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FERMIONIC SEMICLASSICAL L^p ESTIMATES

Ngoc Nhi Nguyen^{*†}

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Abstract

We generalize the semiclassical L^p estimates of Koch, Tataru and Zworski in the setting of Schrödinger operators with confining potentials to density matrices. This is motivated by the problem of the concentration of free fermionic particles in a trapping potential. Our proof relies on semiclassical and many-body tools. As an application, we provide bounds on spectral clusters. We also discuss the optimality of the one-body and many-body bounds through explicit examples of quasimodes.

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^{*}Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, 91405 Orsay, France.

[†]Current affiliation: Università degli Studi di Milano, Dipartimento di Matematica, Via Cesare Saldini 50, 20133 Milano. Email: ngoc.nguyen@unimi.it

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I INTRODUCTION

This article is devoted to estimates related to the spatial concentration of orthonormal families of eigenfunctions of Schrödinger operators

$$P := -h^2\Delta + V$$

in the semiclassical regime $h \rightarrow 0$. Here $-\Delta := \sum_{j=1}^d \partial_{x_j}^2$ is the standard Laplacian, $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is the potential and $d \geq 1$ is the spatial dimension. We work in the convenient setting where V is smooth and $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, so that P is a well-defined self-adjoint, bounded from below operator on $L^2(\mathbb{R}^d)$ with a discrete spectrum going to $+\infty$. This model traps quantum systems, where particles are confined to live in an essentially bounded region of space. We are interested in the concentration properties of L^2 normalized eigenfunctions $\{u_h\}$ of P associated to an eigenvalue E (we have in mind the case where E is essentially h -independent or bounded in h). More precisely, we aim to determine how “concentrate” such an eigenfunction may be. To do so, we choose to measure concentration through the possible growth of the norms $\|u_h\|_{L^q(\mathbb{R}^d)}$ as $h \rightarrow 0$, for various choices of $q \in [2, \infty]$. Indeed, in the extreme case where u_h is essentially constant (meaning that it is close to an h -independent function), then the norms $\|u_h\|_{L^q(\mathbb{R}^d)}$ do not grow in h . On the contrary, if all the L^2 -mass of u_h is concentrated in a region $\Omega_h \subset \mathbb{R}^d$ with $|\Omega_h| \rightarrow 0$ as $h \rightarrow 0$, then we typically have $\|u_h\|_{L^q(\mathbb{R}^d)} \sim |\Omega_h|^{1/q-1/2} \rightarrow \infty$ as $h \rightarrow 0$, when $q > 2$. In the following, we will consider estimates of the type

$$\|u_h\|_{L^q(\mathbb{R}^d)} \leq Ch^{-s}, \tag{I.1}$$

for some $s > 0$ and $C > 0$ depending only on V and E , which we interpret as a measure of the highest “rate” of concentration of eigenfunctions of P as $h \rightarrow 0$. Notice that Sobolev embeddings imply the previous estimate with $s = d(\frac{1}{2} - \frac{1}{q})$, and we will see several cases where this exponent can be improved. Of particular importance is the determination of the optimal value of the exponent s , which amounts to construct explicit examples of u_h for which the upper bound (I.1) is also a lower bound (with possibly a different value of C). Ideally, it would be much clearer if we had a pointwise description of the function u_h . But it is very hard to obtain for general potentials in any dimensions. That is why we settle for this rarer version of the concentration with the estimation of L^q norms.

This strategy to study concentration of functions via L^q norms was invented by Sogge, first in the context of spherical harmonics [30], and then in the context of general compact Riemannian manifolds without boundary (where P is replaced by $-h^2\Delta_g$, the Laplace-Beltrami operator) [31]. He not only considered eigenfunctions but more generally functions u_h in spectral clusters, i.e. that satisfy $u_h = \mathbb{1}(|P - E| \leq h) u_h$. Furthermore, he managed to find the optimal exponent $s = s(q)$ on any manifold, and proved that for high values of q , the highest rate of concentration was attained for specific functions concentrating around a point (generalizing the zonal spherical harmonics) while for low values of q , it was attained for functions concentrating around a curve (generalizing the Gaussian beams on spheres). These results were later extended to the case of Schrödinger operators with confining potentials on \mathbb{R}^d (which is the case that we consider here) by Koch and Tataru in [21], and their method was revisited from the point of view of semiclassical analysis by Koch, Tataru and Zworski in [22]. This last article also treats the more general case of quasimodes u_h , i.e. that satisfy $(P - E)u_h = \mathcal{O}_{L^2}(h)$, and we will follow on their approach. Notice that many works were devoted to the improvement of Sogge's estimates for eigenfunctions, in specific geometries (typically with negative curvature, see for instance [15, 18, 34, 2]). We will not pursue this direction here. The case of confining potentials is more complicated than the one of compact manifolds without potentials, due to the presence of a transition region between the classically allowed region $\{V \leq E\}$ and the classically forbidden region $\{V > E\}$. Indeed, Koch and Tataru discovered specific concentration phenomena in the transition region that did not appear in Sogge's work. This can be first understood in the one dimensional case, as we will explain below.

In this article, we investigate the more general situation of concentration of orthonormal families of quasimodes. This is motivated by the study of fermionic systems in quantum mechanics, where a well-known example of systems of N uncorrelated fermionic states are Slater determinants $u_h^1 \wedge \dots \wedge u_h^N(x^1, \dots, x^N) = \frac{1}{\sqrt{N!}} \det((u_h^j(x_i))_{1 \leq i, j \leq N})$, which are associated to N orthonormal functions $\{u_h^j\}_{1 \leq j \leq N}$ in $L^2(\mathbb{R}^d)$. The orthonormality is a manifestation of Pauli's exclusion principle, which states that two fermions cannot occupy the same quantum state. Intuitively, it means that two fermions cannot concentrate in the same region in space. Hence, while a single particle may be localized in a small region, many particles will tend to delocalize by this "repulsion" induced by Pauli's principle. To measure quantitatively the concentration of several particles, it is useful to introduce the spatial density of particles

$$\rho_h = \sum_{j=1}^N |u_h^j|^2,$$

and estimate the growth in h of its $L^{q/2}$ -norms, if each of the u_h^j is a quasimode of P . For $N = 1$, we recover the question mentioned above. This quantity is interesting because it provides, up to a normalization, the spatial repartition of the fermionic system. Note that this density corresponds to the density of the one-body operator of the Slater determinant $\Psi_N = u_h^1 \wedge \dots \wedge u_h^N \in L^2((\mathbb{R}^d)^N)$, defined by its associated integral kernel

$$(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto N \int_{\mathbb{R}^{d(N-1)}} \Psi_N(x, x^2, \dots, x^N) \overline{\Psi_N(y, x^2, \dots, x^N)} dx^2 \dots dx^N.$$

Actually, as we will see below, we instead look at more general densities of the form (I.4). As for the one-body case, the pointwise expression of these objects is not always easy to study, even in the case without interaction¹. That is why we study instead its $L^{q/2}$ -norms. We can also see the

¹The interacting case is even more complicated to deal with. Moreover, it is far from obvious that we can deduce the same type of $L^{q/2}$ -estimates. Indeed, one can expect the interactions affect the concentration of the particles.

measure of the spatial concentration as the quotient of the $L^{q/2}$ -norms by the L^2 one. One can estimate trivially using the triangle inequality and the $N = 1$ estimate (I.1),

$$\|\rho_h\|_{L^{q/2}(\mathbb{R}^d)} \leq \sum_{j=1}^N \|u_h^j\|_{L^q(\mathbb{R}^d)}^2 \leq C^2 h^{-2s} N.$$

Our goal in this work is to prove estimates of the type²

$$\|\rho_h\|_{L^{q/2}(\Omega)} \leq C h^{-2s} N^\theta, \quad (\text{I.2})$$

for some $\theta \in [0, 1]$, in some regions $\Omega \subset \mathbb{R}^d$. Notice that this estimate reduces to (I.1) in the case $N = 1$, and that it is a strict improvement of (I.1) only if $\theta < 1$ by the above argument. In this paper, we have made the choice to prove many-body estimates so that we recover the best possible exponent s (in our case the existing exponents s in [22]) for $N = 1$, and then we try to obtain the smallest θ possible with respect to this constraint. However, one can have different values of s and α , as pointed out in Remark 14 (when we can have a better α but a worse s). An other approach would be to prove the estimates for a large N_h (for instance in power of h^{-1}) with the smallest exponent θ possible, and then interpolate these estimates with for instance the one obtained for smaller range $N = 1$. It is worth to mentioning in our proof we do not really prove the case $N = 1$, except for the exponent $q = \infty$. But in this case, it is because the proof for any given N does not need anything new compared to the existing estimates (I.1). In the case of the Laplace-Beltrami operator on compact manifolds, it was done in [12], where the sharp exponent $\theta = \theta(q)$ was found. Here, we generalize their work to the case of confining potentials. From the point of view of physics, the statistical properties of systems of non-interacting trapped fermions and in particular of their possible scales of concentration has attracted some attention recently [4, 5, 6, 3], and our work goes in a similar direction.

The fact that enough fermions tend to delocalize can be understood by the pointwise Weyl law, which informally states that

$$\rho_h(x) \sim_{h \rightarrow 0} \frac{|B_{\mathbb{R}^d}(0, 1)|}{(2\pi h)^d} (E - V(x))_+^{d/2}, \quad (\text{I.3})$$

when the u_h^j are chosen to be an orthonormal family of eigenfunctions associated to all the eigenvalues less than E of P . For this choice of $\{u_h^j\}_j$, the $L^{q/2}$ -norms of ρ are of the same order Ch^{-d} for all q , which underlines delocalization. Actually, this delocalization also occurs for a much lower number of functions. Indeed, if one does not consider all the eigenvalues less than E of P , but only the eigenvalues between $E - h$ and $E > \min V$, one can show (for $d \geq 2$, see Section VIII.2) that all the $L^{q/2}$ -norms are also of the same order ($Ch^{-(d-1)}$ in this case), so that delocalization is also true for this much smaller spectral window. We will see below that this example is very important to prove the sharpness of the exponent θ that we obtain in our estimates of the type (I.2). This is why (I.2) measures the transition between the localization for small N and the delocalization for N large enough: when $N = 1$, it is saturated by concentrated functions while for N large, it is saturated by a delocalized system of functions. On compact manifolds with $V = 0$, (I.3) was made rigorous by Avakumovic [1], Levitan [24] and Hörmander [19, Thm. 1.1]. For confining potentials, this asymptotic fails close to the transition region $\{V = E\}$ and a pointwise Weyl law was proved for $V(x) = |x|^2$ in [20] and more recently for general potentials in [7].

²See below Theorems 1 and 2 for more precise and general statements. We have actually also an other factor $1/\log(1/h)$ at some power $t \geq 0$, but we omit it in the first part of the introduction.

To understand what happens in the transition region $\{V = E\}$, and also to illustrate the transition between localization and delocalization, it is useful to consider the case of the harmonic oscillator $V(x) = x^2$ in $d = 1$, for which many explicit computations are available. For instance, asymptotics as $h \rightarrow 0$ of individual eigenfunctions u_h associated to an eigenvalue $E > 0$ (independent of h) are very well understood using Wentzel-Kramers-Brillouin (WKB) methods (see for instance [27]) as depicted in Figure 1: in the classically allowed region $\{V < E\}$, u_h has size 1 (and oscillates, which is not measured by L^q -norms) and in the classically forbidden region $\{V > E\}$, it is exponentially decaying (both in h and $|x|$). An interesting phenomenon appears in the transition region $\{V = E\}$, since u_h has size $h^{-1/6}$ in a neighborhood of size $h^{2/3}$ of this region. One can thus see a concentration phenomenon which does not happen in the absence of a potential. Notice also that the concentration is only visible at the level of L^q -norms for large q because $\|u_h\|_{L^q(\mathbb{R}^d)} \sim 1$ for $2 \leq q \leq 4$ and $\|u_h\|_{L^q(\mathbb{R}^d)} \sim h^{-\frac{1}{6} + \frac{2}{3q}}$ for $q \geq 4$. Asymptotics of ρ_h , when u_h^j fill all the energy levels up to E , are also well-known by the same method as depicted in Figure 2: in the classically allowed region $\{V < E\}$, ρ_h has size h^{-1} and in the classically forbidden region $\{V > E\}$, it is also exponentially decaying. In the transition region, it displays some concentration, but contrary to the case of individual eigenfunctions, it is too small compared to the bulk $\{V < E\}$, so it is invisible in the $L^{q/2}$ -norms. In this case, all the $L^{q/2}$ -norms are of the same order h^{-1} . Of course, such a precise pointwise information is very specific to the one-dimensional case and one cannot hope to extend it to higher dimensions. The results of [21, 22] cover the higher dimension case using L^q -norms, at the level of eigenfunctions/quasimodes. We extend their results to the case of several functions. These one-dimensional examples also show the different behavior according to the different regions $\{V < E\}$, $\{V > E\}$ and $\{V = E\}$, and the higher dimensional results will also take into account these differences.

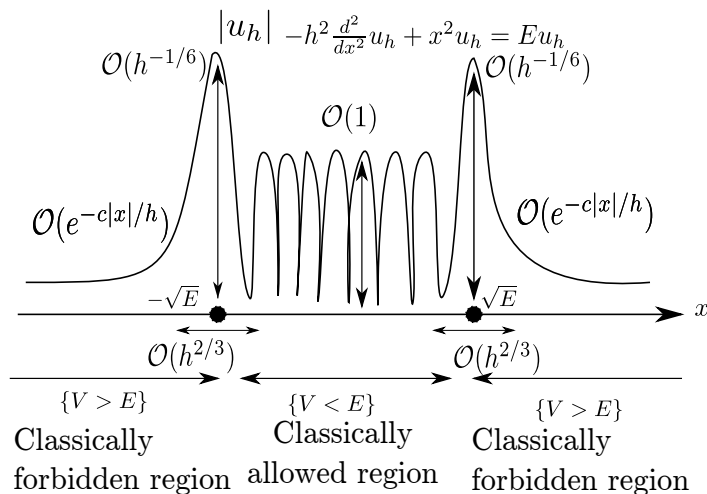


Figure 1: Eigenfunction of the scalar harmonic oscillator associated to the eigenvalue E .

Let us now summarize (in a simplified way) our main results, which precise statements are in Theorem VII.2. First, we show that for any $E > \min V$ and for any $\varepsilon \in (0, E - \min V)$, one has

$$\|\rho_h\|_{L^{q/2}(\{V < E - \varepsilon\})} \leq Ch^{-2s} N^\theta,$$

for any orthonormal systems $\{u_h^j\}_{1 \leq j \leq N}$ of eigenfunctions associated to eigenvalues in $[E - h, E]$, for any N , with sharp values of s and θ (which are the same as on compact manifolds without potential). The sharp value of s is obtained for $N = 1$, while the sharp value of θ is obtained choosing the maximal number of such $\{u_h^j\}_j$. This case is the same as what happens on compact

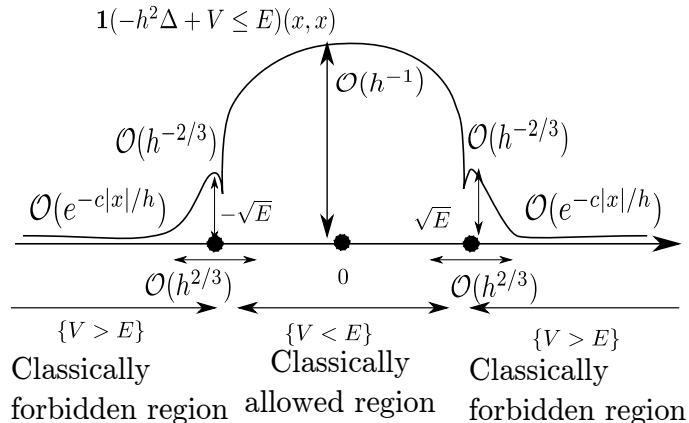


Figure 2: Spectral projector of the scalar harmonic oscillator.

manifolds since we are far from the transition region. Around the transition region, we obtain a similar estimate

$$\|\rho_h\|_{L^{q/2}(\{|V-E|\leq\varepsilon\})} \leq Ch^{-2s}N^\theta,$$

with different values of exponents s and $\theta < 1$, and under the important assumption that $\nabla_x V \neq 0$ on $\{V = E\}$. These estimates are typically not sharp, even for $N = 1$, as noticed in [21], because they are obtained by summing rescaled estimates on multiple scales intermediate between a neighborhood of size $h^{2/3}$ of the transition region and the bulk. It is rather the estimates on each of these individual intermediate scales that are sharp (that is, the value of s is sharp), as proved in [21]. The sharpness of the exponent θ in these scales is an open question. Finally, we obtain estimates of the type

$$\|\rho_h\|_{L^{q/2}(\mathbb{R}^d)} \leq Ch^{-2s}N^\theta,$$

without any assumption on E or on the behavior of V on $\{V = E\}$. This is useful for instance in the case where $E = \min V$, in which there is no bulk. The exponent s is also sharp using again $N = 1$ (the saturation happening for the ground state of P), while the sharpness of the exponent θ is also open.

Let us now comment on the methods of proof and detail the structure of the paper. As we already said, we use the strategy of [22] based on microlocal analysis, mixed with the many-body tools of [12]. First, we notice (as we will see in Section VII) that it is enough to estimate functions u_h or u_h^j that are microlocalized, meaning that they “live” in a compact region in the phase space $\mathbb{R}^d \times \mathbb{R}^d$. This is because spectral localization implies microlocalization for elliptic operators (see for instance (VII.8)). By compactness, it is thus enough to treat functions that are microlocalized around a point. Then, the properties of the classical symbol of P ,

$$p_E(x, \xi) = |\xi|^2 + V(x) - E, \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

at this point intervene. In Section III, we will treat elliptic points where $p_E \neq 0$. There, the main tools are Sobolev embeddings in the one-body case and the Kato-Seiler-Simon inequalities in the many-body case. In the region where $p_E = 0$, several cases are distinguished: that satisfy some geometric assumptions (*cond*) with respect to the energy level sets of $p_E(x, \xi) = p(x, \xi) - E$. That can be points out of $\{p_E = 0\}$ or in the level set $\{p_E = 0\}$ under one of the three conditions stated (see below Table 1 for the references to corresponding geometric assumptions)

- In Section IV, we give a general estimate which is valid for any potential V and any energy E . The proof relies on adding an artificial time variable and use many-body Strichartz estimates in the spirit of [9, 11].

- In Section V, we treat the bulk case for which $p_E = 0$ but $V \neq E$. The proof relies on seeing one of the d space variables as a time variable and again use many-body Strichartz estimates.
- In Section VI, we treat the turning point region $p_E = 0$ and $V = E$, under the nondegeneracy assumption that $\nabla_x V \neq 0$ on this set. In a neighborhood of size $h^{2/3}$ of this region, we use a H^1 -estimate of [22] together with the Kato-Seiler-Simon inequality. The remaining region is split into multiple scales $2^j h^{2/3}$, and each of them is treated using the estimates of Section V by rescaling.

Conditions on the symbol $p_E := p(x, \xi) - E$		Statement
<i>Elliptic</i>	(ellip)	$p_E \neq 0$
<i>General</i>	(gene)	Assumption 2
<i>Sogge (for the bulk)</i>	(Sogge)	Assumption 3
<i>Turning points</i>	(TP)	Assumption 4

Table 1: Geometric conditions and their abbreviation in the article.

In Section VII, we gather all the previous estimates to obtain our main results on spectral clusters. Finally, we discuss their optimality in Section VIII.

In the following, we will consider a more general description of many-body states than orthonormal functions. Actually, many more fermionic states are described by a one-body operator than Slater determinants. Namely, we will consider one-body density matrices (nonnegative compact operators on $L^2(\mathbb{R}^d)$). Such an operator γ can be diagonalized in an orthonormal basis $\{u^j\}_j$ with associated eigenvalues $\{\lambda_j\}_j \subset \mathbb{R}_+$, and each λ_j is interpreted as the average number of particles described by the state γ which have wavefunction u^j . In this formalism, the case described above of N orthonormal functions $\{u^j\}_{1 \leq j \leq N}$ corresponds to $\gamma = P_N$, the orthogonal projection on the space generated by the $\{u^j\}_{1 \leq j \leq N}$. The N^θ factor in the right-side of (I.2) is then interpreted as the Schatten norm $\|P_N\|_{\mathfrak{S}^\alpha}$ where $\alpha = 1/\theta$ (see below for the definition of Schatten spaces). Furthermore, to any one-body density matrix γ , one can associate a density of particles

$$\rho_\gamma = \sum_{j=1}^{+\infty} \lambda_j |u^j|^2, \quad (\text{I.4})$$

which measures the spatial repartition of the particles described by γ . We will prove estimates similar to (I.2), where ρ_h in the left-side is replaced by ρ_γ and N^θ in the right-side is replaced by $\|\gamma\|_{\mathfrak{S}^\alpha}$.

The exponents s and α have actually a dependence on d and q , that we sum up in Table 2. As mentioned above, we will consider estimates on microlocalized objects. In the one-body setting, it means that one estimates $\|\chi^w u\|_{L^q}$ instead of $\|u\|_{L^q}$ for a fixed $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, where χ^w denotes the Weyl quantization of the localization function χ (see below for the definition). In the many-body setting, it means that one estimates $\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}}$ instead of $\|\rho_\gamma\|_{L^{q/2}}$. Furthermore, we also mentioned that one could more generally estimate quasimodes u_h (meaning that $\|u_h\|_{L^2(\mathbb{R}^d)} = 1$ and $(P - E)u_h = \mathcal{O}_{L^2}(h)$). An equivalent way to consider estimates for microlocalized quasimodes is to replace (I.1) by

$$\|\chi^w u_h\|_{L^q} \leq Ch^{-s} \left(\|u_h\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|(P - E)u_h\|_{L^2(\mathbb{R}^d)} \right).$$

The generalization to the many-body setting is given by estimates of the type

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}} \leq Ch^{-2s} \left(\|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|(P - E)\gamma(P - E)\|_{\mathfrak{S}^\alpha} \right). \quad (\text{I.5})$$

The advantage of such a formulation is that the microlocalization is imposed by χ^w and the property to be a quasimode is related to the choice of the norm in the right-side. Hence, we may prove (I.5) for general γ , the restriction to be a microlocalized quasimode being included in the form of the inequality. We will prove such estimates in Sections III to VI, with different values of s and α according to the properties of p_E on $\text{supp } \chi$.

Let us group and summarize more precisely the main results informally mentioned before.

Condition	Exponents		
	s_{cond}	t_{cond}	α_{cond}
Comparison when $d \geq 3$	Figure 3	Figure 10	Figure 4
(ellip)	defined in Eq. (III.1), see Figure 5		
(gene)	defined in Theorems IV.1 and IV.2		
(Sogge)	defined in Eq. (V.2)	$t_{\text{Sogge}} = 0$	defined in Eq. (V.3)
(TP)	defined in Theorems VI.1 and VI.2		

Table 2: Reference to all appearing exponents.

Theorem 1 (Spectral cluster L^q estimates, see Theorem VII.2 and Section VIII.2). *Let the symbol $p(x, \xi) = |\xi|^2 + V(x)$ with a confining³ potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$, $E \in \mathbb{R}$, $\varepsilon > 0$ be a small error. Let us denote by P by the Schrödinger operator $-h^2\Delta + V$ and by $\Pi_{E,h}$ the spectral projector on the window $[E - h, E + h]$*

$$\Pi_{E,h} := \mathbf{1}(P \in [E - h, E + h]).$$

Upper bounds. *Assume that $\Omega = \mathbb{R}^d$, or $\Omega = \{V > E + \varepsilon\}$ in the classical forbidden region, $\Omega = \{V < E - \varepsilon\}$ in the bulk, or in a neighborhood of the turning points $\Omega = \{|V - E| < \varepsilon\}$ under the additional assumption that all points $x \in \mathbb{R}^d$ in $\{V = E\}$ must satisfy the condition $\nabla_x V(x) \neq 0$. Then, there exist*

- $h_0 = h_0(E, \varepsilon) > 0$,
- fixed exponents $s \in [-\infty, \infty)$, $t \geq 0$ and $\alpha \in [1, \infty]$ (to be defined below),
- a multiplicative constant $C = C(\Omega, V, E, \varepsilon) > 0$,

such that for any $2 \leq q \leq \infty$, any $h \in (0, h_0]$ and any γ_h such that $\gamma_h = \Pi_{E,h}\gamma_h = \gamma_h\Pi_{E,h}$

$$\|\rho_{\gamma_h}\|_{L^{q/2}(\Omega)} \leq Ch^{-2s(q,d)} \log(1/h)^{2t(q,d)} \|\gamma_h\|_{\mathfrak{S}^{\alpha(q,d)}(L^2(\mathbb{R}^d))}.$$

with

$$(s(q, d), t(q, d), \alpha(q, d)) = \begin{cases} (s_{\text{gene}}(q, d), t_{\text{gene}}(q, d), \alpha_{\text{gene}}(q, d)) & \text{for any } \Omega \subset \mathbb{R}_x^d, \\ (-\infty, 0, \infty) & \text{if } \Omega = \{V > E + \varepsilon\}, \\ (s_{\text{Sogge}}(q, d), t_{\text{Sogge}}(q, d), \alpha_{\text{Sogge}}(q, d)) & \text{if } \Omega = \{V < E - \varepsilon\}, \\ (s_{\text{TP}}(q, d), t_{\text{TP}}(q, d), \alpha_{\text{TP}}(q, d)) & \text{if } \Omega = \{|V - E| < \varepsilon\}. \end{cases}$$

Optimality. *Moreover, under additive assumption of “flatness”⁴ around the energy $E \in \mathbb{R}$, one has the optimality of the exponent α_{Sogge} in a classically allowed region. There exist a multiplicative constant $C = C(d, h_0, V, \varepsilon) > 0$, $h_0 > 0$, a sequence of energies $E_h \in \mathbb{R}$ in a neighborhood of E and of density matrix γ_h such that for any $2 \leq q \leq \infty$, any $h \in (0, h_0]$*

$$\|\rho_{\gamma_h}\|_{L^{q/2}(\{V < E_h - \varepsilon\})} \geq Ch^{-2s_{\text{Sogge}}(q,d)} \log(1/h)^{2t_{\text{Sogge}}(q,d)} \|\gamma_h\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}(L^2(\mathbb{R}^d))}.$$

³to be defined in Section VII

⁴see Assumption (VIII.3)

We also sum up Theorems III.1, IV.2, V.2 and VI.2 in the case of Schrödinger operators. Actually, the results are stated for a more general class of pseudodifferential operators.

Theorem 2 (Microlocalized L^q estimates). *Let the symbol $p(x, \xi) = |\xi|^2 + V(x)$ with a confining⁵ potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$, $E \in \mathbb{R}$, $\varepsilon > 0$ be a small error. Let us denote by P by the operator $-h^2\Delta + V$. For any point $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ that satisfies one of the geometric conditions in Table 1: (cond) = (ellip), (gene), (Sogge) or (TP), there exist fixed $s_{\text{cond}} \geq 0$, $t_{\text{cond}} \geq 0$ and $\alpha_{\text{cond}} \in [1, \infty]$ associated to (cond) (see Table 2 above) and*

- an open bounded neighborhood $\mathcal{U} \times \mathcal{V}$ of (x_0, ξ_0) ,
- $h_0 > 0$,

such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported into $\mathcal{U} \times \mathcal{V}$, there exists a multiplicative constant $C = C(d, \chi, h_0) > 0$, such that for any $2 \leq q \leq \infty$, any $h \in (0, h_0]$ and any non-negative self-adjoint operator γ on $L^2(\mathbb{R}^d)$, we have

$$\|\rho_{\chi^w(x, hD)} \gamma \chi^w(x, hD)\|_{L^{q/2}(\mathbb{R}^d)} \leq Ch^{-2s_{\text{cond}}(q, d)} \log(1/h)^{2t_{\text{cond}}(q, d)} \times \left(\|\gamma\|_{\mathfrak{S}^{\alpha_{\text{cond}}(q, d)}(L^2(\mathbb{R}^d))} + \frac{1}{h^2} \|(P - E)\gamma(P - E)\|_{\mathfrak{S}^{\alpha_{\text{cond}}(q, d)}(L^2(\mathbb{R}^d))} \right).$$

Implicitly, we can deduce L^q bounds for density of family of non-negative bounded operators $\{\gamma_h\}_{h \in (0, h_0]}$ with an integral kernel, that are “many-body quasimode” of P in nuclear type norm for any $h \in (0, h_0]$

$$\forall \alpha \geq 1, \quad \gamma_h = \mathcal{O}_{\mathfrak{S}^\alpha}(1) \quad \text{and} \quad (P - E)\gamma_h, \gamma_h(P - E) = \mathcal{O}_{\mathfrak{S}^\alpha}(h),$$

with a spectral or a phase-space localization assumption

- $\gamma_h = \Pi_{E, h} \gamma_h \Pi_{E, h} + \mathcal{O}_{\mathfrak{S}_{P, E}^\alpha}(h^\infty)$,
- or such that there exists a compact $K \subset \mathbb{R}^d \times \mathbb{R}^d$ such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in K , $\gamma_h = \chi^w(x, hD) \gamma_h \chi^w(x, hD) + \mathcal{O}_{\mathfrak{S}_{P, E}^\alpha}(h^\infty)$.

Here, we denote $\gamma = \mathcal{O}_{\mathfrak{S}_{P, E}^\alpha}(h^\infty)$ if the weighted norm $\|\gamma\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))} + \frac{1}{h^2} \|(P - E)\gamma(P - E)\|_{\mathfrak{S}^\alpha}$ or $\|(1 + (P - E)^2/h^2)^{1/2} \gamma (1 + (P - E)^2/h^2)^{1/2}\|_{\mathfrak{S}^\alpha}$ are $\mathcal{O}(h^\infty)$. Note that one can also replace $\mathcal{O}_{\mathfrak{S}_{P, E}^\alpha}(h^\infty)$ with $\mathcal{O}_{\mathfrak{S}_{P, E}^\alpha}(h^N)$ for a large enough fixed $N > 0$. For now, the question of getting rid of the localization property is still opened, even in the one-body case.

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II REVIEW OF SEMICLASSICAL ANALYSIS AND DENSITY MATRICES

Before going to the main results and their proofs, we recall the results of semiclassical analysis and density matrix analysis that we will use. We refer to [8], [37] and [28] for further details.

⁵We will later that the condition of “almost polynomial growth” (Definition II.3) is enough.

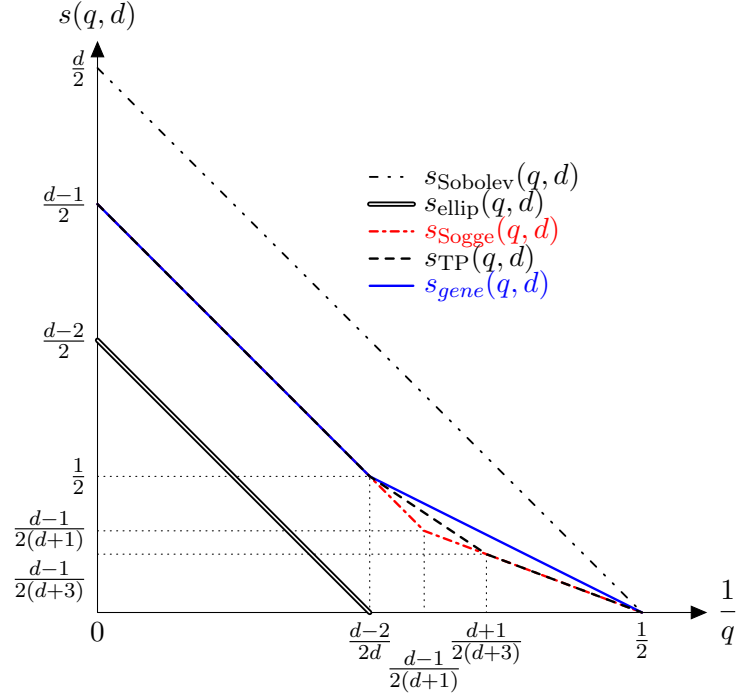


Figure 3: Exponent $s(q, d)$ of microlocalized estimates when $d \geq 2$.

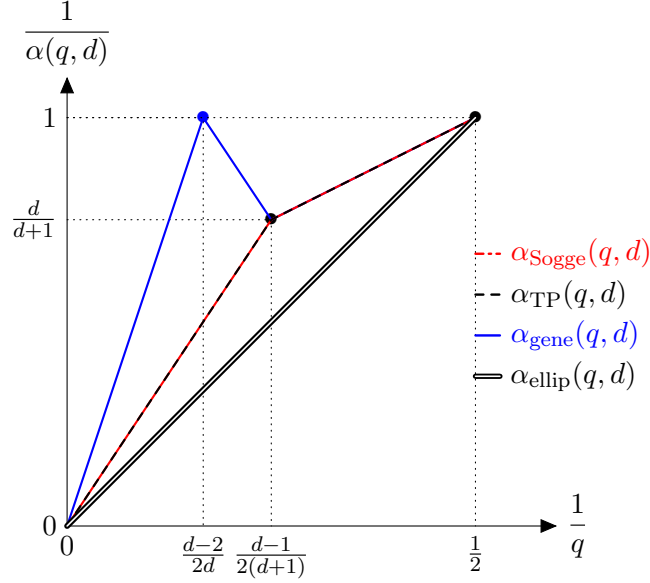


Figure 4: Schatten exponent $\alpha(q, d)$ when $d \geq 3$.

II.1 Symbol classes and quantization

Let us recall the definitions of order functions and symbol classes. In the following, we will fix $d \in \mathbb{N}^*$ and we will use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$, for $x \in \mathbb{R}^d$.

Definition II.1 (Order functions [37, Sec. 4.4.1]). *A function $m \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, [0, \infty[)$ is called*

an order function on $\mathbb{R}^d \times \mathbb{R}^d$ if there exist $C, N > 0$ such that

$$\forall (x, \xi), (y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \quad m(x, \xi) \leq C(1 + |x - y| + |\xi - \eta|)^N m(y, \eta).$$

Remark 1. Relevant examples of order functions are $(x, \xi) \mapsto \langle x \rangle^k \langle \xi \rangle^\ell$ for any $k, \ell \in \mathbb{R}$.

Definition II.2 (Symbols [37, Sec. 4.4.1]). Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$. A function $a \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a symbol in the class $S(m)$ if for all $\alpha \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_\alpha > 0$ such that

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |\partial_{x, \xi}^\alpha a(x, \xi)| \leq C_\alpha m(x, \xi).$$

Remark 2. In the following, we will consider symbols a which will depend on an external parameter (which can be h). In that case, it will be important that the constants C_α are independent of the parameter.

Remark 3. Below, we will encounter symbols belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, which is equivalent to belong to the symbol classes $S(\langle x, \xi \rangle^{-k})$ for all $k \in \mathbb{N}$.

Notation. Let $N \in \mathbb{R}$ and $a \in S(m)$.

- We write $a = \mathcal{O}_{S(m)}(h^N)$ if for any $\alpha \in \mathbb{N}^{2d}$, there exists $C_{\alpha, N} > 0$, independent of h , such that

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |\partial_{x, \xi}^\alpha a(x, \xi)| \leq C_{\alpha, N} h^N m(x, \xi).$$

- We denote $a = \mathcal{O}_{S(m)}(h^\infty)$ if $a = \mathcal{O}_{S(m)}(h^N)$ for any $N \in \mathbb{N}$.

Similarly, for $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, we write $a = \mathcal{O}_{\mathcal{S}}(h^N)$ (resp. $a = \mathcal{O}_{\mathcal{S}}(h^\infty)$) if $a = \mathcal{O}_{S(\langle x, \xi \rangle^{-k})}(h^N)$ (resp. $a = \mathcal{O}_{S(\langle x, \xi \rangle^{-k})}(h^\infty)$) for all $k \in \mathbb{N}$.

It will be important for us that the classical symbol $p(x, \xi) = |\xi|^2 + V(x)$ is a symbol in the sense of the above definition. This motivates the following definition of the class of potentials that we consider.

Definition II.3. A potential $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ has at most polynomial growth if there exists $k \in \mathbb{N}^*$ such that

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, \forall x \in \mathbb{R}^d, |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^k. \quad (\text{II.1})$$

Remark 4. In the above definition, (II.1) implies that $p(x, \xi) = |\xi|^2 + V(x)$ is in the symbol class $S(m)$ for $m(x, \xi) := \langle \xi \rangle^2 \langle x \rangle^k$.

We will always assume that the potential V is bounded from below. That ensures that the operator $P = h^2 \Delta + V$ is also bounded from below.

Definition II.4 (Boundedness from below). The potential V is bounded from below, more exactly that there exists $C \in \mathbb{R}$ such that for any $x \in \mathbb{R}^d$, there exists $C \in \mathbb{R}$ such that for any $x \in \mathbb{R}^d$, one has $V(x) \geq C$.

Definition II.5 (Quantization, [37, Thm. 4.16]). Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$ and $a \in S(m)$. Let $h > 0$. Let $\mathfrak{t} \in [0, 1]$. The \mathfrak{t} -quantization of a , denoted by $\text{Op}_h^\mathfrak{t}(a)$, is the linear continuous operator $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ defined by the formula

$$\text{Op}_h^\mathfrak{t}(a)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\frac{\xi \cdot (x-y)}{h}} a(\mathfrak{t}x + (1-\mathfrak{t})y, \xi) u(y) d\xi dy$$

for any $x \in \mathbb{R}^d$ and any $u \in \mathcal{S}(\mathbb{R}^d)$. For $\mathfrak{t} = 1/2$, $\text{Op}_h^{1/2}(a)$ is called the Weyl quantization of a and is also denoted by $a^w(x, hD)$. For $\mathfrak{t} = 1$, $\text{Op}_h^1(a)$ is called the right quantization of a and is also denoted by $a^R(x, hD)$.

II.2 Semiclassical pseudodifferential calculus

In the following, we list some standard operations on pseudodifferential operators. We only state them for the Weyl quantization to keep a light notation, but they are still valid for other quantizations (in Definition II.5).

Proposition II.6 (Composition of pseudodifferential operators [37, Thm. 4.12 and 4.18]). *Let m_1 and m_2 two order functions on $\mathbb{R}^d \times \mathbb{R}^d$. Let $a \in S(m_1)$ and $b \in S(m_2)$.*

1) *Then, there exists a symbol in $S(m_1 m_2)$, that we denote by $a_{\#} b$, such that*

$$a^w(x, hD) b^w(x, hD) = (a_{\#} b)^w(x, hD).$$

2) *Furthermore, there exists a unique family $\{c_j\}_{j \in \mathbb{N}} \subset S(m_1 m_2)$ supported in $\text{supp } a \cap \text{supp } b$ such that for any $N \in \mathbb{N}^*$, there exists $r_N \in S(m_1 m_2)$ such that*

$$a_{\#} b = \sum_{j=0}^{N-1} h^j c_j + h^N r_N.$$

Moreover, we have $c_0 = ab$.

This result has two important corollaries.

Corollary II.7 (Disjoint supports [37, Thm. 4.12]). *Let $a \in S(m_1)$ and $b \in S(m_2)$ be such that $\text{supp } a \cap \text{supp } b = \emptyset$. Then,*

$$a_{\#} b = \mathcal{O}_{S(m_1 m_2)}(h^\infty).$$

Corollary II.8 (Commutator). *Let $a \in S(m_1)$ and $b \in S(m_2)$. Then, there exists $r \in S(m_1 m_2)$ such that for any $h > 0$*

$$[a^w(x, hD), b^w(x, hD)] = h r^w(x, hD).$$

The following proposition quantifies the difference between two quantizations of the same symbol.

