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Trung-Hieu Vu

► **To cite this version:**

Trung-Hieu Vu. On the Nonemptiness and Boundedness of Solution Sets of Weakly Homogeneous Optimization Problems. *Set-Valued and Variational Analysis*, 2022, 30 (3), pp.1105-1116. 10.1007/s11228-022-00637-0 . hal-03654665

HAL Id: hal-03654665

<https://hal.science/hal-03654665v1>

Submitted on 28 Apr 2022

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On the Nonemptiness and Boundedness of Solution Sets of Weakly Homogeneous Optimization Problems

Vu Trung Hieu

Received: date / Accepted: date

Abstract In this paper, we introduce a new class of optimization problems whose objective functions are weakly homogeneous relative to the constraint sets. By using the normalization argument in asymptotic analysis, we prove two criteria for the nonemptiness and boundedness of the solution set of a weakly homogeneous optimization problem. Moreover, the normalization argument enables us to study the existence and stability of the solution sets of linearly parametric optimization problems. Several examples are given to illustrate the derived results.

Keywords Weakly homogeneous optimization problem · Asymptotic cone · Pseudoconvex function · Nonemptiness · Boundedness · Upper semicontinuity

Mathematics Subject Classification (2000) 90C30 · 90C31 · 90C46

1 Introduction

The class of weakly homogeneous functions, which contains the subclass of all polynomial functions, has been introduced in the setting of variational inequality problems by Gowda and Sossa in [6], and studied recently by Ma, Zheng, and Huang in [11]. This setting is a simpler case of the one used initially by Gowda and Pang [7], and later by Flores-Bazán [5].

In the present paper, we introduce a new class of weakly homogeneous functions (see Definition 2.1 in the next section), which is stronger in some sense than that has been introduced in [6], and investigate the existence as well as the stability of the solution sets of *weakly homogeneous optimization problems*. Asymptotic analysis plays an important role and the normalization argument (see, e.g., [2, 3, 10]) is used effectively for our study.

Vu Trung Hieu
LIP6, Sorbonne University, Paris, France
E-mail: trung-hieu.vu@lip6.fr

For a given weakly homogeneous optimization problem, its *kernel* is defined by the solution set of the corresponding asymptotic optimization problem. The kernel is always a closed cone if it is nonempty. Based on the kernel notion, we establish two criteria for the nonemptiness and boundedness of the solution sets of weakly homogeneous optimization problems. The first one says that if the kernel is the trivial cone then the solution set is nonempty and bounded. The second one concerns the pseudoconvexity of the objective function when the kernel is non-trivial and the constraint set is convex. These results are considered as extensions of the Frank-Wolfe type theorem and the Eaves type theorem for polynomial optimization problems in [9].

The kernel and the domain of an affine variational inequality were introduced in [4,8]. They are useful in the study of the existence and stability of solution sets of parametric affine variational inequality problems. For weakly homogeneous optimization problems, these notions can be developed to study the existence and stability of solutions to *linearly parametric optimization problems*.

The organization of the paper is as follows. Section 2 gives a brief introduction to asymptotic cones, weakly homogeneous functions, pseudoconvexity, and optimization problems. Asymptotic problems are discussed in Section 3 while two results on the nonemptiness and boundedness of solution sets are shown in Section 4. Sections 5 and 6 are devoted to investigate the existence and stability of solutions to linearly parametric optimization problems, respectively.

2 Preliminaries

Let S be a nonempty closed set in \mathbb{R}^n , its *asymptotic cone* S^∞ is defined [2] by

$$S^\infty = \left\{ v \in \mathbb{R}^n : \exists t_k \rightarrow +\infty, \exists x^k \in S \text{ with } \lim_{k \rightarrow \infty} \frac{x^k}{t_k} = v \right\}.$$

It is well-known that S^∞ is a closed cone and that S^∞ is the trivial cone if and only if S is bounded. Besides, if S is convex then S^∞ coincides with the recession cone of S containing vectors $v \in \mathbb{R}^n$ such that $x + tv \in S$ for any $x \in S$ and $t \geq 0$, i.e., $S = S + S^\infty$.

