihf$  then  $\det(M) \notin L$ , so  $\det(M) \notin \mathfrak{p}$ . Thanks to (\*), this means that  $L_1, L_2, L_3 \in \mathfrak{p}$ , i.e.  $\mathfrak{p} = L$ . So  $L$  is a minimal prime among the prime ideals containing  $I$ .  $\square$

So now we need to show that  $J$  also describes schematically an irreducible component of  $L_{3,A}$ . In order to do this, we use liaison theory.

**5.2.6. Definition.** Let  $J$  and  $L$  be two ideals in a ring  $R$ . We say that  $J$  and  $L$  are *linked* in  $R$  by an ideal  $I$  if  $L = [I : J]$  and  $J = [I : L]$ .

**5.2.7. Lemma.** *The sequences  $(L_1, L_2, L_3)$  and  $(Q_1, Q_2, Q_3)$  are regular in  $A$ .*

*Proof.* Left to the reader. □

**5.2.8. Corollary.** *Let us denote by  $M^t$  the multiplication by the matrix  $M^t : D^3 \rightarrow D^3$ . The two regular sequences  $\{L_1, L_2, L_3\}$  and  $\{Q_1, Q_2, Q_3\}$  define two Koszul complexes and we have a morphism of Koszul complexes between them:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{d_3^Q} & D[2]^3 & \xrightarrow{d_2^Q} & D[4]^3 & \xrightarrow{d_1^Q} & D[6] & \longrightarrow & 0 \\
& & \downarrow \det(M^t) & & \downarrow \wedge^2 M^t & & \downarrow M^t & & \downarrow \text{id} & & \\
0 & \longrightarrow & D & \xrightarrow{d_3^L} & D[2]^3 & \xrightarrow{d_2^L} & D[4]^3 & \xrightarrow{d_1^L} & D[6] & \longrightarrow & 0.
\end{array}$$

Here:  $D[n]$  is the graded ring  $D$  where we shift the graduation  $n$  times, in order to have a morphism of graded rings (i.e. a polynomial of degree  $d$  in  $D$  is seen in the  $(d - n)$ -th graduation of  $D[n]$ ).

*Proof.* This diagram comes from the definition of the Koszul complex (see for example Eisenbud's book [E95], Section 17, Subsection 17.2) and the functoriality of the Koszul complex: indeed we have  $\wedge^0(D^3) = \wedge^3(D^3) = D$  and  $\wedge^1(D^3) = \wedge^2(D^3) = D^3$ , and the morphism  $\text{id}$  is just the morphism  $\wedge^0 M^t$ , the morphism  $M^t$  is  $\wedge^1 M^t$  and the morphism  $\det(M^t)$  is  $\wedge^3 M^t$ . □

**5.2.9. Remark.** In the following, we will not need the graduation of our complex, so we will be writing and using it without specifying the graduation.

**5.2.10. Corollary.** *A projective resolution of  $[I : L]$  can be obtained by taking the mapping cone of the map of Koszul complexes  $(M^t)^\vee : K.[L_1, L_2, L_3]^\vee \rightarrow K.[Q_1, Q_2, Q_3]^\vee$ .*

*Proof.* This is straightforward from Proposition 2.6 in [PS74]. □

**5.2.11. Corollary.** *The ideal  $[I : L]$  is perfect of height 3. Moreover,*

$$[I : L] = I + \det(M) = J.$$

*Proof.* Using the notations of Corollary 5.2.8, we can see that the mapping cone of  $M^t$  is the following complex:

$$0 \rightarrow D \oplus 0 \xrightarrow{u} D^3 \oplus D \rightarrow D^3 \oplus D^3 \rightarrow D \oplus D^3 \xrightarrow{f} 0 \oplus D \rightarrow 0$$

where the morphism  $f$  is defined by:

$$\begin{aligned}
f : D \oplus D^3 &\rightarrow D \\
(a, b, c, d) &\mapsto a + bL_1 + cL_2 + dL_3.
\end{aligned}$$

Then, dualizing the complex, we obtain:

$$0 \rightarrow (D)^\vee \xrightarrow{f^\vee} (D \oplus D^3)^\vee \rightarrow (D^3 \oplus D^3)^\vee \rightarrow (D \oplus D^3)^\vee \rightarrow (D)^\vee \rightarrow 0$$

where the morphism  $f^\vee$  is defined by:

$$\begin{aligned}
f^\vee : (D)^\vee &\rightarrow (D \oplus D^3)^\vee \\
\phi &\mapsto ((a, b, c, d) \mapsto \phi(a + bL_1 + cL_2 + dL_3)).
\end{aligned}$$

Replacing the morphism  $f^\vee$  with its image, and showing this image is projective, we manage to reduce the length of this resolution. Indeed, let us denote by  $H$  the kernel of  $f$ :

$$H := \{(a, b, c, d) \in D^4, a + bL_1 + cL_2 + dL_3 = 0\}.$$

Then let us show

$$\text{im}(f^\vee) = (D^4/H)^\vee \simeq (D^3)^\vee = \{\psi \in (D^4)^\vee, \psi|_H \equiv 0\}.$$

Let  $\psi$  be a form on  $D^4$  such that  $\psi(H) = 0$ . Then for all  $b, c, d \in D$ ,

$$\psi(-bL_1 - cL_2 - dL_3, b, c, d) = 0.$$

Let  $(a, b, c, d) \in D^4$ . Then

$$\begin{aligned} \psi(a, b, c, d) &= \psi(a + bL_1 + cL_2 + dL_3, 0, 0, 0) + \psi(-bL_1 - cL_2 - dL_3, b, c, d) \\ &= \psi(a + bL_1 + cL_2 + dL_3, 0, 0, 0). \end{aligned}$$

Let us set  $\phi : D \rightarrow D, x \mapsto \psi(x, 0, 0, 0)$ . Then we obtain  $\psi = f^\vee(\phi)$ . The other inclusion is trivial.

Then,  $\text{im}(f)$  is free over  $D$ , so the projective resolution given by Corollary 5.2.10 can be changed into this one:

$$0 \rightarrow \text{im}(f^\vee) \rightarrow (D^3 \oplus D^3)^\vee \rightarrow (D^3 \oplus D)^\vee \rightarrow (D)^\vee \rightarrow 0$$

which is a projective resolution of length 3 of  $D/[I : L]$ . Then  $\text{projdim}([I : L]) \leq 3$ . But because  $I \subset [I : L]$ , we know that  $\text{grade}([I : L]) \geq 3$ . Hence  $[I : L]$  is perfect of grade 3.

Now in order to show  $[I : L] = I + \det(M)$ , we will see that the resolution found above is actually a resolution of  $I + \det(M)$ . Let us calculate the cokernel of the dual of this map:

$$\begin{aligned} u : D &\rightarrow D^3 \oplus D \\ 1 &\mapsto (-Q_3, Q_2, -Q_1, \det(M)). \end{aligned}$$

Then the dual map is given by

$$\begin{aligned} u^\vee : (D^3 \oplus D)^\vee &\rightarrow D^\vee \\ \phi &\mapsto (1 \mapsto \phi(-Q_3, Q_2, -Q_1, \det(M))). \end{aligned}$$

Then the cokernel of this morphism is given by  $D/(I + \det(M))$ . So by uniqueness of the cokernel, we have  $[I : L] = I + \det(M) = J$ .  $\square$

In order to show that the ideals  $L$  and  $J$  are linked, it remains to show that  $L = [I : J]$ . It is one of the purposes of the following proposition, which is the main result of liaison theory that we will be using in this article. It will give us powerful tools to understand the ideal  $J$  thanks to the ideal  $L$ . For more convenience, let us denote by  $\underline{L} := L/I$  and  $\underline{J} := J/I$  the two quotient ideals in the quotient ring  $R_{3,A}$ . We have seen that  $\underline{J}$  is the annihilator of  $\underline{L}$  in  $R_{3,A}$ .