Proposition II.9 (Change of quantization [37, Thm. 4.13]). *Let $a \in S(m)$ and $\mathfrak{t}, \mathfrak{s} \in [0, 1]$. Then, there exists $\tilde{a}_{\mathfrak{t}, \mathfrak{s}} \in S(m)$ such that for any $h > 0$*

$$\text{Op}_h^{\mathfrak{t}}(a) - \text{Op}_h^{\mathfrak{s}}(a) = h \text{Op}_h^{\mathfrak{t}}(\tilde{a}_{\mathfrak{t}, \mathfrak{s}}).$$

Let us now recall the definition of locally elliptic symbols.

Definition II.10 (Elliptic symbol). *Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$. A symbol $a \in S(m)$ is elliptic on $U \subset \mathbb{R}^d \times \mathbb{R}^d$ if there exists $C > 0$ such that $|a(x, \xi)| \geq m(x, \xi)/C$ for all $(x, \xi) \in U$.*

The following lemma gives local left and right inverses for quantization of locally elliptic symbols. These microlocal equalities will be very useful in the proof of Theorem VI.2.

Lemma II.11 ([22, Lem. 2.1]). *Let $\chi \in S(1)$, m be an order function and $a \in S(m)$ elliptic on $\text{supp } \chi$. Then, for any $\mathfrak{t} \in [0, 1]$, there exist $b_{\mathfrak{t}} \in S(1/m)$, $r_{1, \mathfrak{t}}, r_{2, \mathfrak{t}} \in S(1)$ such that*

$$\begin{aligned} \text{Op}_h^{\mathfrak{t}}(b_{\mathfrak{t}}) \text{Op}_h^{\mathfrak{t}}(a) \text{Op}_h^{\mathfrak{t}}(\chi) &= \text{Op}_h^{\mathfrak{t}}(\chi) + \text{Op}_h^{\mathfrak{t}}(r_{1, \mathfrak{t}}), \\ \text{Op}_h^{\mathfrak{t}}(a) \text{Op}_h^{\mathfrak{t}}(b_{\mathfrak{t}}) \text{Op}_h^{\mathfrak{t}}(\chi) &= \text{Op}_h^{\mathfrak{t}}(\chi) + \text{Op}_h^{\mathfrak{t}}(r_{2, \mathfrak{t}}), \end{aligned}$$

where $r_{1, \mathfrak{t}}, r_{2, \mathfrak{t}} = \mathcal{O}_{S(1)}(h^\infty)$. If $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ then $r_{1, \mathfrak{t}}, r_{2, \mathfrak{t}} = \mathcal{O}_{\mathcal{S}}(h^\infty)$.

For any real elliptic $p \in S(m)$ (when $m \geq 1$), the operator $P = p^w(x, hD)$ is self-adjoint on a suitable domain ([37, Sec. 10.1.2]) so that $f(P)$ is well-defined by functional calculus, for any $f \in C_c^\infty(\mathbb{R})$. The next theorem states that such a $f(P)$ is actually a pseudodifferential operator and provides us information on its associated symbol. This result is crucial for justifying the application of microlocalized estimates (in Sections III, IV, V and VI) to spectral clusters in Section VII.

Theorem II.12 ([8, Thm. 8.7]). *Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $m \geq 1$, $p \in S(m)$ be a symbol such that $p + i$ is elliptic on $\mathbb{R}^d \times \mathbb{R}^d$ and $P := p^w(x, hD)$. Let $f \in C_c^\infty(\mathbb{R})$. Then, there exists $a \in \cap_{k \in \mathbb{N}} S(m^{-k})$ such that $f(P) = a^w(x, hD)$. Moreover, there exist functions $\{a_j\}_{j \in \mathbb{N}} \subset \cap_{k \in \mathbb{N}} S(m^{-k})$ supported in $\text{supp}(f \circ p)$ such that for all $N \geq 1$ there exists $r_N \in \cap_{k \in \mathbb{N}} S(m^{-k})$ such that*

$$a = \sum_{j=0}^{N-1} h^j a_j + h^N r_N.$$

In particular, the principal symbol a_0 is equal to $f \circ p$.

II.3 Semiclassical bounds

In this section, almost all the results are stated accordingly to symbols $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$. But, for the reader's convenience we voluntarily write the dimension n instead of d in Lemma II.19, Theorem II.20, Theorem II.26 and Theorem II.27 to draw the attention that they can be different objects. Actually, we essentially will apply these results to $n = d$ (see Section IV) and $n = d - 1$ (see Section V).

We state now a natural property on quantizations of Schwartz functions, that is very useful to prove that the density $\rho_{\chi^w \gamma \chi^w}$ is well-defined (Lemma II.22) and a corollary of Mercer theorem (Remark 7).

Lemma II.13. *If $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then the integral kernel $\text{Op}_h^\dagger(\chi)$ is also in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. As a consequence, this operator is trace-classes and Hilbert-Schmidt.*

When a is in the Schwartz space $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, the operator $a^w(x, hD)$ not only preserves continuously $\mathcal{S}(\mathbb{R}^d)$ (it is still valid for any symbol $a \in S(m)$), but it has the good property of extending to a regularizing operator.

Proposition II.14 ([37, Thm. 4.1]). *Let $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Then, for any $h > 0$, the operator $a^w(x, hD)$ maps continuously $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.*

Let us recall the Calderon-Vaillancourt theorem, which implies the L^2 -boundness of the quantizations of symbols in $S(1)$.

Proposition II.15 (Calderon-Vaillancourt [37, Thm. 4.23]). *Let $a \in S(1)$. Then, for any $h > 0$, the operator $a^w(x, hD)$ extends to a bounded linear operator on $L^2(\mathbb{R}^d)$, with operator norm bounded uniformly in $h \in (0, 1]$.*

Let us state now basic semiclassical L^q estimates, from which one can deduce a semiclassical version of Sobolev embedding for microlocalized functions.

Lemma II.16 (Basic L^q estimates, [22, Lemma 2.2]). *Let $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Then, there exists $C > 0$ such that for any $h > 0$ and any $1 \leq p \leq q \leq \infty$*

$$\|a^w(x, hD)u\|_{L^q(\mathbb{R}^d)} \leq Ch^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p(\mathbb{R}^d)}.$$

The exponent $d(1/p - 1/q)$ on the semiclassical parameter in the previous estimate can be indeed improved in the elliptic setting for $p = 2$.

Lemma II.17 (One-body elliptic estimates, [22, Thm. 3]). *Let $d \geq 1$. Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $p \in S(m)$ and $P := p^w(x, hD)$ (or any other quantization). Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ such that*

$$p(x_0, \xi_0) \neq 0.$$

Then, there exists a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for all $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in \mathcal{V} , for any $0 < h \leq h_0$, there exists $C > 0$ such that for all $t \in [0, 1]$ and for all $2 \leq q \leq \infty$,

$$\|\text{Op}_h^t(\chi)u\|_{L^q(\mathbb{R}^d)} \leq Ch^{1-d(\frac{1}{2}-\frac{1}{q})} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right).$$

Equivalently, for all $2 \leq q \leq \infty$

$$\text{Op}_h^t(\chi)(1 + P^*P/h^2)^{-1/2} = \mathcal{O} \left(h^{1-d(\frac{1}{2}-\frac{1}{q})} \right) : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d). \quad (\text{II.2})$$

Remark 5. *The bound of Lemma II.17 is one power of h better than the one in Lemma II.16 (for $p = 2$, c.f. Figure 3), thanks to the term involving the operator P on the right-side. This is particularly relevant for quasimodes u of P , since they satisfy $Pu = \mathcal{O}_{L^2}(h)\|u\|_{L^2}$.*

Here, we state the integrated form of Weyl's law, which gives an asymptotic of the number of eigenfunctions of a pseudodifferential operator in a fixed interval as $h \rightarrow 0$.

Proposition II.18 (Integrated Weyl law, [8, Chap. 9]). *Let m be an order function such that $m(x, \xi) \rightarrow +\infty$ when $|(x, \xi)| \rightarrow +\infty$ and let $p \in S(m)$ be real valued such that $p + i$ is elliptic on $\mathbb{R}^d \times \mathbb{R}^d$. Let $a < b$ be two real numbers. For any $h > 0$, define $P = p^w(x, hD)$ and denote by $N_h([a, b])$ the number of eigenvalues of P in the interval $[a, b]$. Then, we have*

$$N_h([a, b]) = \frac{1}{(2\pi h)^d} \left[|p^{-1}([a, b])| + o_{h \rightarrow 0}(1) \right].$$

We now review well-known results on quantum dynamics and their propagators.

Lemma II.19 (Properties of the propagator $F(t, t_0)$, [37, Thm. 10.1]). *Let $n \in \mathbb{N}^*$ and $h > 0$. Let $t_0 \in \mathbb{R}$ and $a \in C^\infty(\mathbb{R}_t, S_{\mathbb{R}^n \times \mathbb{R}^n}(1))$. The equation*

$$\begin{cases} [hD_t - a^w(t, x, hD_x)]F(t, t_0) = 0, & t \in \mathbb{R}, \\ F(t_0, t_0) = \text{Id}, \end{cases}$$

has a unique solution $\{F(t, t_0)\}_{t \in \mathbb{R}}$ in $\mathcal{C}(\mathbb{R}, \mathcal{B}(L^2(\mathbb{R}^n)))$, which is a family of unitary operators. Furthermore,

- (i) *For any compact $J \subset \mathbb{R}$ and any $k, s \in \mathbb{N}$, there exists $C > 0$ (independent of $h \in (0, h_0)$) such that for any $t, t_0 \in J$,*

$$\left\| (hD_t)^k F(t, t_0) \right\|_{H_h^s(\mathbb{R}^n) \rightarrow H_h^s(\mathbb{R}^n)} \leq C. \quad (\text{II.3})$$

- (ii) *For any $\psi_1 \in C_c^\infty(\mathbb{R})$, the operator $\psi_1(t)F(t, t_0)$ maps continuously $H_h^s(\mathbb{R}^n)$ into $H_h^s(\mathbb{R}^{n+1})$ for all $s \in \mathbb{N}$ (with an operator norm independent of h).*

(iii) Let us define the operator T_F , which acts on functions on \mathbb{R}^{n+1} , by

$$T_F : u(t, x) \mapsto \psi_1(t) \int_{t_0}^t (F(t, s)u(s))(x) ds.$$

Then, T_F maps continuously $H_h^s(\mathbb{R}^{n+1})$ into $H_h^s(\mathbb{R}^{n+1})$ for all $s \in \mathbb{N}$ (with a bound independent of h).

Remark 6. The proof of (i) is done in [37, Thm. 10.1] in the case $k = 0$. The bounds for higher values of k can be obtained by induction using the equation satisfied by $F(t, t_0)$. The proofs of (ii) and (iii) follow from (i) by elementary arguments. But for more completeness, we detail these points in the proof below.

Proof of Lemma II.19.

(i) Assume that we have the induction until the index $k \in \mathbb{N}$ for (II.3). We now check that the relation is still valid for $k + 1$. By using the evolution equation satisfied by the propagator, one has

$$\begin{aligned} (hD_t)^{k+1}F(t, t_0) &= (hD_t)^k(hD_tF(t, t_0)) = (hD_t)^k(a^w(t, x, hD_x)F(t, t_0)) \\ &= \sum_{\ell+\ell'=k} h^\ell D_t^\ell a^w(t, x, hD_x)(hD_t)^{\ell'}F(t, t_0). \end{aligned}$$

Moreover, any operator $D_t^\ell a^w(t, x, hD_x)$ can be written as the Weyl quantization of a symbol $b_{\ell, t}$, that satisfies the relation $b_{\ell, t}(x/2, \xi) = \partial_t^\ell(a_t(x/2, \xi))$ for any $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. By the definition of the function a , for any $\ell \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^n \times \mathbb{N}^n$, there exists $C_{\ell, \alpha} > 0$ such that

$$\forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \quad \left| \partial_t^k \partial_{x, \xi}^\alpha a_t(x, \xi) \right| \leq C_{\ell, \alpha}.$$

As a consequence $b_{\ell, t} \in S(1)$. Eventually, one gets the desired uniform bound (II.3) for the index $k + 1$ by the compactness of the interval J , the Calderon-Vaillancourt theorem and the induction relation.

(ii) Let $\psi_1 \in C_c^\infty(\mathbb{R})$ and let $s \in \mathbb{N}$. Let $t_0 \in J$ and $v \in L^2(\mathbb{R}^n)$. We write the H_h^s -norm of $\psi_1(t)F(t, t_0)v$ as a finite sum of

$$\| (hD_t)^k (hD_x)^\ell (\psi_1(t)F(t, t_0)v(x)) \|_{L^2(\mathbb{R}^{n+1})}$$

on all multiindexes $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^n$ such that $|k| + |\ell| = s$. By using the bound (II.3), that the intervals $\text{supp } \psi_1$ and J are compact, there exists $C > 0$ such that one has, for any $(k, \ell) \in \mathbb{N} \times \mathbb{N}^n$ as above, any $t, t_0 \in J$ and any $v \in L^2(\mathbb{R}^n)$

$$\begin{aligned} \left\| (hD_t)^k (hD_x)^\ell (\psi_1(t)F(t, t_0)v(x)) \right\|_{L^2_{t,x}(\mathbb{R}^{n+1})} &\leq \left\| (hD_x)^\ell (hD_t)^k (\psi_1(t)F(t, t_0)v(x)) \right\|_{L^2_t(\text{supp } \psi_1, L^2_x(\mathbb{R}^n))} \\ &\leq \left\| (hD_t)^k (\psi_1(t)F(t, t_0)v(x)) \right\|_{L^2(\mathbb{R}, H_h^s(\mathbb{R}^n))} \\ &\leq C \|v\|_{H_h^s(\mathbb{R}^n)}. \end{aligned}$$

Finally, for any $s \in \mathbb{N}$, there exists $C > 0$ such that for any $t_0 \in J$

$$\|\psi_1(t)F(t, t_0)\|_{H_h^s(\mathbb{R}^n) \rightarrow H_h^s(\mathbb{R}^{n+1})} \leq C.$$

(iii) Let $s \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^{n+1})$. One has

$$\|T_F u\|_{H_h^s(\mathbb{R}^{n+1})} \leq \sum_{0 \leq k+\ell \leq s} \left\| (hD_t)^k \left(\int_{t_0}^t \psi_1(t) F(t, r) u(r) dr \right) \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))}.$$

Let $k, \ell \in \mathbb{N}$ such that $k + \ell \leq s$. We apply the following equality (that can be proved by induction on $k \in \mathbb{N}$)

$$\partial_t^k \left(\int_{t_0}^t v(t, r) dr \right) = \int_{t_0}^t \partial_t^k v(t, r) dr + \sum_{k_1+k_2=k-1} \partial_t^{k_1} (t \mapsto \partial_1^{k_2} v(t, t)) - k \partial_t^{k-1} v(t, t_0),$$

to the function $v(t, r) = \psi_1(t) F(t, r) u(r, \cdot)$ and we have then

$$\begin{aligned} & \left\| (hD_t)^k \left(\int_{t_0}^t \psi_1(t) F(t, r) u(r) dr \right) \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))} \\ & \lesssim \sum_{0 \leq k+\ell \leq s} \left(\left\| \int_{t_0}^t (hD_t)^k \psi_1(t) F(t, r) u(r) dr \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))} \right. \\ & \quad \left. + h \sum_{k_1+k_2=k-1} \left\| (hD_t)^{k_1} (t \mapsto (hD_1)^{k_2} \psi_1(t) F(t, t) u(t)) \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))} \right. \\ & \quad \left. + h \left\| (hD_t)^{k-1} \psi_1(t) F(t, t_0) u(t_0) \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))} \right). \end{aligned} \quad (\text{II.4})$$

First, notice that

$$\left\| \int_{t_0}^t (hD_t)^k \psi_1(t) F(t, r) u(r) dr \right\|_{L_t^2(\mathbb{R}, H_h^\ell(\mathbb{R}^n))} \leq \sum_{m=1}^k \left\| \int_{t_0}^t (hD_t)^m F(t, r) u(r) dr \right\|_{L_t^2(\text{supp } \psi_1, H_h^\ell(\mathbb{R}^n))}.$$

By the bound (II.3) applied to $J = \text{supp } \psi_1$, there exists $C > 0$ such that for any $t, r \in \text{supp } \psi_1$

$$\left\| \partial_t^k F(t, r) u(r) \right\|_{H_h^\ell(\mathbb{R}^n)} \leq C \|u(r)\|_{H_h^\ell(\mathbb{R}^n)},$$

so that for any $m \leq k$

$$\begin{aligned} & \left\| \int_{t_0}^t (hD_t)^m F(t, r) u(r) dr \right\|_{L_t^2(\text{supp } \psi_1, H_h^\ell(\mathbb{R}^n))} \\ & \leq |\text{supp } \psi_1|^{1/2} \sup_{t \in \text{supp } \psi_1} \left| \int_{t_0}^t \sup_{\tau \in \text{supp } \psi_1} \left\| (hD_\tau)^m F(\tau, r) u(r) \right\|_{H_h^\ell(\mathbb{R}^n)} dr \right| \\ & \leq C \int_{\text{supp } \psi_1} \|u(r)\|_{H_h^\ell(\mathbb{R}^n)} dr \leq C \|u\|_{H_h^\ell(\mathbb{R}^{n+1})} \\ & \leq C \|u\|_{H_h^s(\mathbb{R}^{n+1})}. \end{aligned}$$

Finally, we obtain that for any $s \in \mathbb{N}$, there exists $C > 0$ such that for any $u \in L^2(\mathbb{R}^{n+1})$

$$\|T_F u\|_{H_h^s(\mathbb{R}^{n+1})} \leq C \|u\|_{H_h^s(\mathbb{R}^{n+1})},$$

which is the desired estimate. \square

Let us now give a statement of semiclassical dispersive estimates, which are crucial ingredients in the proof of our results. The obtention of these dispersive bounds is based on the semiclassical parametrix construction of the propagator $F(t, r)$, using WKB method. The decay estimates then follows from the stationary phase formula. More precisely, this propagator is approximated by a Fourier integral operator. It is done in [37, Thm. 10.4 and 10.8] or in [8, Chapter 10].

Theorem II.20 (Semiclassical dispersive bounds). *Let $n \in \mathbb{N}$ be such that $n \geq 1$. Let $a = a_t(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}_t, \mathcal{S}_{\mathbb{R}^n \times \mathbb{R}^n}(1))$. Let $r \in \mathbb{R}$. Let $\{F(t, r)\}_{t \in \mathbb{R}}$ be the propagator of the Schrödinger evolution equation*

$$\begin{cases} [hD_t - a^w(t, x, hD_x)]F(t, r) = 0 & t \in \mathbb{R} \\ F(r, r) = \text{Id}. \end{cases}$$

For any $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, let us define the microlocalized propagator $U(t, r)$ of the previous equation by

$$U(t, r) := \psi(t - r)F(t, r)\chi^w(x, hD).$$

Let $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $I \subset \mathbb{R}$ a compact interval of \mathbb{R} , such that for all $t \in I$

$$\partial_\xi^2 a_t(x_0, \xi_0) \text{ is non-singular.} \quad (\text{II.5})$$

Then, for every open interval J such that $J \subset I$, there exist $\delta > 0$ independent of h and a neighborhood $U \times V$ of (x_0, ξ_0) such that for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ supported in $(-\delta, \delta)$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ supported in $U \times V$, we have the uniform bounds for all $t, s \in \mathbb{R}$

$$\begin{cases} \sup_{r \in J} \|U(t, r)U(s, r)^*\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} & \leq C \\ \sup_{r \in J} \|U(t, r)U(s, r)^*\|_{L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} & \leq Ch^{-n/2}(h + |t - s|)^{-n/2}. \end{cases} \quad (\text{II.6})$$

II.4 Density matrices

We finally review some definitions and standard results on Schatten spaces.

Definition II.21 (Schatten spaces). *Let $\alpha \geq 1$. For any Hilbert spaces \mathcal{H} and \mathcal{H}' , we define the Schatten space $\mathfrak{S}^\alpha(\mathcal{H}, \mathcal{H}')$ for any $\alpha \in [1, +\infty)$ the set*

$$\mathfrak{S}^\alpha(\mathcal{H}, \mathcal{H}') = \{A : \mathcal{H} \rightarrow \mathcal{H}' \text{ compact operator} : \text{Tr}_{\mathcal{H}}((A^*A)^{\alpha/2}) < \infty\}.$$

Endowed with the norm

$$\|A\|_{\mathfrak{S}^\alpha(\mathcal{H}, \mathcal{H}')} := \left(\text{Tr}_{\mathcal{H}}((A^*A)^{\alpha/2}) \right)^{1/\alpha},$$

it is a Banach space. Let us call $\mathfrak{S}^\infty(\mathcal{H}, \mathcal{H}')$, the space of compact operators. In the following, when it appears, the \mathfrak{S}^∞ -norm denotes the operator norm of compact operators that maps $L^2(\mathcal{H})$ into $L^2(\mathcal{H}')$.

We first state below the good properties of a microlocalized operator of the form $\chi^w \gamma \chi^w$.

Lemma II.22. *Assume that γ is a non-negative operator on $L^2(\mathbb{R}^d)$. Then, if $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and γ is bounded, then the operator $\chi^w(x, hD)\gamma\chi^w(x, hD)$ is non-negative, compact, and trace-class. In particular, its kernel is continuous, bounded and is in $L^p(\mathbb{R}^d \times \mathbb{R}^d)$ for any $p \geq 1$.*

A first consequence is that the L^q norm of the restriction of the integral kernel to the diagonal $\rho_{\chi^w \gamma \chi^w}$ of the integral kernel of $\chi^w \gamma \chi^w$ is always well-defined.

In the proofs of the L^q bounds

$$\|\rho_{A\gamma A^*}\|_{L^q} \leq Ch^{-s} \log(1/h)^t \|W\|_{L^{2(q/2)'}} \left(\|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|(P - E)\gamma(P - E)\|_{\mathfrak{S}^\alpha} \right), \quad (\text{II.7})$$

for a compact operator A mapping $L^2(\mathbb{R}^d)$ into $L^{2(q/2)' }(\mathbb{R}^d)$, we will see that it is enough to prove a kind of “dual form”

$$\|WA\sqrt{\gamma}\|_{\mathfrak{S}^2} \leq Ch^{-s} \log(1/h)^t \|W\|_{L^{2(q/2)' }} \left(\|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|(P-E)\gamma(P-E)\|_{\mathfrak{S}^\alpha} \right), \quad (\text{II.8})$$

for any $W \in L^{2(q/2)' } \cap C^0(\mathbb{R}^d)$.

Notation. Here and in the following, for an exponent $p \in [1, \infty]$, we define its conjugated exponent $p' := p/(p-1)$.

Notice that the implication (II.8) \implies (II.7) is similar to [11, Lem. 3] but with the Hilbert-Schmidt norm instead of the $\mathfrak{S}^{2\alpha'}$ norm and a weighted Schatten norm in the right-hand side. The conjugate exponent of the exponent α will appear naturally when we write Hölder inequality on $\|WA_j\sqrt{\gamma}\|_{\mathfrak{S}^2}$ for well-chosen A_j such that $A = \sum_j A_j$

$$\|WA_j\|_{\mathfrak{S}^{2\alpha'}} \leq Ch^{-s} \log(1/h)^t \|W\|_{L^{2(q/2)' }}.$$

In this paper, A is essentially χ^w or $\mathbf{1}$ ($P \in [E-h, E+h]$).

We state now a version of Mercer theorem (originally in [26] for kernels on compact sets), that allows to write the implication (II.8) \implies (II.7).

Theorem II.23 (Mercer theorem). *Let a bounded non-negative operator γ on $L^2(\mathbb{R}^d)$ associated to an integral kernel continuous on Ω . Assume that the restriction of the kernel to the diagonal $\rho_\gamma \in L^1(\mathbb{R}^d)$ and that the kernel is square integrable. Then, there exists a orthonormal basis of continuous eigenfunctions $\{u_j\}_{j \in \mathbb{N}}$ of γ with corresponding non-negative eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ such that*

$$\forall x, y \in \mathbb{R}^d, \quad \gamma(x, y) = \sum_{j \in \mathbb{N}} \lambda_j u_j(x) \overline{u_j(y)},$$

with a convergence of the series on L^2 -norm and an uniform convergence in all compacts of \mathbb{R}^d .

Remark 7. A consequence of Lemma II.23 is that for any function $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and any non-negative bounded γ , the operator $\chi^w(x, hD)\gamma\chi^w(x, hD)$ is non-negative and trace-class. It naturally follows from Mercer theorem that for any continuous functions $W : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\psi \in L^\infty(\mathbb{R}^d, \mathbb{R})$

$$\text{Tr}_{L^2}(W\psi\chi^w\gamma\chi^w\psi W) = \int_{\mathbb{R}^d} (W\psi\chi^w\gamma\chi^w\psi W)(x, x) dx = \int_{\mathbb{R}^d} \rho_{\psi\chi^w\gamma\chi^w\psi}(x) W(x)^2 dx.$$

Then, for $q = 2$

$$\|\psi^2 \rho_{\psi\chi^w\gamma\chi^w\psi}\|_{L^1(\mathbb{R}^d)} = \text{Tr}_{L^2}(\psi\chi^w\gamma\chi^w\psi) = \|\psi\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2}^2.$$

As well, for $q \in (2, \infty)$

$$\begin{aligned} & \|\psi^2 \rho_{\psi\chi^w\gamma\chi^w\psi}\|_{L^{q/2}(\mathbb{R}^d)} \\ & \leq \sup_{W \in L^{2(q/2)' } \cap C^0(\mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} \rho_{\psi\chi^w\gamma\chi^w\psi}(x) W(x)^2 dx}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2} = \sup_{W \in L^{2(q/2)' } \cap C^0(\mathbb{R}^d)} \frac{\text{Tr}_{L^2}(W\psi\chi^w\gamma\chi^w\psi W)}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2} \\ & \leq \sup_{W \in L^{2(q/2)' } \cap C^0(\mathbb{R}^d)} \frac{\|W\psi\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2}^2}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2}. \end{aligned}$$

In the following $\psi = 1$ or a localization function on a region of \mathbb{R}^d .

We can now state the Kato-Seiler-Simon inequalities, which are very useful tools in the many-body setting.

Lemma II.24 (Kato-Seiler-Simon,[28, Thm. 4.1]). *Let $2 \leq \alpha < \infty$. Then, for all functions $f, g \in L^\alpha(\mathbb{R}^d)$, the operator $f(x)g(-i\nabla)$ is in $\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))$ and*

$$\|f(x)g(-i\nabla)\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))} \leq (2\pi)^{-d/\alpha} \|f\|_{L^\alpha(\mathbb{R}^d)} \|g\|_{L^\alpha(\mathbb{R}^d)}.$$

As a corollary, this implies a version of semiclassical Sobolev embedding estimates $H_h^m \hookrightarrow L^q$ for operators.

Lemma II.25 (Semiclassical Schatten Sobolev estimates). *Let $m \geq 0$ and $q > 2$. Then, if $\frac{1}{q} > \frac{1}{2} - \frac{m}{d}$, we have for all $W \in L^{2(q/2)'}(\mathbb{R}^d)$*

$$\left\| W(1 - h^2 \Delta)^{-m/2} \right\|_{\mathfrak{S}^{2(q/2)'}(L^2(\mathbb{R}^d))} \leq Ch^{-d\left(\frac{1}{2} - \frac{1}{q}\right)} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}.$$

We will use the following complex interpolation result in Schatten spaces, which can be found in [11, Prop. 1] (see also [28, Thm. 2.9]).

Theorem II.26 (Complex interpolation in Schatten spaces). *Let $n \geq 1$. Let $a_0 < a_1$ be two real numbers. Let T be an application which maps the strip $S = \{z \in \mathbb{C}, a_0 \leq \Re z \leq a_1\}$ into bounded operators on $L^2(\mathbb{R}^{n+1})$. Moreover, let us assume that the family of operators $\{T_z\}_{z \in S}$ is analytic in the sense of Stein i.e.*

$$z \in S \mapsto \langle f, T_z g \rangle_{L^2(\mathbb{R}^{n+1})} \text{ is continuous for all simple functions } f, g.$$

and

$$z \in \overset{\circ}{S} \mapsto \langle f, T_z g \rangle_{L^2(\mathbb{R}^{n+1})} \text{ is analytic for all simple functions } f, g.$$

If there exist $C_0, C_1, b_0, b_1 > 0$, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $1 \leq r_0, r_1 < \infty$ such that for all $\sigma \in \mathbb{R}$, one has for all simple functions W_1, W_2 on \mathbb{R}^{n+1}

$$\forall j = 0, 1, \quad \|W_1 T_{a_j + i\sigma} W_2\|_{\mathfrak{S}^{r_j}(L^2(\mathbb{R}^{n+1}))} \leq C_j e^{b_j |\sigma|} \|W_1\|_{L_t^{p_j} L_x^{q_j}(\mathbb{R}^{n+1})} \|W_2\|_{L_t^{p_j} L_x^{q_j}(\mathbb{R}^{n+1})}.$$

Then, for all $0 \leq \theta \leq 1$

$$\|W_1 T_{a_\theta} W_2\|_{\mathfrak{S}^{r_\theta}(L^2(\mathbb{R}^{n+1}))} \leq C_0^{1-\theta} C_1^\theta \|W_1\|_{L_t^{p_\theta} L_x^{q_\theta}(\mathbb{R}^{n+1})} \|W_2\|_{L_t^{p_\theta} L_x^{q_\theta}(\mathbb{R}^{n+1})},$$

where $a_\theta, r_\theta, p_\theta$ and q_θ are defined by

$$a_\theta = (1 - \theta)a_0 + \theta a_1, \quad \frac{1}{r_\theta} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Actually, when one of the exponents $r_j = \infty$, the \mathfrak{S}^∞ norm on the left-hand side of the two previous bounds can be replaced by the operator norm.

II.5 Strichartz estimates for density matrices

In this section, we provide Strichartz estimates in Schatten spaces, which will be a key ingredient for our proof. They generalize the one-body Strichartz estimates [22, Prop. 4.3], which were also the key ingredient of the proof of Koch-Tataru-Zworski. Such many-body Strichartz estimates were discovered in [9], and later generalized in [11]. Our proof is inspired by the one in [11], and provides a way to obtain the full range of Strichartz estimates in Schatten spaces under the general assumption that the propagator satisfies dispersive estimates such as the one in Theorem

II.20. In the one-body case, the fact that Strichartz estimates follow abstractly from dispersive bounds were discovered by Ginibre and Velo [14], and we generalize the results to the many-body case. Interestingly, our many-body proof uses complex interpolation in the spirit of the original proof of Strichartz [35] rather than the direct approach using the Hardy-Littlewood-Sobolev inequality of [14].

Theorem II.27. *Assume the same hypotheses as in Theorem II.20. Let $2 \leq q \leq \frac{2(n+1)}{n-1}$. Let $p(q) := \frac{2(\frac{q}{2})'}{1+(\frac{2}{q-2}-\frac{n}{2})_-}$. Then, there exist $C > 0$ and $h_0 > 0$, such that for any $0 < h \leq h_0$, we have, for any $W \in L_t^{p(q)} L_x^{2(q/2)'}(\mathbb{R}^{n+1})$*

$$\sup_{r \in J} \|WU(t, r)\|_{\mathfrak{S}^2(\frac{2q}{q+2})'(L^2(\mathbb{R}^n), L^2(\mathbb{R}^{n+1}))} \leq C \|W\|_{L_t^{p(q)} L_x^{2(q/2)'}(\mathbb{R}^{n+1})} \times \begin{cases} h^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})} & \text{if } 2 \leq q < \frac{2(n+1)}{n-1}, \\ \log(1/h)^{\frac{1}{n+1}} h^{-\frac{1}{n+1}} & \text{if } q = \frac{2(n+1)}{n-1}. \end{cases}$$

Proof of Theorem II.27. Fix $r \in J$. Let $z \in \mathbb{C}$. For all $t, s \in \mathbb{R}$, let us define the operator $T_z(t, s, r)$ on $L^2(\mathbb{R}^n)$ by

$$T_z(t, s, r) := (t - s + i0)^z U(t, r) U(s, r)^*.$$

Let the operator $T_z(r)$ acting on functions on \mathbb{R}^{n+1} be defined by

$$\forall (f, g) \quad \langle f, T_z(r)g \rangle_{L^2(\mathbb{R}^{n+1})} = \int_{\mathbb{R} \times \mathbb{R}} \langle f(t), T_z(t, s, r)g(s) \rangle_{L_x^2(\mathbb{R}^n)} dt ds.$$

Defining $A = WU(t, r) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{n+1})$, we notice that we have $AA^* = WT_0(r)\bar{W}$ so that the bound in the theorem will follow from estimating $WT_0(r)\bar{W}$ in $\mathfrak{S}^2(\frac{2q}{q+2})'(L^2(\mathbb{R}^{n+1}))$. We will use the following properties of the distribution $m_z(t) = (t + i0)^z$, which can be found in [13, Chap. I, Sec. 3.6]:

- ◇ the family $\{m_z\}_{z \in \mathbb{C}} \subset \mathcal{S}'(\mathbb{R})$ is analytic and for any $z \in \mathbb{C}$ each m_z admits a Fourier transform with this expression

$$\mathcal{F}((t + i0)^z)(\omega) = \frac{\sqrt{2\pi} e^{iz\pi/2}}{\Gamma(-z)} \omega_+^{-z-1}. \quad (\text{II.9})$$

- ◇ Let $z = -1 + i\sigma$ with $\sigma \in \mathbb{R}$, then \hat{m}_z is bounded and

$$\|\hat{m}_{-1+i\sigma}\|_{L^\infty(\mathbb{R})} \leq \sqrt{\frac{2}{\pi}} e^{\sigma\pi/2}. \quad (\text{II.10})$$

- ◇ When the real part of z is strictly greater than -1 , m_z is in $L_{\text{loc}}^1(\mathbb{R})$ and

$$|m_z(t)| \leq |t|^{\Re z}.$$

We will obtain bounds on $W_1 T_0(r) W_2$ for all simple functions W_1 and W_2 using Theorem II.26, estimating the operator $W_1 T_z(r) W_2$ in

- the operator norm for $\Re z = -1$,
- the \mathfrak{S}^2 -norm for $\Re z = \beta \geq \frac{n-1}{2}$.

Step 1. Operator norm bounds. Let us prove that there exists $C > 0$ such that for any simple functions W_1, W_2 on \mathbb{R}^{n+1} and for any $\sigma \in \mathbb{R}$

$$\sup_{r \in J} \|W_1 T_{-1+i\sigma}(r) W_2\|_{L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1})} \leq C e^{\sigma\pi/2} \|W_1\|_{L^\infty(\mathbb{R}^{n+1})} \|W_2\|_{L^\infty(\mathbb{R}^{n+1})}. \quad (\text{II.11})$$

Let $\sigma \in \mathbb{R}$ and F, G be functions $\mathcal{C}_c^\infty(\mathbb{R}^{n+1}) \subset L^2(\mathbb{R}^{n+1})$. We can write

$$\begin{aligned} & \langle F, T_{-1+i\sigma}(r) G \rangle_{L_{t,x}^2} \\ &= \int_{\mathbb{R} \times \mathbb{R}} \langle F(t), T_{-1+i\sigma}(t, s, r) G(s) \rangle_{L_x^2} dt ds \\ &= \int_{\mathbb{R} \times \mathbb{R}} \langle F(t), (t-s+i0)^{-1+i\sigma} U(t, r) U(s, r)^* G(s) \rangle_{L_x^2} dt ds \\ &= \int_{\mathbb{R} \times \mathbb{R}} (t-s+i0)^{-1+i\sigma} \langle U(t, r)^* F(t), U(s, r)^* G(s) \rangle_{L_x^2} dt ds. \end{aligned}$$

Define the functions f, g by

$$f(t, x; r) := (U(t, r)^* F(t, \cdot))(x) \quad \text{and} \quad g(t, x; r) := (U(t, r)^* G(t, \cdot))(x).$$

By the $L^2 \rightarrow L^2$ bound of (II.6)

$$\sup_{t \in \mathbb{R}} \sup_{r \in J} \|U(t, r)^*\|_{L_x^2 \rightarrow L_x^2} \lesssim 1,$$

so that the previous functions satisfy the bounds

$$\begin{aligned} \sup_{r \in J} \|f(t, x; r)\|_{L_{t,x}^2(\mathbb{R}^{n+1})} &\lesssim \|F\|_{L_{t,x}^2(\mathbb{R}^{n+1})}, \\ \sup_{r \in J} \|g(t, x; r)\|_{L_{t,x}^2(\mathbb{R}^{n+1})} &\lesssim \|G\|_{L_{t,x}^2(\mathbb{R}^{n+1})}. \end{aligned}$$

We now write everything with the Fourier transform in the time variable, with $m_z(t) := (t+i0)^z$

$$\begin{aligned} & \langle F, T_{-1+i\sigma}(r) G \rangle_{L_{t,x}^2} \\ &= \int_{\mathbb{R} \times \mathbb{R}} m_{-1+i\sigma}(t-s) \langle f(t; r), g(s; r) \rangle_{L_x^2} dt ds \\ &= \sqrt{2\pi} \int_{\mathbb{R}} \hat{m}_{-1+i\sigma}(\omega) \langle \hat{f}(\omega; r), \hat{g}(\omega; r) \rangle_{L_x^2} d\omega. \end{aligned}$$

Hence, by the Cauchy-Schwarz inequality

$$\begin{aligned} \forall r \in J \quad & \left| \langle F, T_{-1+i\sigma}(r) G \rangle_{L_{t,x}^2} \right| \\ & \leq \sqrt{2\pi} \|\hat{m}_{-1+i\sigma}\|_{L^\infty(\mathbb{R})} \|\mathcal{F}_t(f)\|_{L_{\omega,x}^2(\mathbb{R}^{n+1})} \|\mathcal{F}_t(g)\|_{L_{\omega,x}^2(\mathbb{R}^{n+1})} \\ & \leq \sqrt{2\pi} \|\hat{m}_{-1+i\sigma}\|_{L^\infty(\mathbb{R})} \|f\|_{L_{t,x}^2(\mathbb{R}^{n+1})} \|g\|_{L_{t,x}^2(\mathbb{R}^{n+1})} \\ & \leq C \|\hat{m}_{-1+i\sigma}\|_{L^\infty(\mathbb{R})} \|F\|_{L_{t,x}^2(\mathbb{R}^{n+1})} \|G\|_{L_{t,x}^2(\mathbb{R}^{n+1})}. \end{aligned}$$

Finally, $\hat{m}_{-1+i\sigma} \in L^\infty(\mathbb{R})$ and we have the bound (II.10). Hence, we deduce a bound on $T_{-1+i\sigma} : L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1})$, from which we deduce (II.11).