For an arbitrary cone C in \mathbb{R}^n , we recall that the *dual cone* C^* of C is given by

$$C^* = \{ u \in \mathbb{R}^n : \langle u, v \rangle \geq 0, \forall v \in C \}$$

and that u belongs to the interior of C^* , denoted by $\text{int}(C^*)$, if and only if $\langle u, v \rangle > 0$ for all $v \in C$ and $v \neq 0$.

Assume that C is a closed cone in \mathbb{R}^n , K is an unbounded and closed subset of C , and $f : C \rightarrow \mathbb{R}$ is a continuous function on C . Clearly, K^∞ is a subset of C .

Definition 2.1 The function f is said to be *weakly homogeneous of degree α relative to K* if there exists a positively homogeneous continuous function h of degree $\alpha > 0$ on C , i.e. $h(tx) = t^\alpha h(x)$ for all $x \in C$ and $t > 0$, such

that $f(x) - h(x) = o(\|x\|^\alpha)$ on K . Then, one says that h is an *asymptotically homogeneous function* of f relative to K .

Assume that f is weakly homogeneous of degree α relative to K . The asymptotically homogeneous function h in Definition 2.1 may be not unique (see Example 2.1). We denote by $[f_{\alpha,K}^\infty]$ the set of asymptotically homogeneous functions of f relative to K , and by \mathcal{O}_K^α the space of all continuous functions p on C , such that $p(x) = o(\|x\|^\alpha)$ on K . Clearly, if $p \in \mathcal{O}_K^\alpha$, then $f + p$ is also weakly homogeneous of degree α relative to K .

Remark 2.1 The notion of weakly asymptotic function in Definition 2.1 is different from the notion in the monograph by Auslender and Teboulle [2, Definition 2.5.1] as well as that of Gowda and Sossa [6]. In some sense, the notion in Definition 2.1 is stronger (see Example 2.1 and Remark 4.1) than that of [6] in which it is required that $f(x) - h(x) = o(\|x\|^\alpha)$ on the whole cone C instead of only on the subset K . Hence, in Definition 2.1 we emphasize that f is weakly homogeneous (of degree α) *relative to K* .

Example 2.1 Consider the cone $C := \mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, the set $K := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 = 0\}$, and the function $f(x_1, x_2) := x_1 x_2 + \sqrt{x_1}$. It is clear that $K^\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$ and $h(x_1, x_2) = \sqrt{x_1}$ is an asymptotically homogeneous function of f relative to K , i.e., f is weakly homogeneous of degree $\alpha = 1/2$ relative to K . We can see that $\bar{h}(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ is also a different asymptotically homogeneous function of f relative to K ; Clearly, h is different from \bar{h} on \mathbb{R}_+^2 but they have the same values on K^∞ . Meanwhile, the function f is a weakly homogeneous function of degree $\alpha = 2$ in the sense of [6], in which the involved asymptotic function is $\hat{h}(x_1, x_2) = x_1 x_2$.

Example 2.2 Let K be an unbounded and closed subset of \mathbb{R}^n and f be a polynomial of degree d in n variables. Then, f is weakly homogeneous of degree α relative to K , where α is a positive integer, $\alpha \leq d$. Indeed, assume that $f = \sum_{\beta \in \mathcal{B}} a_\beta x^\beta$, where \mathcal{B} is a finite subset of \mathbb{N}^n , $\beta = (\beta_1, \dots, \beta_n)$, $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $|\beta| = \beta_1 + \dots + \beta_n \leq d$, $a_\beta \in \mathbb{R}$. Let $I := \{i : \exists x \in K^\infty, x_i \neq 0\}$ and $\mathcal{B}_I := \{(\beta_1, \dots, \beta_n) \in \mathcal{B} : \beta_j = 0 \text{ or } j \in I, \forall j = 1, \dots, n\}$. We can check that the following homogeneous polynomial is an asymptotically homogeneous function of f relative to K :

$$h = \begin{cases} \sum_{\beta \in \mathcal{B}_I, |\beta| = \alpha} a_\beta x^\beta & \text{if } \mathcal{B}_I \neq \emptyset, \\ 0 & \text{if } \mathcal{B}_I = \emptyset, \end{cases}$$

where $\alpha := \max\{|\beta| : \beta \in \mathcal{B}_I\}$.