**5.2.12. Proposition.** *The ideal  $\underline{L}$  is the annihilator of  $\underline{J}$ , and  $R_{3,A}/\underline{J}$  has Cohen-Macaulay geometric fibers of dimension 6.*

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(R_{3,A})$  and let us denote  $R := (R_{3,A})_{\mathfrak{p}}$ . Let us denote  $I_1 := (\underline{L})_{\mathfrak{p}}$  and  $I_2 := (\underline{J})_{\mathfrak{p}}$ . We would like to apply Proposition 1.3 from [PS74]. Let us show that we are in good conditions:

1.  $R$  is a Gorenstein local ring: indeed,  $A[a, \dots, i]$  is regular hence Gorenstein, but because  $I = (Q_1, Q_2, Q_3)$  is a regular sequence, then  $A[a, \dots, i]/I$  is also Gorenstein. Hence  $R$  is Gorenstein as a localisation of a Gorenstein ring.
2.  $I_2 = \text{ann}(I_1)$ : indeed  $I_2 = (\underline{J})_{\mathfrak{p}} = (\text{ann}(\underline{L}))_{\mathfrak{p}} = \text{ann}((\underline{L})_{\mathfrak{p}})$  because  $R$  is Noetherian.
3.  $\dim(R) = \dim(R/I_1)$  because as  $R$  is Gorenstein hence Cohen-Macaulay, so we can apply Proposition 2.15 d) in Chapter 8, Section 8.2.2 in Liu's book [L02], using  $I_1$  as prime ideal which has height 0 (because  $L$  is a minimal prime thanks to Lemma 5.2.5).

4.  $R/I_1$  is regular hence Cohen-Macaulay. Then using Proposition 1.3 in [PS74], we obtain that

$$(\underline{L})_{\mathfrak{p}} = [0 : (\underline{J})_{\mathfrak{p}}] = [0 : \underline{J}]_{\mathfrak{p}}.$$

Because we obtain this result for all  $p \in \text{Spec}(R)$  and because we already know the inclusion  $\underline{L} \subset [0 : \underline{J}]$ , then we have the equality not only locally but globally

$$\underline{L} = [0 : \underline{J}].$$

Hence  $L$  is an associate prime of  $I$ , and  $L$  is the annihilator of  $J$  in  $R_{3,A}$ .

The end of the lemma follows from Proposition 1.3 in [PS74].  $\square$

Then now we know that the ideals  $J$  and  $L$  are linked, and thanks to this the previous proposition says that because  $D/L$  is Cohen-Macaulay, then  $D/J$  is Cohen-Macaulay as well. Thanks to this, we will prove that, for any algebraically closed field  $k$ , the ideal  $J$  is prime in  $R_{3,k}$ , then it describes schematically the second irreducible component of  $L_{3,k}$ . We need this preliminary lemma first.

**5.2.13. Lemma.** *Let  $k$  be a field. The scheme  $L_{3,k}^{(2)} = \text{Spec}(k[a, \dots, i]/J)$  has a smooth point.*

*Proof.* Actually, we will find a  $\mathbb{Z}$ -point of  $L_3$  along which  $L_3$  is smooth. Let  $t \in \mathbb{Z}$  and let  $\mathfrak{g}_t : \text{Spec}(\mathbb{Z}) \rightarrow \text{Spec}(\mathbb{Z}[a, \dots, i]/J)$  given on the rings by  $a, c, d, e, g, h, i \mapsto 0, b \mapsto 1$  and  $f \mapsto t$ . Let us recall that  $\det(M) = -ch^2 + gbf + ei^2 - gec + hbi - ihf$ .

Then,

$$\mathfrak{g}_t^*(\Omega_{L_3^{(2)}}^1) = \mathfrak{g}_t^*\left(\frac{\mathbb{Z} \cdot da \oplus \dots \oplus \mathbb{Z} \cdot di}{dQ_1, dQ_2, dQ_3, d \det(M)}\right)$$

and we have the following equalities:

$$\begin{aligned} dQ_1 &= a(dh) + h(da) + d(di) + i(dd) - f(dg) - g(df) - b(dg) - g(db) \\ dQ_2 &= i(de) + e(di) + b(dd) + d(db) - f(dh) - h(df) - a(de) - e(da) \\ dQ_3 &= h(dc) + c(dh) + d(dc) + c(dd) - a(df) - f(da) - b(di) - i(db) \\ d \det(M) &= (fg + hi)db + (-eg - h^2)dc + (-cg + i^2)de + (bg - hi)df \\ &\quad + (-ce + bf)dg + (-2ch + bi - fi)dh + (bh - fh + 2ei)di. \end{aligned}$$

Then,

$$\mathfrak{g}_t^*(\Omega_{L_3^{(2)}}^1) = \frac{\mathbb{Z} \cdot da \oplus \dots \oplus \mathbb{Z} \cdot di}{(t+1)dg, dd - tdh, di + tdf, tdg} = \frac{\mathbb{Z} \cdot da \oplus \dots \oplus \mathbb{Z} \cdot di}{dg, dd - tdh, di + tdf}$$

so it is a free  $\mathbb{Z}$ -module of rank 6. But thanks to Proposition 5.2.12, we know that  $L_3^{(2)}$  has dimension 6, thus this  $\mathbb{Z}$ -point is smooth and the proof is done.  $\square$

**5.2.14. Corollary.** *The ideal  $J$  is prime in  $R_{3,k}$  for any algebraically closed field  $k$ , and the scheme  $\text{Spec}(k[a, \dots, i]/J)$  is integral.*

*Proof.* Let  $k$  be an algebraically closed field. We have proved in the Lemma 5.1.11 that  $V(J) = \overline{o(\mathfrak{g}_T)} \subset R_{3,k}$  which is irreducible. Moreover, we saw in Lemma 5.2.13 that it has a smooth point. But because we know from Proposition 5.2.12 that it is Cohen-Macaulay, then without associated points, so because it is generically reduced, it is reduced. Hence  $V(J)$  is integral and  $J$  is prime.  $\square$

**5.2.15. Proposition.** *In the polynomial ring  $A[a, \dots, i]$ , we have the equality*

$$I = J \cap L.$$

*Proof.* Thanks to Corollary 3.5 in Section 3, Subsection 3.2 of [E95], it is sufficient to show that  $(J \cap L)_{\mathfrak{p}} \subset I_{\mathfrak{p}}$  for all primes  $\mathfrak{p}$  associated to  $I$ . Thanks to Proposition 5.2.12, we know that  $L$  is an associated prime of  $I$ .

Now let  $\mathfrak{p}$  be such a prime ideal. As  $(J \cap L)_{\mathfrak{p}} \subset J_{\mathfrak{p}} \cap L_{\mathfrak{p}}$ , it is sufficient to prove that  $J_{\mathfrak{p}} \cap L_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ .

- If  $\mathfrak{p} = L$ , we will show that  $I_{\mathfrak{p}} = L_{\mathfrak{p}}$ . Then let  $\frac{a}{b} \in L_{\mathfrak{p}}$ , with  $a \in L$  and  $b \notin L$ . Then, because  $\det(M) \notin L$ ,  $\frac{a}{b} = \frac{a \det(M)}{b \det(M)} \in I_{\mathfrak{p}}$  because we showed in Lemma 5.2.5 that  $\det(M)L_1 \in I$ . Then  $I_{\mathfrak{p}} = L_{\mathfrak{p}}$ , so  $J_{\mathfrak{p}} \cap L_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ .
- If  $\mathfrak{p} \neq L$ , we will show that  $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ . Let  $\frac{\det(M)}{b} \in J_{\mathfrak{p}}$ . As  $L \neq \mathfrak{p}$  and  $\mathfrak{p}$  is minimal, we have  $L \not\subset \mathfrak{p}$ , so we can suppose  $L_1 \notin \mathfrak{p}$ . Then because  $\det(M)L_1 \in I$  and  $\frac{\det(M)}{b} = \frac{\det(M)L_1}{bL_1}$ , we have the equality  $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ . Hence  $J_{\mathfrak{p}} \cap L_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ .

□

**5.2.16. Corollary.** *The ideal  $J$  is minimal among the prime ideals containing  $I$ .*

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(A[a, \dots, i])$  such that  $I \subset \mathfrak{p} \subset J$ . If  $\det(M) \in \mathfrak{p}$ , then  $J = \mathfrak{p}$ . Otherwise, we have  $L \subset \mathfrak{p} \subset J$ , hence  $I = J \cap L = L$  which is impossible. Then  $\mathfrak{p} = J$  and  $J$  is a minimal prime among the ones containing  $I$ . □

Now we have our two relative irreducible components denoted by  $L_{3,A}^{(1)}$  and  $L_{3,A}^{(2)}$ , we still have to prove the flatness of  $L_3^{(2)}$  over the ring of integer  $\mathbb{Z}$ . We need a preliminary lemma first.

**5.2.17. Lemma.** *Let  $R$  be any commutative ring with unit, and  $d \geq 1$ . Let*

$$0 \rightarrow P_{d+1} \rightarrow P_d \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$$

*be an exact sequence of  $R$ -modules such that all  $P_1, \dots, P_d$  are  $R$ -flat and we suppose that this exact sequence is still exact after any base change  $R \rightarrow R/I$  where  $I$  is an ideal of  $R$ . Then,  $M$  is also  $R$ -flat.*

*Proof.* We do an induction on the integer  $d$ . If  $d = 1$ , then this is classic. If  $d > 1$ , let

$$0 \rightarrow P_{d+2} \xrightarrow{\phi_{d+2}} P_{d+1} \xrightarrow{\phi_{d+1}} P_d \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$$

be an exact sequence with  $d + 3$  terms, which is like in the statement. Let us define

$$C := \text{coker}(\phi_{d+2}) = P_{d+1}/\text{im}(\phi_{d+2}) = P_{d+1}/\ker(\phi_{d+1}) = \text{im}(\phi_{d+1}) = \ker(\phi_d).$$

Because the inclusion  $P_{d+2} \xrightarrow{\phi_{d+2}} P_{d+1}$  is universally injective by hypothesis, the cokernel  $C$  is flat over  $R$ . Then the following exact sequence:

$$0 \rightarrow C \xrightarrow{\phi_d} P_d \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$$

is an exact sequence of  $d + 2$  terms which is like in the statement. Then by induction, we conclude that  $M$  is  $R$ -flat. □

**5.2.18. Corollary.** *The scheme  $L_3^{(2)}$  is flat over  $\text{Spec}(\mathbb{Z})$ .*

*Proof.* Let  $B$  be any ring. Then, the resolution found in Corollary 5.2.10 is a resolution of the ideal  $J$  in  $B[a, \dots, i]$ , so it is universal. Then we can apply the previous Lemma 5.2.17. □

**5.2.19. Proposition.** *The entire scheme  $L_3$  is flat over  $\mathbb{Z}$ , and the ideal  $I$  is radical.*

*Proof.* Because we have proved  $I = L \cap J$  in Proposition 5.2.15, the following map is injective:

$$\begin{aligned} \mathbb{Z}[a, \dots, i]/I &\hookrightarrow \mathbb{Z}[a, \dots, i]/L \times \mathbb{Z}[a, \dots, i]/J \\ \bar{a} &\mapsto (\bar{a}, \bar{a}). \end{aligned}$$

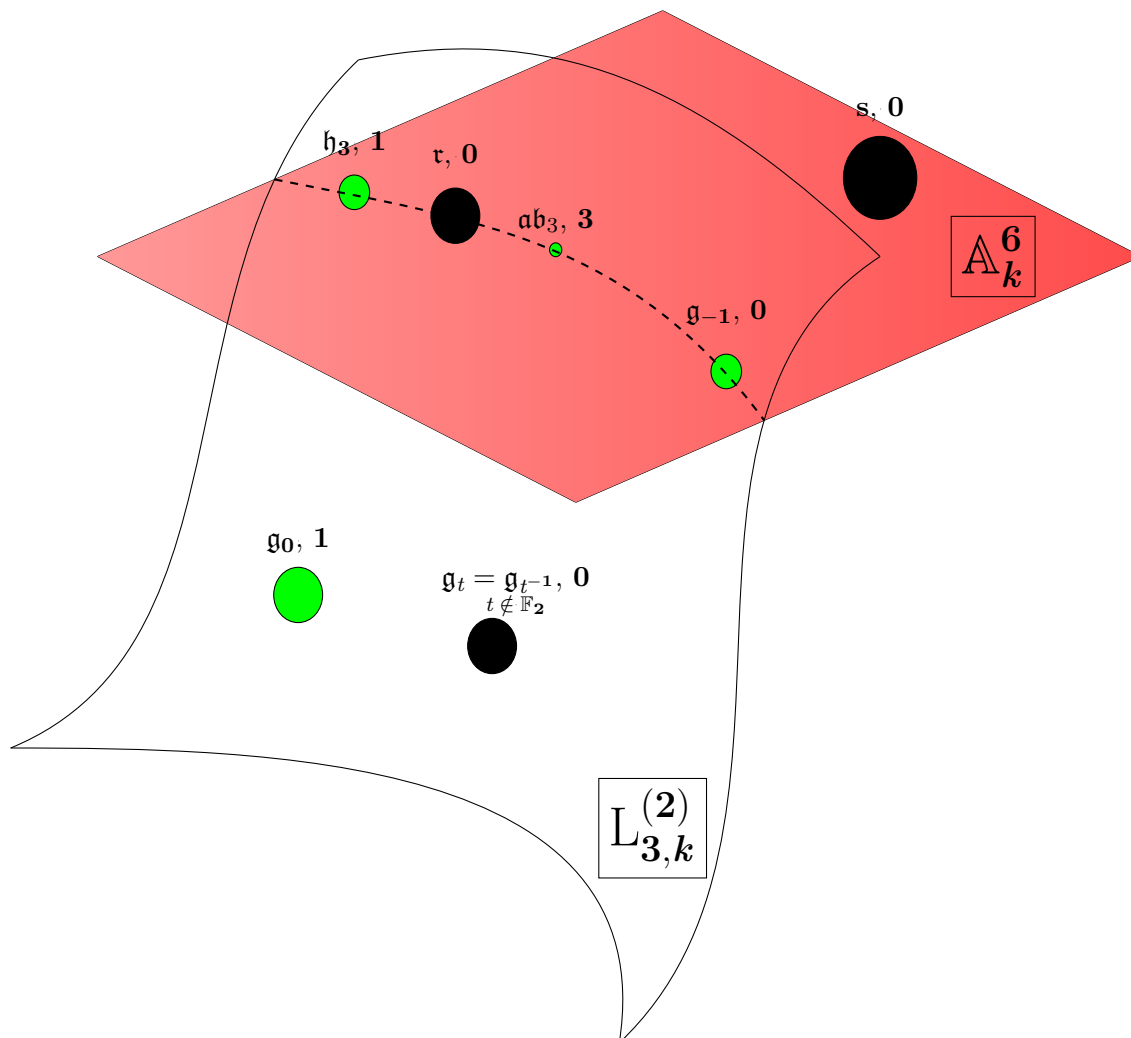
But both of the rings that appear on the right-hand side are flat over  $\mathbb{Z}$ , then without torsion, so  $\mathbb{Z}[a, \dots, i]/I$  is without  $\mathbb{Z}$ -torsion, then it is  $\mathbb{Z}$ -flat.