Step 2. Schatten \mathfrak{S}^2 -bounds. Let $\beta \geq \frac{n-1}{2}$. Let us prove that there exists $C > 0$ such that for any $z \in \mathbb{C}$ with $\Re z = \beta$, and any simple functions W_1, W_2 on \mathbb{R}^{n+1}

$$\begin{aligned} & \sup_{r \in J} \|W_1 T_z(r) W_2\|_{\mathfrak{S}^2(L^2(\mathbb{R}^{n+1}))} \\ & \leq C \|W_1\|_{L_t^{\frac{2}{1+(\beta-\frac{n}{2})^-}} L_x^2(\mathbb{R}^{n+1})} \|W_2\|_{L_t^{\frac{2}{1+(\beta-\frac{n}{2})^-}} L_x^2(\mathbb{R}^{n+1})} \times \\ & \quad \times \begin{cases} \log(1/h)^{1/2} h^{-n/2} & \text{if } \beta = \frac{n-1}{2}, \\ h^{-n/2} & \text{if } \frac{n-1}{2} < \beta \leq \infty. \end{cases} \end{aligned} \quad (\text{II.12})$$

By the $L^1 \rightarrow L^\infty$ -bound of (II.6), the integral kernel $T_z(t, s, r)(x, y)$ of $T_z(t, s, r)$ satisfies

$$\begin{aligned} \forall t, s \in \mathbb{R} \quad \sup_{r \in J} \|T_z(t, s, r)(x, y)\|_{L_{x,y}^\infty(\mathbb{R}^n \times \mathbb{R}^n)} &= \sup_{r \in J} \|T_z(t, s, r)\|_{L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \\ &\lesssim h^{-n/2} |t-s|^{\Re z} (h+|t-s|)^{-n/2}. \end{aligned}$$

Thus, we obtain a bound on the \mathfrak{S}^2 -norm of $W_1 T_z(r) W_2$ for any $\beta := \Re z \geq 0$

$$\begin{aligned} \forall r \in J, \quad & \|W_1 T_z(r) W_2\|_{\mathfrak{S}^2(L^2(\mathbb{R}^{n+1}))}^2 \\ &= \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} |W_1(t, x) T_z(t, x, s, y; r) W_2(s, y)|^2 dt dx ds dy \\ &\lesssim h^{-n} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}(|t-s| < 2\delta) \frac{|t-s|^{2\beta}}{(h+|t-s|)^n} \|W_1(t)\|_{L^2(\mathbb{R}^n)}^2 \|W_2(s)\|_{L^2(\mathbb{R}^n)}^2 dt ds \\ &\lesssim h^{-n} \begin{cases} \|W_1\|_{L_{t,x}^2(\mathbb{R}^{n+1})}^2 \|W_2\|_{L_{t,x}^2(\mathbb{R}^{n+1})}^2 & \text{if } \beta \geq \frac{n}{2}, \\ \|W_1\|_{L_t^{\frac{2}{1+\beta-\frac{n}{2}}} L_x^2(\mathbb{R}^{n+1})}^2 \|W_2\|_{L_t^{\frac{2}{1+\beta-\frac{n}{2}}} L_x^2(\mathbb{R}^{n+1})}^2 & \text{if } \frac{n-1}{2} < \beta < \frac{n}{2}, \\ \log(1/h) \|W_1\|_{L_t^{\frac{2}{1+\beta-\frac{n}{2}}} L_x^2(\mathbb{R}^{n+1})}^2 \|W_2\|_{L_t^{\frac{2}{1+\beta-\frac{n}{2}}} L_x^2(\mathbb{R}^{n+1})}^2 & \text{if } \beta = \frac{n-1}{2}. \end{cases} \end{aligned}$$

In the first line, we used $|t-s|^{2\beta} (h+|t-s|)^{-n} \lesssim 1$ for $|t-s| < 2\delta$. In the second line, we used $|t-s|^{2\beta} (h+|t-s|)^{-n} \lesssim |t-s|^{2\beta-n}$ and the Hardy-Littlewood-Sobolev inequality (see for instance [25, Thm. 4.3] applied to the functions $|W_1|^2$ and $|W_2|^2$, and to the exponents $p = r = 2/(2\beta - n)$ and $\lambda = n - 2\beta$). In the third line, we used the Young inequality (see for instance [25, Thm. 4.2] applied to $(f, g, h) = (|W_1|^2, |W_2|^2, t \mapsto \mathbf{1}(|t| < 2\delta)) |t|^{2\beta} (h+|t|)^{-n}$) and to the corresponding exponents $p = q = 1/(1 + \beta - n/2) = 2$ and $r = 1$) and that

$$\int_{-2\delta}^{2\delta} \frac{|t|^{n-1}}{(h+|t|)^n} dt \lesssim \log(1/h).$$

That ends the proof of (II.12).

Step 3. Conclusion. Interpolating $z = 0$ between $\Re z = -1$ and $\Re z = \beta \geq \frac{n-1}{2}$, by Theorem II.26, we get

$$\begin{aligned} \sup_{r \in J} \|W_1 T_0(r) W_2\|_{\mathfrak{S}^{2(\beta+1)}(L^2(\mathbb{R}^{n+1}))} &\lesssim \|W_1\|_{L_t^{\frac{2(\beta+1)}{1+(\beta-\frac{n}{2})^-}} L_x^{2(\beta+1)}(\mathbb{R}^{n+1})} \|W_2\|_{L_t^{\frac{2(\beta+1)}{1+(\beta-\frac{n}{2})^-}} L_x^{2(\beta+1)}(\mathbb{R}^{n+1})} \times \\ &\quad \times \begin{cases} h^{-\frac{n}{2(\beta+1)}} & \text{if } \beta > \frac{n-1}{2}, \\ \log(1/h)^{\frac{1}{n+1}} h^{-\frac{1}{n+1}} & \text{if } \beta = \frac{n-1}{2}. \end{cases} \end{aligned}$$

Defining $q \geq 2$ such that $2(q/2)' = 2(\beta+1)$, we have the desired estimates for all $2 \leq q \leq \frac{2(n+1)}{n-1}$. That ends the proof of Theorem II.27. \square

II.6 Relations between various estimates on quasimodes

Below, we will see several estimates of type (I.5) depending on how the phase space localization is made. Here, we explain how to relate these different estimates.

Let $d \geq 1$. Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $p \in S(m)$ and $P := p^w(x, hD)$ (or any other quantization). In the following, we will consider parameters $q \in [2, \infty]$, $s, t \geq 0$ and $\alpha \geq 1$ satisfying

$$s \geq d \left(\frac{1}{2} - \frac{1}{q} \right) - 1 \quad \text{and} \quad \alpha \leq \frac{q}{2}. \quad (\text{II.13})$$

Remark 8. *The previous assumption states that an estimate with a bound*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}} \leq Ch^{-2s} \log(1/h)^{2t} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}$$

for γ a bounded non-negative operator and $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, with (q, s, t, α) satisfying (II.13), is worse than the elliptic one, i.e. for which there is equality case of (II.13). Such an elliptic estimate is proved in Theorem III.1. We should insist on the fact that the density $\rho_{\chi^w \gamma \chi^w}$ is well-defined, thanks to the assumptions of γ and χ (see Lemma II.22 below). Furthermore, these results will be applied in this paper to values (s, α) that always satisfy the relation (II.13), as evidenced by Figures 3 and 4.

Notice also the equivalence of the following weighted L^2 -norms

$$\|(1 + P^*P/h^2)^{1/2} u\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2} \|(1 + P^*P/h^2)^{1/2} u\|_{L^2(\mathbb{R}^d)}.$$

can be extended to density matrices.

Lemma II.28. *Let m be an order function. Let $p \in S(m)$ and $P := p^w(x, hD)$. For any $\alpha \geq 1$ and non-negative density matrix γ on $L^2(\mathbb{R}^d)$*

$$\begin{aligned} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha} &\leq \|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|P^* \gamma P\|_{\mathfrak{S}^\alpha} \\ &\leq 2 \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

Proof of Lemma II.28. Let γ be a bounded self-adjoint non-negative operator on $L^2(\mathbb{R}^d)$ and let $\alpha \geq 1$. On the one hand, by cyclicity of the trace and the triangle inequality

$$\begin{aligned} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha} &= \left\| \sqrt{\gamma} (1 + P^*P/h^2) \sqrt{\gamma} \right\|_{\mathfrak{S}^\alpha} \\ &= \left\| \gamma + \sqrt{\gamma} \frac{P^*P}{h^2} \sqrt{\gamma} \right\|_{\mathfrak{S}^\alpha} \\ &\leq \|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|\sqrt{\gamma} P^* P \sqrt{\gamma}\|_{\mathfrak{S}^\alpha} = \|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|P^* \gamma P\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

On the other hand, since $0 \leq \gamma \leq \sqrt{\gamma} (1 + P^*P/h^2) \sqrt{\gamma}$ and $0 \leq \sqrt{\gamma} P^* P/h^2 \sqrt{\gamma} \leq \sqrt{\gamma} (1 + P^*P/h^2) \sqrt{\gamma}$, we have that

$$\begin{aligned} \|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|P^* \gamma P\|_{\mathfrak{S}^\alpha} &= \|\gamma\|_{\mathfrak{S}^\alpha} + \frac{1}{h^2} \|\sqrt{\gamma} P^* P \sqrt{\gamma}\|_{\mathfrak{S}^\alpha} \\ &\leq 2 \left\| \sqrt{\gamma} (1 + P^*P/h^2) \sqrt{\gamma} \right\|_{\mathfrak{S}^\alpha} = 2 \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

□

The following is an assumption on $S \subset \mathbb{R}^d \times \mathbb{R}^d$, $q \in [2, \infty]$, $s, t \geq 0$ and $\alpha \geq 1$.

Assumption 1 (Microlocalization around points). *The parameters $S \subset \mathbb{R}^d \times \mathbb{R}^d$, $q \in [2, \infty]$, $s, t \geq 0$ and $\alpha \geq 1$ satisfy Assumption 1 if they satisfy the hypothesis (II.13) and if for all $(x_0, \xi_0) \in S$, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$ and any bounded non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}.$$

Theorem II.29 (Microlocalization in a compact). *Let $S \subset \mathbb{R}^d \times \mathbb{R}^d$, $q \in [2, \infty]$, $s, t \geq 0$ and $\alpha \geq 1$ be such that Assumption 1 holds. Then, for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in S , there exists $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$ and any bounded non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}.$$

Proof of Theorem II.29. Since $\text{supp } \chi$ is compact and is contained on S , there exist open sets $\{\mathcal{V}_j\}_{j=1}^M$ given by Assumption 1 such that

$$\text{supp } \chi \subset \bigcup_{j=1}^M \mathcal{V}_j.$$

Moreover, one can find a partition of unity $1 = \sum_{j=1}^M \varphi_j$ on $\text{supp } \chi$ with $\text{supp } \varphi_j \subset \mathcal{V}_j$. Let us treat separately the different possible cases $q = 2$, $q \in (2, \infty)$ and $q = \infty$. Note that

$$\rho_{\chi^w \gamma \chi^w} = \sum_{j=1}^M \rho_{(\varphi_j \chi)^w \gamma (\chi \varphi_j)^w} + 2 \sum_{1 \leq \ell < k \leq M} \rho_{(\varphi_\ell \chi)^w \gamma (\chi \varphi_k)^w}.$$

By assumption, one has the bound for the L^∞ norm of $\rho_{(\varphi_\ell \chi)^w \gamma (\chi \varphi_k)^w}$ when $\ell = k$. Let us show that it is also true when $k \neq \ell$.

$$\begin{aligned} & (\varphi_\ell \chi)^w \gamma (\chi \varphi_k)^w \\ &= (\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1/2} (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} (1 + P^* P/h^2)^{-1/2} (\chi \varphi_k)^w \\ &\leq \left\| (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} \right\|_{L^2 \rightarrow L^2} \left\| (\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1} (\chi \varphi_k)^w \right\|. \end{aligned}$$

Furthermore, one has for any $x \in \mathbb{R}^d$, by the one-body L^∞ version of Assumption 1

$$\begin{aligned} & \rho_{(\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1} (\chi \varphi_k)^w}(x) \\ &= \int_{\mathbb{R}^d} ((\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1/2})(x, y) \overline{((\chi \varphi_k)^w (1 + P^* P/h^2)^{-1/2})(x, y)} dy \\ &\leq \left\| (\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1/2} \right\|_{L_x^\infty L_y^2} \left\| (\varphi_k \chi)^w (1 + P^* P/h^2)^{-1/2} \right\|_{L_x^\infty L_y^2} \\ &\leq \left\| (\varphi_\ell \chi)^w (1 + P^* P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^\infty} \left\| (\varphi_k \chi)^w (1 + P^* P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^\infty} \\ &\leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} \right\|_{L^2 \rightarrow L^2}. \end{aligned}$$

Then,

$$\|\rho_{(\varphi_\ell \chi)^w \gamma (\chi \varphi_k)^w}\|_{L^\infty(\mathbb{R}^d)} \leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^* P/h^2)^{1/2} \gamma (1 + P^* P/h^2)^{1/2} \right\|_{L^2 \rightarrow L^2}.$$

By the triangle inequality, one has the desired bound for the L^∞ norm. Furthermore, we have for all $j \in \{1, \dots, M\}$, bounds on $\|W(\chi\varphi_j)^w \sqrt{\gamma}\|_{\mathfrak{S}^2}$,

$$\begin{aligned} & \|W(\chi\varphi_j)^w \sqrt{\gamma}\|_{\mathfrak{S}^2} \\ & \leq C \log(1/h)^t h^{-s} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}^{1/2} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}, \end{aligned}$$

for

$$\begin{cases} W = 1 & \text{if } q = 2, \\ W \in L^{2(q/2)'(\mathbb{R}^d)} \cap C^0(\mathbb{R}^d) & \text{if } q \in (2, +\infty). \end{cases}$$

Hence, by the triangle inequality, we deduce the bound on $\|W\chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2}$ and then we recover the one on $\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)}$ with the Mercer theorem (Remark 7). That ends the proof of Theorem II.29. \square

Theorem II.30 (Microlocalization and localization in space). *Let $S \subset \mathbb{R}^d \times \mathbb{R}^d$, $q \in [2, \infty]$, $s, t \geq 0$ and $\alpha \geq 1$ be such that Assumption 1 holds. Then, for all $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for all set $\Omega \subset \mathbb{R}^d$ such that*

$$\text{supp } \chi \cap \Omega \times \mathbb{R}^d \subset \overset{\circ}{S},$$

there exist $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$ and any bounded non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\Omega)} \leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}.$$

Proof of Theorem II.30. Let $\tilde{\Omega} \subset \mathbb{R}^d$ an open bounded set such that $\Omega \subset \bar{\Omega} \subset \tilde{\Omega}$ and such that

$$\{(x, \xi) \in \text{supp } \chi : x \in \tilde{\Omega}\} \subset S.$$

Let $\chi_\Omega \in C_c^\infty(\mathbb{R}^d, [0, 1])$ be a function supported in $\tilde{\Omega}$ such that $\chi_\Omega = 1$ on Ω . We have by the Mercer theorem (more precisely Remark 7) when $q \in (2, \infty)$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\Omega)} \leq \|\rho_{\chi_\Omega \chi^w \gamma \chi^w \chi_\Omega}\|_{L^{q/2}(\mathbb{R}^d)} \leq \begin{cases} \|\chi_\Omega \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2}^2 & \text{if } q = 2, \\ \sup_{W \in L^{2(q/2)'(\mathbb{R}^d)} \cap C^0(\mathbb{R}^d)} \frac{\|W \chi_\Omega \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2}^2}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2} & \text{if } q \in (2, \infty). \end{cases}$$

There exists $r \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\chi_\Omega \chi^w = (\chi_\Omega \chi)^w + h r^w.$$

On the one hand, by the Hölder and Kato-Seiler-Simon inequalities (Lemma II.25) for $m \in \mathbb{N}$ such that $m > \frac{d}{2(q/2)'}$, for any $N \in \mathbb{N}$

$$\begin{aligned} & h \|W r^w \sqrt{\gamma}\|_{\mathfrak{S}^2} \\ & \leq h \|W \chi_\Omega (1 - h^2 \Delta)^{-m}\|_{\mathfrak{S}^{2(q/2)'}} \|(1 - h^2 \Delta)^m r^w\|_{L^2 \rightarrow L^2} \|\sqrt{\gamma}\|_{\mathfrak{S}^q} \\ & \leq C h^{1-d(\frac{1}{2}-\frac{1}{q})} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}} \|\gamma\|_{\mathfrak{S}^{q/2}}^{1/2}. \end{aligned}$$

On the other hand, by Theorem II.29 applied to S and (q, s, t, α) , there exist $C > 0$ and h_0 such that for any $0 < h \leq h_0$ and any non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\begin{aligned} \sup_{W \in L^{2(q/2)'(\mathbb{R}^d)} \cap C^0(\mathbb{R}^d)} \frac{\|W(\chi_\Omega \chi)^w \sqrt{\gamma}\|_{\mathfrak{S}^2}^2}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2} &= \|\rho_{(\chi_\Omega \chi)^w \gamma (\chi_\Omega \chi)^w}\|_{L^{q/2}(\mathbb{R}^d)} \\ &\leq C \log(1/h)^{2t} h^{-2s} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}^2. \end{aligned}$$

Finally, by the triangle inequality, we get the desired inequality. \square

Remark 9. The above proof shows that the result of Theorem II.30 also holds when $\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\Omega)}$ is replaced by $\|\rho_{\chi^w \chi \Omega \gamma \chi \Omega \chi^w}\|_{L^{q/2}(\mathbb{R}^d)}$ or $\|\rho_{\chi \Omega \chi^w \gamma \chi^w \chi \Omega}\|_{L^{q/2}(\mathbb{R}^d)}$.

III ELLIPTIC ESTIMATES

In this section, we state and prove estimates in the elliptic region where $p \neq 0$. In the one-body case (rank $\gamma = 1$), one recovers Lemma II.17.

Theorem III.1 (Many-body elliptic estimates). *Let $d \geq 2$ and $2 \leq q \leq \infty$. Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$ and $p \in S(m)$. Let $P := p^w(x, hD)$ (or any other quantization). Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point such that*

$$p(x_0, \xi_0) \neq 0.$$

Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$, for any bounded non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq Ch^{-2s(q,d)} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha(q,d)}},$$

where the exponents s and α are given by

$$s(q, d) = d \left(\frac{1}{2} - \frac{1}{q} \right) - 1, \quad \alpha(q, d) = \frac{q}{2}. \quad (\text{III.1})$$

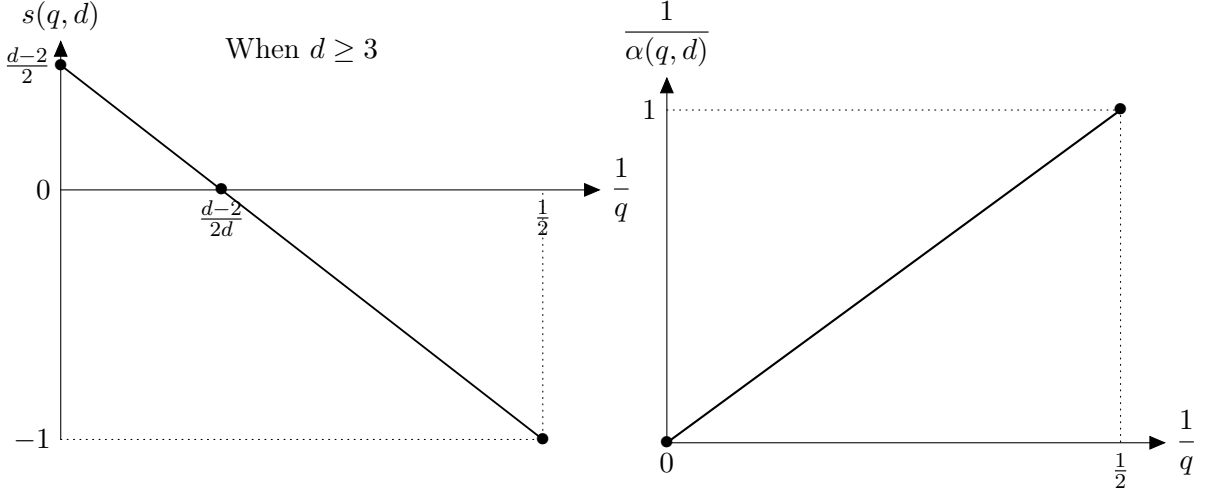


Figure 5: Exponent $s(q, d)$ and $\alpha(q, d)$ for elliptic estimates.

Proof of Theorem III.1. There exists a neighborhood \mathcal{V} of (x_0, ξ_0) where p is non-zero. We have by Mercer theorem

$$\begin{aligned} & \|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \\ & \leq \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\text{Tr}_{L^2}(W \chi^w \gamma \chi^w W)}{\|W\|_{L^{2(q/2)'}(\mathbb{R}^d)}^2} \\ & \leq \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\|W \chi^w (1 + P^*P/h^2)^{-1/2}\|_{\mathfrak{S}^{2(q/2)'}}^2}{\|W\|_{L^{2(q/2)'}(\mathbb{R}^d)}^2} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}}. \end{aligned}$$

It remains to prove for any $2 \leq q \leq \infty$

$$\left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{\mathfrak{S}^{2(q/2)'}} \lesssim h^{1-d\left(\frac{1}{2}-\frac{1}{q}\right)} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}},$$

i.e. for any $2 \leq \alpha \leq \infty$

$$\left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{\mathfrak{S}^\alpha} \lesssim h^{1-d/\alpha} \|W\|_{L^\alpha(\mathbb{R}^d)}.$$

Let us show the previous bound with $\alpha = 2$ and $\alpha = \infty$. The proof of Lemma II.17 indeed shows (II.2), that we recall:

$$\forall 2 \leq q \leq \infty, \quad \chi^w(1 + P^*P/h^2)^{-1/2} = \mathcal{O}\left(h^{1-d\left(\frac{1}{2}-\frac{1}{q}\right)}\right) : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d).$$

The case $\alpha = \infty$ is given by the one function's estimate (II.2) applied to $q = 2$

$$\begin{aligned} \left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2} &= \left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2} \\ &\leq \left\| \chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2} \|W\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim h \|W\|_{L^\infty(\mathbb{R}^d)}. \end{aligned}$$

Suppose that $\alpha = 2$. We write the \mathfrak{S}^2 norm with respect to the integral kernel and use the one function's estimate (II.2) applied to $q = \infty$

$$\begin{aligned} \left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{\mathfrak{S}^2}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \left(W\chi^w(1 + P^*P/h^2)^{-1/2} \right) (x, y) \right|^2 dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |W(x)|^2 \left| \left(\chi^w(1 + P^*P/h^2)^{-1/2} \right) (x, y) \right|^2 dx dy \\ &\leq \|W\|_{L^2(\mathbb{R}^d)}^2 \sup_{x \in \mathbb{R}^d} \left\| \left(\chi^w(1 + P^*P/h^2)^{-1/2} \right) (x, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \|W\|_{L^2(\mathbb{R}^d)}^2 \left\| \left(\chi^w(1 + P^*P/h^2)^{-1/2} \right) (x, y) \right\|_{L_x^\infty L_y^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 \\ &\leq \|W\|_{L^2(\mathbb{R}^d)}^2 \left\| \chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^\infty}^2 \\ &\lesssim h^{2-d} \|W\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

We may thus write

$$\left\| W\chi^w(1 + P^*P/h^2)^{-1/2} \right\|_{\mathfrak{S}^2} \lesssim h^{1-d/2} \|W\|_{L^2(\mathbb{R}^d)}.$$

Then, by interpolation between the two previous bounds we get the bounds for all the exponents $2 \leq \alpha \leq \infty$. Finally, for any $2 \leq q \leq \infty$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq Ch^{2-2d\left(\frac{1}{2}-\frac{1}{q}\right)} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}}.$$

□

IV MORE GENERAL L^p ESTIMATES

We now turn to the region $p = 0$. We give a general first estimate which holds under the sole assumption that $\partial_\xi^2 p$ is not degenerate. This is particularly useful in the context of Schrödinger operators, because this assumption holds without any hypothesis on the potential V . In the one-body case ($\text{rank } \gamma = 1$), we recover [22, Thm. 6] (up to logarithmic factors which appear in few cases).

IV.1 Statement of the result

Let $d \geq 1$. Let m be an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $p \in S(m)$ be real-valued and $P := p^w(x, hD)$ (the following theorem are also true for any other quantization P of p).

Assumption 2. A point $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfies the general non-degeneracy condition for the symbol p if

$$\partial_{\xi}^2 p(x_0, \xi_0) \text{ is non-degenerate.}$$

Remark 10. For Schrödinger operators $p(x, \xi) = \xi^2 + V(x) - E$ with $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ bounded from below and satisfying Definition II.3, the previous assumption is satisfied for all $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$.

Recall first the one-body result.

Theorem IV.1 (General one-body estimates, [22, Thm. 6]). *Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying Assumption 2. Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$, for any $2 \leq q \leq \infty$ and $u \in L^2(\mathbb{R}^d)$,*

$$\|\chi^w u\|_{L^q(\mathbb{R}^d)} \leq C \log(1/h)^{t(q,d)} h^{-s(q,d)} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right)$$

where $s(q, d)$ and $t(q, d)$ and are given by the formulas

- when $d = 1$:

$$t(q, 1) = 0 \quad \text{and} \quad s(q, 1) = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{q} \right), \quad (\text{IV.1})$$

- when $d = 2$:

$$t(q, 2) = \begin{cases} 0 & \text{if } 2 \leq q < \infty, \\ \frac{1}{2} & \text{if } q = \infty. \end{cases} \quad \text{and} \quad s(q, 2) = \frac{1}{2} - \frac{1}{q}, \quad (\text{IV.2})$$

- when $d \geq 3$: $t(q, d) = 0$ and

$$s(q, d) = \begin{cases} \frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right) & \text{if } 2 \leq q \leq \frac{2d}{d-2}, \\ d \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2} & \text{if } \frac{2d}{d-2} \leq q \leq \infty. \end{cases} \quad (\text{IV.3})$$

Equivalently, one has for all $2 \leq q \leq \infty$

$$\chi^w (1 + P^* P / h^2)^{-1/2} = \mathcal{O} \left(\log(1/h)^{t(q,d)} h^{-s(q,d)} \right) : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d).$$

Remark 11. The exponent s_{gene} defined on Theorem IV.1 is larger or equal to the one in Sobolev estimates $s_{Sobolev}(q, d) = d(1/2 - 1/q)$ ($s_{gene}(q, d) > s_{Sobolev}(q, d)$ for $2 < q \leq \infty$ and they are equal for $q = 2$). It is also strictly smaller than the exponent of the elliptic estimates $s_{ellip}(q, d) = d(1/2 - 1/q) - 1$ (see Figure 3).

Theorem IV.2 (General many-body estimates). *Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying Assumption 2. Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $2 \leq q \leq \infty$, for any $0 < h \leq h_0$, for any bounded non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq C \log(1/h)^{2t(q,d)} h^{-2s(q,d)} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha(q,d)}$$

where $s(q, d)$ is given by the formula (IV.3) and $t(q, d)$, $\alpha(q, d)$ are given by

- when $d = 1$:

$$t(q, 1) = 0 \quad \text{and} \quad \alpha(q, 1) = \frac{q}{2}. \quad (\text{IV.4})$$

- when $d = 2$:

$$t(q, 2) = \begin{cases} 0 & \text{if } 2 \leq q < 6, \\ \frac{1}{2} - \frac{1}{q} & \text{if } 6 \leq q \leq \infty, \end{cases} \quad (\text{IV.5})$$

and

$$\alpha(q, 2) = \begin{cases} \frac{2q}{q+2} & \text{if } 2 \leq q \leq 6, \\ \frac{q}{4} & \text{if } 6 \leq q \leq \infty, \end{cases} \quad (\text{IV.6})$$

- when $d \geq 3$:

$$t(q, d) = \begin{cases} 0 & \text{if } 2 \leq q < \frac{2(d+1)}{d-1} \\ \frac{d}{q} - \frac{d-2}{2} & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \frac{2d}{d-2} \\ 0 & \text{if } \frac{2d}{d-2} \leq q \leq \infty, \end{cases} \quad (\text{IV.7})$$

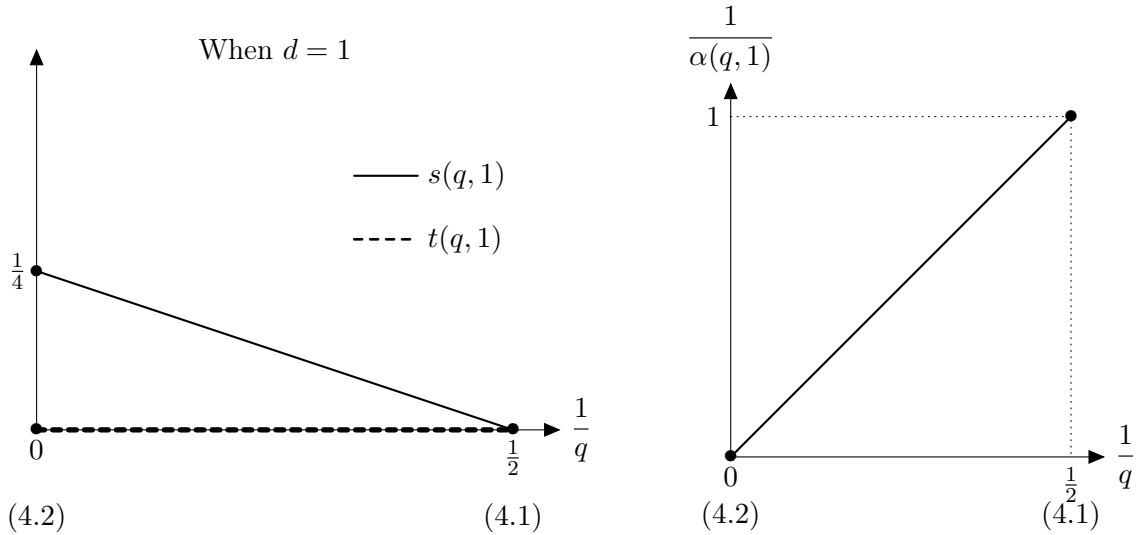
and

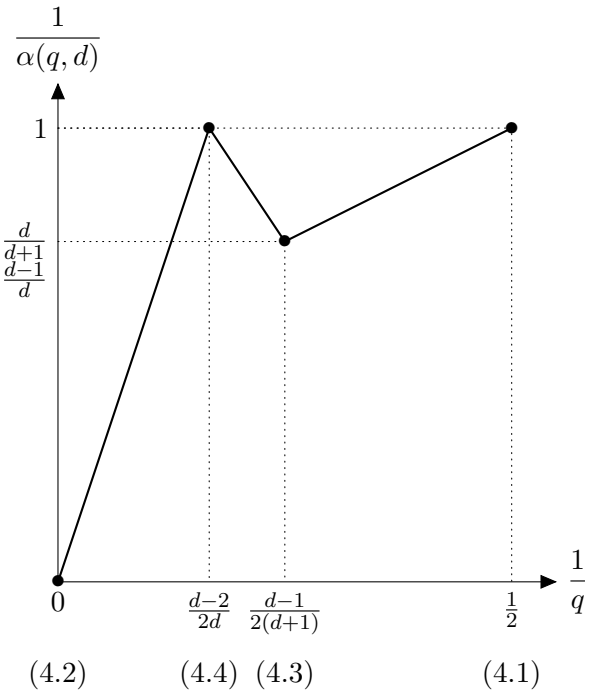
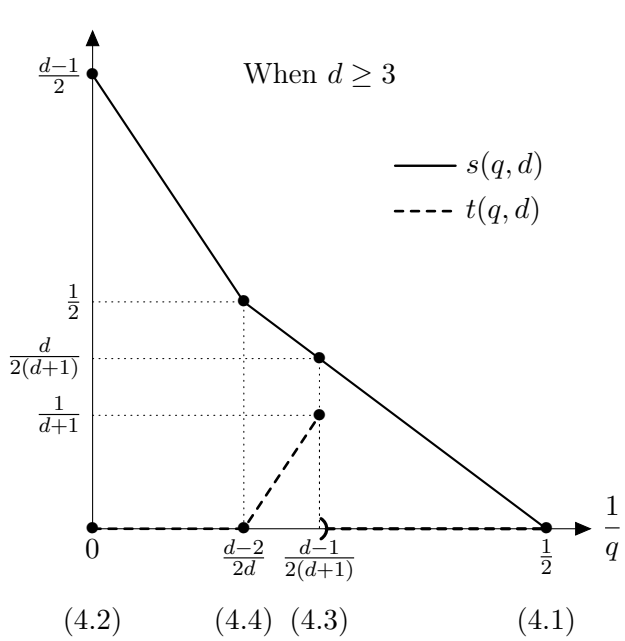
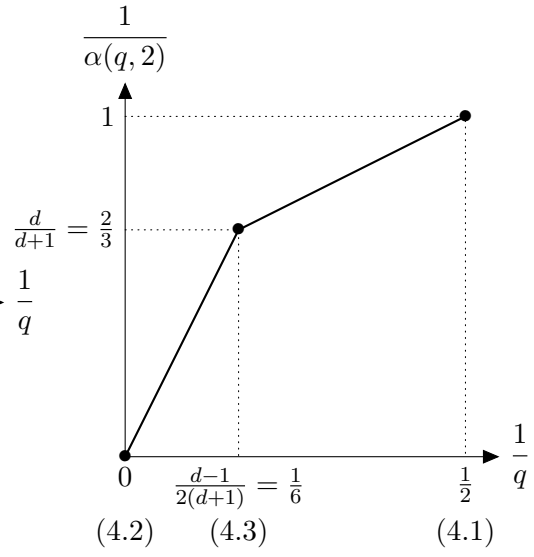
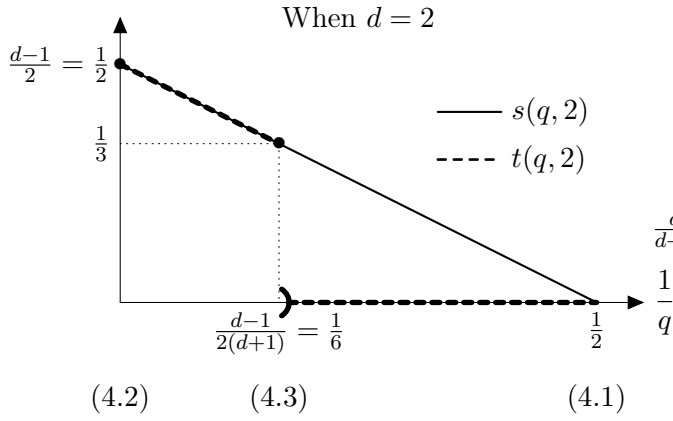
$$\alpha(q, d) = \begin{cases} \frac{2q}{q+2} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1} \\ \frac{2q}{d(q-2)} & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \frac{2d}{d-2} \\ \frac{(d-2)}{2d}q & \text{if } \frac{2d}{d-2} \leq q \leq \infty. \end{cases} \quad (\text{IV.8})$$

We write the different end-points with what we obtain the estimates by interpolation.

End-point q	2	∞	$\frac{2(d+1)}{(d-1)}$	$\frac{2d}{(d-2)}$
Referring name	(4.1)	(4.2)	(4.3)	(4.4)

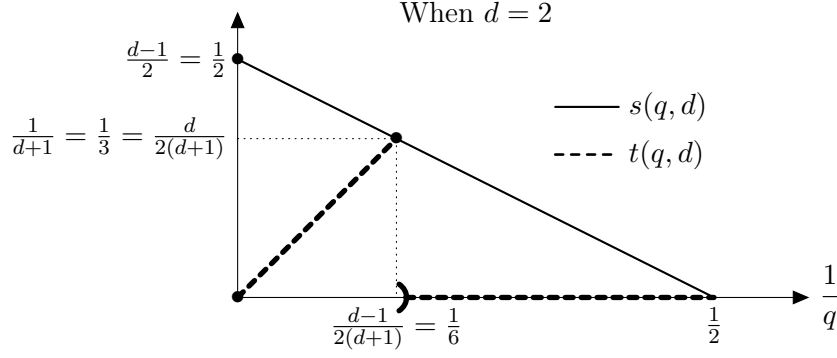
Table 3: References of the end-points' labels in the general case.





Some comments on improvement

Remark 12. In dimension 2, in the case of Schrödinger operators for the symbols $p(x, \xi) = |\xi|^2 + V(x)$ satisfying Definition II.3, Smith and Zworski [29] proved that Theorem IV.1 is true with $t(\infty, 2) = 0$ (which means that we can get rid of the logarithm in dimension 2 for $q = \infty$). This implies that we can set $t(\infty, 2) = 0$ in Theorem IV.2 as well, and by interpolation Theorem IV.2 holds for $t(q, 2) = \frac{2}{q}$ for all $q \in [6, \infty]$.



Remark 13. When $d \geq 3$, the Schatten exponent that we obtain in Theorem IV.2 for the Keel-Tao endpoint $q = 2d/(d-2)$ is $\alpha = 1$. Frank and Sabin [10, Lem. 2] proved that this Schatten exponent is sharp in the related context of the Strichartz estimates associated to the propagator $e^{it\Delta}$. We expect that this result extends to our context, however it is not straightforward to adapt their proof. Indeed, their strategy amounts to showing that the dual operator is not compact, while here the dual operator is compact. Hence, we would rather need to quantify the “loss of compactness” of our dual operator as $h \rightarrow 0$, which is a very interesting problem.

IV.2 Notation for the proof of Theorem IV.2

- Let $\delta \in]0, 1[$ and $I = [-\delta/2, \delta/2]$.
- Let $J = [-1, 1]$.
- Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\text{supp } \psi \subset \overset{\circ}{I}$ and $\psi \neq 0$.
- Let $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\tilde{\psi} = 1$ on I and $\text{supp } \tilde{\psi} \subset [-\delta, \delta]$.
- Let $\mathcal{V} = U \times V$ a bounded open neighborhood of (x_0, ξ_0) .
- Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be such that $\text{supp } \chi \subset \mathcal{V}$.
- Let $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d, [0, 1])$ be such that $\tilde{\chi} = 1$ on $\text{supp } \chi$ and such that $\text{supp } \tilde{\chi} \subset \mathcal{V}$.
- Let $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\chi_0 = 1$ on a neighborhood of (x_0, ξ_0) .

We will add constraints on δ and \mathcal{V} along the proof.

IV.3 Proof of Theorem IV.2

End-points (4.1) and (4.2) of Table 3. We start to give the extremal estimates for $q = 2$ and $q = \infty$.

- By the Mercer theorem (Remark 7) and the Calderon-Vaillancourt theorem (Theorem II.15), we have

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^1(\mathbb{R}^d)} = \text{Tr}_{L^2}(\chi^w \gamma \chi^w) \lesssim \|\gamma\|_{\mathfrak{S}^1}.$$

- Similarly as in the proof of Theorem II.29, by the one function L^∞ estimate (c.f. Theorem IV.1)

$$\begin{aligned} \|\rho_{\chi^w} \gamma \chi^w\|_{L^\infty(\mathbb{R}^d)} &\leq \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{L^2 \rightarrow L^2} \|\rho_{\chi^w (1 + P^2/h^2)^{-1} \chi^w}\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{L^2 \rightarrow L^2} \begin{cases} h^{-1/2} & \text{if } d = 1, \\ \log(1/h)/h & \text{if } d = 2, \\ h^{-(d-1)} & \text{if } d > 2. \end{cases} \end{aligned}$$

We thus have the bounds for $q = 2$ and $q = \infty$. For $d = 1$, we interpolate between them and get

$$\|\rho_{\chi^w} \gamma \chi^w\|_{L^{q/2}(\mathbb{R})} \lesssim h^{\frac{1}{q} - \frac{1}{2}} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}))},$$

which is exactly Theorem IV.2 in the case $d = 1$.