Let K be a nonempty closed subset in \mathbb{R}^n and $f : K \rightarrow \mathbb{R}$ be a continuous function. The minimization problem with the constraint set K and the objective function f is written formally as follows:

$$\text{OP}(K, f) : \text{ minimize } f(x) \text{ subject to } x \in K.$$

The solution set of $\text{OP}(K, f)$ is denoted by $\text{Sol}(K, f)$. By the closedness of K and the continuity of f , $\text{Sol}(K, f)$ is closed. Clearly, if $\text{Sol}(K, f)$ is nonempty then f is bounded from below on K . If K is bounded, then $\text{Sol}(K, f)$ is nonempty by the Bolzano-Weierstrass theorem.

Remark 2.2 Suppose that K is a cone and f is positively homogeneous function of degree $\alpha > 0$ relative to K . We can see that if $\text{Sol}(K, f)$ is nonempty then this set is a closed cone. Besides, it is not difficult to check that the following conditions are equivalent: $\text{Sol}(K, f)$ is nonempty; f is bounded from below on K ; 0 is a solution of $\text{OP}(K, f)$; f is non-negative on K . Furthermore, if one of these conditions is satisfied, then $\text{Sol}(K, f)$ is the set of zero points of f in K , i.e. $\text{Sol}(K, f) = \{x \in K : f(x) = 0\}$.

Assume that f is differentiable on an open subset $U \subset \mathbb{R}^n$. The gradient of f is denoted by ∇f . The function f is said to be *pseudoconvex* on U if, for any $x, y \in U$ such that $\langle \nabla f(x), y - x \rangle \geq 0$, we have $f(y) \geq f(x)$. Recall that f is pseudoconvex on U if and only if ∇f is *pseudomonotone* on U , i.e., if for any $x, y \in U$ such that $\langle \nabla f(x), y - x \rangle \geq 0$ then $\langle \nabla f(y), y - x \rangle \geq 0$ (see, e.g., [1, Theorem 4.4]). If f is convex on U then f is pseudoconvex on U . Note that if f, g are two convex functions on a convex set K then the sum $f + g$ is convex on K .

Remark 2.3 Assume that K is convex and f is pseudoconvex on an open set U containing K . If $x^0 \in \text{Sol}(K, f)$ then $\langle \nabla f(x), x - x^0 \rangle \geq 0$ for all $x \in K$. Indeed, since x^0 is a solution of $\text{OP}(K, f)$, one has $\langle \nabla f(x^0), x - x^0 \rangle \geq 0$ for all $x \in K$ (see, e.g., [1, Proposition 5.2]). The pseudomonotonicity of the gradient implies that $\langle \nabla f(x), x - x^0 \rangle \geq 0$ for all $x \in K$. Conversely, if the point $x^0 \in K$ satisfied $\langle \nabla f(x^0), x - x^0 \rangle \geq 0$ for all $x \in K$ then $x^0 \in \text{Sol}(K, f)$ (see, e.g., [1, Proposition 5.3]).

Let V be a closed cone contained in K^∞ . The following lemma provides a necessary condition for a vector to belong to the interior of $V^* - \nabla f(K)$. This lemma will be used in the proof of Theorems 5.2 and 6.2.

Lemma 2.1 *Assume that f is differentiable on an open set U containing K . If $u \in \text{int}(V^* - \nabla f(K)) \neq \emptyset$, then for each $v \in V \setminus \{0\}$ there is $x \in K$ such that $\langle \nabla f(x) + u, v \rangle > 0$.*

Proof Assume that $\text{int}(V^* - \nabla f(K))$ is nonempty and u belongs to this set. There exists an $\varepsilon > 0$ such that $\mathbb{B}(u, \varepsilon) \subset V^* - \nabla f(K)$, where $\mathbb{B}(u, \varepsilon)$ is the open ball of radius ε centered at u . Assume on the contrary that there is $\bar{v} \in V \setminus \{0\}$ such that $\langle \nabla f(x) + u, \bar{v} \rangle \leq 0$ for all $x \in K$. Let a be an arbitrary point in $\mathbb{B}(u, \varepsilon) \subset \text{int}(V^* - \nabla f(K))$. Then, there exist $w \in V^*$ and $\bar{x} \in K$ such that $a = w - \nabla f(\bar{x})$. This leads to

$$\langle a - u, \bar{v} \rangle = \langle w, \bar{v} \rangle - \langle \nabla f(\bar{x}) + u, \bar{v} \rangle \geq 0.$$

As the latter holds for any $a \in \mathbb{B}(u, \varepsilon)$, it follows that $\bar{v} = 0$; hence one gets a contradiction. \square

The following assumptions will be essential throughout the paper: K is nonempty, closed, and unbounded; f is continuous on the cone C containing K ; f is weakly homogeneous of degree $\alpha > 0$ relative to K .