Moreover, because  $L$  and  $J$  are both radical and  $I = J \cap L$ , then  $J$  is radical. □

### 5.3 Summary: a picture of a geometric fiber of our moduli space

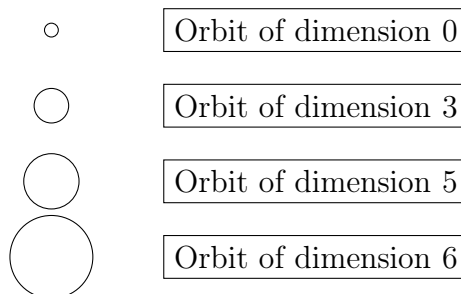
#### 5.3.1. In characteristic $p = 2$ .

Let  $k$  be an algebraically closed field of characteristic  $p = 2$ . Here is a picture representing the two irreducible components of  $L_{3,k}$ . The points correspond to the different orbits on it, and we specify the restrictable ones. We also write on it the dimension of the center of those Lie algebras.



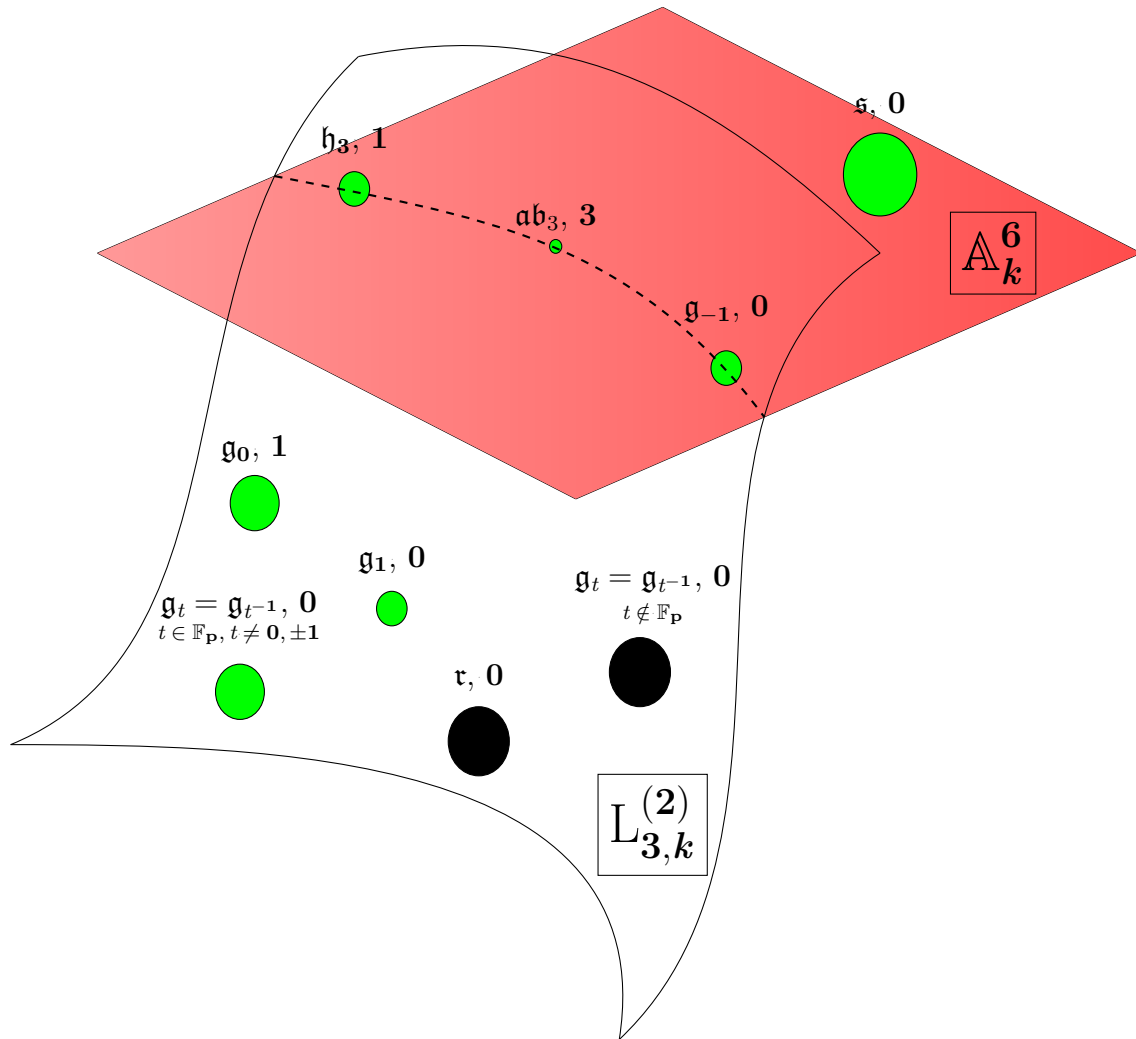
**Caption:**

- Restrictable orbit.
- Non-restrictable orbit



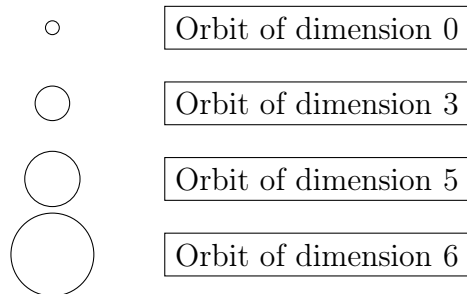
5.3.2. In characteristic  $p \neq 2$ .

Let  $k$  be an algebraically closed field of characteristic  $p > 2$ . As on the previous page, here is a picture representing the two irreducible components of  $L_{3,k}$  with the orbits on it, and the dimension of the center of those Lie algebras.



**Caption:**

- Restrictable orbit.
- Non-restrictable orbit





## 6 Smoothness of $L_3^{\text{res}}$ on the flattening stratification of the center

We did not study all the equations of the singular locus of  $L_3$ , but using [Macaulay2], we can see that the singular locus of  $L_{3,\mathbb{Q}}^{(2)}$  is given by an ideal, whose radical is  $I_2(M) + L$ , where  $I_2(M)$  is the ideal generated by the two-minors of the matrix  $M$ , the one introduced in Subsection 5.2. For the rest of the article, let us denote by  $\mathbb{L}_n := \mathbb{A}_{\mathbb{L}_n}^n$  the universal Lie algebra of rank  $n$  over  $\mathbb{L}_n$ . Then in the following, we will study the smoothness of the restricted locus  $L_3^{\text{res}} \hookrightarrow L_3$  of the universal Lie algebra  $\mathbb{L}_3 \rightarrow L_3$ . As said before, we know from Theorem 4.1.8 that it is interesting to study it after passing to the flattening stratification of the center. So for all this section, let us denote  $k = \mathbb{F}_p$ . All the schemes are understood as  $k$ -schemes.