End-point (4.4) of Table 3. For $d \geq 3$, by the triangle inequality at the Keel-Tao endpoint, that is the one-body estimate for $q = \frac{2d}{d-2}$ (end-point (4.3)) of Theorem IV.1, we have

$$\|\rho_{\chi^w} \gamma \chi^w\|_{L^{d/(d-2)}(\mathbb{R}^d)} \lesssim h^{-1} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^1(L^2(\mathbb{R}^d))}.$$

Interpolating this bound with the bound for $q = \infty$ proves Theorem IV.2 in the case $2d/(d-2) \leq q \leq \infty$, $d \geq 3$. The next step is to get estimates for $2 \leq q \leq 2(d+1)/(d-1)$ with $d \geq 2$. The remaining estimates in the range $2(d+1)/(d-1) \leq q \leq 2d/(d-2)$, $d \geq 3$ are then obtained by interpolating the estimates for $q = 2(d+1)/(d-1)$ and $q = 2d/(d-2)$.

End-point (4.3) of Table 3 (and other points between (4.1) and (4.3)). We thus now fix $d \geq 2$ and $2 \leq q \leq 2(d+1)/(d-1)$. The idea is to introduce a new variable $t \in \mathbb{R}$. Then, we have by Mercer theorem (Remark 7)

$$\begin{aligned} \|\psi\|_{L^2(I)}^2 \|\rho_{\chi^w} \gamma \chi^w\|_{L^{q/2}(\mathbb{R}^d)} &= \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\|\psi\|_{L^2(I)}^2 \int_{\mathbb{R}^d} \rho_{\chi^w} \gamma \chi^w(x) |W(x)|^2 dx}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2} \\ &\leq \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\|\psi\|_{L^2(I)}^2 \|W(x) \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d))}^2}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2} \\ &\leq \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\|\psi(t) W(x) \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))}^2}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2}. \end{aligned}$$

In the last inequality, we use that for any bounded operator A on $L^2(\mathbb{R}^d)$ and any $\psi \in L^2(\mathbb{R})$, we have (as can be seen by computing $(\psi(t)A)^* \psi(t)A$)

$$\|\psi(t)A\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} = \|\psi\|_{L^2(\mathbb{R})} \|A\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d))}.$$

We define the operator B_h by $\frac{1}{h}P$. Let $u \in L^2(\mathbb{R}^d)$. Then $v(t, x) := u(x)$ satisfies

$$\begin{cases} (hD_t + P)v &= hB_h v, \\ v(0, x) &= u(x). \end{cases}$$

Let $R_h \in \text{Op}_h^w(\mathcal{S})$ be such that

$$\chi_0^w(x, hD)P = (\chi_0 p)^w(x, hD) - hR_h.$$

Defining the unitary operators $\{F(t)\}_{t \in \mathbb{R}}$ on $L^2(\mathbb{R}^d)$ such that

$$\begin{cases} (hD_t + (\chi_0 p)^w)F(t) = 0, & \forall t \in \mathbb{R}, \\ F(0) = \text{Id}, \end{cases}$$

we have by the Duhamel formula

$$\begin{aligned} \chi^w(x, hD) &= F(t)\chi^w(x, hD) - i \int_0^t F(t)F(r)^*(\chi_0^w(x, hD)B_h + R_h)\chi^w(x, hD)dr \\ &= F(t)F(0)^*\chi^w - i \int_0^t F(t)F(r)^*\tilde{\chi}^w(x, hD)(\chi_0^w(x, hD)B_h + R_h)\chi^w(x, hD)dr \\ &\quad - i \int_0^t F(t)F(r)^*(1 - \tilde{\chi})^w(x, hD)(\chi_0^w(x, hD)B_h + R_h)\chi^w(x, hD)dr, \end{aligned}$$

as an identity between bounded operators on $L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$.

Definition IV.3. For all $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, let $U_\varphi(t, r) = U_{\varphi,+}(t, r) - U_{\varphi,-}(t, r)$ where

$$U_{\varphi,+}(t, r) := \mathbf{1}(r \geq 0) \mathbf{1}(t \geq r) \psi(t)\tilde{\psi}(t-r)F(t)F(r)^*\varphi^w(x, hD)$$

and

$$U_{\varphi,-}(t, r) := \mathbf{1}(r < 0) \mathbf{1}(t \leq r) \psi(t)\tilde{\psi}(t-r)F(t)F(r)^*\varphi^w(x, hD).$$

Let us also define S by

$$S = -i \int_0^t \psi(t)F(t)F(r)^*(1 - \tilde{\chi}^w(x, hD))(\chi_0^w(x, hD)B_h + R_h)\chi^w(x, hD)dr. \quad (\text{IV.9})$$

By multiplying by ψ on the left of the previous Duhamel formula, we have

$$\psi(t)\chi^w(x, hD) = U_\chi(t, 0) - i \int_J U_{\tilde{\chi}}(t, r)(\chi_0^w B_h \chi^w + R_h \chi^w) dr + S.$$

By the triangle inequality and Hölder inequality, we get for all $\alpha \geq 1$

$$\begin{aligned} &\|\psi(t)W(x)\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \\ &\leq \|W(x)U_\chi(t, 0)\|_{\mathfrak{S}^{2\alpha'}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \|\sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\quad + |J| \sup_{r \in J} \|W(x)U_{\tilde{\chi}}(t, r)\|_{\mathfrak{S}^{2\alpha'}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \|(\chi_0^w B_h \chi^w + R_h \chi^w)\sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\quad + \|W(x)S\|_{\mathfrak{S}^{(q/2)'}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \|\sqrt{\gamma}\|_{\mathfrak{S}^q(L^2(\mathbb{R}^d))}. \end{aligned}$$

There exists $\mathfrak{r} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $[B_h, \chi^w] = \frac{1}{h} [P, \chi^w] = \mathfrak{r}^w$. Thus

$$\begin{aligned} &\|(\chi_0^w B_h \chi^w + R_h \chi^w)\sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\leq \|\chi_0^w \chi^w B_h \sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} + \|\chi_0^w [B_h, \chi^w] \sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} + \|R_h \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\leq \|\chi_0^w \chi^w\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \|B_h \sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\quad + \left(\|\chi_0^w [B_h, \chi^w]\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} + \|R_h \chi^w\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \right) \|\sqrt{\gamma}\|_{\mathfrak{S}^{2\alpha}(L^2(\mathbb{R}^d))} \\ &\lesssim \frac{1}{h} \|P\gamma P\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))}^{1/2} + \|\gamma\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))}^{1/2}. \end{aligned}$$

We only need to prove the Schatten estimates for the operators $U_\chi(t, r)$, $U_{\tilde{\chi}}(t, r)$ and S .

Proposition IV.4. *Recall that S is the operator defined by (IV.9). Let $\beta \geq 2$. Then, we have the bound for all $W \in L^\beta(\mathbb{R}^{d+1})$*

$$\|W(t, x)S\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} = \mathcal{O}(h^\infty)\|W\|_{L^\beta(I \times \mathbb{R}^d)}.$$

Before proving this proposition, let us conclude the proof of Theorem IV.2. Given Assumption 2, we apply Theorem II.27 to $n = d$, $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $a = \chi_0 p \in \mathcal{C}_t^\infty(\mathbb{R}, S_{(x, \xi)}(1))$ and $J = [-1, 1]$. Thus, there exist $\delta > 0$ and $\mathcal{V} = U \times V$ neighborhood of (x_0, ξ_0) , so that we have for any $2 \leq q \leq \frac{2(d+1)}{d-1}$

$$\begin{aligned} & \|W(x)U_\chi(t, 0)\|_{\mathfrak{S}^{2\alpha(q, d)'}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} + \sup_{r \in J} \|W(x)U_{\tilde{\chi}}(t, r)\|_{\mathfrak{S}^{2\alpha(q, d)'}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \\ & \lesssim \log(1/h)^{t(q, d)} h^{-s(q, d)} \|W\|_{L^{2(q/2)'}(\mathbb{R}^d)} \|\psi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

for $\alpha(q, d) = \frac{2q}{q+2}$, $s(q, d) = \frac{d}{2}(\frac{1}{2} - \frac{1}{q})$ and $t(q, d) = 0$ for $2 \leq q < \frac{2(d+1)}{d-1}$ and $t(q, d) = \frac{1}{d+1}$ if $q = \frac{2(d+1)}{d-1}$. Hence, for any $2 \leq q \leq \frac{2(d+1)}{d-1}$

$$\begin{aligned} & \|\psi(t)W(x)\chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \\ & \leq C_I \log(1/h)^{t(q, d)} h^{-s(q, d)} \|W\|_{L^{2(q/2)'}(\mathbb{R}^d)} \times \\ & \quad \times \left(\|\gamma\|_{\mathfrak{S}^{\alpha(q, d)}(L^2(\mathbb{R}^d))}^{1/2} + \frac{1}{h} \|P\gamma P\|_{\mathfrak{S}^{\alpha(q, d)}(L^2(\mathbb{R}^d))}^{1/2} + \|\gamma\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}^d))}^{1/2} \right) \\ & \leq C'_I \log(1/h)^{t(q, d)} h^{-s(q, d)} \|W\|_{L^{2(q/2)'}(\mathbb{R}^d)} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha(q, d)}(L^2(\mathbb{R}^d))}^{1/2}, \end{aligned}$$

which finishes the proof of Theorem IV.2 (for all points between (4.1) and (4.3) of Table 3).

Let us now prove Proposition IV.4.

Proof of Proposition IV.4. We only need to show the estimates

$$\forall M \in \mathbb{N}, \quad \|S\|_{L^2(\mathbb{R}^d) \rightarrow H_h^M(\mathbb{R}^{d+1})} = \mathcal{O}(h^\infty). \quad (\text{IV.10})$$

Note that $S = \mathbf{1}(t \in I)S$. Then, by Kato-Seiler-Simon Lemma II.24 applied to $M \in \mathbb{N}$ such that $M\beta > d + 1$

$$\begin{aligned} & \|W(t, x)S\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))} \\ & \leq \left\| W(t, x) \mathbf{1}(t \in I) (1 - h^2 \Delta_{t, x})^{-M/2} \right\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^{d+1}))} \left\| (1 - h^2 \Delta_{t, x})^{M/2} S \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{d+1})} \\ & = \mathcal{O}(h^\infty) \|W\|_{L^\beta(I \times \mathbb{R}^d)}. \end{aligned}$$

That gives us the desired estimates.

Let us prove now the bounds (IV.10). On the one hand, there exists $r \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $(1 - \tilde{\chi}^w(x, hD))B_h \chi^w(x, hD) = \mathfrak{r}^w(x, hD)$. Since $\text{supp}(1 - \tilde{\chi})$ and $\text{supp} \chi$ are disjoint

$$\mathfrak{r}^w(x, hD) = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d).$$

On the other hand, by Lemma II.19

$$\forall M \in \mathbb{N}, \exists C > 0 \quad \sup_{r \in J} \|\psi(t)F(t)F(r)^*\|_{H_h^M(\mathbb{R}^d) \rightarrow H_h^M(\mathbb{R}^{d+1})} \leq C.$$

Then, by composition we have for all $M \in \mathbb{N}$

$$\sup_{r \in J} \|\mathbf{1}(r \geq 0) \mathbf{1}(t \geq r) \psi(t)F(t)F(r)^* \mathfrak{r}^w\|_{L^2(\mathbb{R}^d) \rightarrow H_h^M(\mathbb{R}^{d+1})} = \mathcal{O}(h^\infty),$$

and

$$\sup_{r \in J} \|\mathbf{1}(r \leq 0) \mathbf{1}(t \leq r) \psi(t) F(t) F(r)^* \mathbf{r}^w\|_{L^2(\mathbb{R}^d) \rightarrow H_h^M(\mathbb{R}^{d+1})} = \mathcal{O}(h^\infty).$$

Finally, we get (IV.10), that ends the proof of Proposition IV.4. \square

Remark 14. Notice that (II.12) allows to treat the case $\beta \in [0, \frac{d-1}{2})$ leading to better values $\alpha(q, d)$ and $t(q, d)$ but also a worse value of $s(q, d)$ for $d \geq 2$ and $\frac{2(d+1)}{d-1} < q \leq \infty$:

$$s(q, d) = \left(d + \frac{1}{2} - \frac{2}{q-2}\right) \left(\frac{1}{2} - \frac{1}{q}\right), \quad t(q, d) = 0, \quad \alpha(q, d) = \frac{2q}{q+2}.$$

We discard these estimates because we always want to keep the same exponent $s(q, d)$ as in the one-body case (so that our many-body estimates imply the one-body estimates). However, we can discuss that for $\gamma = \sum_{j=1}^{N_h} \lambda_j |u_j| \langle u_j |$, how to write the best estimates (to choose the best exponent possible) for according to the range of N_h . For instance, for $N_h > (h^{-1})^{\min_q \frac{s(q) - s_{\text{gene}}(q)}{1/\alpha_{\text{gene}}(q) - 1/\alpha(q)}}$, the concentration is better with the above estimated rather than the one of Theorem IV.2. But, the discussion is quite complex in the general case, without any specific given situation.

Remark 15. Let us comment on why the many-body case has an additional transition point $q = 2(d+1)/(d-1)$ compared to the one-body case. Let $K_{h,\beta}$ be defined by

$$\forall t \in \mathbb{R} \quad K_{h,\beta}(t) := h^{-d/2} |t|^\beta (h + |t|)^{-d/2}.$$

We have in the one body case (which can be proved with the complex interpolation).

$$\sup_{r \in J} \|T_0(r)\|_{L_t^2 L_x \left(\frac{2(\beta+1)}{\beta}\right)'(\mathbb{R}^{d+1}) \rightarrow L_t^2 L_x \frac{2(\beta+1)}{\beta}(I \times \mathbb{R}^d)} \leq C \left(\int_{-1}^1 K_{h,\beta}(t) dt \right)^{\frac{1}{\beta+1}},$$

while as the above proof shows, we have in the many body case

$$\begin{aligned} & \sup_{r \in J} \|WT_0(r)W\|_{\mathfrak{S}^2(L^2(\mathbb{R}^{d+1}))} \\ & \leq C \|W\|_{L_t^\infty L_x^{2(\beta+1)}(I \times \mathbb{R}^d)}^2 \left(\int_{-1}^1 K_{h,\beta}(t)^2 dt \right)^{\frac{1}{2(\beta+1)}}. \end{aligned}$$

Note that

$$\int_{-1}^1 K_{h,\beta}(t) dt = \begin{cases} h^{\beta-d+1} & \text{if } \beta < \frac{d-2}{2}, \\ h^{-d/2} \log(1/h) & \text{if } \beta = \frac{d-2}{2}, \\ h^{-d/2} & \text{if } \beta > \frac{d-2}{2}. \end{cases}$$

and

$$\left(\int_{-1}^1 K_{h,\beta}(t)^2 dt \right)^{1/2} = \begin{cases} h^{\beta-d+1/2} & \text{if } \beta < \frac{d-1}{2}, \\ h^{-d/2} \log(1/h)^{1/2} & \text{if } \beta = \frac{d-1}{2}, \\ h^{-d/2} & \text{if } \beta > \frac{d-1}{2}, \end{cases}$$

so that the one-body and many-body constants coincide for $\beta > (d-1)/2$ (which corresponds to $2 \leq q < 2(d+1)/(d-1)$) but differ for $\beta \leq (d-1)/2$ (which corresponds to $q > 2(d+1)/(d-1)$). We expect that it is not a technical artefact of the proof, but rather that this transition point does appear in the many-body case. Indeed, a similar phenomenon exists for Strichartz estimates [9] where the existence of a transition is shown at this point $q = 2(d+1)/(d-1)$. It is a challenging problem to adapt their result to our setting. A related problem would be to get rid of the logarithm in our many-body estimates at $q = 2(d+1)/(d-1)$.

V SOGGE'S L^p ESTIMATES

We now treat the case $p = 0$ and $\nabla_\xi p \neq 0$. In the case of Schrödinger operators, it means that we are away from the turning point region $\{V = E\}$. This setting corresponds to the one of Sogge without potential on a compact manifold. In the one-body case ($\text{rank } \gamma = 1$), we recover [22, Thm. 5].

V.1 Statement of the result

Let $d \geq 2$. For $x \in \mathbb{R}^d$, we denote by x' the $d - 1$ last variables of x

$$x' := (x_2, \dots, x_d) \in \mathbb{R}^{d-1}.$$

Let m an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $p \in S(m)$ be real-valued and $P := p^w(x, hD)$ (but the following theorems are true for any other quantization).

Assumption 3. *A point $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfies the Sogge non-degeneracy conditions for the symbol p if*

$$p(x_0, \xi_0) = 0, \quad \nabla_\xi p(x_0, \xi_0) \neq 0,$$

and if

$$\begin{aligned} & \text{the second fundamental form of } \{\xi \in \mathbb{R}^d : p(x_0, \xi) = 0\} \\ & \text{is non-degenerate at } \xi_0. \end{aligned} \tag{V.1}$$

First recall the one-body result.

Theorem V.1 (Sogge one-body estimates, [22, Thm. 5]). *Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying Assumption 3. Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$ and for any $2 \leq q \leq \infty$*

$$\|\chi^w u\|_{L^q(\mathbb{R}^d)} \leq Ch^{-s(q,d)} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right),$$

where

$$s(q, d) = \begin{cases} \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right) & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ d \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2} & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases} \tag{V.2}$$

Equivalently, one has for all $2 \leq q \leq \infty$

$$\chi^w (1 + P^*P/h^2)^{-1/2} = \mathcal{O}(h^{-s(q,d)}) : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d).$$

Remark 16. *The exponent s_{Sogge} , defined in (V.2), is always larger than the elliptic one s_{ellip} for any $d \geq 2$ and $2 \in [2, \infty]$. Moreover, it is strictly smaller than s_{gene} for any $q \in (2, 2d/(d-2))$ and they coincide when $q \in \{2\} \cup [2d/(d-2), \infty]$. (c.f. Figure 3).*

Theorem V.2 (Sogge many-body estimates). *Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying Assumption 3. Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$, for any $2 \leq q \leq \infty$ and for any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq Ch^{-2s(q,d)} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha(q,d)}}$$

where $s(q, d)$ is given by the formula (V.2) and $\alpha(q, d)$ is given by

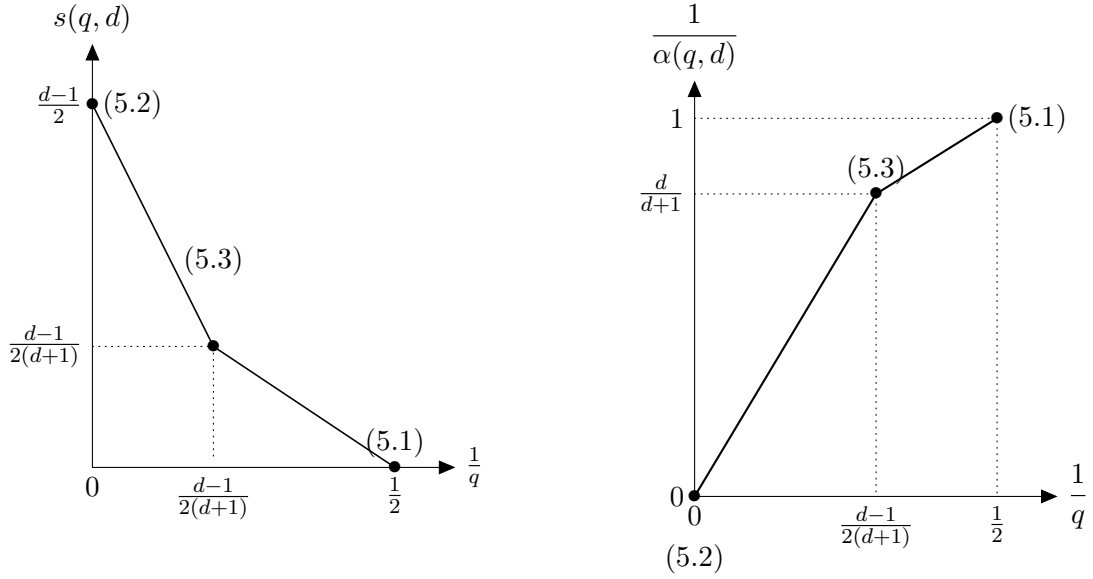
$$\alpha(q, d) = \begin{cases} \frac{2q}{q+2} & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ \frac{q(d-1)}{2d} & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases} \tag{V.3}$$

Remark 17. For $d \geq 3$, the exponent $\alpha_{Sogge} > \alpha_{gene}$ for $q \in (2d/(d-2), \infty)$ and $\alpha_{Sogge} = \alpha_{gene}$ for $q = [2, 2d/d-2]$. It is strictly larger than the one $\alpha_{ellip}(q, d) = q/2$ in the elliptic estimates for any $q \in (2, \infty)$ and they coincide for $q = 2$ or $q = \infty$ (c.f Figure 4).

We write the different end-points with what we obtain the estimates by interpolation.

End-point q	2	∞	$\frac{2d}{(d-2)}$
Referring name	(5.1)	(5.2)	(5.3)

Table 4: References of the end-points' labels in the Sogge's case.



V.2 Definitions and notation for the proof of Theorem V.2

- Let $I_0, I \subset \mathbb{R}$ be open intervals which contain $(x_0)_1$ such that

$$I_0 \subset \bar{I}_0 \subset I \subset [(x_0)_1 - 1, (x_0)_1 + 1].$$

- Let $\psi_1 \in \mathcal{C}_c^\infty(\mathbb{R}, [0, 1])$ such that $\psi_1 = 1$ on \bar{I}_0 and such that $\text{supp } \psi_1 \subset I$.
- Let $t_0 \in I \setminus I_0$.
- Let $J \subset \mathbb{R}$ a bounded open interval which contains $(\xi_0)_1$.
- Let $R := |I| > 0$. Note that $R \leq 2$.
- We choose $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $[-R, R]$ and $\psi = 0$ outside $[-2R, 2R]$. Note that $\text{supp } \psi \subset [-4, 4]$.
- Let us define $I_r := [(x_0)_1 - 5, (x_0)_1 + 5]$.
- Let $\mathcal{V}'_0 = U'_0 \times V'_0$ and $\mathcal{V}' = U' \times V' \subset \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ be bounded open neighborhood of (x'_0, ξ'_0) such that

$$\mathcal{V}'_0 \subset \bar{\mathcal{V}}'_0 \subset \mathcal{V}'.$$

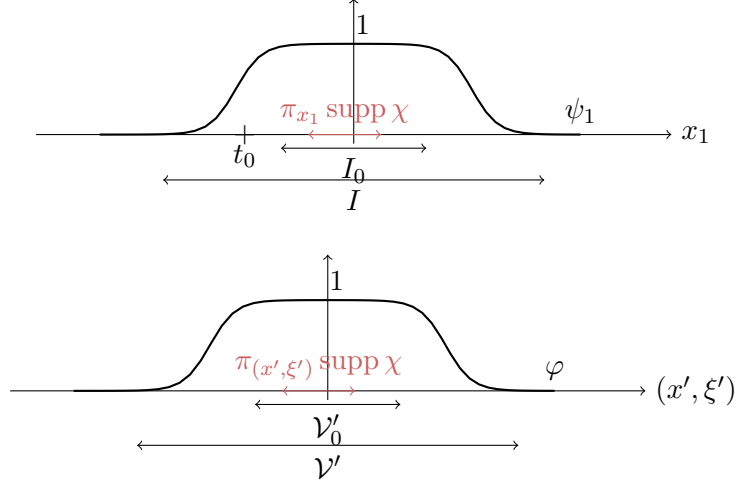
- Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2(d-1)}, [0, 1])$ such that $\varphi = 1$ on $\bar{\mathcal{V}}'_0$ and $\text{supp } \varphi \subset \mathcal{V}'$.

- We define $\mathcal{V} := I_0 \times J \times \mathcal{V}'_0$ and $\mathcal{W} := I \times J \times \mathcal{V}'$. We will add constraints on the size of \mathcal{W} along the proof.
- Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\text{supp } \chi \subset \mathcal{V}$.

By construction this implies that

$$\pi_{x_1} \text{supp } \chi \subset I_0, \quad \pi_{(x', \xi')} \text{supp } \chi \subset \mathcal{V}'_0,$$

then $\text{supp}(1 - \psi_1) \cap \pi_{x_1} \text{supp } \chi = \emptyset$ and $\text{supp}(1 - \varphi) \cap \pi_{(x', \xi')} \text{supp } \chi = \emptyset$.



V.3 Proof of Theorem V.2

First and foremost, the bounds at the two points $q = 2$ and $q = \infty$ follow from Theorem IV.2. Since we have the bounds for $q = 2$ and $q = \infty$, we now show it for $q = 2(d+1)/(d-1)$, which implies the theorem by interpolation.

Let us first explain why we will focus our proof on $\|\rho_{\psi_1 \chi^w} \gamma \chi^w \psi_1\|_{L^{q/2}(\mathbb{R}^d)}$. Recall with Mercer theorem (Remark 7) that

$$\|\rho_{\chi^w} \gamma \chi^w\|_{L^{q/2}(\mathbb{R}^d)} = \sup_{W \in L^{2(q/2)' }(\mathbb{R}^d) \cap \mathcal{C}^0(\mathbb{R}^d)} \frac{\|W(x) \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(\mathbb{R}^d)}^2}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2}.$$

By the triangle inequality, up to a multiplicative factor, it is bounded by

$$\sup_{W \in L^{2(q/2)' }(\mathbb{R}^d)} \frac{\|W(x) \psi_1(x_1) \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(\mathbb{R}^d)}^2}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2}$$

and

$$\sup_{W \in L^{2(q/2)' }(\mathbb{R}^d)} \frac{\|W(x) (1 - \psi_1(x_1)) \chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(\mathbb{R}^d)}^2}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2}.$$

By construction $\text{supp}(1 - \psi_1)$ and $\pi_{x_1} \text{supp } \chi$ are disjoint. Then

$$(1 - \psi_1(x_1)) \chi^w = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d).$$

By the Hölder and Kato-Seiler-Simon inequalities (Lemma II.24), with $M \in \mathbb{N}$ such that $2M(q/2)' > d$

$$\begin{aligned} & \|W(x)(1 - \psi_1(x_1))\chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(\mathbb{R}^d)} \\ & \leq \|W(x)(1 - h^2\Delta)^{-M}\|_{\mathfrak{S}^{2(q/2)'}(L^2(\mathbb{R}^d))} \|(1 - h^2\Delta)^M(1 - \psi_1(x_1))\chi^w\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \|\sqrt{\gamma}\|_{\mathfrak{S}^q(L^2(\mathbb{R}^d))} \\ & = \mathcal{O}(h^{N-d/2}) \|W\|_{L^{2(q/2)' }(\mathbb{R}^d)} \|\gamma\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}^d))}^{1/2} \quad \forall N \in \mathbb{N}. \end{aligned}$$

Hence, the crucial part of the proof relies on the estimation of

$$\sup_{W \in L^{2(q/2)' }(\mathbb{R}^d)} \frac{\|W(x)\psi_1(x_1)\chi^w \sqrt{\gamma}\|_{\mathfrak{S}^2(\mathbb{R}^d)}^2}{\|W\|_{L^{2(q/2)' }(\mathbb{R}^d)}^2} = \|\rho_{\psi_1\chi^w\gamma\chi^w\psi_1}\|_{L^{q/2}(\mathbb{R}^d)}.$$

The main idea now is to reduce the problem to an evolution equation in $d - 1$ variables. Up to a permutation of coordinates, by the implicit functions theorem, there exist a neighborhood \mathcal{U} of (x_0, ξ_0) , functions $e \in S(1)$ and $a \in C_c^\infty(\mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$, such that

- the Hessian

$$\partial_{\xi'}^2 a(x_0, \xi'_0) \text{ is non-degenerate,} \quad (\text{V.4})$$

- $\inf |e| > 0$,
- for all $(x, \xi) \in \mathcal{U}$

$$p(x, \xi) = e(x, \xi)(\xi_1 - a(x_1, x', \xi')).$$

We thus assume that $\mathcal{W} \subset \mathcal{U}$. Then, since $\text{supp } \chi \subset \mathcal{W}$, we have

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad p(x, \xi)\chi(x, \xi) = e(x, \xi)(\xi_1 - a(x, \xi'))\chi(x, \xi).$$

We can write $P\chi^w$ as

$$\begin{aligned} P\chi^w &= (p\chi)^w(x, hD) + hr_1^w(x, hD) \\ &= e^w(hD_{x_1} - a^w(x_1, x', hD_{x'}))\chi^w(x, hD) + hr_2^w(x, hD) + hr_1^w(x, hD) \\ &= e^w(hD_{x_1} - a^w(x_1, x', hD_{x'}))\chi^w(x, hD) + hr^w(x, hD). \end{aligned}$$

By symbolic calculus $r \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Let B_h be the following pseudodifferential operator

$$B_h := \frac{1}{h} e^w(x, hD_x)^{-1} (P\chi^w(x, hD_x) - hr^w(x, hD)).$$

By definition B_h satisfies $(hD_{x_1} - a^w)\chi^w = hB_h$. In other terms, we have for all $u \in L^2(\mathbb{R}^d)$

$$[hD_{x_1} - a^w(x, hD_{x'})]\chi^w u = hB_h u.$$

Furthermore, we have

Lemma V.3. *The operator B_h satisfies*

(i) *the localization property*

$$(1 - \varphi^w(x', hD_{x'}))B_h = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d), \quad (\text{V.5})$$

(ii)

$$B_h(1 + P^2/h^2)^{-1/2} = \mathcal{O}(1) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d). \quad (\text{V.6})$$

Proof of Lemma V.3.

(i) Since B_h satisfies

$$B_h = \frac{1}{h}(hD_{x_1} - a^w(x_1, x', hD_{x'}))\chi^w(x, hD)$$

and $\text{supp}(1 - \varphi)$ and $\pi_{(x', \xi')} \chi$ are disjoint, then

$$(1 - \varphi^w(x', hD_{x'}))B_h = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d).$$

(ii) Besides recalling the definition of B_h , $e \in S(1)$ and $r \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$

$$\begin{aligned} B_h(1 + P^2/h^2)^{-1/2} &= \left(\frac{1}{h}(e^w)^{-1}P - (e^w)^{-1}r^w \right) (1 + P^2/h^2)^{-1/2} \\ &= (e^w)^{-1} \left(\frac{1}{h}P \right) (1 + P^2/h^2)^{-1/2} \\ &\quad - (e^w)^{-1}r^w(1 + P^2/h^2)^{-1/2}. \end{aligned}$$

We obtain (V.6) using that $(e^w)^{-1}$, r^w , $(P/h)(1 + P^2/h^2)^{-1/2}$ and $(1 + P^2/h^2)^{-1/2}$ are $\mathcal{O}(1) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$. \square

Let $r \in \mathbb{R}$. The following evolution equation

$$\begin{cases} [hD_t - a^w(t, x', hD_{x'})]F(t, r) = 0 & t \in \mathbb{R}, \\ F(r, r) = \text{Id}, \end{cases}$$

is solved by a unique family of unitary operators $\{F(t, r)\}_{t \in \mathbb{R}}$ on $L^2(\mathbb{R}^{d-1})$ (we refer the reader to [37, Thm. 10.1]). Recall the Duhamel's formula satisfied by all $u \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}$

$$\chi^w u(t) = F(t, t_0)(\chi^w u)(t_0) + i \int_{t_0}^t F(t, s)(B_h u)(s) ds \quad \text{in } L^2_{x'}(\mathbb{R}^{d-1}).$$

Defining by $\text{ev}_{x_1=t_0}$ the operator of evaluation in $t_0 \in \mathbb{R}$ of the first variable, which maps functions on \mathbb{R}^d to functions on \mathbb{R}^{d-1}

$$\text{ev}_{x_1=t_0} u(x) = u(t_0, x')$$

and $U(t, r)$ the microlocalized operator on $L^2_{x'}(\mathbb{R}^{d-1})$

$$U(t, r) := \psi(t - r)F(t, r)\varphi^w(x', hD_{x'}),$$

given the support property of ψ_1 and ψ , we get the decomposition

$$\begin{aligned} \psi_1(t)\chi^w(t, x', hD_{t, x'}) &= \psi_1(t) \left(F(t, t_0) \text{ev}_{x_1=t_0} \chi^w + i \left(\int_{t_0}^t F(t, s)(1 - \varphi^w) \text{ev}_{x_1=s} B_h ds \right) \right) \\ &\quad + i\psi_1(t) \left(\int_{t_0}^t U(t, s) \text{ev}_{x_1=s} B_h ds \right). \end{aligned}$$

We notice that each term of this operator maps functions on \mathbb{R}^d into functions on \mathbb{R}^d . Define S by

$$S := \psi_1(t)F(t, t_0) \text{ev}_{x_1=t_0} \chi^w + i\psi_1(t) \left(\int_{t_0}^t F(t, s)(1 - \varphi^w) \text{ev}_{x_1=s} B_h ds \right)$$

We introduce the operator T_U which acts on functions on \mathbb{R}^d

$$T_U f(t) := \psi_1(t) \int_{t_0}^t U(t, s)f(s) ds \quad \text{in } L^2(\mathbb{R}^d).$$

We then have the following results.

Proposition V.4. *Let $\beta \geq 2$. We have the bound*

$$\|WS\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d))} = \mathcal{O}(h^\infty)\|W\|_{L^\beta(\mathbb{R}^d)}.$$

Proposition V.5. *If \mathcal{W} is a small enough neighborhood of (x_0, ξ_0) , then the operator T_U satisfies the dual estimates*

$$\|WT_U\|_{\mathfrak{S}^{2\alpha(q,d)}(L^2(\mathbb{R}^d))} \leq Ch^{-s(q,d)}\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}$$

for all $W \in L^{2(q/2)'(\mathbb{R}^d)}$, where $s(q, d)$ and $\alpha(q, d)$ are defined in the statement of Theorem V.2.

Remark 18. *Recall that the operator T_U depends on \mathcal{W} through the functions ψ_1 and φ .*

Before proving the previous propositions, we use them to complete the proof of Theorem V.2. By the decomposition of the operator $\psi_1(t)\chi^w$ and by the triangle inequality

$$\|W\psi_1(t)\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2} \leq \|WS\sqrt{\gamma}\|_{\mathfrak{S}^2} + \|WT_UB_h\sqrt{\gamma}\|_{\mathfrak{S}^2}.$$

Using the Hölder inequality and Proposition V.4, we have

$$\begin{aligned} \|WS\sqrt{\gamma}\|_{\mathfrak{S}^2} &\leq \|WS\|_{\mathfrak{S}^{2(q/2)'}} \|\sqrt{\gamma}\|_{\mathfrak{S}^q} \\ &= \mathcal{O}(h^\infty)\|W\|_{L^{2(q/2)'}} \|\gamma\|_{\mathfrak{S}^{q/2}}^{1/2}. \end{aligned}$$

By the Hölder inequality, Lemma V.3 and Proposition V.5

$$\begin{aligned} \|WT_UB_h\sqrt{\gamma}\|_{\mathfrak{S}^2} &\leq \|WT_U\|_{\mathfrak{S}^{2\alpha'}} \left\| B_h(1 + P^2/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2} \left\| (1 + P^2/h^2)^{1/2} \sqrt{\gamma} \right\|_{\mathfrak{S}^{2\alpha}} \\ &\lesssim h^{-s(q,d)} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}^{1/2}. \end{aligned}$$

Then, we have the same bound for the norm $\|W(x)\psi_1(t)\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2}$.

Finally, we obtain for $q = 2(d+1)/d - 1$

$$\|\rho\chi^w\gamma\chi^w\|_{L^{q/2}(\mathbb{R}^d)} \lesssim h^{-2s(q,d)} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha(q,d)}},$$

which is the desired bound for the end-point (5.2).

V.3.1 Proof of Proposition V.4

Let $\beta \geq 2$ and $W \in L^\beta(\mathbb{R}^d)$. In order to prove the desired inequality, we prove

$$\forall k \in \mathbb{N}, \quad S = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow H_h^k(\mathbb{R}^d). \quad (\text{V.7})$$

Assuming this result, we only need to choose $k \in \mathbb{N}$ so that $x \mapsto \langle x \rangle^{-k}$ be $L^\beta(\mathbb{R}^d)$. By the Hölder, the previous inequality (V.7) and Kato-Seiler-Simon inequality (Lemma II.24)

$$\begin{aligned} &\|W(t, x')S\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d))} \\ &\leq \left\| W(t, x')(1 - h^2\Delta_{t,x'})^{-k/2} \right\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d))} \left\| (1 - h^2\Delta_{t,x'})^{k/2} S \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \\ &\leq C_{d,k,N} h^{N-d/2} \|W\|_{L^\beta(L^2(\mathbb{R}^d))} \quad \forall N \in \mathbb{N}. \end{aligned}$$

Hence,

$$\|WS\|_{\mathfrak{S}^\beta(L^2(\mathbb{R}^d))} = \mathcal{O}(h^\infty)\|W\|_{L^\beta(L^2(\mathbb{R}^d))}.$$

Let us now prove (V.7).

Since $t_0 \notin \pi_{x_1} \text{supp } \chi$, we have

$$\text{ev}_{x_1=t_0} \chi^w(x, hD) = \mathcal{O}(h^\infty) : \mathcal{S}'_x(\mathbb{R}^d) \rightarrow \mathcal{S}'_{x'}(\mathbb{R}^{d-1}),$$

which together with Lemma II.19, implies that

$$\forall k \in \mathbb{N}, \quad \psi_1(t)F(t, t_0) \text{ev}_{x_1=t_0} \chi^w = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow H_h^k(\mathbb{R}^d).$$

Let us prove the same equality for the second term $\psi_1(t) \int_{t_0}^t F(t, s)(1 - \varphi^w(x, hD_{x'})) \text{ev}_{x_1=s} B_h ds$. Recall T_F is the operator which acts on functions in \mathbb{R}^d defined by

$$T_F : u = u(t, x') \mapsto \psi_1(t) \int_{t_0}^t (F(t, s)u(s))(x') ds. \quad (\text{V.8})$$

By Lemma II.19 (to $n = d - 1$, $t = x_1$ and $x = x'$), the operator T_F which maps to $H_h^k(\mathbb{R}^d)$ into $H_h^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$.

Then, since $1 - \varphi^w$ commutes with $\text{ev}_{x_1=s}$ (because φ^w only acts on the variables (x', ξ')) and given Lemma V.3

$$\begin{aligned} \psi_1(t) \int_{t_0}^t F(t, s)(1 - \varphi^w) \text{ev}_{x_1=s} B_h ds &= \psi_1(t) \int_{t_0}^t F(t, s) \text{ev}_{x_1=s}(1 - \varphi^w) B_h ds \\ &= T_F \circ ((1 - \varphi^w) B_h) \\ &= \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow H_h^k(\mathbb{R}^d) \quad \forall k \in \mathbb{N}. \end{aligned}$$

Finally, for all $k \in \mathbb{N}$

$$S = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \rightarrow H_h^k(\mathbb{R}^d),$$

what is exactly (V.7).