3 Properties of the Asymptotic Optimization Problem

This section shows some basic properties of the asymptotic optimization problem associated to $\text{OP}(K, f)$. The asymptotic optimization problem plays a vital role to investigate the behavior of the original problem at infinity.

The following proposition says that asymptotically homogeneous functions in $[f_{\alpha, K}^{\infty}]$ have the same values on the asymptotic cone K^{∞} ; hence the asymptotic optimization problem is defined uniquely.

Proposition 3.1 *Suppose that h, \bar{h} are in $[f_{\alpha, K}^{\infty}]$. Then, $h = \bar{h}$ on K^{∞} ; hence $\text{Sol}(K^{\infty}, h) = \text{Sol}(K^{\infty}, \bar{h})$.*

Proof Let h, \bar{h} be two asymptotically homogeneous functions of degree α of f relative to K . Assume that $f = h + g$ and $f = \bar{h} + \bar{g}$, where $g, \bar{g} \in \mathcal{O}_K^{\alpha}$. Let $v \in K^{\infty}$ be given. There exist two sequences $\{t_k\} \subset \mathbb{R}_+$ and $\{x^k\} \subset K$ such that $t_k \rightarrow +\infty$ and $\lim_{k \rightarrow \infty} t_k^{-1} x^k = v$. Taking $y_k = t_k v$, one has $y_k \in K^{\infty}$ and

$$h(t_k v) + g(t_k v) = f(y_k) = \bar{h}(t_k v) + \bar{g}(t_k v), \quad (3.1)$$

for any k . By dividing the equations in (3.1) by t_k^{α} and letting $k \rightarrow +\infty$, we get $h(v) = \bar{h}(v)$. This conclusion holds for any $v \in K^{\infty}$; hence $h = \bar{h}$ on K^{∞} . \square

According to Proposition 3.1, $\text{Sol}(K^{\infty}, h)$ does not depend on the choice of h in $[f_{\alpha, K}^{\infty}]$. We can write a member of $[f_{\alpha, K}^{\infty}]$ simply by f^{∞} when no confusion arises. We denote

$$\mathcal{K}(K, f) := \text{Sol}(K^{\infty}, f^{\infty}).$$

From Remark 2.2, one can see that if $\mathcal{K}(K, f)$ is nonempty then $\mathcal{K}(K, f)$ is the set of zero points of f^{∞} in K^{∞} , i.e., $\mathcal{K}(K, f) = \{x \in K^{\infty} : f^{\infty}(x) = 0\}$. So, we call $\mathcal{K}(K, f)$ the *kernel* of the weakly homogeneous optimization problem $\text{OP}(K, f)$.

Proposition 3.2 *Assume that K is convex. One has the following inclusion*

$$\bigcup_{p \in \mathcal{O}_K^{\alpha}} \text{Sol}(K, f + p)^{\infty} \subset \mathcal{K}(K, f). \quad (3.2)$$

Furthermore, if K is a cone then the reverse inclusion of (3.2) holds.

Proof Assume $f = f^{\infty} + g$, where $g \in \mathcal{O}_K^{\alpha}$. Let $p \in \mathcal{O}_K^{\alpha}$ be given. Suppose that $\bar{x} \in \text{Sol}(K, f + p)^{\infty}$. There exist two sequences $\{x^k\} \subset \text{Sol}(K, f + p)$ and $\{t_k\} \subset \mathbb{R}_+$ such that $t_k \rightarrow +\infty$ and $t_k^{-1} x^k \rightarrow \bar{x}$. By assumptions, for each x^k , one has