Thanks to the theory of Fitting ideals (the reader can look at [SP22, Tag 0C3C] for more details), we can have an explicit description of the different strata. We write  $Z(\mathbb{L}_n)$  the center of the universal Lie algebra. Let

$$L_n =: Z_{-1} \supset Z_0 \supset Z_1 \supset \dots$$

be the closed subschemes defined by the Fitting ideals of  $Z(\mathbb{L}_n)$ . Then, for  $r \geq 0$ , let us define  $L_{n,r} := Z_{r-1} \setminus Z_r$  is the locally closed subscheme of  $L_n$  where  $Z(\mathbb{L}_n)$  is locally free of rank  $r$ . Actually in the following, we will not need to calculate explicitly the flattening stratification. We use the notation  $L_{n,r}^{\text{res}}$  for the locally closed subscheme of  $L_n$  where the center  $Z(\mathbb{L}_n)$  is locally free of rank  $r$ , and  $\mathbb{L}_n$  is restrictable, i.e.  $L_{n,r}^{\text{res}} := L_{n,r} \cap L_n^{\text{res}}$ .

### 6.1 Correspondence between the center of the group and the one of the Lie algebra

In the following, we will extend the classical equivalence of categories between locally free Lie  $p$ -algebras of finite rank with finite locally free group schemes of height 1, showing that the centers of those objects correspond to each other. This is remarkable because the centers are not flat in general. In order to do this we will use the functor denoted by  $\text{Spec}^*$  in [SGA3], Tome 1, exposé VII<sub>A</sub>, §3.1.2. We need first a preliminary lemma. For all this section, let  $S$  be a scheme of characteristic  $p > 0$ .

**6.1.1. Lemma.** *Let  $G \rightarrow S$  a group scheme. Let  $R$  be any ring. Then the following morphism:*

$$\begin{aligned} \text{Lie}(G)(R) &\xrightarrow{\text{exp}} G(R[\alpha, \beta]/(\alpha^2, \beta^2)) \\ x &\longmapsto \exp(\alpha\beta x) \end{aligned}$$

*is injective.*

*Proof.* Let us write  $R(\alpha, \beta) := R[\alpha, \beta]/(\alpha^2, \beta^2)$  and let  $f : \text{Spec}(R(\alpha, \beta)) \rightarrow \text{Spec}(R[\epsilon]/(\epsilon^2))$  be the scheme morphism coming from this injective ring morphism:  $R[\epsilon]/(\epsilon^2) \hookrightarrow R(\alpha, \beta)$ ,  $\epsilon \mapsto \alpha\beta$ . Then  $f$  is surjective as a topological map, so because  $f^\#$  is injective,  $f$  is an epimorphism in the category of schemes. Then this gives an injective morphism  $G(R(\epsilon)) \hookrightarrow G(R(\alpha, \beta))$  which gives by restriction to the Lie algebras an injective morphism

$$\text{Lie}(G)(R) \hookrightarrow G(R(\epsilon)) \hookrightarrow G(R(\alpha, \beta)).$$

□

**6.1.2. Proposition.** *Let  $G \rightarrow S$  be a finite locally free group scheme of height 1. Let  $Z(G)$  denote its center. Then*

$$Z(\text{Lie}(G)) = \text{Lie}(Z(G)).$$

*Proof.* For more convenience, let us write  $\mathfrak{g} := \text{Lie}(G)$  and  $\mathfrak{z} := Z(\mathfrak{g})$ . When  $\mathfrak{l}$  is a Lie  $p$ -algebra, we use the notation  $G_p(\mathfrak{l}) := \text{Spec}^*(U_p(\mathfrak{l}))$  where  $U_p(\mathfrak{l})$  is the universal restricted enveloping algebra of  $\mathfrak{l}$  and where the notation  $\text{Spec}^*$  comes from [SGA3], exposé VII<sub>A</sub>, §3.1.2. and is defined for any  $S$ -scheme  $T \rightarrow S$  by:

$$G_p(\mathfrak{l})(T) := \text{Spec}^*(U_p(\mathfrak{l})) = \{x \in U_p(\mathfrak{l}) \otimes \mathcal{O}_T(T), \epsilon(x) = 1 \text{ and } \Delta(x) = x \otimes x\}.$$

Let us show  $\mathfrak{z} \subset \text{Lie}(Z(G))$ . The inclusion  $\mathfrak{z} \subset \mathfrak{g}$  gives a bialgebra inclusion of universal restricted enveloping algebras  $U_p(\mathfrak{z}) \subset U_p(\mathfrak{g})$ , and looking at the definition, we see that this gives an inclusion of functors:

$$G_p(\mathfrak{z}) \subset G_p(\mathfrak{g}) = G$$

where the last equality is because  $G$  is of height 1. But actually, this subfunctor takes its values in the center of  $G$ : indeed, because  $\mathfrak{z}$  is an abelian Lie algebra, the bialgebra  $U_p(\mathfrak{z})$  is commutative (because for all  $x, y \in \mathfrak{z}$ , we have  $x \otimes y - y \otimes x = [x, y] = 0$  in  $U_p(\mathfrak{z})$ ). Moreover by definition, we have for any  $S$ -scheme  $T \rightarrow S$ ,

$$G(T) = G_p(\mathfrak{g}) = \{x \in U_p(\mathfrak{g}) \otimes \mathcal{O}_T(T), \epsilon(x) = 1 \text{ and } \Delta(x) = x \otimes x\}$$

where the group law of  $G(T)$  is given by  $(x, y) \mapsto x \otimes y$ . But because the algebra  $U_p(\mathfrak{z})$  is abelian, then if  $x \in G_p(\mathfrak{z})(T)$ , then  $x \in Z(G_p(\mathfrak{g}))$ , i.e.

$$G_p(\mathfrak{z}) \subset Z(G).$$

Applying the functor  $\text{Lie}$  we obtain

$$\text{Lie}(G_p(\mathfrak{z})) \subset \text{Lie}(Z(G))$$

but looking at [SGA3], exposé VII<sub>A</sub>, §3.2.3, we know that  $\text{Lie}(G_p(\mathfrak{z})) = \text{Prim}(\mathbb{W}(U_p(\mathfrak{z})))$  and by definition,  $\mathfrak{z} \subset \text{Prim}(\mathbb{W}(U_p(\mathfrak{z})))$  so we have the inclusion

$$\mathfrak{z} \subset \text{Lie}(Z(G)).$$

Now let us show  $\text{Lie}(Z(G)) \subset \mathfrak{z}$ . Let  $f : Z(G) \hookrightarrow G$  be the closed immersion. It is a monomorphism then it is injective on the functor of points. Let  $R$  be any ring and let us denote by  $R(\alpha, \beta) := R[\alpha, \beta]/(\alpha^2, \beta^2)$ . Then we know from [DG70] Chapitre II, §4, n°3, 3.7 (3), that the following diagram is commutative:

$$\begin{array}{ccc} \text{Lie}(Z(G))(R) & \xrightarrow{\text{exp}} & Z(G)(R(\alpha, \beta)) \\ \text{Lie}(f_R) \downarrow & & \downarrow f_S \\ \text{Lie}(G)(R) & \xrightarrow{\text{exp}} & G(R(\alpha, \beta)) \end{array}$$

and the composed map  $\text{Lie}(Z(G))(R) \rightarrow G(R(\alpha, \beta))$  is injective. Moreover, if  $x \in \text{Lie}(Z(G))(R)$ , then  $\text{exp}(\alpha x) \in Z(G)(R) \subset Z(G)(R(\alpha, \beta))$  hence for all  $y \in \text{Lie}(G)(R)$ ,

$$1 = \text{exp}(\alpha x) \text{exp}(\beta y) \text{exp}(-\alpha x) \text{exp}(-\beta y) = \text{exp}(\alpha \beta [x, y])$$

where the last equality comes from [DG70], Chapitre II, §4, n°4, 4.2 (6), and where  $[x, y]$  is the bracket on  $\text{Lie}(G)(R)$ . But  $x \mapsto \text{exp}(\alpha \beta x)$  is injective thanks to Lemma 6.1.1, then we obtain  $[x, y] = 0$  for all  $y \in \text{Lie}(G)(R)$  then  $x \in Z(\text{Lie}(G))(R)$ .  $\square$