V.3.2 Proof of Proposition V.5

The operator T_U can be split as $T_U = T_+ - T_-$ with

$$T_+ := \mathbf{1}(t \geq t_0) T_U$$

and

$$T_- := -\mathbf{1}(t \leq t_0) T_U.$$

Their dual operators' expressions are the following

$$T_+^* : \begin{cases} L^2(\mathbb{R}^d) & \rightarrow L^2(\mathbb{R}^d) \\ f & \mapsto \mathbf{1}(r \geq t_0) \int_{\mathbb{R}} \mathbf{1}(s \geq t_0) \mathbf{1}(s \geq r) U(s, r)^* \psi_1(s) f(s) ds \end{cases}$$

and

$$T_-^* : \begin{cases} L^2(\mathbb{R}^d) & \rightarrow L^2(\mathbb{R}^d) \\ f & \mapsto \mathbf{1}(r \leq t_0) \int_{\mathbb{R}} \mathbf{1}(s \leq t_0) \mathbf{1}(t \leq r) U(s, r)^* \psi_1(s) f(s) ds. \end{cases}$$

Then the operators $T_\pm T_\pm^*$ can be written as

$$\begin{aligned} T_+ T_+^* f(t) &= \int_{r \geq t_0} dr \int_{\mathbb{R}} ds \mathbf{1}(t \geq t_0) \mathbf{1}(t \geq r) \mathbf{1}(s \geq t_0) \mathbf{1}(s \geq r) \times \\ &\quad \times \psi_1(t) U(t, r) U(s, r)^* \psi_1(s) f(s) \end{aligned}$$

and

$$\begin{aligned} T_- T_-^* f(t) &= \int_{r \leq t_0} dr \int_{\mathbb{R}} ds \mathbf{1}(t \leq t_0) \mathbf{1}(t \leq r) \mathbf{1}(s \leq t_0) \mathbf{1}(s \leq r) \times \\ &\quad \times \psi_1(t) U(t, r) U(s, r)^* \psi_1(s) f(s). \end{aligned}$$

Let $r \in \mathbb{R}$. Let us introduce the operators $A_{r, \pm}$

$$A_{r, +} : \begin{cases} L_{x'}^2(\mathbb{R}^{d-1}) & \rightarrow L_{t, x'}^2(\mathbb{R}^d) \\ g & \mapsto \mathbf{1}(t \geq t_0) \psi_1(t) \mathbf{1}(t \geq r) U(t, r) g \end{cases}$$

and

$$A_{r, -} : \begin{cases} L_{x'}^2(\mathbb{R}^{d-1}) & \rightarrow L_{t, x'}^2(\mathbb{R}^d) \\ g & \mapsto \mathbf{1}(t \leq t_0) \psi_1(t) \mathbf{1}(t \leq r) U(t, r) g. \end{cases}$$

Their dual operators can be written as

$$A_{r, +}^* : \begin{cases} L_{t, x'}^2(\mathbb{R}^d) & \rightarrow L_{x'}^2(\mathbb{R}^{d-1}) \\ f & \mapsto \int_{\mathbb{R}} \mathbf{1}(s \geq t_0) \mathbf{1}(s \geq r) U(s, r)^* \psi_1(s) f(s) ds \end{cases}$$

and

$$A_{r, -}^* : \begin{cases} L_{t, x'}^2(\mathbb{R}^d) & \rightarrow L_{x'}^2(\mathbb{R}^{d-1}) \\ f & \mapsto \int_{\mathbb{R}} \mathbf{1}(s \leq t_0) \mathbf{1}(s \leq r) U(s, r)^* \psi_1(s) f(s) ds. \end{cases}$$

The operators $A_{r, \pm} A_{r, \pm}^*$ acts on $L^2(\mathbb{R}^d)$. This gives

$$\begin{aligned} T_+ T_+^* &= \int_{r \geq t_0} A_{r, +} A_{r, +}^* dr = \int_{I_r} A_{r, +} A_{r, +}^* dr, \\ T_- T_-^* &= \int_{r \leq t_0} A_{r, -} A_{r, -}^* dr = \int_{I_r} A_{r, -} A_{r, -}^* dr. \end{aligned}$$

Moreover

$$T_U T_U^* = T_+ T_+^* + T_- T_-^*.$$

Thus, for any $\alpha \geq 1$

$$\begin{aligned} &\|W T_U\|_{\mathfrak{S}^{2\alpha'}(L^2(\mathbb{R}^d))}^2 \\ &= \|W T_U T_U^* \bar{W}\|_{\mathfrak{S}^{\alpha'}(L^2(\mathbb{R}^d))} \\ &\leq |I_r| \left(\sup_{r \in I_r} \|W A_{r, +} A_{r, +}^* \bar{W}\|_{\mathfrak{S}^{\alpha'}(L^2(\mathbb{R}^d))} + \sup_{r \in I_r} \|W A_{r, -} A_{r, -}^* \bar{W}\|_{\mathfrak{S}^{\alpha'}(L^2(\mathbb{R}^d))} \right) \\ &\leq |I_r| \left(\sup_{r \in I_r} \|W A_{r, +}\|_{\mathfrak{S}^{2\alpha'}(L^2(\mathbb{R}^{d-1}), L^2(\mathbb{R}^d))}^2 + \sup_{r \in I_r} \|W A_{r, -}\|_{\mathfrak{S}^{2\alpha'}(L^2(\mathbb{R}^{d-1}), L^2(\mathbb{R}^d))}^2 \right). \end{aligned}$$

Now, notice that

$$\|W A_{r, \pm}\|_{\mathfrak{S}^{2\alpha'}} \leq C \|W U(t, r)\|_{\mathfrak{S}^{2\alpha'}}.$$

Given (V.4), we can apply Theorem II.27 to $n = d-1$, $(x'_0, \xi'_0) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$, $a \in \mathcal{C}_{x_1}^\infty(\mathbb{R}, S_{(x', \xi')}(1))$ and $J = I_r$. This defines $\delta > 0$ and $U'_1 \times V'_1$ a neighborhood of (x'_0, ξ'_0) (which corresponds to the neighborhood $U \times V$ in Theorem II.20). Thus, imposing the following constraints on \mathcal{W} :

- $|I| < \frac{\delta}{2}$,
- $\mathcal{V}' \subset U'_1 \times V'_1$.

we obtain

$$\sup_{r \in I_r} \|W U(t, r)\|_{\mathfrak{S}^{d+1}(L^2(\mathbb{R}^{d-1}), L^2(\mathbb{R}^d))} \lesssim h^{-\frac{d-1}{2(d+1)}} \|W\|_{L^{d+1}(\mathbb{R}^d)}.$$

That ends the proof of Proposition V.5.

VI L^p ESTIMATES AROUND TURNING POINTS

We now treat the turning point region $\{V = E\}$, under the assumption $\nabla_x V \neq 0$ on this set. In the one-body case ($\text{rank } \gamma = 1$), we recover [37, Thm. 7].

VI.1 Statement of the result

Let $d \geq 2$.

Assumption 4. A point $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfies the following turning point conditions for a symbol p if

$$p(x_0, \xi_0) = 0, \quad \nabla_{\xi} p(x_0, \xi_0) = 0, \quad \nabla_x p(x_0, \xi_0) \neq 0, \quad \partial_{\xi}^2 p(x_0, \xi_0) \text{ is positive definite.}$$

Remark 19. For Schrödinger operators $p(x, \xi) = \xi^2 + V(x) - E$ with $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ bounded from below satisfying Definition II.3, the previous assumption is equivalent to:

$$\xi_0 = 0, \quad V(x_0) = E, \quad \nabla_x V(x_0) \neq 0.$$

First recall the individual function result.

Theorem VI.1 (Improved one-body estimates, [22, Thm. 7]). *Let $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ bounded and below and satisfying Definition II.3, define $p(x, \xi) := |\xi|^2 + V(x)$ and $P := p^w(x, hD)$ (or any other quantization). Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying the Assumption 4 for the symbol p . Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$, for any $2 \leq q \leq \infty$ and for any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\chi^w u\|_{L^q(\mathbb{R}^d)} \leq C \log(1/h)^{t(q,d)} h^{-s(q,d)} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right)$$

where $t(q, d)$ and $s(q, d)$ are given by the following formulas

$$t(q, d) = \begin{cases} \frac{d+1}{2(d+3)} & \text{if } q = \frac{2(d+3)}{d+1}, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{VI.1})$$

and

- when $d = 2$:

$$s(q, 2) = \begin{cases} \frac{1}{4} - \frac{1}{2q} & \text{if } 2 \leq q \leq \frac{10}{3}, \\ \frac{1}{2} - \frac{4}{3} \frac{1}{q} & \text{if } \frac{10}{3} \leq q \leq \infty, \end{cases} \quad (\text{VI.2})$$

- when $d \geq 3$:

$$s(q, d) = \begin{cases} \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{q} \right) & \text{if } 2 \leq q \leq \frac{2(d+3)}{d+1}, \\ \frac{2d}{3} \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{1}{6} & \text{if } \frac{2(d+3)}{d+1} \leq q \leq \frac{2d}{d-2}, \\ d \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2} & \text{if } \frac{2d}{d-2} \leq q \leq \infty. \end{cases} \quad (\text{VI.3})$$

Equivalently, one has for all $2 \leq q \leq \infty$

$$\chi^w (1 + P^* P / h^2)^{-1/2} = \mathcal{O}(\log(1/h)^{t(q,d)} h^{-s(q,d)}) : L^2(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d).$$

Remark 20. The exponent s_{TP} , defined in Theorem VI.1 satisfies $s_{Sogge} \leq s_{TP} \leq s_{gene}$ for any $d \geq 1$ and $q \in [2, \infty]$. They are all equal for $q = \{2\} \cup [2d/(d-2), \infty]$. Furthermore $s_{TP} = s_{Sogge}$ when $q \in [2, 2(d+3)/(d+1)]$. Otherwise, the inequalities are strict. (c.f. Figure 3).

Remark 21. Note that the previous result has been proved in [22, Thm. 7] for slightly more general symbols

$$p(x, \xi) = \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j + V(x),$$

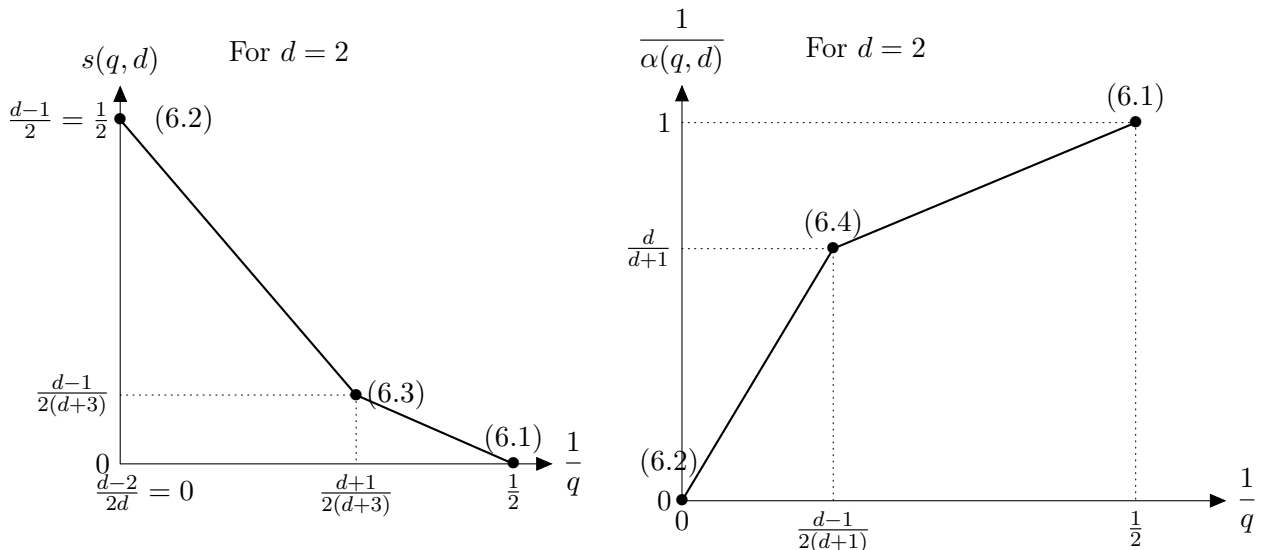
where $\{a_{ij}\}_{1 \leq i,j \leq d} \subset C^\infty(\mathbb{R}^d, \mathbb{R})$ is a positive definite Riemannian metric on \mathbb{R}^d and $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ follows Definition II.4 and Definition II.3. Our results can also be generalized to this case.

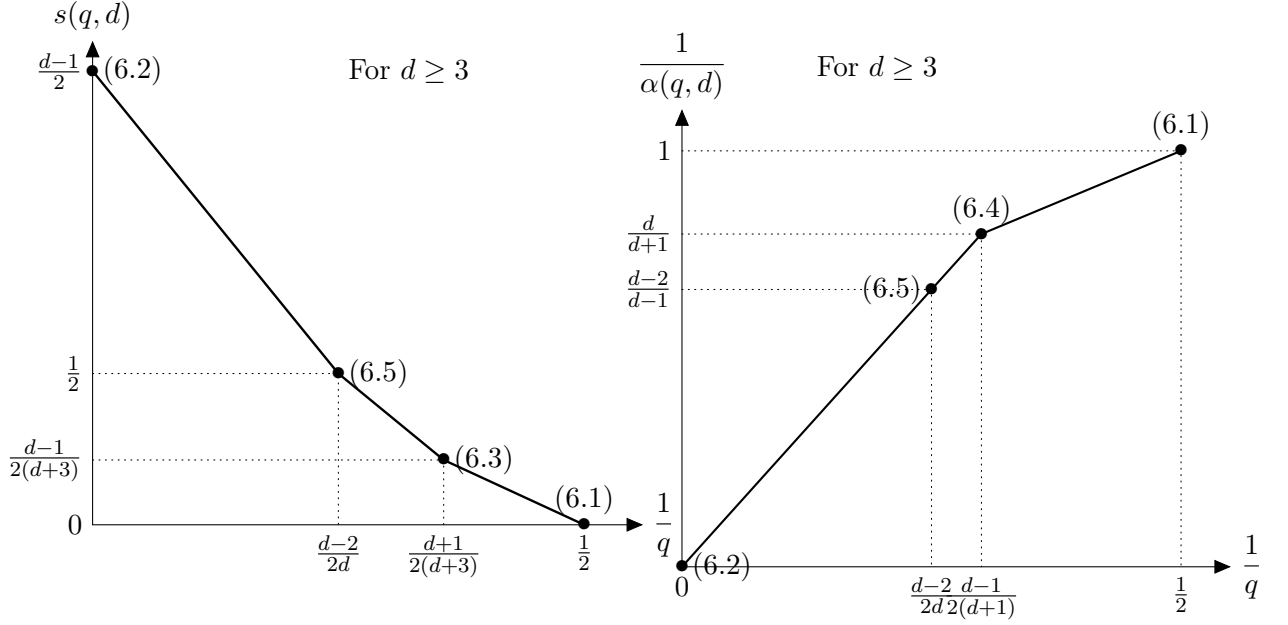
Theorem VI.2 (Improved many-body estimates). *Let $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ bounded from below and satisfying Definition II.3, define $p(x, \xi) := |\xi|^2 + V(x)$ and $P := p^w(x, hD)$. Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be a point satisfying the Assumption 4 for the symbol p . Then, there exist a neighborhood \mathcal{V} of (x_0, ξ_0) and $h_0 > 0$, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with support contained in \mathcal{V} , there exists $C > 0$ such that for any $0 < h \leq h_0$, for any $2 \leq q \leq \infty$ and for any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)} \leq C \log(1/h)^{2t(q,d)} h^{-2s(q,d)} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha(q,d)}}$$

where $t(q, d)$, $s(q, d)$ are given by the formulas (VI.1), (VI.2), (VI.3), and $\alpha(q, d)$ is given by the formula of (V.3).

Remark 22. The proof of [22] gives the exponent $q = \frac{2d}{d-2}$ as a threshold for the exponent $s(q, d)$. Actually, their proof shows that this threshold can be improved to $q = \frac{2d}{d-4}$ (because they only use a control in H^1 while their proof also provides a control in H^2). We choose to keep this weaker statement because it only applies to functions which are microlocalized around a turning point $(x_0, \xi_0 = 0)$ ($p(x_0, \xi_0) = 0$). In our application to spectral clusters we will also need to deal with points such that $p(x_0, \xi_0) = 0$ and $\xi_0 \neq 0$, where only the Sogge estimates are available.





We write the different end-points with what we obtain the estimates by interpolation.

End-point q	2	∞	$\frac{2(d+3)}{d+1}$	$\frac{2(d+1)}{d-1}$	$\frac{2d}{d-2}$
Referring name	(6.1)	(6.2)	(6.3)	(6.4)	(6.5)

Table 5: References of the end-points' labels in the turning points case.

VI.2 Proof of Theorem VI.2

As argued in [22], we may reduce the problem to the case

$$p(x, \xi) = \xi_1^2 + \sum_{i,j=2}^d a_{ij}(x) \xi_i \xi_j + V(x), \quad V(x) = -c(x)x_1,$$

where $(a_{ij}(x))_{i,j} \in \mathcal{C}^\infty(\mathbb{R}^d)$ is positive definite uniformly, $c \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $c(0) > 0$, and

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, \forall i, j = \{2, \dots, d\}, \forall x \in \mathbb{R}^d, \quad |\partial^\alpha a_{ij}(x)| + |\partial^\alpha V(x)| \leq C_\alpha.$$

Notation

- Let $\delta > 0$ such that $\inf_{x \in B_\delta} c(x) > 0$, where $B_\delta := \{x \in \mathbb{R}^d : |x| < \delta\}$.
- Let $\mathcal{V} \subset \mathbb{R}^d \times \mathbb{R}^d$ a bounded open neighborhood of $(x_0, \xi_0) = (0, 0)$ such that $\pi_x \mathcal{V}$ is contained in $B_{\delta/4}$ and such that $\mathcal{V} \subset \mathcal{V}_0$ where \mathcal{V}_0 is given by Corollary VI.4.
- Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\text{supp } \chi \subset \mathcal{V}$.
- Let $M \geq 1$ be larger than $M_0 \geq 1$ given by Corollary VI.4. An other constraint will be given in the proof.
- For all $\varepsilon > 0$, let us define

$$\Omega_\varepsilon := \{x \in \mathbb{R}^d : x_1 < \varepsilon\}.$$

- Let us define $\chi_\varepsilon := \chi_0(\cdot/\varepsilon)$ where $\chi_0 \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ is a nonnegative function equal to 1 on $]-\infty, 1]$ and equal to 0 on $[2, \infty[$.
- Let s_{Sogge} and α_{Sogge} be given by the formulas in the statement of Theorem V.2.

As in the proof of Theorem IV.2, we have by the one-body estimates (Theorem VI.1)

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^\infty(\mathbb{R}^d)} \lesssim h^{1-d} \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h^2)^{1/2} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}.$$

We prove the estimates of Theorem VI.2 for low regime $2 \leq q \leq 2d/(d-2)$. The remaining estimates for $2d/(d-2) < q < \infty$ are then obtained by interpolating between $q = 2d/(d-2)$ and $q = \infty$. We now fix $2 \leq q \leq 2d/(d-2)$. The strategy to estimate $\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\mathbb{R}^d)}$ is to estimate $\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\Omega)}$ on various regions Ω that cover \mathbb{R}^d (c.f. Figure 6), and then sum the obtained estimates.

Before going into the proof, let us recall some key estimates of Koch-Tataru-Zworski [22].

Lemma VI.3 ([22, Lem. 7.3 and Sec. 7]). *Let $d \geq 1$. Then, there exist $M \geq 1$, $h_0 > 0$ and a bounded neighborhood $\mathcal{V} \subset \mathbb{R}^d \times \mathbb{R}^d$ of 0 such that, for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in \mathcal{V} , any $h \in (0, h_0]$, and any $\varepsilon \geq Mh^{2/3}$ we have for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq 2$*

$$\|(hD)^\alpha \chi^w u\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}\left(\varepsilon^{\frac{1}{4} + \frac{|\alpha|}{2}}\right) \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right). \quad (\text{VI.4})$$

Remark 23. *In [22], the estimate (VI.4) is proved only for $|\alpha| \leq 1$ and for $|\alpha| = 2$ when $d = 2$ and $\varepsilon = Mh^{2/3}$. Their method allows to treat the case $|\alpha| = 2$ without the restrictions $d = 2$ and $\varepsilon = Mh^{2/3}$.*

Remark 24. *In the case $\varepsilon = Mh^{2/3}$, the estimate (VI.4) reduces to*

$$\sum_{|\alpha| \leq 2} \|(h^{2/3}D)^\alpha \chi^w u\|_{L^2(\Omega_{Mh^{2/3}})} \leq Ch^{1/6} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right),$$

which, by Sobolev embeddings, imply that for all $2 \leq q \leq \frac{2d}{(d-4)_+}$ (excluding $q = \infty$ for $d = 4$)

$$\|\chi^w u\|_{L^q(\Omega_{Mh^{2/3}})} \leq Ch^{\frac{1}{6} - \frac{2d}{3}(\frac{1}{2} - \frac{1}{q})} \left(\|u\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|Pu\|_{L^2(\mathbb{R}^d)} \right).$$

In dimension $d = 1$, the above estimates match the L^q norms of the normalized Hermite functions. One can even get rid of the microlocalization χ^w in the argument of [22]. Indeed, the two normalized functions u_h such that $-h^2 u_h'' + x^2 u_h = u_h$ satisfy

$$\|u_h\|_{L^q(\mathbb{R})} \lesssim \begin{cases} 1 & \text{if } 2 \leq q \leq 4, \\ h^{-\frac{1}{6} + \frac{2}{3q}} & \text{if } 4 \leq q \leq \infty. \end{cases}$$

These bounds are equivalent to [22, Cor. 3.2] for (ϕ, λ^2) the eigenfunction-eigenvalue pair of the Hermite operator $-\frac{d^2}{dx^2} + x^2$, by the scaling $u_h = h^{-1/4} \phi(h^{-1/2} \cdot)$ and $h = \lambda^{-2}$. For $\alpha = 0$, this estimate is sharp for this potential $V(x) = x^2 - 1$ and $x_0 = 1$ (and $x_0 = -1$), because Hermite functions behave like $h^{-1/6} \text{Ai}(h^{-2/3}(x-1))$ close to $x_0 = 1$ (c.f. also Figure 1 for $E = 1$). In particular, the contribution of the L^q norm around in a turning point's neighborhood of size $h^{2/3}$ is $h^{-\frac{1}{6} + \frac{2}{3q}}$, that saturates the bound for $q \geq 4$. More precisely, according to [27, Chap. 6

and 11], there exist a normalization constant $C > 0$ such that the normalized eigenfunction u_h of $-h^2 \frac{d^2}{dx^2} + x^2 - 1$ satisfy, for any $x \in \mathbb{R}$

$$u_h(x) = Ch^{-1/6} \frac{\zeta(x)^{1/4}}{(x^2 - 1)^{1/4}} \text{Ai} \left(h^{-2/3} \zeta(x) \right) \left(1 + r_h \left(h^{-2/3} \zeta(x) \right) \right),$$

with

$$\forall x \in \mathbb{R}^\pm, \quad \zeta(x) := \pm \text{sgn}(x \mp 1) \left| \frac{3}{2} \int_{\pm 1}^x \sqrt{t^2 - 1} dt \right|^{2/3},$$

and r_h a smooth function such that for any $h > 0$

$$\forall x \in \mathbb{R}, \quad |r_h(x)| \leq h(1 + |x|^{1/4})^{-1} \exp \left(-\frac{2}{3}(x_+)^{3/2} \right).$$

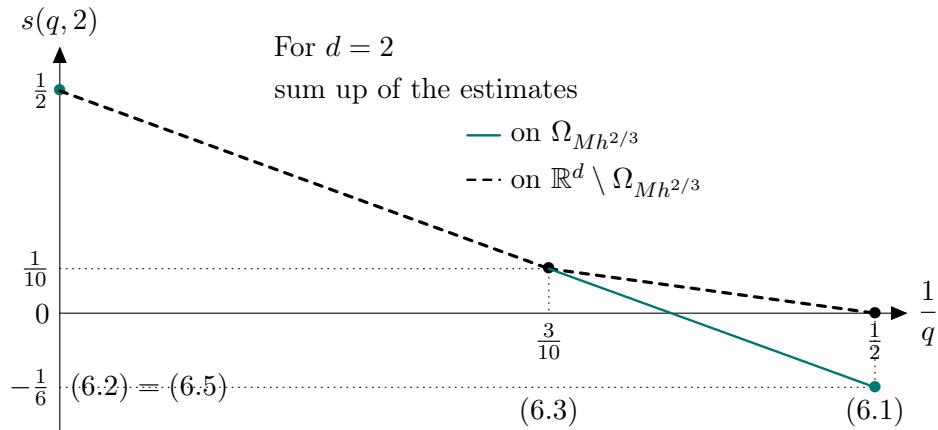
We obtain the saturated L^q bounds using the asymptotics of the Airy function

$$\begin{aligned} \text{Ai}(x) &=_{x>0} \frac{\exp \left(-\frac{2}{3}x^{3/2} \right)}{2\pi^{1/2}x^{1/4}} \left[1 + \mathcal{O} \left(\frac{1}{x^{3/2}} \right) \right], \\ \text{Ai}(x) &=_{x<0} \frac{\cos \left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4} \right)}{\pi^{1/2}(-x)^{1/4}} \left[1 + \mathcal{O} \left(\frac{1}{(-x)^{3/2}} \right) \right]. \end{aligned}$$

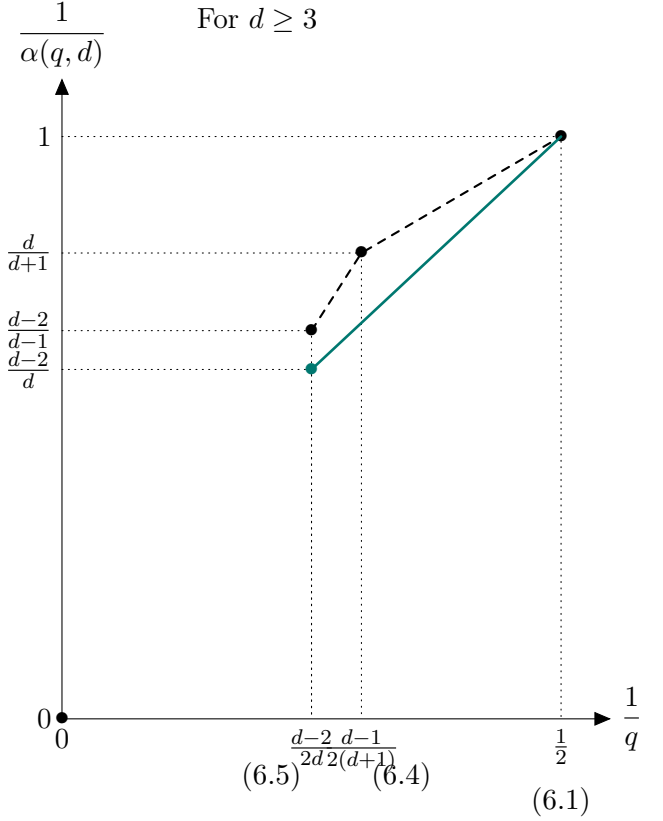
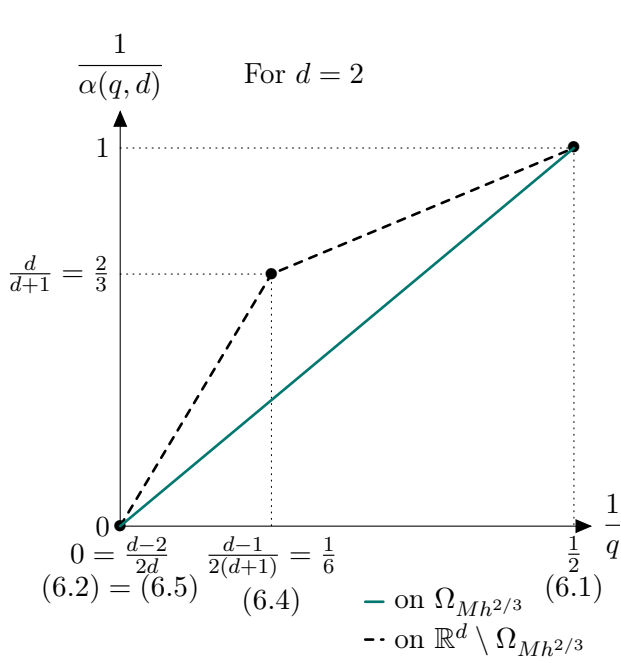
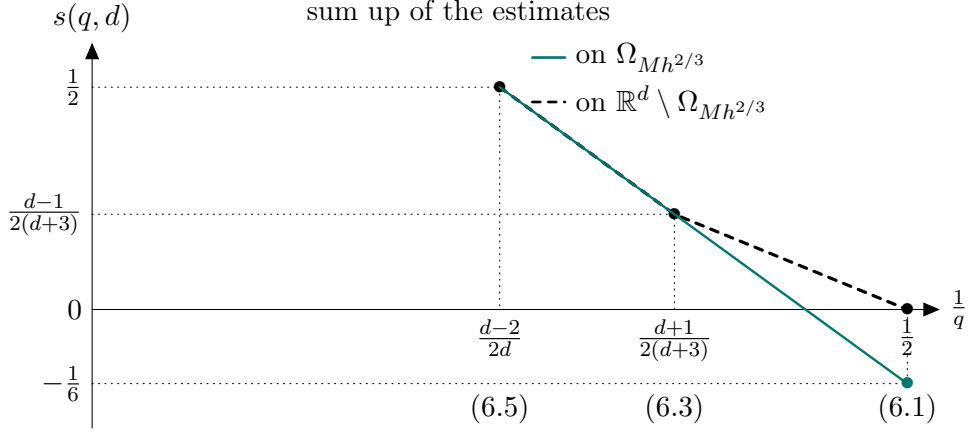
and suitable changes of variables in the three types of regions. Note that [23] provides also equivalents of L^q norms of Hermite functions with respect to a gaussian weight. The sharpness in higher dimension seems open to us.

Corollary VI.4. *Let $d \geq 1$. Then, there exist $M_0 \geq 1$, $h_0 > 0$ and a bounded neighborhood $\mathcal{V}_0 \subset \mathbb{R}^d \times \mathbb{R}^d$ of 0, such that for any $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ supported in \mathcal{V}_0 , for any $h \in (0, h_0]$ and any $\varepsilon \in [M_0 h^{2/3}, 1]$*

$$(1 - h^2 \Delta) \chi_\varepsilon \chi^w (1 + P^2/h^2)^{-1/2} = \mathcal{O}_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \left(\varepsilon^{1/4} \right).$$



For $d \geq 3$



VI.2.1 Estimates on $\Omega_{Mh^{2/3}}$

Lemma VI.5. Let $d \geq 1$. For $2 \leq q \leq \frac{2d}{(d-2)_+}$ and any bounded self-adjoint operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(\Omega_{Mh^{2/3}})} \leq Ch^{\frac{1}{3} - \frac{4d}{3}(\frac{1}{2} - \frac{1}{q})} \left\| \left(1 + \frac{1}{h^2} P^2\right)^{1/2} \gamma \left(1 + \frac{1}{h^2} P^2\right)^{1/2} \right\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}^d))}.$$

Remark 25. The previous estimates are also true for all $2 \leq q < \frac{2d}{(d-4)_+}$, but we will not need it (see Remark 22).

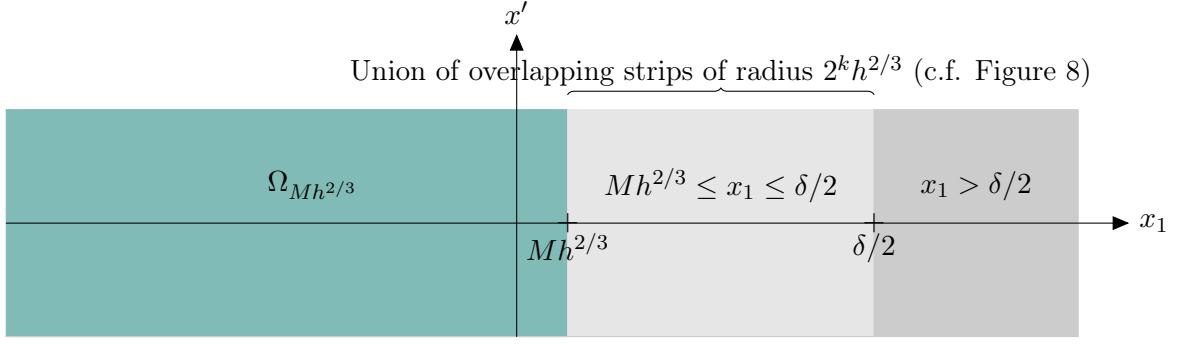


Figure 6: Different regions of \mathbb{R}^d .

Proof of Lemma VI.5. By Corollary VI.4 with $\varepsilon = Mh^{2/3}$

$$(1 - h^{4/3}\Delta)\chi_{Mh^{2/3}}\chi^w(1 + P^2/h^2)^{-1/2} = \mathcal{O}_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}(h^{1/6}),$$

On the one hand, by the Mercer theorem

$$\begin{aligned} \|\rho_{\chi^w\gamma\chi^w}\|_{L^1(\Omega_{Mh^{2/3}})} &= \text{Tr}_{L^2}(\chi_{Mh^{2/3}}\chi^w\gamma\chi^w\chi_{Mh^{2/3}}) \\ &\leq \left\| (1 - h^{4/3}\Delta)^{-1} \right\|_{L^2 \rightarrow L^2}^2 \left\| (1 - h^{4/3}\Delta)\chi_{Mh^{2/3}}\chi^w(1 + P^2/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2}^2 \times \\ &\quad \times \left\| (1 + P^2/h^2)^{1/2}\gamma(1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^1}^{1/2} \\ &\lesssim h^{1/6} \left\| (1 + P^2/h^2)^{1/2}\gamma(1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^1}^{1/2}. \end{aligned}$$

Let us consider now $2 < q \leq \frac{2d}{(d-2)_+}$. By the Kato-Seiler-Simon bound applied to $m = 2$, which is true if and only if $2 < q < \frac{2d}{d-4}$ (see Lemma II.25)

$$\left\| W(1 - h^{4/3}\Delta)^{-1} \right\|_{\mathfrak{S}^{2(q/2)'}(L^2(\mathbb{R}^d))} \leq Ch^{-\frac{2d}{3}(\frac{1}{2}-\frac{1}{q})} \|W\|_{L^{2(q/2)'}(L^2(\mathbb{R}^d))},$$

we have for $2 < q \leq \frac{2d}{(d-2)_+}$ and $W \in L^{2(q/2)'(\mathbb{R}^d)}$

$$\begin{aligned} &\|W\chi_{Mh^{2/3}}\chi^w\sqrt{\gamma}\|_{\mathfrak{S}^2} \\ &\leq \left\| W(1 - h^{4/3}\Delta)^{-1} \right\|_{\mathfrak{S}^{2(q/2)'}} \left\| (1 - h^{4/3}\Delta)\chi_{Mh^{2/3}}\chi^w(1 + P^2/h^2)^{-1/2} \right\|_{L^2 \rightarrow L^2} \times \\ &\quad \times \left\| (1 + P^2/h^2)^{1/2}\sqrt{\gamma} \right\|_{\mathfrak{S}^q} \\ &\lesssim h^{\frac{1}{6}-\frac{2d}{3}(\frac{1}{2}-\frac{1}{q})} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}} \left\| (1 + P^2/h^2)^{1/2}\gamma(1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}}^{1/2}, \end{aligned}$$

which by duality (Remark 7) ends the proof. \square

VI.2.2 Estimates on $\{x \in \mathbb{R}^d : x_1 > \delta/2\}$

Since $\pi_x \text{supp } \chi$ and $\{x \in \mathbb{R}^d : x_1 > \delta/2\}$ are disjoint, we deduce as in the beginning of the proof of Theorem V.2 that for any $2 \leq q \leq \infty$ and any bounded self-adjoint operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\chi^w\gamma\chi^w}\|_{L^{q/2}(\{x \in \mathbb{R}^d : x_1 > \delta/2\})} = \mathcal{O}(h^\infty) \left\| (1 + P^2/h^2)^{1/2}\gamma(1 + P^2/h^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}}.$$

VI.2.3 Estimates on $\{x \in \mathbb{R}^d : Mh^{2/3} \leq x_1 \leq \delta/2\}$

In the following, we assume $h_0 < (\delta/(2M))^{3/2}$ to ensure $Mh^{2/3} < \delta/2$ for $M > 0$ be defined later.

Lemma VI.6. *Let $d \geq 2$ and $2 \leq q \leq \frac{2d}{d-2}$. Then, there exists $M > 0$ such that for all bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$ we have*

$$\|\rho_{\chi^w} \gamma \chi^w\|_{L^{q/2}(\Omega_{\delta/2} \setminus \Omega_{Mh^{2/3}})} \lesssim C_h^2 \left\| (1 + P^2/h^2)^{1/2} \gamma (1 + P^2/h)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}(L^2(\mathbb{R}^d))},$$

where

$$C_h := \begin{cases} h^{-\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q})} & \text{if } 2 \leq q < \frac{2(d+3)}{d+1}, \\ \log^{\frac{d+1}{2(d+3)}}(1/h) h^{-\frac{d-1}{2(d+3)}} & \text{if } q = \frac{2(d+3)}{d+1}, \\ h^{\frac{1}{6}-\frac{2d}{3}(\frac{1}{2}-\frac{1}{q})} & \text{if } \frac{2(d+3)}{d+1} < q \leq \frac{2d}{d-2}. \end{cases}$$

Remark 26. *The previous estimates are also true for all $2 \leq q < \frac{2d}{(d-4)_+}$.*

Proof of Lemma VI.6.

We prove the result for P which is the right quantization of the symbol p (so that $P = h^2 D_{x_1}^2 + \sum_{2 \leq i, j \leq d} a_{ij}(x)(hD_{x_i})(hD_{x_j}) - c(x)x_1$ is a differential operator). While the final result does not depend on the choice of quantization, the proof will rely on several space localizations so that it will be useful that the operator P is local.

Let us now fix some notation. Let $\varepsilon > 0$ and let us define the strips by

$$A_\varepsilon := \{x \in \mathbb{R}^d : |x_1 - \varepsilon| < \varepsilon/2\} \quad \text{and} \quad \tilde{A}_\varepsilon := \{x \in \mathbb{R}^d : |x_1 - \varepsilon| < 3\varepsilon/4\}.$$

Each strip A_ε or \tilde{A}_ε can be decomposed into an union of boxes of size ε (see for instance Figure 7)

$$A_\varepsilon = \bigcup_{k \in \mathbb{Z}^{d-1}} A_\varepsilon^k, \quad \tilde{A}_\varepsilon = \bigcup_{k \in \mathbb{Z}^{d-1}} \tilde{A}_\varepsilon^k$$

defined by

$$\begin{aligned} A_\varepsilon^k &:= \{x \in \mathbb{R}^d : |x_1 - \varepsilon| < \varepsilon/2, \quad |x' - \varepsilon k|_{l^\infty} < \varepsilon/2\} \\ \tilde{A}_\varepsilon^k &:= \{x \in \mathbb{R}^d : |x_1 - \varepsilon| < 3\varepsilon/4, \quad |x' - \varepsilon k|_{l^\infty} < 3\varepsilon/4\}, \\ \tilde{\tilde{A}}_\varepsilon^k &:= \{x \in \mathbb{R}^d : |x_1 - \varepsilon| < 4\varepsilon/5, \quad |x' - \varepsilon k|_{l^\infty} < 4\varepsilon/5\}. \end{aligned}$$

Finally, the set $\Omega_{\delta/2} \setminus \Omega_{Mh^{2/3}} = \{x \in \mathbb{R}^d : Mh^{2/3} \leq x_1 \leq \delta/2\}$ can be covered by an union of $\sim \log(1/h)$ overlapping strips $\bigcup_{k=\lfloor \log_2(M) \rfloor}^{K(h)} A_{2^k h^{2/3}}$ where $K(h) = \lceil \log_2(\delta h^{-2/3}/2) \rceil$ (see for instance Figure 8).