Thanks to this result, we can count the number of centerless finite locally free group schemes of order  $p^3$  of height 1 on an algebraically closed field:

**6.1.3. Proposition.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Up to isomorphism,*

- if  $p = 2$ , there is only 1 such group scheme.
- if  $p \neq 2$ , there are  $(p + 3)/2$  such group schemes.

*Proof.* It suffices to count the centerless restrictable Lie algebras of rank 3, classified in 5.1.8. Indeed, because they are centerless, they have only one structure of Lie  $p$ -algebra so there is only one algebraic group scheme corresponding to it. For  $p \neq 2$ , it is useful to remember that, with our notations, the Lie algebras  $\mathfrak{g}_t$  and  $\mathfrak{g}_{t^{-1}}$  are in same orbit when  $t \neq 0$ .  $\square$

This extended equivalence allows us to study the properties of  $L_n^{\text{res}}$  to deduce properties on the moduli space of finite locally free group schemes killed by Frobenius. That is, let  $S$  be a scheme of characteristic  $p > 0$ , and for  $r \leq n$ , let us recall the notations  $\text{Lie}_{n,r}^p(S)$  for the category of  $n$ -dimensional restrictable  $\mathcal{O}_S$ -Lie algebras whose center is locally free of rank  $r$ , and  $G_{n,r}^p(S)$  the category of finite locally free group schemes of order  $p^n$ , of height 1, whose center is locally free of rank  $p^r$ . With these notations and using the previous results, we know that the functor Lie gives us an equivalence of categories:

$$\text{Lie} : G_{n,r}^p(S) \xrightarrow{\sim} \text{Lie}_{n,r}^p(S).$$

Moreover, because  $\text{GL}_n$  is smooth, the quotient map  $L_n \rightarrow \text{Lie}_n$  is smooth, so studying the smoothness of  $L_n$  is equivalent to study the one of  $\text{Lie}_n$ . In all the article, we denote by  $L_n^p := X(\mathbb{L}_n)$  the set of  $p$ -mappings on  $\mathbb{L}_n$ . Then the quotient map  $L_n^p \rightarrow \text{Lie}_n^p$  is smooth, and

$$L_n^p \xrightarrow{\text{Forgetful}} L_n^{\text{res}}$$

is an affine fibration, and if for  $r \leq n$ , we denote by  $L_{n,r}^p := L_n^p \cap L_{n,r}$ , we know that

$$L_{n,r}^p \xrightarrow{\text{Forgetful}} L_{n,r}^{\text{res}}$$

is smooth for all  $r \leq n$ . This is the reason why in the following, we will study the smoothness of  $L_{3,r}^{\text{res}}$  for  $r \leq 3$ .

## 6.2 In the stratum $L_{3,0}$

### 6.2.1. Study of $L_{3,0}^{\text{res}}$ in the whole scheme $L_3$ .

Thanks to the results we have established before, we can imagine all the  $k$ -point which are in the orbit of  $\mathfrak{g}_{-1}$  are singular in  $L_{3,0}^{\text{res}}$  because this orbit is in the intersection of two irreducible components. Actually, thanks to a calculation of tangent space, we will see that they are the only singular ones.

**6.2.2. Theorem.** *If  $\text{char}(k) \neq 2$ , the singular locus of  $L_{3,0}^{\text{res}}$  is the orbit of  $\mathfrak{g}_{-1}$ . If  $\text{char}(k) = 2$ , the scheme  $L_{3,0}^{\text{res}}$  is smooth.*

*Proof.* - If  $p \neq 2$ . We see in the classification Theorem 5.1.4 that the points of  $L_{3,0}^{\text{res}}$  are the points which are in the orbit of  $\mathfrak{g}_t$  with  $t \in \mathbb{F}_p$  and  $t \neq 0$ , and the points in the orbit of  $\mathfrak{s}$ . Let us start with  $\mathfrak{g}_t$ , i.e. let us denote for  $t \in \mathbb{F}_p^*$  as before the  $k$ -point  $\mathfrak{g}_t := \text{Spec}(k) \rightarrow L_{3,k}$ . We need to calculate the local ring of this point. We will show that  $\mathcal{O}_{L_{3,0}^{\text{res}}, \mathfrak{g}_t} = \mathcal{O}_{L_{3,k}^{\text{res}}, \mathfrak{g}_t}$  is smooth.

Let us compute  $T_{L_3^{\text{res}}, \mathfrak{g}_t}$ . Let us denote by  $N$  the  $\mathbb{F}_p[\varepsilon]$ -module  $\mathbb{F}_p[\varepsilon]x \oplus \mathbb{F}_p[\varepsilon]y \oplus \mathbb{F}_p[\varepsilon]z$ . Then we can write

$$T_{L_3^{\text{res}}, \mathfrak{g}_t} = \{\text{Structures of Lie algebra on } N, \text{ restrictable, such that } N \otimes \mathbb{F}_p = \mathfrak{g}_t\}.$$

Let us use again the matrix notation for a Lie algebra structure over  $N$ . We denote it by

$$\mathfrak{g}_{t,\varepsilon} := \begin{pmatrix} a\varepsilon & d\varepsilon & g\varepsilon \\ 1 + b\varepsilon & e\varepsilon & h\varepsilon \\ c\varepsilon & t + f\varepsilon & i\varepsilon \end{pmatrix}.$$

First of all,  $\mathfrak{g}_{t,\varepsilon}$  is a Lie algebra structure if and only if its coefficients satisfy the conditions denoted by  $Q_1, Q_2$  and  $Q_3$  above, that is if and only if

$$\begin{cases} (1+t)g = 0 \\ d - th = 0 \\ ta + i = 0. \end{cases}$$

Then,  $\mathfrak{g}_{t,\varepsilon}$  is in  $T_{L_3^{\text{res}}, \mathfrak{g}_t}$  if and only if it is restrictable. In order to see the conditions to be restrictable we will calculate  $\text{ad}_x^p, \text{ad}_y^p$  and  $\text{ad}_z^p$  for any  $p$  prime. Let us denote

$$\beta := 1 + t + \dots + t^{p-1} = \begin{cases} 0 & \text{if } t = 1 \\ 1 & \text{if } t \neq 1. \end{cases}$$

Now, using the matrix notation in the basis  $\{x, y, z\}$ , we have  $\text{ad}_x = \begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1 + b\varepsilon & e\varepsilon \\ 0 & c\varepsilon & t + f\varepsilon \end{pmatrix}$

then for all  $p$  prime  $\text{ad}_x^p = \begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1 & \beta e\varepsilon \\ 0 & \beta c\varepsilon & t \end{pmatrix}$ .

Likewise,  $\text{ad}_y = \begin{pmatrix} -a\varepsilon & 0 & g\varepsilon \\ -1 - b\varepsilon & 0 & h\varepsilon \\ -c\varepsilon & 0 & i\varepsilon \end{pmatrix}$  hence  $\text{ad}_y^2 = \begin{pmatrix} 0 & 0 & 0 \\ a\varepsilon & 0 & -g\varepsilon \\ 0 & 0 & 0 \end{pmatrix}$  and for all  $p > 2$ ,  $\text{ad}_y^p \equiv 0$ .

Likewise,  $\text{ad}_z = \begin{pmatrix} -d\varepsilon & -g\varepsilon & 0 \\ -e\varepsilon & -h\varepsilon & 0 \\ -t - f\varepsilon & -i\varepsilon & 0 \end{pmatrix}$ , hence  $\text{ad}_z^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ td\varepsilon & tg\varepsilon & 0 \end{pmatrix}$  and for all  $p > 2$ ,  $\text{ad}_z^p \equiv 0$ .

Then  $\text{ad}_x^p$  is a linear combination of  $\text{ad}_x, \text{ad}_y$  and  $\text{ad}_z$  if and only if it exists  $\lambda = \lambda_0 + \lambda_1\varepsilon$ ,  $\mu = \mu_0 + \mu_1\varepsilon$  and  $\nu = \nu_0 + \nu_1\varepsilon$  such that:

$$\begin{pmatrix} 0 & a\varepsilon & d\varepsilon \\ 0 & 1 & \beta e\varepsilon \\ 0 & \beta c\varepsilon & t \end{pmatrix} = \begin{pmatrix} (-\mu_0 a - \nu_0 d)\varepsilon & (\lambda_0 a - \nu_0 g)\varepsilon & (\lambda_0 d + \mu_0 g)\varepsilon \\ -\mu_0 - (b\mu_0 + \mu_1 + \nu_0 e)\varepsilon & \lambda_0 + (b\lambda_0 + \lambda_1 - \nu_0 h)\varepsilon & (\lambda_0 e + \mu_0 h)\varepsilon \\ -t\nu_0 - (\mu_0 c + s\nu_0 + t\nu_1)\varepsilon & (\lambda_0 c - \nu_0 i)\varepsilon & t\lambda_0 + (f\lambda_0 + t\lambda_1 + \mu_0 i)\varepsilon \end{pmatrix}.$$

Then  $\text{ad}_x^p$  is a linear combination of  $\text{ad}_x, \text{ad}_y$  and  $\text{ad}_z$  if and only if

$$\begin{cases} bt = f \\ c = \beta c \\ e = \beta e. \end{cases}$$

Because  $\text{ad}_y^p \equiv \text{ad}_z^p \equiv 0$ , they are always linear combination of  $\text{ad}_x, \text{ad}_y$  and  $\text{ad}_z$ .

Hence we obtain the following conditions:

$$(*) \begin{cases} (1+t)g = d - th = i + ta = 0 \\ bt - f = c - \beta c = e - \beta e = 0. \end{cases}$$

So we have to distinguish different cases. First let us suppose  $t = -1$ . Then the conditions (\*) are equivalent to:

$$\begin{cases} d + h = 0 \\ a - i = 0 \\ b + f = 0. \end{cases}$$

Hence  $\dim(T_{L_3^{\text{res}}, \mathfrak{g}_{-1}}) = 6$ . But  $\dim(o(\mathfrak{g}_{-1})) = 5$  from Theorem 5.1.4, hence the local ring of  $\mathfrak{g}_{-1}$  is singular.

Let us suppose  $t = 1$ . Then the conditions (\*) are equivalent to

$$\begin{cases} g = d - h = a + i = 0 \\ b - f = c = e = 0 \end{cases}$$

so  $\dim(T_{L_3^{\text{res}}, \mathfrak{g}_1}) = 3 = \dim(o(\mathfrak{g}_1))$ . Then the point  $\mathfrak{g}_1$  is smooth.

Then let us suppose  $t \neq 1$  and  $t \neq -1$ . Then the conditions (\*) are equivalent to:

$$\begin{cases} g = 0 \\ d - th = i + ta = bt - f = 0 \end{cases}$$

Hence  $\dim(T_{L_3^{\text{res}}, \mathfrak{g}_t}) = 5$ . But  $\dim(o(\mathfrak{g}_t)) = 5$ , hence the local ring of  $\mathfrak{g}_t$  is regular.

Doing the same calculations for  $\mathfrak{s} \in L_{3,0}^{\text{res}}$ , we obtain these conditions:

$$\begin{cases} a - i = 0 \\ b + f = 0 \\ d + h = 0. \end{cases}$$

Hence  $\dim(T_{L_3^{\text{res}}, \mathfrak{s}}) = 6$ . But  $\dim(o(\mathfrak{s})) = 6$ , hence the local ring of  $\mathfrak{s}$  is regular.

- Let us suppose  $p = 2$ . Then the only point in  $L_{3,0}^{\text{res}}$  is  $\mathfrak{g}_1$ . So using the same notations as before, we see in this case, the conditions are equivalent to

$$\begin{cases} d - h = a + i = b - f = 0 \\ c = e = g = 0 \end{cases}$$

so  $\dim(T_{L_3^{\text{res}}, \mathfrak{g}_1}) = 3$ . But  $\dim(o(\mathfrak{g}_1)) = 3$ , hence the local ring of  $\mathfrak{g}_1 = \mathfrak{g}_{-1}$  is regular.  $\square$

**6.2.3. Study of  $L_{3,0}^{\text{res}}$  in the first irreducible component.** We start by establishing a result on the scheme structure of  $L_{3,0}^{\text{res}}$  in the first irreducible component, in the case we choose a field  $k$  of characteristic  $p \neq 2$ .

**6.2.4. Proposition.** *The scheme  $L_{3,0} \cap L_3^{(1)}$  is reduced. Moreover, if  $\text{char}(k) \neq 2$ ,*

$$L_{3,0}^{\text{res}} \cap L_3^{(1)} \simeq L_{3,0} \cap L_3^{(1)} \text{ as schemes.}$$

*Proof.* Because  $L_{3,0}$  is open in  $L_3$ ,  $L_{3,0} \cap L_3^{(1)}$  is open in the reduced irreducible component  $L_3^{(1)}$ , then it is reduced. Moreover,

$$\begin{aligned} |L_{3,0}^{\text{res}} \cap L_3^{(1)}| &= \bigcup_{R_k \rightarrow k = \bar{k}} (L_{3,0}^{\text{res}} \cap L_3^{(1)})(k) = \bigcup_{R_k \rightarrow k = \bar{k}} L_3^{\text{res}}(k) \times_{S(k)} L_{3,0}(k) \times_{S(k)} L_3^{(1)}(k) \\ &= \bigcup_{R_k \rightarrow k = \bar{k}} L_{3,0}(k) \times_{S(k)} L_3^{(1)}(k) = |L_{3,0} \cap L_3^{(1)}|. \end{aligned}$$

Then  $L_{3,0}^{\text{res}} \cap L_3^{(1)}$  is a closed subscheme of the reduced scheme  $L_{3,0} \cap L_3^{(1)}$  with the same underlying set. Then they are equal as schemes.  $\square$

Now we study the  $k$ -points of this intersection of schemes. We have to do exactly the same calculus we have done in the previous subsection, but we have to change the conditions  $Q_1, Q_2$  and  $Q_3$  for the conditions  $L_1, L_2$  and  $L_3$ . Then we find:

**6.2.5. Proposition.** *If  $\text{char}(k) \neq 2$ , in  $L_{3,0}^{\text{res}} \cap L_3^{(1)}$ , the  $k$ -points  $\mathfrak{g}_{-1}$  is singular, and  $\mathfrak{s}$  is regular. If  $\text{char}(k) = 2$ , the scheme  $L_{3,0}^{\text{res}} \cap L_3^{(1)}$  is smooth.*

*Proof.* - Let us suppose  $p \neq 2$ . We first look at the point  $\mathfrak{s}$ . We obtain, as before, these conditions:

$$\begin{cases} a - i = 0 \\ b + f = 0 \\ d + h = 0. \end{cases}$$

Hence  $\dim(T_{L_{3,0}^{\text{res}}, \mathfrak{s}}) = 6$ , so the local ring of  $\mathfrak{s}$  is regular.