The main steps for the proof of L^q estimates in $\Omega_{\delta/2} \setminus \Omega_{Mh^{2/3}}$ are the following:

0. obtain the estimates on a box A of size 1,
1. obtain the estimates on the boxes A_ε^k of size ε by scaling the previous one,
2. obtain the estimates on the strips A_ε by summing the estimates on the ε -boxes,
3. conclude by summing the estimates on the strips.

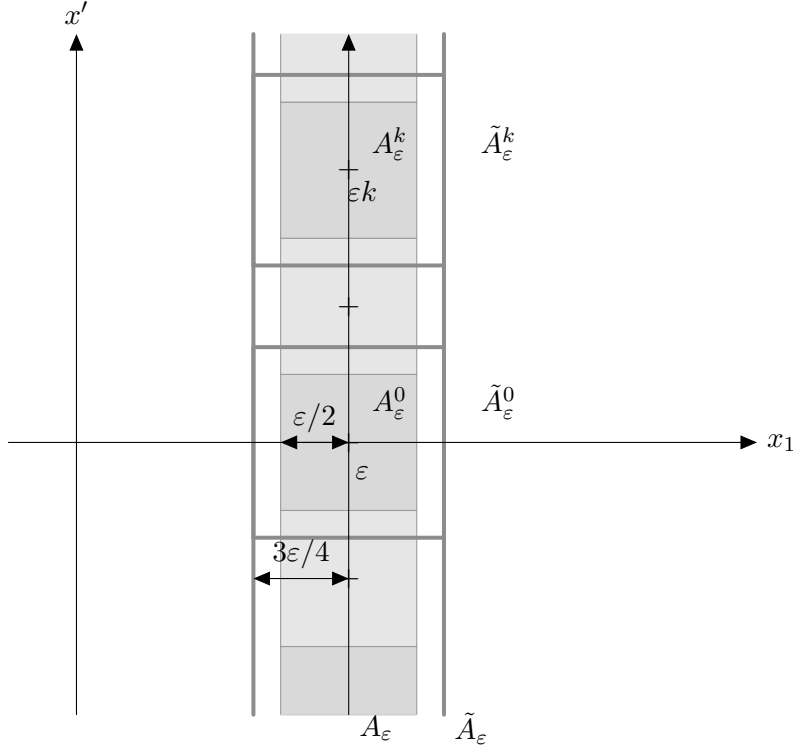


Figure 7: Boxes A_ε^k and \tilde{A}_ε^k .

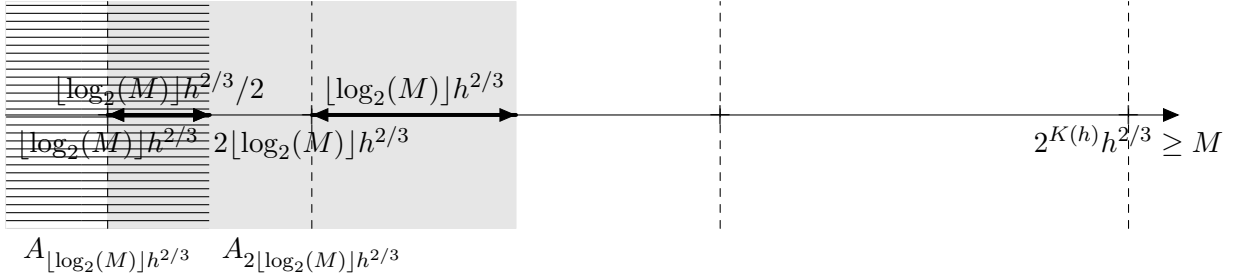


Figure 8: Strips A_ε .

Step 0. Estimates on a box of size 1. We prove estimates on the boxes

$$\begin{aligned} A &:= \{\tilde{x} \in \mathbb{R}^d : |\tilde{x}_1 - 1| < 1/2, \quad |\tilde{x}'|_{l^\infty} < 1/2\}, \\ \tilde{A} &:= \{\tilde{x} \in \mathbb{R}^d : |\tilde{x}_1 - 1| < 3/4, \quad |\tilde{x}'|_{l^\infty} < 3/4\}, \\ \tilde{\tilde{A}} &:= \{x \in \mathbb{R}^d : |\tilde{x}_1 - 1| < 4/5, \quad |\tilde{x}'|_{l^\infty} < 4/5\}. \end{aligned}$$

Lemma VI.7. Let $d \geq 2$ and $2 \leq q \leq \frac{2d}{d-2}$. Let $\varepsilon_0 > 0$. Let $\tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) := \langle \tilde{\xi}, B_\varepsilon(\tilde{x})\tilde{\xi} \rangle + V_\varepsilon(\tilde{x})$ with $\{(B_\varepsilon, V_\varepsilon)\}_{\varepsilon \in [0, \varepsilon_0]} \subset \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_{\text{sym}}^{d \times d} \times \mathbb{R})$ and $(\varepsilon, \tilde{x}) \mapsto (B_\varepsilon(\tilde{x}), V_\varepsilon(\tilde{x})) \in \mathcal{C}([0, \varepsilon_0] \times \mathbb{R}^d)$ such that

- for any $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$

$$\forall x \in \mathbb{R}^d \quad |\partial^\alpha B_\varepsilon(\tilde{x})| + |\partial^\alpha V_\varepsilon(\tilde{x})| \leq C_\alpha, \quad (\text{VI.5})$$

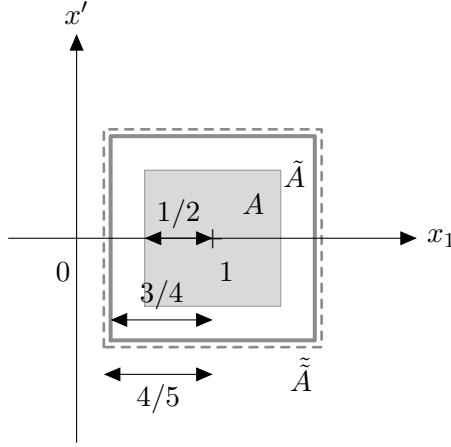


Figure 9: Boxes of size 1

- there exists $c > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$

$$B_\varepsilon \geq c, |V_\varepsilon| \geq c \quad \text{on} \quad \{\tilde{x} \in \mathbb{R}^d : |\tilde{x}_1 - 1| < 9/10, \quad |\tilde{x}'|_{L^\infty} < 9/10\}. \quad (\text{VI.6})$$

Let $\tilde{P} := \tilde{p}_\varepsilon^w(\tilde{x}, \tilde{h}D_{\tilde{x}})$ (or any other quantization) and let $\chi_A \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d, [0, 1])$ such that $\chi_A = 1$ on A and $\text{supp } \chi_A \subset \tilde{A}$. Then, there exist $C > 0$ and $\tilde{h}_0 > 0$ such that for all $0 < \tilde{h} \leq \tilde{h}_0$, for all $0 \leq \varepsilon \leq \varepsilon_0$ and all bounded self-adjoint non-negative operator $\tilde{\gamma}$ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\tilde{\gamma}}(\tilde{x})\|_{L^{q/2}(A)} \leq C \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^\alpha \text{Sogge}(q,d)}.$$

Proof of Lemma VI.7. Let $\chi_{\tilde{A}} \in C_c^\infty(\mathbb{R}^d, [0, 1])$ be a cut-off function equal to 1 on \tilde{A} and supported into $\tilde{\tilde{A}}$. It follows from the equality $\chi_{\tilde{A}}^2(\tilde{x}) \rho_{\tilde{\gamma}}(\tilde{x}, \tilde{x}) = \rho_{\chi_{\tilde{A}} \tilde{\gamma} \chi_{\tilde{A}}}(\tilde{x}, \tilde{x})$ that

$$\|\rho_{\tilde{\gamma}}\|_{L^{q/2}(A)} \leq \|\chi_{\tilde{A}}^2 \rho_{\tilde{\gamma}}\|_{L^{q/2}(\mathbb{R}^d)} = \|\rho_{\chi_{\tilde{A}} \tilde{\gamma} \chi_{\tilde{A}}}\|_{L^{q/2}(\mathbb{R}^d)}.$$

Let

$$\tilde{K} := \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \left\{ (\tilde{x}, \tilde{\xi}) \in \tilde{\tilde{A}} \times \mathbb{R}^d : \langle \tilde{\xi}, B_\varepsilon(\tilde{x}) \tilde{\xi} \rangle + V_\varepsilon(\tilde{x}) \leq \frac{1}{2} (1 + \langle \tilde{\xi}, B_\varepsilon(\tilde{x}) \tilde{\xi} \rangle) \right\}.$$

Note that each \tilde{K} is a closed set of \mathbb{R}^d (here we use the continuity of $(B_\varepsilon(\tilde{x}), V_\varepsilon(\tilde{x}))$ in (ε, \tilde{x})). It is also bounded, since it is contained into the bounded set

$$\tilde{\tilde{A}} \times \left\{ \tilde{\xi} \in \mathbb{R}^d : |\tilde{\xi}| \leq c^{-1/2} \sqrt{1 + 2 \sup_{0 \leq \varepsilon \leq \varepsilon_0} \|V_\varepsilon\|_{L^\infty(\mathbb{R}^d)}} \right\}.$$

Let $\tilde{\psi} \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a function $0 \leq \tilde{\psi} \leq 1$, such that $\tilde{\psi} = 1$ in the compact \tilde{K} . The operator is composed of three parts

$$\chi_A \tilde{\gamma} \chi_A = \tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\gamma}_3,$$

defined by

$$\begin{aligned} \tilde{\gamma}_1 &:= \tilde{\psi}^w \chi_A \tilde{\gamma} \chi_A \tilde{\psi}^w, \\ \tilde{\gamma}_2 &:= (1 - \tilde{\psi}^w) \chi_A \tilde{\gamma} \chi_A (1 - \tilde{\psi}^w), \\ \tilde{\gamma}_3 &:= \underbrace{(1 - \tilde{\psi}^w) \chi_A \tilde{\gamma} \chi_A \tilde{\psi}^w}_{=:\tilde{\gamma}_{3,1}} + \underbrace{\tilde{\psi}^w \chi_A \tilde{\gamma} \chi_A (1 - \tilde{\psi}^w)}_{=:\tilde{\gamma}_{3,2}}. \end{aligned}$$

Let us prove that for any $i \in \{1, 2, 3\}$

$$\|\rho_{\tilde{\gamma}_i}\|_{L^{q/2}(\mathbb{R}^d)} \lesssim \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}}.$$

These above bounds together with the triangle inequality prove the lemma.

Estimate of the term $\tilde{\gamma}_1$. Note that

$$\text{supp } \tilde{\psi} \cap \tilde{A} \times \mathbb{R}^d \subset S = \{\tilde{x} \in \mathbb{R}^d : |\tilde{x}_1 - 1| < 9/10, \quad |\tilde{x}'|_{l^\infty} < 9/10\} \times \mathbb{R}^d.$$

Every point $(\tilde{x}, \tilde{\xi})$ of S satisfies $\tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) \neq 0$ or Sogge curvature conditions (Assumption 3). Indeed, if $(\tilde{x}, \tilde{\xi})$ satisfies $\tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) = 0$, then uniformly in ε , by (VI.6)

$$\left| \nabla_{\tilde{\xi}} \tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) \right| = 2 \left| B_\varepsilon(\tilde{x}) \tilde{\xi} \right| = 2\sqrt{c} \sqrt{|V_\varepsilon(\tilde{x})|} \geq 2c.$$

Furthermore, for all ε and \tilde{x}

$$\{\tilde{\xi} \in \mathbb{R}^d : \tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) = 0\} = \{\tilde{\xi} \in \mathbb{R}^d : \langle \tilde{\xi}, B_\varepsilon(\tilde{x}) \tilde{\xi} \rangle = |V_\varepsilon(\tilde{x})|\}$$

has a positive curvature which is bounded and bounded away from 0 uniformly in ε . Hence, by Theorem III.1 and Theorem V.2, Assumption 1 holds for q , $s = s_{\text{Sogge}}(q, d)$, $t = 0$ and $\alpha = \alpha_{\text{Sogge}}(q, d)$ on the set S . By Remark 9 applied to S , to $\Omega = \tilde{A}$, to $\chi = \tilde{\psi}$, to $(q, s_{\text{Sogge}}, 0, \alpha_{\text{Sogge}})$ and to the operator $\gamma = \chi_A \tilde{\gamma} \chi_A$

$$\begin{aligned} \|\rho_{\tilde{\gamma}_1}\|_{L^{q/2}(\mathbb{R}^d)} &= \|\rho_{\tilde{\psi}^w \chi_A \tilde{\gamma} \chi_A \tilde{\psi}^w}\|_{L^{q/2}(\mathbb{R}^d)} = \|\rho_{\tilde{\psi}^w \chi_{\tilde{A}} \chi_A \tilde{\gamma} \chi_A \chi_{\tilde{A}} \tilde{\psi}^w}\|_{L^{q/2}(\mathbb{R}^d)} \\ &\lesssim \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}}. \end{aligned}$$

Estimate of the term $\tilde{\gamma}_2$. We write with the Mercer theorem (Remark 7) that

$$\|\rho_{\tilde{\gamma}_2}\|_{L^{q/2}(\mathbb{R}^d)} = \begin{cases} \left\| (1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^2}^2 & \text{if } q = 2, \\ \sup_{W \in L^{2(q/2)'} \cap C^0(\mathbb{R}^d)} \frac{\left\| W(1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^2}^2}{\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2} & \text{if } 2 \leq q \leq \frac{2d}{d-2}. \end{cases}$$

By the Hölder and Kato-Seiler-Simon inequalities (Lemma II.25) then applied successively, we have for any $2 \leq q \leq \frac{2d}{d-2}$ (since $\frac{2d}{d-2} < \frac{2d}{(d-4)_+}$)

$$\begin{aligned} \left\| W(1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^2} &\leq \left\| W(1 - \tilde{h}^2 \Delta)^{-1} \right\|_{\mathfrak{S}^{2(q/2)'}} \left\| (1 - \tilde{h}^2 \Delta)(1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^{q/2}}^{1/2} \\ &\lesssim h^{-d(\frac{1}{2} - \frac{1}{q})} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}} \left\| (1 - \tilde{h}^2 \Delta)(1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^{q/2}}^{1/2}. \end{aligned}$$

Implicitly, when $q = 2$, we write $W = 1$ and the operator norm instead of the $\mathfrak{S}^{2(q/2)'}$ norm. Now let us give a estimate of the right side of the previous bound.

Fact VI.8. *There exist $C > 0$ and $\tilde{h}_0 > 0$ such that for any $0 < \tilde{h} \leq \tilde{h}_0$ and any $0 \leq \varepsilon \leq \varepsilon_0$, one has for all $\alpha \geq 1$*

$$\left\| (1 - \tilde{h}^2 \Delta)(1 - \tilde{\psi}^w) \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} \leq C \left(\left\| \tilde{P} \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} + \tilde{h} \left\| \chi_A \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} \right).$$

Proof of Fact VI.8. Let $\alpha \geq 1$.

1) Let us first show that it is enough to do everything with right quantization by replacing $(1 - \tilde{\psi}^w)$ by $(1 - \tilde{\psi}^R)$ into Fact VI.8. By Proposition II.9 applied to the symbol $\tilde{\psi}$, there exists $\tilde{r} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(1 - \tilde{\psi}^w) - (1 - \tilde{\psi}^R) = \tilde{\psi}^R - \tilde{\psi}^w = \tilde{h}\tilde{r}^w.$$

We can now write

$$\begin{aligned} (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^w)\chi_A\sqrt{\tilde{\gamma}} &= (1 - \tilde{h}^2\Delta) \left((1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} + \text{Op}_h^{1/2} \left(\mathcal{O}_{\mathcal{S}}(\tilde{h}) \right) \chi_A\sqrt{\tilde{\gamma}} \right) \\ &= (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} + \mathcal{O}_{L^2 \rightarrow L^2}(\tilde{h})\chi_A\sqrt{\tilde{\gamma}}. \end{aligned}$$

2) It remains to show fact VI.8 by replacing $\tilde{\psi}^w$ by $\tilde{\psi}^R$.

Let $\tilde{m}_0(\tilde{x}, \tilde{\xi}) = \langle \tilde{\xi} \rangle^2$. Let us first notice that the operator \tilde{p}_ε is elliptic on the support of $(1 - \tilde{\psi})\chi_{\tilde{A}}$. Indeed, the function $(1 - \tilde{\psi})\chi_{\tilde{A}}$ is supported into $\tilde{K}^c \cap (\tilde{A} \times \mathbb{R}^d)$ and by definition of \tilde{K} , for all $\varepsilon \in [0, \varepsilon_0]$ and for all $(\tilde{x}, \tilde{\xi}) \in \tilde{K}^c \cap (\tilde{A} \times \mathbb{R}^d)$ we have

$$\tilde{p}_\varepsilon(\tilde{x}, \tilde{\xi}) > \frac{\min(1, c)}{2} \tilde{m}_0(\tilde{x}, \tilde{\xi}).$$

Thus, we get the ellipticity (uniform in ε) of \tilde{p}_ε on $\tilde{K}^c \cap (\tilde{A} \times \mathbb{R}^d)$ and thus on $\text{supp}((1 - \tilde{\psi})\chi_{\tilde{A}})$. By Lemma II.11 applied to $(1 - \tilde{\psi})\chi_{\tilde{A}} \in S(1)$ and to $p \in S(\tilde{m}_0)$, there exist $\tilde{b} \in S(\tilde{m}_0^{-1})$ and $\tilde{r} = \mathcal{O}_{S(1)}(\tilde{h}^\infty)$ such that

$$(1 - \tilde{\psi})^R\chi_{\tilde{A}} = \tilde{b}^R\tilde{P}(1 - \tilde{\psi}^R)\chi_{\tilde{A}} + \tilde{r}^R.$$

Note that $\chi_A = \chi_{\tilde{A}}\chi_A$. We compose by $(1 - \tilde{h}^2\Delta)$ on the left and $\chi_A\sqrt{\tilde{\gamma}}$ on the right to infer

$$\begin{aligned} (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi})^R\chi_A\sqrt{\tilde{\gamma}} &= (1 - \tilde{h}^2\Delta)\tilde{b}^R\tilde{P}(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} + (1 - \tilde{h}^2\Delta)\tilde{r}^R(1 - \tilde{h}^2\Delta)^{-1}(1 - \tilde{h}^2\Delta)\chi_A\sqrt{\tilde{\gamma}}. \\ &\quad \underbrace{= \mathcal{O}_{L^2 \rightarrow L^2}(1)} \qquad \underbrace{= \mathcal{O}_{L^2 \rightarrow L^2}(\tilde{h}^\infty)} \end{aligned}$$

Let us write $\tilde{P}(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}}$ in two parts

$$\tilde{P}(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} = \underbrace{(1 - \tilde{\psi}^R)}_{= \mathcal{O}_{L^2 \rightarrow L^2}(1)} \tilde{P}\chi_A\sqrt{\tilde{\gamma}} - \underbrace{\left[\tilde{P}, \tilde{\psi}^R \right]}_{= \mathcal{O}_{L^2 \rightarrow L^2}(\tilde{h})} \chi_A\sqrt{\tilde{\gamma}}.$$

In addition, the term $(1 - \tilde{h}^2\Delta)\chi_A\sqrt{\tilde{\gamma}}$ is also divided into two parts

$$(1 - \tilde{h}^2\Delta)\chi_A\sqrt{\tilde{\gamma}} = \underbrace{(1 - \tilde{h}^2\Delta)\tilde{\psi}^R}_{= \mathcal{O}_{L^2 \rightarrow L^2}(1)}\chi_A\sqrt{\tilde{\gamma}} + (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}}.$$

Putting everything together and moving to the Schatten norm

$$\begin{aligned} \left\| (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} &\lesssim \left\| \tilde{P}\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} + \tilde{h} \left\| \chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} \\ &\quad + \mathcal{O}(\tilde{h}^\infty) \left\| (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

Then, we conclude by taking \tilde{h} small enough

$$\left\| (1 - \tilde{h}^2\Delta)(1 - \tilde{\psi}^R)\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} \lesssim \left\| \tilde{P}\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} + \tilde{h} \left\| \chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha}.$$

That ends the proof of Fact VI.8. \square

We apply it to $\alpha = q/2$ and we apply Lemma II.28. Hence, we get for any $2 \leq q \leq \frac{2d}{d-2}$

$$\begin{aligned} \|\rho_{\tilde{\gamma}_2}\|_{L^{q/2}(\mathbb{R}^d)} &\lesssim \tilde{h}^{2-2d(\frac{1}{2}-\frac{1}{q})} \left(\|\chi_A \tilde{\gamma} \chi_A\|_{\mathfrak{S}^{q/2}} + \frac{1}{\tilde{h}^2} \left\| \tilde{P}^* \chi_A \tilde{\gamma} \chi_A \tilde{P} \right\|_{\mathfrak{S}^{q/2}} \right) \\ &\lesssim \tilde{h}^{2-2d(\frac{1}{2}-\frac{1}{q})} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{q/2}}. \end{aligned}$$

Estimate of the crossed terms $\tilde{\gamma}_{3,1}$ and $\tilde{\gamma}_{3,2}$. We deduce the estimates on the crossed terms $\tilde{\gamma}_{3,1}$ and $\tilde{\gamma}_{3,2}$ from those of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. For example for $\tilde{\gamma}_{3,1}$, noting $C := \sqrt{\tilde{\gamma}} \chi_A (1 - \tilde{\psi}^w)$ and $B := \sqrt{\tilde{\gamma}} \chi_A \tilde{\psi}^w$

$$\begin{aligned} &|\mathrm{Tr}_{L^2}(W \tilde{\gamma}_{3,1} W)| \\ &= \left| \mathrm{Tr}_{L^2} \left(W (1 - \tilde{\psi}^w) \chi_A \tilde{\gamma} \chi_A \tilde{\psi}^w W \right) \right| = |\mathrm{Tr}_{L^2}(W C^* B W)| \\ &\leq |\mathrm{Tr}_{L^2}(W C^* C W)|^{1/2} |\mathrm{Tr}_{L^2}(W B^* B W)|^{1/2} \\ &\leq \left(\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2 \|\rho_{C^* C}\|_{L^{q/2}(\mathbb{R}^d)} \right)^{1/2} \left(\|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2 \|\rho_{B^* B}\|_{L^{q/2}(\mathbb{R}^d)} \right)^{1/2} \\ &\leq \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2 \|\rho_{(1-\tilde{\psi}^w) \chi_A \tilde{\gamma} \chi_A (1-\tilde{\psi}^w)}\|_{L^{q/2}(\mathbb{R}^d)}^{1/2} \|\rho_{\tilde{\psi}^w \chi_A \tilde{\gamma} \chi_A \tilde{\psi}^w}\|_{L^{q/2}(\mathbb{R}^d)}^{1/2} \\ &\lesssim \tilde{h}^{1-d(\frac{1}{2}-\frac{1}{q})-s_{\mathrm{Sogge}}(q,d)} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}}^2 \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\mathrm{Sogge}}(q,d)}}. \end{aligned}$$

For any $2 \leq q \leq \frac{2d}{d-2}$

$$\|\rho_{\tilde{\gamma}_3}\|_{L^{q/2}(\mathbb{R}^d)} \lesssim \tilde{h}^{1-d(\frac{1}{2}-\frac{1}{q})-s_{\mathrm{Sogge}}(q,d)} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\mathrm{Sogge}}(q,d)}}.$$

□

Step 1. The scaling. Let us deduce from Lemma VI.7 the same kind of result but on the boxes A_ε^k by a scaling argument. The following lemma controls the $L^{q/2}$ norm of the density on the boxes A_ε^k .

Lemma VI.9. *Let $d \geq 2$ and $2 \leq q \leq \frac{2d}{d-2}$. Then, there exists $C > 0$ such that for any $0 < h \leq h_0$, for any $\varepsilon \in [Mh^{2/3}, \delta/2]$ and for any $k \in \mathbb{Z}^{d-1}$ such that $|k|_\infty < \delta/\varepsilon - 1$, there exists $\chi_{\tilde{A}_\varepsilon^k} \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d, [0, 1])$ supported into \tilde{A}_ε^k , such that any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$*

$$\|\rho_\gamma\|_{L^{q/2}(A_\varepsilon^k)} \leq Ch^{-2s_{\mathrm{Sogge}}(q,d)} \varepsilon^{-2\mu(q,d)} \left(\left\| \chi_{\tilde{A}_\varepsilon^k} \gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{\mathfrak{S}^{\alpha_{\mathrm{Sogge}}(q,d)}} + \frac{\varepsilon}{h^2} \left\| \chi_{\tilde{A}_\varepsilon^k} P \gamma P^* \chi_{\tilde{A}_\varepsilon^k} \right\|_{\mathfrak{S}^{\alpha_{\mathrm{Sogge}}(q,d)}} \right),$$

where $\mu(q, d)$ is given by the formula

$$\mu(q, d) := d \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{3s_{\mathrm{Sogge}}(q, d)}{2}. \quad (\text{VI.7})$$

Proof of Lemma VI.9. Recall that h_0 and M were already fixed above. We will make additional constraints on them along this proof. Let $h \in (0, h_0]$, $\varepsilon \in [Mh^{2/3}, \delta/2]$ and $k \in \mathbb{Z}^{d-1}$ such that $|k|_\infty < \delta/\varepsilon - 1$. Let $\chi_{\tilde{A}} \in \mathcal{C}_c^\infty(\mathbb{R}^d, [0, 1])$ be a cut-off function equal to 1 on \tilde{A} and supported into $\tilde{\tilde{A}}$. Let $\chi_{\tilde{A}} \in \mathcal{C}_c^\infty(\mathbb{R}^d, [0, 1])$ be a cut-off function equal to 1 on $\mathrm{supp} \chi_{\tilde{A}}$ and supported into

\tilde{A} . Define for any $\tilde{x} \in \mathbb{R}^d$

$$B_\varepsilon(\tilde{x}) := \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \\ 0 & (a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k))_{2 \leq i, j \leq d} \end{array} \right), \quad V_\varepsilon(\tilde{x}) := -\tilde{x}_1 c(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k) \chi_{\tilde{A}}(\tilde{x}).$$

By our assumptions on $(a_{ij})_{2 \leq i, j \leq d}$ and c , $\{(B_\varepsilon, V_\varepsilon)\}_\varepsilon$ satisfies the assumptions of Lemma VI.7 with $\varepsilon_0 = \delta/2$ (notice that for our choice of ε and k , $|(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k)| < \delta$ for all $\tilde{x} \in \tilde{A}$). Hence, let $C > 0$ and $\tilde{h}_0 > 0$ such that for all $0 < \tilde{h} \leq \tilde{h}_0$ and all bounded self-adjoint non-negative operator $\tilde{\gamma}$ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\tilde{\gamma}}(\tilde{x})\|_{L^{q/2}(A)} \leq C \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}},$$

where $\tilde{P} = p_\varepsilon^R(\tilde{x}, \tilde{h}D_{\tilde{x}})$ and $p_\varepsilon(\tilde{x}, \tilde{\xi}) = \langle \tilde{\xi}, B_\varepsilon(\tilde{x}) \tilde{\xi} \rangle + V_\varepsilon(\tilde{x})$. Moreover, we have the following fact (which proof is given below).

Fact VI.10. *Let $\alpha \geq 1$. Then, there exists $C > 0$ such for all $\tilde{h} \in (0, \tilde{h}_0]$, for all $\varepsilon \in [0, \varepsilon_0]$ and for all bounded self-adjoint non-negative operator $\tilde{\gamma}$*

$$\left\| (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \chi_A \tilde{\gamma} \chi_A (1 + \tilde{P}^* \tilde{P} / \tilde{h}^2)^{1/2} \right\|_{\mathfrak{S}^\alpha} \leq C \left(\|\chi_{\tilde{A}} \tilde{\gamma} \chi_{\tilde{A}}\|_{\mathfrak{S}^\alpha} + \frac{1}{\tilde{h}^2} \|\chi_{\tilde{A}} \tilde{P} \tilde{\gamma} \tilde{P}^* \chi_{\tilde{A}}\|_{\mathfrak{S}^\alpha} \right).$$

We deduce that for all bounded self-adjoint non-negative operator $\tilde{\gamma}$ on $L^2(\mathbb{R}^d)$, we have

$$\|\rho_{\tilde{\gamma}}(\tilde{x})\|_{L^{q/2}(A)} \leq C \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \left(\|\chi_{\tilde{A}} \tilde{\gamma} \chi_{\tilde{A}}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} + \frac{1}{\tilde{h}^2} \|\chi_{\tilde{A}} \tilde{P} \tilde{\gamma} \tilde{P}^* \chi_{\tilde{A}}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} \right).$$

Since $\chi_{\tilde{A}} \chi_{\tilde{A}} = \chi_{\tilde{A}}$, the same bound holds when there is no factor $\chi_{\tilde{A}}$ in V_ε . We still denote by \tilde{P} the resulting operator. Now, let γ be a bounded self-adjoint non-negative operator on $L^2(\mathbb{R}^d)$. We apply the above bound to $\tilde{\gamma} = (U_\varepsilon^k)^* \gamma U_\varepsilon^k$, where U_ε^k is the unitary transformation defined by

$$U_\varepsilon^k : \quad L^2(\mathbb{R}^d) \quad \rightarrow \quad L^2(\mathbb{R}^d) \\ f \quad \quad \quad \mapsto \left(x \mapsto \varepsilon^{-d/2} f \left(\frac{x_1}{\varepsilon}, \frac{x' - \varepsilon k}{\varepsilon} \right) \right).$$

Since we have

$$\|\rho_{\tilde{\gamma}}\|_{L^{q/2}(A)} = \varepsilon^{2d(1/2-1/q)} \|\rho_\gamma\|_{L^{q/2}(A_\varepsilon^k)},$$

and $U_\varepsilon^k \tilde{P} (U_\varepsilon^k)^* = P/\varepsilon$, we deduce that

$$\begin{aligned} & \|\rho_\gamma\|_{L^{q/2}(A_\varepsilon^k)} \\ & \leq C \tilde{h}^{-2s_{\text{Sogge}}(q,d)} \varepsilon^{-2d(1/2-1/q)} \left(\|\chi_{\tilde{A}_\varepsilon^k} \gamma \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} + \frac{1}{(\varepsilon \tilde{h})^2} \|\chi_{\tilde{A}_\varepsilon^k} P \gamma P^* \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} \right), \end{aligned}$$

with $\chi_{\tilde{A}_\varepsilon^k} = U_\varepsilon^k \chi_{\tilde{A}} (U_\varepsilon^k)^*$, i.e. $\chi_{\tilde{A}_\varepsilon^k}(x) = \chi_{\tilde{A}} \left(\frac{x_1}{\varepsilon}, \frac{x' - \varepsilon k}{\varepsilon} \right)$. Now assume that $M \geq \tilde{h}_0^{-2/3}$ and let $h \in (0, h_0]$ (where we recall that $h_0 > 0$ was chosen such that $M h_0^{2/3} < \delta/2$). We apply the above bound to $\tilde{h} = h/\varepsilon^{3/2}$, which indeed satisfies $\tilde{h} \leq \tilde{h}_0$ since $\varepsilon \geq M h^{2/3}$ implies $\tilde{h} \leq M^{-3/2} \leq \tilde{h}_0$. Finally, we obtain

$$\begin{aligned} & \|\rho_\gamma\|_{L^{q/2}(A_\varepsilon^k)} \\ & \leq C (h/\varepsilon^{3/2})^{-2s_{\text{Sogge}}(q,d)} \varepsilon^{-2d(1/2-1/q)} \left(\|\chi_{\tilde{A}_\varepsilon^k} \gamma \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} + \frac{\varepsilon}{h^2} \|\chi_{\tilde{A}_\varepsilon^k} P \gamma P^* \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} \right) \\ & = C h^{-2s_{\text{Sogge}}(q,d)} \varepsilon^{-2\mu(q,d)} \left(\|\chi_{\tilde{A}_\varepsilon^k} \gamma \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} + \frac{\varepsilon}{h^2} \|\chi_{\tilde{A}_\varepsilon^k} P \gamma P^* \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}} \right). \end{aligned}$$

□

We now give the missing proof of Fact VI.10.

Proof of Fact VI.10.

• It is essentially enough to understand why this inequality is true for the one-body case. The bound to prove is

$$\|\tilde{P}\chi_A\tilde{u}\|_{L^2(\mathbb{R}^d)} \lesssim \|\chi_{\tilde{A}}\tilde{P}\tilde{u}\|_{L^2(\mathbb{R}^d)} + \tilde{h}\|\chi_{\tilde{A}}\tilde{u}\|_{L^2(\mathbb{R}^d)}.$$

We already notice that

$$\begin{aligned} \tilde{P}\chi_A &= \chi_A\tilde{P} - \tilde{h}^2(\partial_{\tilde{x}_1}^2\chi_A) - 2\tilde{h}(\partial_{\tilde{x}_1}\chi_A)(\tilde{h}\partial_{\tilde{x}_1}) \\ &\quad - \tilde{h}^2 \sum_{2 \leq i, j \leq d} (\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\chi_A)a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k) - \tilde{h} \left\langle \nabla\chi_A, (a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k))_{i,j}\tilde{h}\nabla \right\rangle. \end{aligned}$$

On the one hand, since $a_{ij} \in L^\infty(\mathbb{R}^d)$

$$\begin{aligned} &\|\tilde{h}^2(\partial_{\tilde{x}_1}^2\chi_A)\tilde{u}\|_{L^2(\mathbb{R}^d)} + \|\tilde{h}^2 \sum_{2 \leq i, j \leq d} (\partial_{\tilde{x}_i}\partial_{\tilde{x}_j}\chi_A)a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k)\tilde{u}\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \tilde{h}^2\|\chi_{\tilde{A}}\tilde{u}\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

On the other hand, since \tilde{P} is elliptic in the sense of [22, Lem. 2.6], we have

$$\sum_{|\alpha|=1} \|(\tilde{h}D)^\alpha\tilde{u}\|_{L^2(\tilde{A})} \lesssim \|\tilde{u}\|_{L^2(\tilde{A})} + \|\tilde{P}\tilde{u}\|_{L^2(\tilde{A})},$$

(by using again that $a_{ij} \in L^\infty(\mathbb{R}^d)$) so that

$$\begin{aligned} &\|\tilde{h}(\partial_{\tilde{x}_1}\chi_A)(\tilde{h}\partial_{\tilde{x}_1})\|_{L^2(\mathbb{R}^d)}^2 + \|\tilde{h} \left\langle \nabla\chi_A, (a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k))_{i,j}(\tilde{h}\nabla\tilde{u}) \right\rangle\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim \tilde{h}^2\|\tilde{h}D_{\tilde{x}_1}\tilde{u}\|_{L^2(\tilde{A})}^2 + \tilde{h}^2\|(a_{ij}(\varepsilon\tilde{x}_1, \varepsilon\tilde{x}' + \varepsilon k))_{i,j}(\tilde{h}\nabla\tilde{u})\|_{L^2(\tilde{A})}^2 \\ &\lesssim \tilde{h}^2 \left(\|\tilde{u}\|_{L^2(\tilde{A})}^2 + \|\tilde{P}\tilde{u}\|_{L^2(\tilde{A})}^2 \right) \\ &\lesssim \tilde{h}^2 \left(\|\chi_{\tilde{A}}\tilde{u}\|_{L^2(\mathbb{R}^d)}^2 + \|\chi_{\tilde{A}}\tilde{P}\tilde{u}\|_{L^2(\mathbb{R}^d)}^2 \right). \end{aligned}$$

Hence,

$$\begin{aligned} \|\tilde{P}\chi_A\tilde{u}\|_{L^2(\mathbb{R}^d)} &\lesssim \|\chi_{\tilde{A}}\tilde{P}\tilde{u}\|_{L^2(\mathbb{R}^d)} + \tilde{h} \left(\|\chi_{\tilde{A}}\tilde{P}\tilde{u}\|_{L^2(\mathbb{R}^d)} + \|\chi_{\tilde{A}}\tilde{u}\|_{L^2(\mathbb{R}^d)} \right) \\ &\lesssim \|\chi_{\tilde{A}}\tilde{P}\tilde{u}\|_{L^2(\mathbb{R}^d)} + \tilde{h}\|\chi_{\tilde{A}}\tilde{u}\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

• Let us now extend the result to density matrices. We have shown the inequality of operators

$$\left(\tilde{P}\chi_A\right)^* \left(\tilde{P}\chi_A\right) \lesssim \tilde{h}^2\chi_{\tilde{A}}^2 + \left(\chi_{\tilde{A}}\tilde{P}\right)^* \chi_{\tilde{A}}\tilde{P}.$$

In other words

$$\chi_A\tilde{P}^*\tilde{P}\chi_A \lesssim \tilde{h}^2\chi_A^2 + \tilde{P}^*\chi_A^2\tilde{P}.$$

Then for all $\alpha \geq 1$ and all bounded self-adjoint non-negative operator $\tilde{\gamma}$ on $L^2(\mathbb{R}^d)$

$$\left\| \sqrt{\tilde{\gamma}}\chi_A(1 + \tilde{P}^*\tilde{P}/\tilde{h}^2)\chi_A\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} \lesssim \left\| \sqrt{\tilde{\gamma}}\chi_A^2\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha} + \frac{1}{\tilde{h}^2} \left\| \sqrt{\tilde{\gamma}}\tilde{P}^*\chi_A^2\tilde{P}\sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^\alpha}.$$

This concludes the proof of Fact VI.10. \square

Step 2. The summation of the boxes. With the results on boxes of size ε , we will now have the following result.

Lemma VI.11. *Let $d \geq 2$ and $2 \leq q \leq \frac{2d}{d-2}$. Then, there exists $C > 0$ such that for any $\varepsilon \in [Mh^{2/3}, \delta/2]$, for any $h \in (0, h_0]$, and for any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$ we have*

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(A_\varepsilon)} \leq Ch^{-2s_{\text{Sogge}}(q,d)} \varepsilon^{1/2-2\mu(q,d)} \left\| \left(1 + \frac{1}{h^2} P^* P\right)^{1/2} \gamma \left(1 + \frac{1}{h^2} P^* P\right)^{1/2} \right\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)},$$

where $\mu(q, d)$ is given by (VI.7).

Proof of Lemma VI.11. Since $\pi_x \text{supp } \chi \subset B_{\delta/2}$, we have for all $2 \leq q \leq \infty$

$$\|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(B_\delta^\varepsilon)} = \mathcal{O}(h^\infty) \|\gamma\|_{\mathfrak{S}^{q/2}}.$$

By Lemma VI.9, we have for non-negative operators γ

$$\begin{aligned} \|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(A_\varepsilon \cap B_\delta)} &= \left(\sum_{|k|_{l^\infty} \leq \delta/\varepsilon - 1} \|\rho_{\chi^w \gamma \chi^w}\|_{L^{q/2}(A_\varepsilon^k)} \right)^{2/q} \\ &\lesssim h^{-2s_{\text{Sogge}}(q,d)} \varepsilon^{-2\mu(q,d)} \times \\ &\quad \times \underbrace{\left(\sum_{k \in \mathbb{Z}^{d-1}} \|\chi_{\tilde{A}_\varepsilon^k} \chi^w \gamma \chi^w \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)} \right)^{2/q}}_{= \|\chi_{\tilde{A}_\varepsilon^k} \chi^w \gamma \chi^w \chi_{\tilde{A}_\varepsilon^k}\|_{l_k^{q/2} \mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}} + \frac{\varepsilon}{h^2} \underbrace{\left(\sum_{k \in \mathbb{Z}^{d-1}} \|\chi_{\tilde{A}_\varepsilon^k} P \chi^w \gamma \chi^w P^* \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)} \right)^{2/q}}_{= \|\chi_{\tilde{A}_\varepsilon^k} P \chi^w \gamma \chi^w P^* \chi_{\tilde{A}_\varepsilon^k}\|_{l_k^{q/2} \mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}}. \end{aligned}$$

▷ The first step consists in showing that for $q \geq 2$, for any bounded self-adjoint operator γ

$$\begin{aligned} &\|\chi_{\tilde{A}_\varepsilon^k} \chi^w \gamma \chi^w \chi_{\tilde{A}_\varepsilon^k}\|_{l_k^{q/2} \mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)} + \frac{\varepsilon}{h^2} \|\chi_{\tilde{A}_\varepsilon^k} P \chi^w \gamma \chi^w P^* \chi_{\tilde{A}_\varepsilon^k}\|_{l_k^{q/2} \mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)} \\ &\lesssim \|\chi_{2\varepsilon} \chi^w \gamma \chi^w \chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)} + \frac{\varepsilon}{h^2} \|\chi_{2\varepsilon} P \chi^w \gamma \chi^w P^* \chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}. \end{aligned}$$

In fact, we prove more precisely that for all non-negative compact operator Γ

$$\sum_{k \in \mathbb{Z}^{d-1}} \|\chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}^{q/2} \leq C \|\chi_{2\varepsilon} \Gamma \chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}^{q/2}.$$

Recall that $\text{supp } \chi_{\tilde{A}_\varepsilon^k} \subset \tilde{A}_\varepsilon^k \times \mathbb{R}^d$, $\chi_{2\varepsilon}(x_1) = 1$ when $x_1 \leq 2\varepsilon$. Hence, we have

$$\chi_{\tilde{A}_\varepsilon^k} = \chi_{\tilde{A}_\varepsilon^k} \chi_{2\varepsilon}$$

and thus we only have to show that

$$\sum_{k \in \mathbb{Z}^{d-1}} \|\chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}^{q/2} \leq C \|\Gamma\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}}(q,d)}^{q/2}.$$

1) Let us check first that

$$\|\chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k}\|_{l_k^1 \mathfrak{S}^1} \lesssim \|\Gamma\|_{\mathfrak{S}^1}.$$

Since $\chi_{\tilde{A}_\varepsilon^k} \in \mathcal{C}^\infty(\mathbb{R}^d, [0, 1])$ and $\text{supp } \chi_{\tilde{A}_\varepsilon^k} \subset \tilde{A}_\varepsilon^k$, there exists $C > 0$ such that for all $\varepsilon > 0$,

$$\sum_{k \in \mathbb{Z}^{d-1}} \chi_{\tilde{A}_\varepsilon^k}^2 \leq C.$$

We recall that:

- if we have two non-negative operators A and B such that $A \leq B$ then $\sqrt{A} \leq \sqrt{B}$,
- for all trace-class operators A and B , $A \leq B \implies \text{Tr}(A) \leq \text{Tr}(B)$.

So if Γ is a non-negative operator

$$\begin{aligned} \left\| \chi_{A_\varepsilon^k} \Gamma \chi_{A_\varepsilon^k} \right\|_{l_k^1 \mathfrak{S}^1} &= \sum_{k \in \mathbb{Z}^{d-1}} \text{Tr}_{L^2} \left(\chi_{A_\varepsilon^k} \Gamma \chi_{A_\varepsilon^k} \right) \\ &= \sum_{k \in \mathbb{Z}^{d-1}} \text{Tr}_{L^2} \left(\sqrt{\Gamma} \chi_{\tilde{A}_\varepsilon^k}^2 \sqrt{\Gamma} \right) \\ &\leq \text{Tr}_{L^2} \left(\sqrt{\Gamma} \sum_{k \in \mathbb{Z}^{d-1}} \chi_{\tilde{A}_\varepsilon^k}^2 \sqrt{\Gamma} \right) \\ &\leq C \text{Tr}_{L^2} \Gamma = C \|\Gamma\|_{\mathfrak{S}^1}. \end{aligned}$$

We can pass to a general trace-class Γ by decomposing $\Gamma = \Gamma_+ - \Gamma_-$ with $\Gamma_+, \Gamma_- \geq 0$ and we obtain

$$\begin{aligned} \left\| \chi_{A_\varepsilon^k} \Gamma \chi_{A_\varepsilon^k} \right\|_{l_k^1 \mathfrak{S}^1} &\leq \left\| \chi_{A_\varepsilon^k} \Gamma_+ \chi_{A_\varepsilon^k} \right\|_{l_k^1 \mathfrak{S}^1} + \left\| \chi_{A_\varepsilon^k} \Gamma_- \chi_{A_\varepsilon^k} \right\|_{l_k^1 \mathfrak{S}^1} \\ &\lesssim \|\Gamma_+\|_{\mathfrak{S}^1} + \|\Gamma_-\|_{\mathfrak{S}^1} \\ &\lesssim \|\Gamma\|_{\mathfrak{S}^1}. \end{aligned}$$

2) Notice that we always have

$$\left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^\infty \mathfrak{S}^\infty} \lesssim \|\Gamma\|_{L^2 \rightarrow L^2}.$$

3) The interpolation of

$$\begin{cases} \left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^1 \mathfrak{S}^1} &\lesssim \|\Gamma\|_{\mathfrak{S}^1} \\ \left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^\infty \mathfrak{S}^\infty} &\lesssim \|\Gamma\|_{L^2 \rightarrow L^2} \end{cases}$$

gives

$$\forall 1 \leq \alpha \leq \infty \quad \left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^\alpha \mathfrak{S}^\alpha} \lesssim \|\Gamma\|_{\mathfrak{S}^\alpha}.$$

Hence, for any $2 \leq q \leq \infty$ and any $1 \leq \alpha \leq q/2$

$$\begin{aligned} \left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^{q/2} \mathfrak{S}^\alpha} &\leq \left\| \chi_{\tilde{A}_\varepsilon^k} \Gamma \chi_{\tilde{A}_\varepsilon^k} \right\|_{l_k^\alpha \mathfrak{S}^\alpha} \\ &\lesssim \|\Gamma\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

Now for any $2 \leq q \leq \infty$, we have $1 \leq \alpha_{\text{Sogge}}(q, d) \leq q/2$, hence for any $2 \leq q \leq \frac{2d}{d-2}$ we have

$$\begin{aligned} &\left\| \rho_{\chi^w \gamma \chi^w} \right\|_{L^{q/2}(A_\varepsilon \cap B_\delta)} \\ &\lesssim h^{-2s_{\text{Sogge}}(q, d)} \varepsilon^{-2\mu(q, d)} \left(\|\chi_{2\varepsilon} \chi^w \gamma \chi^w \chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q, d)}} + \frac{\varepsilon}{h^2} \|\chi_{2\varepsilon} P \chi^w \gamma \chi^w P^* \chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q, d)}} \right). \end{aligned}$$

▷ By the Hölder inequality and Corollary VI.4, we have for any $\alpha \geq 1$,

$$\|\chi_{2\varepsilon}\chi^w\gamma\chi^w\chi_{2\varepsilon}\|_{\mathfrak{S}^\alpha} \lesssim \varepsilon^{1/2} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}.$$

Besides for any $\alpha \geq 1$,

$$\begin{aligned} \|\chi_{2\varepsilon}P\chi^w\gamma\chi^wP^*\chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}} &\lesssim \|P\gamma P^*\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}} \\ &\lesssim \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^\alpha}. \end{aligned}$$

Thus, with the triangle inequality for any $2 \leq q \leq \frac{2d}{d-2}$

$$\begin{aligned} &\|\rho\chi^w\gamma\chi^w\|_{L^{q/2}(A_\varepsilon \cap B_\delta)} \\ &\lesssim h^{-2s_{\text{Sogge}(q,d)}} \varepsilon^{-2\mu(q,d)} \left(\|\chi_{2\varepsilon}\chi^w\gamma\chi^w\chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}} + \frac{\varepsilon}{h^2} \|\chi_{2\varepsilon}P\chi^w\gamma\chi^wP^*\chi_{2\varepsilon}\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}} \right) \\ &\lesssim h^{-2s_{\text{Sogge}(q,d)}} \varepsilon^{1/2-2\mu(q,d)} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}}. \end{aligned}$$

That finishes the proof of Lemma VI.11. \square

Step 3. The final summation. Finally, by Lemma VI.11 we are in position to obtain the estimates

$$\begin{aligned} \|\rho\chi^w\gamma\chi^w\|_{L^{q/2}(\Omega_{\delta/2} \setminus \Omega_{Mh^{2/3}})} &\leq \|\rho\chi^w\gamma\chi^w\|_{L^{q/2}(\cup_k A_{2^k h^{2/3}})} \\ &\leq \left(\sum_{k=\lfloor \log_2(M) \rfloor}^{K(h)} \|\rho\chi^w\gamma\chi^w\|_{L^{q/2}(A_{2^k h^{2/3}})}^{q/2} \right)^{2/q} \\ &\lesssim \left(\sum_{k=\lfloor \log_2(M) \rfloor}^{K(h)} C_{h,2^k h^{2/3}}^q \right)^{2/q} \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}}, \end{aligned}$$

where $C_{h,\varepsilon} = h^{-s_{\text{Sogge}(q,d)}} \varepsilon^{1/4-\mu(q,d)}$. Hence

$$\|\rho\gamma\|_{L^{q/2}(\Omega_{\delta/2} \setminus \Omega_{Mh^{2/3}})} \lesssim C_h^2 \left\| (1 + P^*P/h^2)^{1/2} \gamma (1 + P^*P/h^2)^{1/2} \right\|_{\mathfrak{S}^{\alpha\text{Sogge}(q,d)}}$$

where

$$\begin{aligned} C_h &= \left(\sum_{k=\lfloor \log_2(M) \rfloor}^{K(h)} C_{h,2^k h^{2/3}}^q \right)^{1/q} \\ &= \left(\sum_{k=\lfloor \log_2(M) \rfloor}^{K(h)} h^{-q(s_{\text{Sogge}(q,d)}+2/3(\mu(q,d)-1/4))} 2^{qk(1/4-\mu(q,d))} \right)^{1/q} \\ &\lesssim \begin{cases} h^{-\frac{(d-1)}{2}(\frac{1}{2}-\frac{1}{q})} & \text{if } 2 \leq q < \frac{2(d+3)}{d+1}, \\ \log^{\frac{d+1}{2(d+3)}}(1/h) h^{-\frac{d-1}{2(d+3)}} & \text{if } q = \frac{2(d+3)}{d+1}, \\ h^{\frac{1}{6}-\frac{2d}{3}(\frac{1}{2}-\frac{1}{q})} & \text{if } \frac{2(d+3)}{d+1} < q \leq \frac{2d}{d-2}. \end{cases} \end{aligned}$$

That ends the proof of Lemma VI.6.

VII APPLICATIONS TO SPECTRAL CLUSTERS

In this section, we apply the results of the preceding sections on microlocalized quasimodes to spectral clusters. As we will see, this allows to get rid of the microlocalization and leads to global estimates.

VII.1 Notation

- Let $\varepsilon \in (0, 1)$ and $h_0 \in (0, \varepsilon/2)$.
- Let $E \in \mathbb{R}$ and $h \in (0, h_0]$.
- Let $I_{h,E} := [E - h, E + h]$.
- Let $t_{\text{gene}}, s_{\text{gene}} \geq 0$ and $\alpha_{\text{gene}} \geq 1$ be given by the formulas in the statement of Theorem IV.2.
- Let $s_{\text{Sogge}} \geq 0$ and $\alpha_{\text{Sogge}} \geq 1$ be given by the formulas in the statement of Theorem V.2.
- Let $t_{\text{TP}}, s_{\text{TP}} \geq 0$ and $\alpha_{\text{TP}} \geq 1$ be given by the formulas in the statement of Theorem VI.2.

VII.2 Statement of the results

Let us add an additional assumption on the potential V , that will imply that the operator $-h^2\Delta + V$ has a compact resolvent. Moreover, it also ensures that V is bounded from below.

Definition VII.1 (Polynomial growth). *A potential $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ has a polynomial growth if it satisfies Definition (II.3) and if there exist $k \in \mathbb{N}^*$ and $R > 0$ such that*

$$\forall x \in \mathbb{R}^d, \forall |x| \geq R, \quad V(x) \geq c \langle x \rangle^k \quad (\text{VII.1})$$

Theorem VII.2 (Spectral cluster upper bounds). *(i) Let $d \geq 1$. Let $p(x, \xi) = |\xi|^2 + V(x)$ with $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with a polynomial growth (Definition VII.1). For $h > 0$ and $E \in \mathbb{R}$, let us define $P := p^w(x, hD)$ and the spectral projector Π_h by*

$$\Pi_h := \mathbf{1}(P \in I_{h,E}).$$

Let $E \in \mathbb{R}$. Then, there exist $C > 0$ and $h_0 > 0$ such that, for any $0 < h \leq h_0$, any $2 \leq q \leq \infty$ and any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(\mathbb{R}^d)} \leq C \log(1/h)^{2t_{\text{gene}}(q,d)} h^{-2s_{\text{gene}}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{gene}}(q,d)}(L^2(\mathbb{R}^d))}, \quad (\text{VII.2})$$

(ii) Let $d \geq 2$. There exist $C > 0$ and $h_0 > 0$ such that, for any $0 < h \leq h_0$, any $2 \leq q \leq \infty$ and any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(\{x \in \mathbb{R}^d : |V(x) - E| > \varepsilon\})} \leq C h^{-2s_{\text{Sogge}}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}}, \quad (\text{VII.3})$$

(iii) Let $d \geq 2$. Under the additional assumption that

$$\forall x \in \mathbb{R}^d, \quad V(x) = E \implies \nabla_x V(x) \neq 0, \quad (\text{VII.4})$$

there exist $C > 0$ and $h_0 > 0$ such that, for any $0 < h \leq h_0$, any $2 \leq q \leq \infty$ and any bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(\{x \in \mathbb{R}^d : |V(x) - E| \leq \varepsilon\})} \leq C \log(1/h)^{2t_{\text{TP}}(q,d)} h^{-2s_{\text{TP}}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{TP}}(q,d)}}. \quad (\text{VII.5})$$

Remark 27. The constant C appearing in the above theorem can be chosen to be uniform in the energy level E when it varies in a compact set.

Remark 28. One can split the $L^{q/2}$ -norm of $\rho_{\Pi_h \gamma \Pi_h}$ on $\{x \in \mathbb{R}^d : |V(x) - E| > \varepsilon\}$ into two parts: on the classically allowed region $\{x \in \mathbb{R}^d : V(x) < E - \varepsilon\}$ and on the classically forbidden region $\{x \in \mathbb{R}^d : V(x) > E + \varepsilon\}$. When $d \geq 2$, there exist $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$, any $2 \leq q \leq \infty$ and bounded self-adjoint non-negative operator γ on $L^2(\mathbb{R}^d)$,

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(\{x \in \mathbb{R}^d : V(x) - E < -\varepsilon\})} \leq Ch^{-2s_{\text{Sogge}}(q,d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q,d)}(L^2(\mathbb{R}^d))}. \quad (\text{VII.6})$$

We will see below that, as one can expect, for any $d \geq 1$, there exists $C, c > 0$ such that for any $h \in (0, h_0]$, any $2 \leq q \leq \infty$ and any bounded operator γ

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(\{x \in \mathbb{R}^d : V(x) - E > \varepsilon\})} = \mathcal{O}(h^\infty)(e^{-c/h})^{\frac{1}{q}} \|\gamma\|_{L^2 \rightarrow L^2}. \quad (\text{VII.7})$$

We do not know if one can improve the $\mathcal{O}(h^\infty)(e^{-c/h})^{1/q}$ to $\mathcal{O}(e^{-c/h})$.

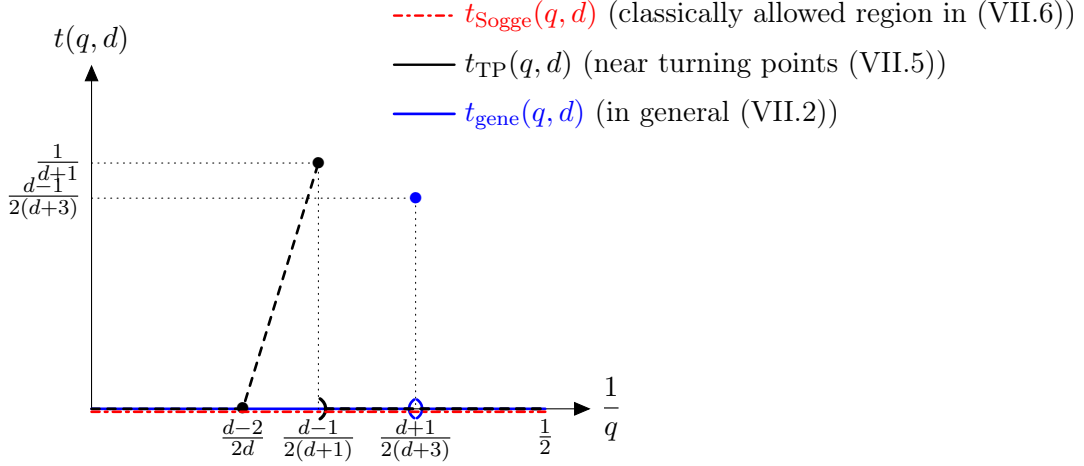


Figure 10: Concentration logarithm exponent $t(q, d)$ when $d \geq 3$.

VII.3 Proof of Theorem VII.2

Microlocalization of γ . We first explain why we only need to consider the microlocalized density matrix $\chi^w \Pi_h \gamma \Pi_h \chi^w$ for some $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ instead of the full one $\Pi_h \gamma \Pi_h$.

Let $f \in \mathcal{C}_c^\infty(\mathbb{R}, [0, 1])$ such that $f = 1$ on $I_{h_0, E}$ and $\text{supp } f \subset I_{\varepsilon/2, E}$. If we assume that $h \in (0, h_0]$, we thus have

$$\Pi_h = f(P)\Pi_h.$$

By functional calculus Theorem II.12, $f(P)$ can be written as the Weyl quantization of a symbol $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, for all $N \in \mathbb{N}$, there exists $\tilde{\chi}_N \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $r_N \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, such that $\text{supp } \tilde{\chi}_N \subset \text{supp } f \circ p$ (note that $\text{supp } f \circ p$ is compact since $p(x, \xi) \rightarrow \infty$ when $|(x, \xi)| \rightarrow \infty$) and

$$\tilde{\chi}(x, \xi) = \tilde{\chi}_N(x, \xi) + h^N r_N(x, \xi).$$

Let us write the decomposition

$$\Pi_h = \tilde{\chi}_N^w \Pi_h + h^N r_N^w \Pi_h. \quad (\text{VII.8})$$

On the one hand, by the fact that $r_N^w = \mathcal{O}_{\mathcal{S}' \rightarrow \mathcal{S}}(1)$ and Kato-Seiler-Simon applied to $k \in \mathbb{N}$ when $q > 2$ (resp. just an estimation of the operator norm of $(1 - h^2 \Delta)^{-k/2}$ and $W = 1$ when $q = 2$) such that $k(q/2)' > d$, for any $W \in L^{2(q/2)'(\mathbb{R}^d)}$

$$\begin{aligned} & \left\| h^N W r_N^w \Pi_h \sqrt{\gamma} \right\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d))} \\ & \leq h^N \left\| W (1 - h^2 \Delta)^{-k/2} \right\|_{\mathfrak{S}^{2(q/2)'(L^2(\mathbb{R}^d))}} \left\| (1 - h^2 \Delta)^{k/2} r_N^w \Pi_h \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \left\| \sqrt{\gamma} \right\|_{\mathfrak{S}^q(L^2(\mathbb{R}^d))} \\ & \lesssim h^{N-d/2} \|W\|_{L^{2(q/2)'(\mathbb{R}^d)}} \|\gamma\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}^d))}^{1/2}. \end{aligned}$$

By duality and Mercer theorem, we deduce that for all $N \in \mathbb{N}$

$$\left\| \rho_{(1 - \tilde{\chi}_N^w) \Pi_h \gamma \Pi_h (1 - \tilde{\chi}_N^w)} \right\|_{L^{q/2}(\mathbb{R}^d)} \leq C h^{2N-d} \|\gamma\|_{\mathfrak{S}^{q/2}(L^2(\mathbb{R}^d))},$$

hence, it remains to estimate $\rho_{\tilde{\chi}_N^w \Pi_h \gamma \Pi_h \tilde{\chi}_N^w}$ on various regions with perhaps additional assumptions on V .

(o) Microlocalized estimates in the classically forbidden region. Notice that since

$$\text{supp}(f \circ p) \cap (\{x \in \mathbb{R}^d : V(x) - E > \varepsilon\} \times \mathbb{R}^d) = \emptyset,$$

we have

$$\left\| \rho_{\tilde{\chi}_N^w \Pi_h \gamma \Pi_h \tilde{\chi}_N^w} \right\|_{L^{q/2}(x \in \mathbb{R}^d : V(x) - E > \varepsilon)} = \mathcal{O}(h^\infty) \|\gamma\|_{\mathfrak{S}^{q/2}}.$$

By Weyl's law (Proposition II.18), we have

$$\left\| \Pi_h \gamma \Pi_h \right\|_{\mathfrak{S}^{q/2}} \leq \|\gamma\|_{L^2 \rightarrow L^2} \left\| \Pi_h \right\|_{\mathfrak{S}^{q/2}} \leq C h^{-2d/q} \|\gamma\|_{L^2 \rightarrow L^2},$$

and thus, we deduce the bound

$$\left\| \rho_{\Pi_h \gamma \Pi_h} \right\|_{L^{q/2}(\{x \in \mathbb{R}^d : V(x) - E > \varepsilon\})} = \mathcal{O}(h^\infty) \|\gamma\|_{L^2 \rightarrow L^2}. \quad (\text{VII.9})$$

Let us write how we can improve this bound in forbidden region. Let us write $\Pi_h = \sum_{j \in J_h} |u_j\rangle \langle u_j|$ with $\{u_j\}_{1 \leq j \leq N_h}$ the orthonormal basis of eigenfunctions of P associated to eigenfunctions in $[E - h, E + h]$. Here, $N_h = \text{Tr}(\Pi_h) = \text{rank}(\Pi_h(L^2(\mathbb{R}^d)))$. By definition, there exists $\{\nu_j\}_{1 \leq j \leq N_h} \subset \mathbb{C}$ such that $\Pi_h \gamma \Pi_h = \sum_{1 \leq j \leq N_h} \nu_j |u_j\rangle \langle u_j|$. Let us introduce the Agmon distance

$$d_{\delta, E} : x \mapsto \delta \text{dist}(x, \{x \in \mathbb{R}^d : V(x) < E + \delta\}).$$

By the Agmon estimates ((c.f. for instance [8, Chap. 6] or [7, Prop. 2.3]), there exists $C = C(\delta, E) > 0$ such that for any normalized eigenfunction v of P less or equal than E_0

$$\|e^{d_{\delta, E}(x)/h} v\|_{L^2(\mathbb{R}^d)} \leq C.$$

Recall that $h_0 \in (0, \varepsilon/2)$. One has, by the triangle inequality, the Agmon estimates and the Weyl's law (Theorem II.18), there exists $C' = C'(h_0, E) > 0$

$$\begin{aligned} & \left\| \rho_{\Pi_h \gamma \Pi_h} \right\|_{L^1(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} \\ & \leq \sum_{j=1}^{N_h} |\nu_j| \|u_j\|_{L^2(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} \\ & \leq \sup_{1 \leq j \leq N_h} |\nu_j| \sum_{j=1}^{N_h} \|u_j\|_{L^2(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} \\ & \leq \left\| \Pi_h \gamma \Pi_h \right\|_{L^2 \rightarrow L^2} \|e^{-d_{h_0, E+h_0}/h}\|_{L^\infty(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} \sum_{j=1}^{N_h} \|e^{d_{h_0, E+h_0}/h} u_j\|_{L^2(\mathbb{R}^d)} \\ & \leq C' e^{-c/h} h^{-d} \|\gamma\|_{L^2 \rightarrow L^2}. \end{aligned}$$

Here $c = \min_{x \in \{V \leq E + \varepsilon\}} \text{dist}(x, \{V \leq E + h_0\}) = \text{dist}(\{V \leq E + \varepsilon\}, \{V \leq E + h_0\}) > 0$, given the choice of $h_0 > 0$. We interpolate then this estimate with the L^∞ estimate (VII.9)

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^\infty(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} = \mathcal{O}(h^\infty) \|\gamma\|_{L^2 \rightarrow L^2}.$$

One has for any $q \in [2, \infty]$, for any $N > 0$, there exists $C > 0$ such that

$$\|\rho_{\Pi_h \gamma \Pi_h}\|_{L^{q/2}(x \in \mathbb{R}^d : V(x) > E + \varepsilon)} \leq C(e^{-c/h})^{\frac{1}{q}} h^N \|\gamma\|_{L^2 \rightarrow L^2}.$$

That proves (VII.7).

(i) General microlocalized estimates. Note that all $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfy the non-degeneracy Assumption 2 for the symbol $p_E := p - E$. By Theorem IV.2, Assumption 1 is thus satisfied for $S = \mathbb{R}^d \times \mathbb{R}^d$, $q \in [2, \infty]$, $s = s_{\text{gene}}(q, d)$, $t = 0$ and $\alpha = \alpha_{\text{gene}}(q, d)$. We apply Theorem II.29 to these parameters $(q, s_{\text{gene}}(q, d), 0, \alpha_{\text{gene}}(q, d))$ and to $\chi = \tilde{\chi}_N$, that gives us

$$\begin{aligned} \|\rho_{\tilde{\chi}_N^w \Pi_h \gamma \Pi_h \tilde{\chi}_N^w}\|_{L^{q/2}(\mathbb{R}^d)} &\leq C \log(1/h)^{2t_{\text{gene}}(q, d)} h^{-2s_{\text{gene}}(q, d)} \times \\ &\quad \times \left(\|\Pi_h \gamma \Pi_h\|_{\mathfrak{S}^{\alpha_{\text{gene}}(q, d)}} + \frac{1}{h^2} \|(P - E) \Pi_h \gamma \Pi_h (P - E)\|_{\mathfrak{S}^{\alpha_{\text{gene}}(q, d)}} \right) \\ &\leq C' \log(1/h)^{2t_{\text{gene}}(q, d)} h^{-2s_{\text{gene}}(q, d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{gene}}(q, d)}}, \end{aligned}$$

which is exactly (VII.2).

(ii) Microlocalized estimates in the classically allowed region. Let $d \geq 2$. Let $S = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |V(x) - E| > \varepsilon\}$. Any $(x_0, \xi_0) \in S$ satisfies either the ellipticity condition $p_E(x_0, \xi_0) \neq 0$ or Assumption 3 for the symbol p_E . By Theorem III.1 and Theorem V.2, the set S thus satisfies Assumption 1 for all $q \in [2, \infty]$, $s = s_{\text{Sogge}}(q, d)$, $t = 0$ and $\alpha = \alpha_{\text{Sogge}}(q, d)$. We apply Theorem II.30 to these parameters, $\Omega = \{x \in \mathbb{R}^d : |V(x) - E| > \varepsilon\}$ and $\chi = \tilde{\chi}_N$. Then, there exist $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$ and $2 \leq q \leq \infty$

$$\|\rho_{\tilde{\chi}_N^w \Pi_h \gamma \Pi_h \tilde{\chi}_N^w}\|_{L^{q/2}(\{|V-E|>\varepsilon\})} \leq C h^{-2s_{\text{Sogge}}(q, d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{Sogge}}(q, d)}},$$

where we got rid of the operator $P - E$ in the Schatten norm by the same method as in the previous step. We thus get (VII.3).

(iii) Microlocalized estimates near the turning points. Let $d \geq 2$. Let $S = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |V(x) - E| \leq \varepsilon\}$. Any $(x_0, \xi_0) \in S$ satisfies either the ellipticity condition $p_E(x_0, \xi_0) \neq 0$, Assumption 3, or Assumption 19 for the symbol p_E . By Theorem III.1, Theorem V.2, and Theorem VI.2, the set S thus satisfies Assumption 1 for all $q \in [2, \infty]$, $s = s_{\text{TP}}(q, d)$, $t = t_{\text{TP}}(q, d)$ and $\alpha = \alpha_{\text{TP}}(q, d)$. We apply Theorem II.30 to these parameters, $\Omega = \{x \in \mathbb{R}^d : |V(x) - E| \leq \varepsilon\}$ and $\chi = \tilde{\chi}_N$. As above, there exist $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$ and $2 \leq q \leq \infty$

$$\|\rho_{\tilde{\chi}_N^w \Pi_h \gamma \Pi_h \tilde{\chi}_N^w}\|_{L^{q/2}(\{|V-E|\leq\varepsilon\})} \leq C \log(1/h)^{2t_{\text{TP}}(q, d)} h^{-2s_{\text{TP}}(q, d)} \|\gamma\|_{\mathfrak{S}^{\alpha_{\text{TP}}(q, d)}},$$

showing (VII.3).

VIII OPTIMALITY

In this section, we discuss about the optimality of the concentration exponents $s(q, d)$ and $\alpha(q, d)$, appearing in the estimates of Theorem VII.2. We first explain why the exponents $s(q, d)$ are sharp for all values of q and d . To do so, we will see that it is enough to consider the one-body case $\text{rank } \gamma = 1$ for which only the exponent $s(q, d)$ appears. In many cases, this optimality was known in the literature, but we provide some details here. On the contrary, the optimality of the exponent $\alpha(q, d)$ is only proved in a restricted range of cases.

VIII.1 One-body optimality

Similarly as [36], that treats the optimality of the Laplacian in the manifolds and submanifolds, we write a survey of the one-body optimality in the Euclidean case. Actually, we will see that the understanding of the proof in Section VIII.1.2 allows us to prove the sharpness of the spectral bound in the bulk. There are reasons to hope that other examples can help to prove the optimality of the exponent α in the other cases.

Let V satisfying Definition VII.1, $h > 0$, $P = -h^2\Delta + V$ and $E \in \mathbb{R}$. In this section, we explain several concentration scenarii of functions u_h , which saturate the various one-body L^q bounds. All the saturating scenarii happen in the bulk $\{V - E < -\varepsilon\}$, meaning that they satisfy lower bounds of the type

$$\|u_h\|_{L^q(\{V-E < -\varepsilon\})} \geq Ch^{-s(q,d)} \left(\|u_h\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|(P-E)u_h\|_{L^2(\mathbb{R}^d)} \right). \quad (\text{VIII.1})$$

The different saturation scenarii according to the values of $s(q, d)$ are summarized in Figure 14 and Figures 11, 12 and 13. The optimality of the estimates in the turning point region $\{|V - E| \leq \varepsilon\}$ is much more delicate. In [21], the optimality in the case $V(x) = |x|^2$ is proved for the estimates in the dyadic regions of size $2^j h^{2/3}$ (see the proof of Theorem VI.2) for fixed j . The optimality in the full region $\{|V - E| \leq \varepsilon\}$ (that is, when we sum over j), as well as the optimality in the small neighborhood $\Omega_{Mh^{2/3}}$ of a turning point seem open to us in dimension $d \geq 2$ (see Remark 24 for the discussion in dimension 1). Indeed, for small regime $2 \leq q \leq 2(d+3)/(d-1)$, the L^q bounds are saturated by Gaussian beams (see Proposition VIII.4); for high regime $2d/(d-2) \leq q \leq \infty$, they are saturated by zonal-type quasimodes (see Proposition VIII.2). This is less obvious for the intermediate regime $2(d+3)/(d-1) \leq q \leq 2d/(d-2)$. By some raw calculations, the estimates cannot be saturated by simple direct products of d scalar gaussian grounds states $\varphi_0(t) = (2\pi\sqrt{h})^{-1/2} e^{-\frac{t^2}{2h}}$ and Hermite functions φ_n associated to the eigenvalue in a neighborhood of E (as in Figure 1). At best, one could expect that well-chosen linear combinaison of direct product scalar Hermite functions can saturate the bounds.

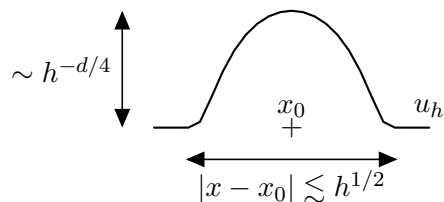


Figure 11: Concentration of a gaussian groundstate.

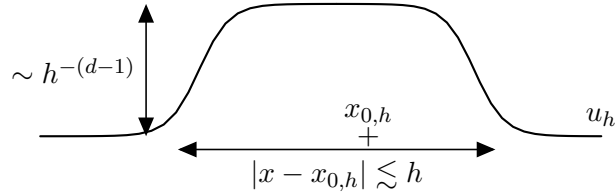


Figure 12: Concentration of a zonal-type quasimode.

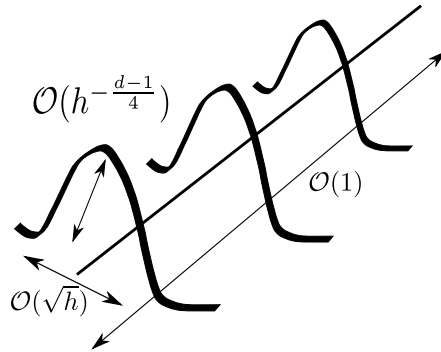


Figure 13: Gaussian beams: concentration around a curve.

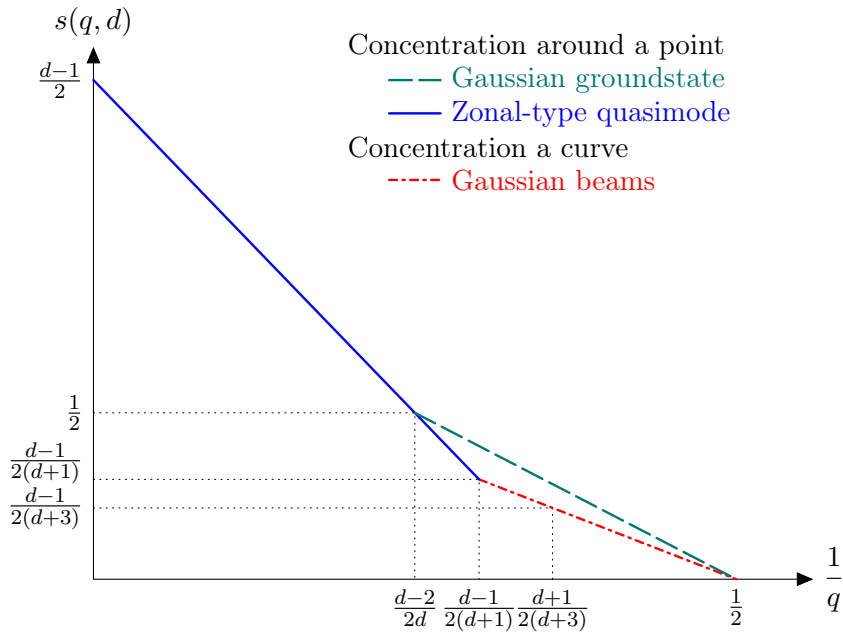


Figure 14: Saturation of $s(q, d)$ for $d \geq 3$.

VIII.1.1 Gaussian groundstate

We begin with explaining why the exponent $s_{\text{gene}}(q, d)$ is sharp (that is, it cannot be lowered). We do so, by exhibiting a family of functions that saturates the inequalities in which $s_{\text{gene}}(q, d)$ appears. We will see that in this case, the saturation scenario happens for functions concentrating

around a non-degenerate local minimum of the potential, exactly like the ground state of a harmonic oscillator. Such a construction is well-known (see for instance [16, Chap. 2], [8, Thm 4.23] and [22, Exemple 2]), but we recall here for completeness.

Proposition VIII.1. *Let $d \geq 1$ and $2 \leq q \leq \infty$. Let $p(x, \xi) := |\xi|^2 + V(x)$ where $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is as in Definition VII.1. For any $h > 0$, define $P = p^w(x, hD)$. Let $E \in \mathbb{R}$ such that there exists $x_0 \in \mathbb{R}^d$ such that*

$$V(x_0) = E, \quad \nabla_x V(x_0) = 0, \quad \partial_x^2 V(x_0) \text{ definite positive.}$$

Then, there exist $C > 0$ and $h_0 > 0$, such that the normalized groundstate of the operator $-h^2\Delta + \langle x - x_0, \sqrt{\partial_x^2 V(x_0)}(x - x_0) \rangle$

$$u_h(x) := (2\pi\sqrt{h})^{-d/2} \det(\partial_x^2 V(x_0))^{-1/4} e^{-\frac{\langle x - x_0, \sqrt{\partial_x^2 V(x_0)}(x - x_0) \rangle}{2h}}$$

associated to the eigenfunction $\lambda_h := h \operatorname{Tr}_{\mathbb{R}^d} \sqrt{\partial_x^2 V(x_0)}$ satisfies the bound for any $h \in (0, h_0]$

$$\|u_h\|_{L^2(\mathbb{R}^d)} = 1, \quad \|(P - E)u_h\|_{L^2(\mathbb{R}^d)} \leq Ch, \quad \|u_h\|_{L^q(\mathbb{R}^d)} \geq (1/C)h^{-\frac{d}{2}\left(\frac{1}{2} - \frac{1}{q}\right)}. \quad (\text{VIII.2})$$

The function u_h concentrates around x_0 at a scale \sqrt{h} with a height $\sim h^{-d/4}$ (c.f. Figure 11).

Remark 29. *The previous proposition gives the optimality of the exponent $s_{\text{gene}}(q, d)$*

- *when $d = 1$: for $2 \leq q \leq \infty$,*
- *when $d = 2$: for $2 \leq q < \infty$,*
- *when $d \geq 3$: for $2 \leq q \leq 2d/(d - 2)$.*

Indeed, recall that in these cases we have $s_{\text{gene}}(q, d) = \frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right)$, which is the exponent appearing in (VIII.2). More precisely, this proves that one cannot take a smaller $s(q, d)$ in Theorem IV.1. Notice that this theorem applies to microlocalized functions, which is not the case for our quasimode u_h above. However, since u_h is an eigenfunction of a Schrödinger operator with a quadratic potential, there exists $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $u_h = \chi^w u_h + \mathcal{O}(h^\infty)$ as in the proof of Theorem VII.2. Any point (x_0, ξ_0) in the support of χ satisfies either $p(x_0, \xi_0) \neq 0$ or Assumption 2. Hence, it shows that $\{u_h\}_{h \in (0, h_0]}$ actually saturates the bound of Theorem IV.1 but in its version of Theorem II.29.