Let us do the same for the point  $\mathfrak{g}_{-1}$ . Doing the same calculations we obtain  $\dim(T_{L_{3,0}^{\text{res}}, \mathfrak{g}_{-1}}) = 6$ , so the local ring of  $\mathfrak{g}_{-1}$  is singular.

- If  $p = 2$ , we have  $\dim(T_{L_{3,0}^{\text{res}}, \mathfrak{g}_1}) = 3$  so  $\mathfrak{g}_1$  is regular.  $\square$

### 6.2.6. Study of $L_{3,0}^{\text{res}}$ in the second irreducible component.

**6.2.7. Theorem.** *In the second irreducible component, all the  $k$ -points of  $L_{3,0}^{\text{res}} \cap L_3^{(2)}$  are smooth.*

*Proof.* We can do the same proof as before, we just need to add the condition  $\det(M) = 0$ . That is, if we keep the same notations as before, we need to add to the system (\*) the condition  $gt = 0$ . Hence the new system is given by

$$\begin{cases} g = d - th = i + ta = 0 \\ bt - f = c - \beta c = e - \beta e = 0. \end{cases}$$

So in this case, we obtain

$$\dim(T_{L_{3,0}^{\text{res}}, \mathfrak{g}_t}) = \begin{cases} 3 & \text{if } t = 1 \\ 5 & \text{if } t \neq 1. \end{cases}$$

$\square$

**6.2.8. Remark.** By a simple computation, we can see that any deformation of Lie algebras which are in the stratum  $L_{3,0}$  is centerless without any condition. It is because the stratum  $L_{3,0}$  is open in  $L_3$ .

### 6.3 In the stratum $L_{3,1}$

Let us do the same calculations for the points of  $L_{3,1}$ .

#### 6.3.1. Study of $L_{3,1}^{\text{res}}$ in $L_3$ .

**6.3.2. Proposition.** *The  $k$ -point  $\mathfrak{h}_3$  is singular in  $L_{3,1}^{\text{res}}$ , and  $\mathfrak{g}_0$  is smooth.*

*Proof.* - For the point  $\mathfrak{h}_3$ , as in the previous section, let us denote by  $\mathfrak{h}_{3,\varepsilon}$  a deformation of the Lie algebra  $\mathfrak{h}_3$ :

$$\mathfrak{h}_{3,\varepsilon} := \begin{pmatrix} a\varepsilon & d\varepsilon & 1 + g\varepsilon \\ b\varepsilon & e\varepsilon & h\varepsilon \\ c\varepsilon & f\varepsilon & i\varepsilon \end{pmatrix}.$$

Then,  $\mathfrak{h}_{3,\varepsilon}$  gives the constants of structure of a Lie algebra if and only if  $b + f = 0$ . Moreover,  $\mathfrak{h}_{3\varepsilon}$  is restrictable if and only if:

- if  $p = 2$ :  $b = c = e = 0$
- if  $p = 3$ : there is no condition
- if  $p > 3$ : there is no condition.

For the end, the center  $Z(\mathfrak{h}_{3,\varepsilon})$  is locally free of rank 1 if and only if:

- if  $p = 2$ : there is no condition
- if  $p = 3$ :  $b = c = e = 0$
- if  $p > 3$ :  $b = c = e = 0$

So to conclude we use the fact that  $\dim(o(\mathfrak{h}_3)) = 3$ .

- For the point  $\mathfrak{g}_0$  let us do the same. Then using the same notations,  $\mathfrak{g}_{0,\varepsilon}$  gives the constants of structure of a Lie algebra if and only if  $d = g = i = 0$ . Moreover,  $\mathfrak{g}_{0,\varepsilon}$  is restrictable if and only if  $f = 0$ . For the end, the center  $Z(\mathfrak{g}_{0,\varepsilon})$  is always locally free of rank 1. So we conclude using the fact that  $\dim(o(\mathfrak{g}_0)) = 5$ .

□

#### 6.3.3. Study of $L_{3,1}^{\text{res}}$ in $L_3^{(1)}$ .

**6.3.4. Proposition.** *The  $k$ -point  $\mathfrak{h}_3$  is smooth in  $L_{3,1}^{\text{res}} \cap L_3^{(1)}$ .*

*Proof.* We have to add to the conditions found before the conditions  $a = i$  and  $d = -h$ . □

#### 6.3.5. Study of $L_{3,1}^{\text{res}}$ in $L_3^{(2)}$ .

**6.3.6. Proposition.** *The  $k$ -point  $\mathfrak{h}_3$  is singular in  $L_{3,1}^{\text{res}} \cap L_3^{(2)}$  and the point  $\mathfrak{g}_0$  is smooth.*

*Proof.* For both of those points, the condition  $\det(M) = 0$  is always satisfied for any deformation. □

## 6.4 In the stratum $L_{3,3}$

This case is really simple because the condition "to be in the stratum  $L_{3,3}$ " implies, using the same notations as in the previous subsections, that all the coefficient of the matrix  $\mathbf{ab}_{3,\varepsilon}$  are 0. Then,  $\dim(T_{L_{3,3}^{\text{res}}, \mathbf{ab}_3}) = 0$ , in the whole scheme and in the irreducible components. Then the point  $\mathbf{ab}_3$  is smooth seen in  $L_{3,3}^{\text{res}}$ .

As stated in the introduction, we can apply the previous results of smoothness to the moduli space  $G_{3,r}^p$ , and this gives the following result:

**6.4.1. Corollary.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Then  $G_3^p(k)$  splits in two irreducible components that we denote by  $G_{3,r}^{p,(1)}$  and  $G_{3,r}^{p,(2)}$ , and we have:*

- *If  $p \neq 2$ ,  $G_{3,0}^p(k)$  is singular, but becomes smooth after intersection with  $G_{3,0}^{p,(2)}$ , if  $p \neq 2$ ,  $G_{3,0}^p(k)$  is smooth.*
- *$G_{3,1}^p(k)$  is singular but becomes smooth when we intersect it with  $G_{3,1}^{p,(1)}$ .*
- *$G_{3,2}^p(k)$  is empty and  $G_{3,3}^p(k)$  is smooth.*

*Proof.* We have an equivalence of categories given by the functor  $\text{Lie} : G_{3,r}^p(k) \rightarrow \text{Lie}_{3,r}^p(k)$ , moreover, the quotient morphism  $L_{3,r}^p \rightarrow \text{Lie}_{3,r}^p$  is smooth. Then, because  $L_{3,r}^p \rightarrow L_{3,r}^{\text{res}}$  is smooth, we can apply the results of the subsections 6.2, 6.3 and 6.4.  $\square$

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Alice Bouillet, UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE  
 Email address: [alice.bouillet@univ-rennes1.fr](mailto:alice.bouillet@univ-rennes1.fr)