Remark 30. *In the special case where V is quadratic (meaning that $\partial_x^2 V$ is constant), u_h (more precisely $\gamma_h = |u_h\rangle \langle u_h|$) saturates also the bound (VII.2) in Theorem VII.2 because in this case it satisfies $\Pi_h u_h = u_h$. Notice that in this one-body setting, no logarithm appears in this estimate as we recall in Section IV.*

Remark 31. *Note that the assumptions of Proposition VIII.1 hold when the potential $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is a Morse function, i.e. such that all its critical points has a non-degenerate Hessian. Actually, the trapping assumption of polynomial growth forces this critical points to be global minima. Besides, in this case the normalized ground states functions u_h of the resulting Schrödinger operator $P = -h^2\Delta + V$ admits a WKB expansion $h^{-d/4} a_h e^{-\varphi/h} := h^{-d/4} \sum_{j=1}^\infty h^j a_j e^{-\varphi/h}$ (see for instance [17, Sec. 3]), which is a quasimode of P for its lower eigenvalue λ_h . In particular, one has*

$$e^{\varphi/h} (P - \lambda_h) \left(h^{-d/4} a_h e^{-\varphi/h} \right) = \mathcal{O}_{L^2}(h^\infty).$$

and there exists a neighborhood K of the minimum x_0 of V such that one has uniformly in K

$$(P - \lambda_h) \left(h^{-d/4} a_h e^{-\varphi/h} \right) = \mathcal{O}(h^\infty) e^{-\varphi/h}.$$

Here, the phase function φ is smooth on \mathbb{R}^d and positive in K such that for any $x \in K$

$$\varphi(x) = \left\langle x - x_0, \sqrt{\partial_x^2 V(x_0)}(x - x_0) \right\rangle + \mathcal{O}(|x - x_0|^3).$$

One has also the functions $\{a_j\}_{j \in \mathbb{N}} \subset S$ and $a_0(x_0) = (2\pi)^{-d/2} \det(\partial_x^2 V(x_0))^{-1/4}$. Furthermore, $\lambda_h = E + \mathcal{O}(h)$. In particular, this ansatz $h^{-d/4} a_h e^{-\varphi/h}$ behaves like the normalized ground state of $-h^2 \Delta + \left\langle x - x_0, \sqrt{\partial_x^2 V(x_0)}(x - x_0) \right\rangle$ in Proposition VIII.1 and then satisfies (VIII.2). Then, by the triangle inequality, it is also the case for the eigenfunction u_h .

Proof of Proposition VIII.1. Let $2 \leq q \leq \infty$ and $h > 0$. Let φ_1 the normalized gaussian on $L^2(\mathbb{R}^d)$

$$\varphi_1(x) := (2\pi)^{-d/2} e^{-|x|^2/2}.$$

When we replace the potential $|x|^2$ by $V_{x_0}(x) := \frac{1}{2} \left\langle x - x_0, \partial_x^2 V(x_0)(x - x_0) \right\rangle$ in the harmonic oscillator, the normalized ground state associated to the eigenvalue $\lambda_h = h \operatorname{Tr}_{\mathbb{R}^d} \sqrt{\partial_x^2 V(x_0)}$ of $P_0 := -h^2 \Delta + V_{x_0}(x)$ is given by the formula

$$u_h(x) = h^{-d/4} (\det(\partial_x^2 V(x_0)))^{-1/4} \varphi_1(h^{-1/2} (\partial_x^2 V(x_0))^{1/4} (x - x_0)).$$

Moreover, for any compact K neighborhood of x_0 , there exists $C(d, K, h_0) > 0$ such that for $0 < h \leq h_0$ and any $2 \leq q \leq \infty$

$$\begin{aligned} \|u_h\|_{L^q(K)} &= (\det(\partial_x^2 V(x_0)))^{-1/4} h^{-d/4} \|\varphi_1(h^{-1/2} (\partial_x^2 V(x_0))^{1/4} (x - x_0))\|_{L^q(K)} \\ &\geq C(d, K) h^{-\frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right)} \underbrace{\|\varphi_1\|_{L^q(h_0^{-1/2} (\partial_x^2 V(x_0))^{-1/4} (K+x_0))}}_{>0} \\ &\geq C(d, K, h_0) h^{-\frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right)}. \end{aligned}$$

We have for any $h_0 > 0$, $C > 0$ such that for any $h \in (0, h_0]$ and any $2 \leq q \leq \infty$

$$\|u_h\|_{L^q(\mathbb{R}^d)} \geq C h^{-\frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right)} \left(\|u_h\|_{L^2(\mathbb{R}^d)} + \frac{1}{h} \|(P_0 - \lambda_h)u_h\|_{L^2(\mathbb{R}^d)} \right).$$

It remains to show that u_h is a quasimode of the operator $P - E$

$$(P - E)u_h = \mathcal{O}_{L^2}(h).$$

By Taylor formula of V at $x = x_0$, we have for any $x \in \mathbb{R}^d$

$$V(x) = E + V_{x_0}(x) + \mathcal{O}(|x - x_0|^3).$$

Thus, one can estimate $(V - E - V_{x_0})u_h$

$$\begin{aligned} \|(V - E - V_{x_0})u_h\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |(V - E - V_{x_0})(x)u_h(x)|^2 dx \\ &= \int_{\mathbb{R}^d} \left| (V - E - V_{x_0})(x_0 + \sqrt{h}x)u_1(x_0 + \sqrt{h}x) \right|^2 dx. \end{aligned}$$

On the one hand, since $\{y \in \mathbb{R}^d : y = x_0 + \sqrt{h}x \text{ and } x \in \mathbb{R}^d, |x| \leq 1/\sqrt{h}\} \subset \{y \in \mathbb{R}^d : |y| \leq |x_0| + 2\}$ is a compact set of \mathbb{R}^d , we have by Taylor formula and by $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned} & \int_{|x| \leq \frac{1}{\sqrt{h}}} \left| (V - E - V_{x_0})(x_0 + \sqrt{h}x) u_1(x_0 + \sqrt{h}x) \right|^2 dx \\ & \leq Ch^3 \int_{|x| \leq \frac{1}{\sqrt{h}}} |x|^6 \left| u_1(x_0 + \sqrt{h}x) \right|^2 dx \\ & \leq Ch^3 \int_{\mathbb{R}^d} |x|^6 \left| \varphi_1((\partial_x^2 V(x_0))^{1/4} x) \right|^2 (\det((\partial_x^2 V(x_0)))^{-1/2} dx \\ & \leq C' h^3 \underbrace{\int_{\mathbb{R}^d} |x|^6 |\varphi_1(x)|^2 dx}_{< \infty}. \end{aligned}$$

On the other hand, by the growth assumption (II.1) on the potential V

$$\begin{aligned} & \int_{|x| > \frac{1}{\sqrt{h}}} \left| (V - E - V_{x_0})(x_0 + \sqrt{h}x) u_1(x_0 + \sqrt{h}x) \right|^2 dx \\ & = \int_{|x| > \frac{1}{\sqrt{h}}} \left| V(x_0 + \sqrt{h}x) - E + h \langle x, \partial_x^2 V(x_0) x \rangle \right|^2 \left| u_1(x_0 + \sqrt{h}x) \right|^2 dx \\ & \leq C \int_{|x| > \frac{1}{\sqrt{h}}} (1 + |x_0 + \sqrt{h}x|^k + h \langle x, \partial_x^2 V(x_0) x \rangle)^2 \left| u_1(x_0 + \sqrt{h}x) \right|^2 dx \\ & \leq C' \int_{|x| > \frac{1}{\sqrt{h}}} \left| \varphi_1((\partial_x^2 V(x_0))^{1/4} x) \right|^2 (\det((\partial_x^2 V(x_0)))^{-1/2} dx \\ & \leq C'' \int_{|x| > \frac{c'}{\sqrt{h}}} |\varphi_1(x)|^2 dx = \mathcal{O}(e^{-c/h}) = \mathcal{O}(h^\infty). \end{aligned}$$

Thus,

$$\|(V - E - V_{x_0})u_h\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(h^{3/2}).$$

We finally get

$$\begin{aligned} (P - E)u_h &= (P_0 - \lambda_h)u_h + \lambda_h u_h + (V - E - V_{x_0})u_h \\ &= \mathcal{O}_{L^2}(h). \end{aligned}$$

This concludes the proof. □

VIII.1.2 Zonal-type quasimode

We now prove the sharpness of the exponent $s_{\text{Sogge}}(q, d)$ for large values of q . We will see that the saturation phenomenon is obtained for a sequence of functions concentrating around a point (with a rate different from the one of the gaussian groundstate of the preceding section). For the harmonic potential $V(x) = |x|^2$, such functions have already been constructed in [21, Sec. 5.2]. Our example which relies on the Weyl law is valid for a more general class of potentials. It is inspired by the construction in [33, Eq. (5.1.12)].

Proposition VIII.2. *Let $d \geq 2$. Let $E_0 > 0$ such that $E_0 > \min V$. Let $\varepsilon_0 := E_0 - \min V$. Assume that*

$$\lim_{\varepsilon \rightarrow 0} \left| \{x \in \mathbb{R}^d : |V(x) - \lambda| \leq \varepsilon\} \right| = 0 \tag{VIII.3}$$

uniformly for λ in a neighborhood of E_0 . Then, there exist $h_0 > 0$, energies $\{E_h\}_{h \in (0, h_0)} \subset [E_0 - \varepsilon_0/2, E_0 + \varepsilon_0/2]$, $\varepsilon \in (0, \varepsilon_0/4)$ and points $\{x_{0,h}\}_{h \in (0, h_0)} \subset \{V \leq E_h - \varepsilon\}$ such that the family of functions defined by

$$u_h := \Pi_h(x_{0,h}, \cdot)$$

where Π_h denotes the spectral projector

$$\Pi_h := \mathbf{1}(P \in I_{h, E_h}) = \mathbf{1}(P \in [E_h - h, E_h + h]),$$

satisfies along a sequence $h_n \rightarrow 0$ that for any $2 \leq q \leq \infty$

$$\|u_h\|_{L^2(\mathbb{R}^d)} \leq Ch^{-\frac{(d-1)}{2}}, \quad \|u_h\|_{L^q(\{x \in \mathbb{R}^d : V(x) - E_h \leq \varepsilon\})} \geq (1/C)h^{-(d-1) + \frac{d}{q}}.$$

Remark 32. As a consequence, we get that

$$\limsup_{h \rightarrow 0} h^{d(1/2 - 1/q) - 1/2} \frac{\|u_h\|_{L^q(\{x \in \mathbb{R}^d : V(x) - E_h \leq \varepsilon\})}}{\|u_h\|_{L^2(\mathbb{R}^d)}} > 0,$$

which proves the optimality of the exponent $s_{\text{Sogge}}(q, d)$ (which is equal to $d(1/2 - 1/q) - 1/2$ when $2(d+1)/(d-1) \leq q \leq \infty$) in the estimate (VII.3).

Remark 33. Assumption (VIII.3) is satisfied for instance when:

- $\nabla_x V(x_0)$ does not vanish for the points $x_0 \in \{V = \lambda\}$,
- or when the Hessian $\partial_x^2 V(x_0)$ is non-degenerate for the points $x_0 \in \{V = \lambda\}$,

for λ in a neighborhood of E_0 . In the first case, we have $|\{|V - \lambda| \leq \varepsilon\}| = \mathcal{O}(\varepsilon)$. We even expect that the condition holds if for any x_0 such that $V(x_0)$ belongs to a neighborhood of E_0 , the Taylor expansion of V at x_0 is not identically zero. On the contrary, if V is constant in a neighborhood of x_0 , then Assumption (VIII.3) is not satisfied.

Remark 34. The lower bound on the L^q norm of u_h comes the pointwise estimate

$$\forall x \in \mathbb{R}^d, |x - x_{0,h}| \leq ch \implies |u_h(x)| \geq Ch^{1-d}. \quad (\text{VIII.4})$$

In other words, the function u_h concentrates around $x_{0,h}$ at a scale h with a height $\sim h^{-(d-1)}$ (such that in Figure 12). This motivates the name zonal-type quasimode because they concentrate similarly to the zonal harmonics, which are known to saturate the Sogge L^q estimates in the same regime of q . This originally appeared in [30].

Before proving Proposition VIII.2, we first provide a lemma which is an easy consequence of the integrated Weyl law, which is well known in the high energy regime and that we state here in the semiclassical setting. This result gives an interval of size h with the maximal number of eigenvalues inside. It will also be useful for the many-body optimality.

For any $I \subset \mathbb{R}$, recall that $N_h(I)$ denotes the number of eigenvalues of P in I .

Lemma VIII.3. Let $a < b$ such that $|p^{-1}([a, b])| > 0$. Then

$$\limsup_{h \rightarrow 0} \sup_{J_h \subset [a, b], |J_h| = 2h} h^{d-1} N_h(J_h) > 0.$$

Proof of Lemma VIII.3. Assume by contradiction that

$$\limsup_{h \rightarrow 0} \sup_{J_h \subset [a, b], |J_h|=2h} h^{d-1} N_h(J_h) = 0$$

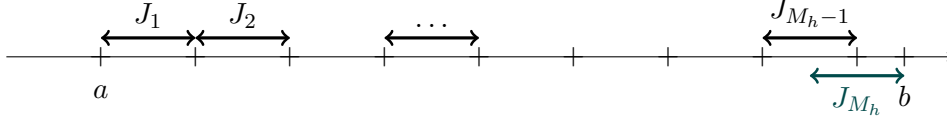
i.e. for all $\varepsilon > 0$, there exists $\tilde{h} > 0$ such that for all $h \in (0, \tilde{h}]$ and for all interval $J_h \subset [a, b]$ such that $|J_h| = 2h$

$$h^{d-1} N_h(J_h) \leq \varepsilon.$$

One can cover $[a, b]$ by a finite set of intervals of length $2h$:

$$\{J_j\}_{j=1}^{M_h} \subset [a, b], \quad [a, b] \subset \bigcup_{j=1}^{M_h} J_j.$$

For example, for $2h < |b - a|$, one can take $M_h = \lfloor \frac{|b-a|}{2h} \rfloor - 1$, define the $M_h - 1$ first intervals by $J_j := [a + 2(j-1)h, a + 2jh]$ and define the M_h -th one $J_{M_h} := [b - 2h, b]$.



Finally,

$$N_h([a, b]) \leq \sum_{j=1}^{M_h} N_h(I_j) \lesssim M_h \varepsilon h^{-(d-1)} \sim \varepsilon |b - a| h^{-d}.$$

When ε goes to 0, we get

$$\lim_{h \rightarrow 0} h^d N_h([a, b]) = 0.$$

However, by the Weyl law (Proposition II.18) $h^d N_h([a, b])$ is equivalent to $(2\pi)^{-d} |p^{-1}([a, b])|$, which is positive. That is in contradiction with the original assumption. \square

Proof of Proposition VIII.2. Let $R > 0$ such that for any $\lambda \in [E_0 - R, E_0 + R]$, we have (VIII.3).

Let $c_0 := \min\left(\frac{1}{4}, \frac{R}{2\varepsilon_0}\right)$. Define the intervals $I_0 := [E_0 - c_0\varepsilon_0, E_0 + c_0\varepsilon_0]$. Given the definition of ε_0 and c_0 , we have $I_0 \subset (\min V, \infty)$. Thus, $|p^{-1}(I_0)| > 0$. We begin to take $h_0 \in (0, c_0\varepsilon_0/2)$ (of course, we will lower it afterwards). By Lemma VIII.3 (up to a sequence $\{h_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+^*$ with $h_n \rightarrow 0$ when $n \rightarrow \infty$) there exists $h_0 > 0$, $\{E_h\}_{h \in (0, h_0]} \subset I_0$ and $C' > 0$ such that for any $h \in (0, h_0]$

$$\|\rho_{\Pi_h}\|_{L^1(\mathbb{R}^d)} \geq C' h^{-(d-1)}.$$

Let $\varepsilon > 0$ such that $\varepsilon < \text{dist}(E_0 - c_0\varepsilon_0, \min V)/2 = (1 - c_0)\varepsilon_0/2$. By the L^∞ estimates (see for instance Theorem VII.2), there exists $C > 0$ such that for any $h \in (0, h_0]$

$$\begin{aligned} \|\rho_{\Pi_h}\|_{L^1(\{|V - E_h| \leq \varepsilon\})} &\leq |\{|V - E_h| \leq \varepsilon\}| \|\rho_{\Pi_h}\|_{L^\infty(\mathbb{R}^d)} \\ &\leq C h^{-(d-1)} |\{|V - E_h| \leq \varepsilon\}|. \end{aligned}$$

By (VIII.3), let us fix $\varepsilon \in (0, (1 - c_0)\varepsilon_0/2)$ such that $C |\{|V - E_h| \leq \varepsilon\}| \leq C'/2$. Besides, by (VII.9), there exists $C_\varepsilon > 0$ such that for any $h \in (0, h_0]$

$$\|\rho_{\Pi_h}\|_{L^1(\{V \geq E_h + \varepsilon\})} \leq C_\varepsilon h^{-(d-2)}.$$

Finally, by the triangle inequality and the previous estimates there exists $h_0 > 0$ such that for any $h \in (0, h_0]$

$$\begin{aligned} \|\rho_{\Pi_h}\|_{L^1(\{V \leq E_h - \varepsilon\})} &= \|\rho_{\Pi_h}\|_{L^1(\mathbb{R}^d)} - \|\rho_{\Pi_h}\|_{L^1(\{|V - E_h| < \varepsilon\})} - \|\rho_{\Pi_h}\|_{L^1(\{V \geq E_h + \varepsilon\})} \\ &\geq \frac{1}{4} C' h^{-(d-1)}. \end{aligned} \quad (\text{VIII.5})$$

For any $h \in (0, h_0]$, let us define $x_{0,h}$ as a maximizer of the function ρ_{Π_h} on the compact set $\{V \leq E_h - \varepsilon\}$. We thus have for any $h \in (0, h_0]$

$$\begin{aligned} \Pi_h(x_{0,h}, x_{0,h}) &= \rho_{\Pi_h}(x_{0,h}) \geq \frac{1}{|\{V \leq E_h - \varepsilon\}|} \|\rho_{\Pi_h}\|_{L^1(\{V \leq E_h - \varepsilon\})} \\ &\geq \frac{C'/4}{|\{V \leq E_0 + c_0\varepsilon_0\}|} h^{-(d-1)} = C'' h^{-(d-1)}. \end{aligned}$$

By the definition of u_h and the L^∞ estimate (see for instance Theorem IV.2) there exists $C > 0$ such that for any $h \in (0, h_0]$

$$\|u_h\|_{L^2(\mathbb{R}^d)} = \rho_{\Pi_h}(x_{0,h}) \leq \|\rho_{\Pi_h}\|_{L^\infty(\mathbb{R}^d)}^{1/2} \leq C h^{-(d-1)/2}.$$

Let us now prove that there exists $C_1 > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0]$

$$\sup_{x, y \in B(x_{0,h}, h)} |\nabla_y \Pi_h(x, y)| \leq C_1 h^{-d}. \quad (\text{VIII.6})$$

This implies that for any $h \in (0, h_0]$ and for any $x \in B(x_{0,h}, \min(1, \frac{C''}{2C_1}h))$

$$\begin{aligned} |u_h(x)| &= |\Pi_h(x_{0,h}, x)| \\ &\geq \Pi_h(x_{0,h}, x_{0,h}) - |\Pi_h(x_{0,h}, x) - \Pi_h(x_{0,h}, x_{0,h})| \\ &\geq \Pi_h(x_{0,h}, x_{0,h}) - \|\nabla_y \Pi_h(x_{0,h}, \cdot)\|_{L^\infty(B(x_{0,h}, h))} |x - x_{0,h}| \\ &\geq \frac{C''}{2} h^{-(d-1)}. \end{aligned}$$

That gives us (VIII.4). Let us prove now the estimate (VIII.6). Denote

$$\Pi_h^{(0)} := \mathbb{1}(-h^2 \Delta + V(x_{0,h}) \in I_{h, E_h}).$$

For all $x, y \in \mathbb{R}^d$, we have

$$\Pi_h^{(0)}(x, y) = \frac{1}{(2\pi h)^d} \int_{E_h - V(x_{0,h}) - h \leq |\xi|^2 \leq E_h - V(x_{0,h}) + h} e^{\frac{i}{h} \langle \xi, x-y \rangle} d\xi.$$

Hence,

$$\begin{aligned} \left| \nabla_y \Pi_h^{(0)}(x, y) \right| &\leq |y| C h^{-d-1} \left((E_h - V(x_{0,h}) + h)^{d/2} - (E_h - V(x_{0,h}) - h)_+^{d/2} \right) \\ &\leq |y| C_\varepsilon h^{-d}. \end{aligned}$$

We next prove that

$$\sup_{x, y \in B(x_{0,h}, h)} \left| \nabla_y \Pi_h(x, y) - \nabla_y \Pi_h^{(0)}(x, y) \right| \leq C h^{-d}$$

by showing that

$$\sup_{x,y \in B(x_{0,h},h)} \left| \nabla_y \Pi_{h,\leq E_h \pm h}(x,y) - \nabla_y \Pi_{h,\leq E_h \pm h}^{(0)}(x,y) \right| \leq Ch^{-d},$$

where for any $E \in \mathbb{R}$, $\Pi_{h,\leq E} := \mathbb{1}(P \leq E)$ and $\Pi_{h,\leq E}^{(0)} := \mathbb{1}(-h^2\Delta + V(x_{0,h}) \leq E)$. Introducing

$$K_{h,\leq E}(x,y) := \Pi_{h,\leq E}(x_{0,h} + hx, x_{0,h} + hy), \quad K_{h,\leq E}^{(0)}(x,y) := \Pi_{h,\leq E}^{(0)}(x_{0,h} + hx, x_{0,h} + hy),$$

it is enough to show that

$$\sup_{x,y \in B(0,1)} \left| \nabla_y K_{h,\leq E_h \pm h}(x,y) - \nabla_y K_{h,\leq E_h \pm h}^{(0)}(x,y) \right| \leq Ch^{-(d-1)}.$$

This is given by [7, Rem. 1.2]. Notice that in this work, the parameter $x_{0,h}$ and E_h are independent of h . However, the result still holds when $x_{0,h}$ and E_h depend on h in such a way that they belong to a h -independent compact set and are such that $V(x_{0,h}) \leq E_h - \delta$ for some h -independent $\delta > 0$. \square

VIII.1.3 Gaussian beams

We now explain the optimality of the exponent $s_{\text{Sogge}}(q,d)$ for low values of q , in the case $V(x) = |x|^2$. The saturating functions already appeared in [21, Sec. 5.1], and we just provide the computational details here. This example only works for the harmonic oscillator since the argument relies on separation of variables. However, we expect that this exponent is also sharp for more general V , using for instance the argument of [32]. Here, the saturation phenomenon happens for a family of functions concentrating around a curve, similarly to Gaussian beams on spheres [30].

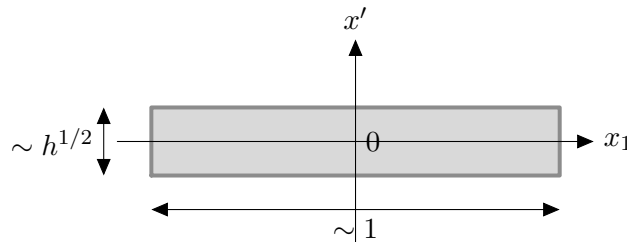
Proposition VIII.4. *Let $d \geq 2$ and $2 \leq q \leq \infty$. Let $p(x,\xi) = |\xi|^2 + |x|^2$. Let $E_{\text{exc}} > 0$. For any $h > 0$, define $P := p^w(x,hD)$ and $E_h := E_{\text{exc}} + (d-1)h$. For any $n \in \mathbb{N}$, let us denote $h_n := E_{\text{exc}}/(2n+1)$ and φ_n the normalized eigenfunction associated to the eigenvalue E_{exc} of the scalar harmonic oscillator $-h_n^2 \frac{d^2}{dy^2} + y^2$ on $L^2(\mathbb{R})$. For any $n \in \mathbb{N}$, define*

$$u_n(x) := (2\pi)^{-(d-1)/2} h_n^{-(d-1)/4} e^{-\frac{|x'|^2}{2h_n}} \varphi_n(x_1), \quad x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Then, for all $n \in \mathbb{N}$, we have $\|u_n\|_{L^2(\mathbb{R}^d)} = 1$, $Pu_n = E_{h_n}u_n$, and there exists $\varepsilon > 0$ and $C > 0$ such that for n large enough

$$\|u_n\|_{L^q(\{x \in \mathbb{R}^d: |x|^2 - E_{h_n} < \varepsilon\})} \geq Ch_n^{-\frac{d-1}{2}(\frac{1}{2} - \frac{1}{q})}.$$

Remark 35. *These eigenfunctions concentrate along the curve $\{x' = 0\}$ at scale \sqrt{h} in the orthogonal direction (c.f. Figure 13). Indeed, due to the well-known asymptotics of the Hermite functions, $\varphi_n(x_1)$ is essentially constant (up to oscillations which average out when taking the L^q norm) for x_1 in a neighborhood of 0.*



Remark 36. *This proves the optimality of the exponent $s_{\text{Sogge}}(q, d)$ for low regime $2 \leq q \leq 2(d+1)/(d-1)$ in the classically allowed region and also the optimality of the exponent $s_{\text{TP}}(q, d) = s_{\text{Sogge}}(q, d)$ for lower regime $2 \leq q \leq 2(d+3)/(d+1)$ around the turning points $\{V = E_{h_n}\}$ on Theorem VII.2 when $V(x) = |x|^2$, by taking $\gamma_{h_n} = |u_n\rangle\langle u_n|$.*

Proof of Proposition VIII.4. Let $\varepsilon = E_{\text{exc}}/2$. First, defining for any $h > 0$

$$\varphi_{h,\text{ground}}(x') := (2\pi)^{-(d-1)/2} h^{-(d-1)/4} e^{-\frac{|x'|^2}{2h}},$$

there exists $C_d > 0$ such that for any $h > 0$

$$\|\varphi_{h,\text{ground}}\|_{L^q(\{x' \in \mathbb{R}^{d-1} : |x'| \leq h^{1/2}\})} = C_d h^{-\frac{d-1}{4} + \frac{d-1}{2q}}.$$

On the other hand, by Liouville-Green asymptotics (see for instance Chapter 6 in [27]), we have for any $x_1 \in [-\sqrt{E_{\text{exc}}}/2, \sqrt{E_{\text{exc}}}/2]$

$$\varphi_n(x_1) = \frac{C_{h_n}}{(E_{\text{exc}} - x_1^2)^{1/4}} \begin{cases} \cos\left(h_n^{-1} \int_0^{x_1} \sqrt{E_{\text{exc}} - t^2} dt\right) (1 + e_{h_n}(x_1)) & \text{if } n \text{ is even,} \\ \sin\left(h_n^{-1} \int_0^{x_1} \sqrt{E_{\text{exc}} - t^2} dt\right) (1 + e_{h_n}(x_1)) & \text{if } n \text{ is odd,} \end{cases}$$

where $C_{h_n} \in \mathbb{R}$ be a normalization constant such that $\lim_{n \rightarrow \infty} C_{h_n} > 0$ and

$$\|e_{h_n}\|_{L^\infty([-\sqrt{E_{\text{exc}}}/2, \sqrt{E_{\text{exc}}}/2])} = \mathcal{O}(h_n).$$

Let us now show that $\liminf_{n \rightarrow \infty} \|\varphi_n\|_{L^q([0, \delta])} > 0$ for $\delta = \sqrt{E_{\text{exc}}}/2$, which then proves the result since for n large enough

$$\begin{aligned} \|u_n\|_{L^q(x \in \mathbb{R}^d : V(x) - E_{h_n} \leq \varepsilon)} &\geq \|u_n\|_{L^q([0, \delta] \times B_{\mathbb{R}^{d-1}}(0, \sqrt{h_n}))} \\ &\geq \|\varphi_n\|_{L^q([0, \delta])} \|\varphi_{h_n,\text{ground}}\|_{L^q(B_{\mathbb{R}^{d-1}}(0, \sqrt{h_n}))} \\ &\gtrsim h_n^{-\frac{d-1}{4} + \frac{d-1}{2q}}. \end{aligned}$$

Suppose for instance that n is even (the cas n odd is similar). For any $n \in \mathbb{N}$ and any $x_1 \in [0, \delta]$, define

$$f_{h_n}(x_1) := (E_{\text{exc}} - x_1^2)^{-1/4} \cos\left(h_n^{-1} \int_0^{x_1} \sqrt{E_{\text{exc}} - t^2} dt\right).$$

Given the estimate on the error term e_{h_n} , we have

$$\|\varphi_n - f_{h_n}\|_{L^q([0, \delta])} = \mathcal{O}(h_n).$$

Let us denote

$$S(x_1) := \int_0^{x_1} \sqrt{E_{\text{exc}} - t^2} dt.$$

Since its derivative S' is positive on $[0, \delta]$, we have

$$\begin{aligned} \|f_{h_n}\|_{L^q([0, \delta])}^q &= \int_0^\delta S'(x_1)^{-q/2} |\cos(h_n^{-1} S(x_1))|^q dx_1 \\ &\gtrsim \int_0^\delta |\cos(h_n^{-1} S(x_1))|^q S'(x_1) dx_1 \\ &= \int_0^{S(\delta)} |\cos(h_n^{-1} y)|^q dy = \int_0^{S(\delta)} (1 + \cos(2h_n^{-1} y))^{q/2} dy. \end{aligned}$$

Since $q/2 \geq 1$, the function $g = (1 + \cos(2\cdot))^{q/2}$ is π -periodic and \mathcal{C}^1 . Hence, its Fourier coefficients

$$c_k(g) := \frac{1}{\pi} \int_0^{h_n\pi} e^{-2ikt} g(t) dt$$

are summable over $k \in \mathbb{Z}$. As a consequence,

$$\int_0^{S(\delta)} g(h_n^{-1}y) dy = \sum_{k \in \mathbb{Z}} c_k(g) \int_0^{S(\delta)} e^{2iky/h_n} dy = S(\delta)c_0(g) + \mathcal{O}(h_n),$$

which finishes the proof. \square

VIII.2 Many-body optimality

We now turn to the optimality of the Schatten exponent α . We show that the exponent $\alpha = \alpha_{Sogge}$ is sharp in the estimates where it appears together with the exponent $s = s_{Sogge}$. The saturation scenario here is a family of operators γ such that ρ_γ is delocalized in the bulk region of the potential. We will see that it happens for the maximal family $\gamma = \Pi_h$, in the same spirit as in [12, Rem. 11]. The optimality of the other Schatten exponents α_{gene} or α_{TP} (when they are not equal to α_{Sogge}) is a very challenging problem since $\gamma = \Pi_h$ does not saturate the inequalities where they appear.

Proposition VIII.5 (Many-body optimality of the Sogge exponent). *Let $d \geq 2$. Let $p(x, \xi) = |\xi|^2 + V(x)$ with $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ satisfying Definition VII.1. Let $E_0 > \min V$ such that we have (VIII.3). Then, there exist $h_0 > 0$, $\{E_h\}_{h \in (0, h_0]}$ in a compact neighborhood of E_0 on $(\inf V, \infty)$, $\varepsilon > 0$ and $C > 0$ such that for any $h \in (0, h_0]$ and any $2 \leq q \leq \infty$ (along a sequence $h_n \rightarrow 0$ when $n \rightarrow \infty$)*

$$\|\rho_{\Pi_h}\|_{L^{q/2}(\{x \in \mathbb{R}^d : V(x) \leq E_h - \varepsilon\})} \geq Ch^{-(d-1)}, \quad (\text{VIII.7})$$

where Π_h denotes the spectral projector $\Pi_h := \mathbf{1}(P \in I_{h, E_h})$.

Remark 37. *Due to the L^∞ estimates (VII.2) in the case $q = \infty$, we always have $\|\rho_h\|_{L^\infty(\mathbb{R}^d)} \leq Ch^{-(d-1)}$. Together with (VIII.7), this shows that all the $L^{q/2}$ norms of ρ_{Π_h} are of the order $h^{-(d-1)}$ in the bulk region $\{V < E_h - \varepsilon\}$, indicating that ρ_{Π_h} behaves like a (large) constant in this region.*

Remark 38. *This result also proves that the Schatten exponent $\alpha_{Sogge}(q, d)$ is optimal for instance in the estimate (VII.3). Indeed, for $\gamma = \Pi_h$, (VIII.7) shows that the left side of (VII.3) is of order $h^{-(d-1)}$, while the right side is order*

$$h^{-2s_{Sogge}(q, d)} \|\Pi_h\|_{\mathfrak{S}^{\alpha_{Sogge}(q, d)}} \lesssim h^{-2s_{Sogge}(q, d)} h^{-(d-1)/\alpha_{Sogge}(q, d)} = h^{-(d-1)}.$$

Here, we used that $\text{rank}(\Pi_h) \leq Ch^{-(d-1)}$ which follows from the fact that

$$\text{rank}(\Pi_h) = \|\rho_{\Pi_h}\|_{L^1(\mathbb{R}^d)} = \|\rho_{\Pi_h}\|_{L^1(\{V \leq E_h + \varepsilon\})} + \|\rho_{\Pi_h}\|_{L^1(\{V > E_h + \varepsilon\})}$$

and the estimates (VII.2) in the case $q = \infty$ and (VII.9).

Proof of Proposition (VIII.5). Let $\varepsilon_0 := E_0 - \min V$. Here, we take $\{E_h\}_{h \in (0, h_0]} \subset [E_0 - \varepsilon_0/2, E_0 + \varepsilon_0/2]$ and $\varepsilon \in (0, \varepsilon_0/4)$ as in Proposition VIII.2. They are chosen to satisfy the lower bound (VIII.5). Thus, there exists $C' > 0$ such that for any $h \in (0, h_0]$ and any $2 \leq q \leq \infty$

$$\begin{aligned} \|\rho_{\Pi_h}\|_{L^{q/2}(\{V \leq E_h - \varepsilon\})} &\geq |\{V \leq E_h - \varepsilon\}|^{-1/(q/2)'} \|\rho_{\Pi_h}\|_{L^1(\{V \leq E_h - \varepsilon\})} \\ &\geq |\{V \leq E_0 + 3\varepsilon_0/4\}|^{-1/(q/2)'} \|\rho_{\Pi_h}\|_{L^1(\{V \leq E_h - \varepsilon\})} \\ &\geq C' h^{-(d-1)}. \end{aligned}$$

\square

REFERENCES

- [1] V. G. AVAKUMOVIĆ, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z., 65 (1956), pp. 327–344.
- [2] M. D. BLAIR AND C. D. SOGGE, *Logarithmic improvements in L^p bounds for eigenfunctions at the critical exponent in the presence of nonpositive curvature*, Invent. Math., 217 (2019), pp. 703–748.
- [3] D. DEAN, P. LE DOUSSAL, G. SCHEHR, ET AL., *Impurities in systems of noninteracting trapped fermions*, SciPost Physics, 10 (2021), p. 082.
- [4] D. S. DEAN, P. LE DOUSSAL, S. N. MAJUMDAR, AND G. SCHEHR, *Universal ground-state properties of free fermions in a d -dimensional trap*, EPL (Europhysics Letters), 112 (2015), p. 60001.
- [5] ———, *Wigner function of noninteracting trapped fermions*, Physical Review A, 97 (2018), p. 063614.
- [6] ———, *Noninteracting fermions in a trap and random matrix theory*, Journal of Physics A: Mathematical and Theoretical, 52 (2019), p. 144006.
- [7] A. DELEPORTE AND G. LAMBERT, *Universality for free fermions and the local weyl law for semi-classical schrödinger operators*, arXiv preprint arXiv:2109.02121, (2021).
- [8] M. DIMASSI AND J. SJOSTRAND, *Spectral asymptotics in the semi-classical limit*, no. 268, Cambridge university press, 1999.
- [9] R. L. FRANK, M. LEWIN, E. H. LIEB, AND R. SEIRINGER, *Strichartz inequality for orthonormal functions*, Journal of the European Mathematical Society, 16 (2014), pp. 1507–1526.
- [10] R. L. FRANK AND J. SABIN, *The Stein-Tomas inequality in trace ideals*, Séminaire Laurent Schwartz—EDP et applications, (2016), pp. 1–12.
- [11] ———, *Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform sobolev estimates*, American Journal of Mathematics, 139 (2017), pp. 1649–1691.
- [12] ———, *Spectral cluster bounds for orthonormal systems and oscillatory integral operators in Schatten spaces*, Advances in Mathematics, 317 (2017), pp. 157–192.
- [13] I. GEL’FAND AND G. E. SHILOV, *Generalized Functions: Properties and operations*, vol. 1, Academic Press, 1964.
- [14] J. GINIBRE AND G. VELO, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Communications in mathematical physics, 144 (1992), pp. 163–188.
- [15] A. HASSELL AND M. TACY, *Improvement of eigenfunction estimates on manifolds of nonpositive curvature*, Forum Math., 27 (2015), pp. 1435–1451.
- [16] B. HELFFER, *Semi-classical analysis for the Schrödinger operator and applications*, vol. 1336, Springer, 2006.
- [17] B. HELFFER AND J. SJÖSTRAND, *Multiple wells in the semiclassical limit. I*, Comm. Partial Differential Equations, 9 (1984), pp. 337–408.
- [18] H. HEZARI AND G. RIVIÈRE, *L^p norms, nodal sets, and quantum ergodicity*, Adv. Math., 290 (2016), pp. 938–966.
- [19] L. HÖRMANDER, *The spectral function of an elliptic operator*, Acta Math., 121 (1968), pp. 193–218.
- [20] G. KARADZHOV, *A complete asymptotic of the spectral function for harmonic oscillator*, Mathematica Balkanica. New Series, 3 (1989).
- [21] H. KOCH AND D. TATARU, *L^p eigenfunction bounds for the Hermite operator*, Duke Mathematical Journal, 128 (2005), pp. 369–392.
- [22] H. KOCH, D. TATARU, AND M. ZWORSKI, *Semiclassical L^p estimates*, 8 (2007), pp. 885–916.
- [23] L. LARSSON-COHN, *L^p -norms of Hermite polynomials and an extremal problem on Wiener chaos*, Ark. Mat., 40 (2002), pp. 133–144.

- [24] B. M. LEVITAN, *On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order*, Izvestiya Akad. Nauk SSSR. Ser. Mat., 16 (1952), pp. 325–352.
- [25] E. H. LIEB AND M. LOSS, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.
- [26] J. MERCER, *Functions of positive and negative type, and their connection the theory of integral equations*, Philosophical transactions of the royal society of London. Series A, containing papers of a mathematical or physical character, 209 (1909), pp. 415–446.
- [27] F. OLVER, *Asymptotics and special functions*, AK Peters/CRC Press, 1997.
- [28] B. SIMON, *Trace ideals and their applications*, no. 120, American Mathematical Soc., 2005.
- [29] H. F. SMITH AND M. ZWORSKI, *Pointwise bounds on quasimodes of semiclassical schrodinger operators in dimension two*, arXiv preprint arXiv:1209.2721, (2012).
- [30] C. D. SOGGE, *Oscillatory integrals and spherical harmonics*, PhD thesis, Princeton University, 1985.
- [31] C. D. SOGGE, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, Journal of functional analysis, 77 (1988), pp. 123–138.
- [32] C. D. SOGGE, *Remarks on L^2 restriction theorems for Riemannian manifolds*, in Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987), vol. 137 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1989, pp. 416–422.
- [33] C. D. SOGGE, *Fourier integrals in classical analysis*, vol. 210, Cambridge University Press, 2017.
- [34] C. D. SOGGE, *Improved critical eigenfunction estimates on manifolds of nonpositive curvature*, Math. Res. Lett., 24 (2017), pp. 549–570.
- [35] R. S. STRICHARTZ, *Restrictions of fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Mathematical Journal, 44 (1977), pp. 705–714.
- [36] M. TACY, *Semiclassical L^p estimates of quasimodes on submanifolds*, Comm. Partial Differential Equations, 35 (2010), pp. 1538–1562.
- [37] M. ZWORSKI, *Semiclassical analysis*, vol. 138, American Mathematical Soc., 2012.