$$f(y) + p(y) \geq f(x^k) + p(x^k) \quad (3.3)$$

for all $y \in K$. Let $u \in K$ be fixed. Since K is convex, K^∞ coincides with the recession cone of K . For every $v \in K^\infty$, one has $u + t_k v \in K$ for any k . Now, (3.3) implies that

$$f^\infty(u + t_k v) + g(u + t_k v) + p(u + t_k v) \geq f^\infty(x^k) + g(x^k) + p(x^k), \quad (3.4)$$

for any k . As $g, p \in \mathcal{O}_K^\alpha$, one can see that the following values: $g(u + t_k v)/t_k^\alpha$, $g(x^k)/t_k^\alpha$, $p(u + t_k v)/t_k^\alpha$, and $p(x^k)/t_k^\alpha$ go to 0 together as $k \rightarrow +\infty$. Therefore, by dividing the inequality in (3.4) by t_k^α and letting $k \rightarrow +\infty$, we obtain $f^\infty(v) \geq f^\infty(\bar{x})$. The assertion holds for every $v \in K^\infty$. We conclude that $\bar{x} \in \mathcal{K}(K, f)$, hence the inclusion (3.2) is proved.

Assume that K is a cone. Clearly, $f^\infty = f + (f^\infty - f)$ and $f^\infty - f \in \mathcal{O}_K^\alpha$. Hence, one has $\mathcal{K}(K, f) = \text{Sol}(K, f^\infty) = \text{Sol}(K, f + (f^\infty - f))$. It follows that the reverse inclusion holds. \square

Proposition 3.3 *If f is bounded from below on K , then $\mathcal{K}(K, f)$ is nonempty.*

Proof Let $v \in K^\infty$ be arbitrary. There exist two sequences $\{t_k\} \subset \mathbb{R}_+$ and $\{x^k\} \subset K$ such that $t_k \rightarrow +\infty$ and $\lim_{k \rightarrow \infty} t_k^{-1} x^k = v$. Assume that γ is a bound from below of f on K . One has $f(x^k) \geq \gamma$, for any k . By dividing the inequality by t_k^α and letting $k \rightarrow +\infty$, we get $f^\infty(v) \geq 0$. This conclusion implies f^∞ is non-negative on K^∞ . According to Remark 2.2, $\text{Sol}(K^\infty, f^\infty)$ is nonempty. \square

4 Nonemptiness and Boundedness of Solution Sets

In this section, we introduce two criteria for the nonemptiness and boundedness of $\text{Sol}(K, f)$ that are considered as extended versions of the Frank-Wolfe type theorem and the Eaves type theorem for polynomial optimization problems in [9]. Their proofs are modified from [9, Theorem 3.1] and [9, Theorem 3.2].

The following theorem asserts that the nonemptiness and boundedness property holds for $\text{Sol}(K, f)$ provided that the kernel is the trivial cone.

Theorem 4.1 *If $\mathcal{K}(K, f) = \{0\}$, then $\text{Sol}(K, f)$ is nonempty and bounded.*

Proof Suppose that $\mathcal{K}(K, f) = \{0\}$, i.e., $f^\infty(v) > 0$ for all $v \in K^\infty$ and $v \neq 0$. Take $x^0 \in K$ and define $M = \{x \in K : f(x^0) \geq f(x)\}$. Clearly, M is nonempty and closed. Besides, it is not difficult to check that $\text{Sol}(K, f) = \text{Sol}(M, f)$.

We claim that M is bounded. Indeed, on the contrary, we suppose that M is unbounded. Then there exists a sequence $\{x^k\} \subset M$ such that $\|x^k\| \rightarrow +\infty$. Assume, without loss of generality, that $\|x^k\|^{-1} x^k \rightarrow v$ and $v \in K^\infty \setminus \{0\}$. For each k , we have $f(x^0) \geq f(x^k)$. Dividing both sides of this inequality by $\|x^k\|^\alpha$ and letting $k \rightarrow +\infty$, we obtain $0 \geq f^\infty(v)$. This is impossible because of $v \neq 0$ and $f^\infty(v) > 0$. Hence, M is bounded.

Since M is compact, the desired result follows from the Bolzano-Weierstrass theorem. \square

The next example illustrates Theorem 4.1.

Example 4.1 Consider the weakly homogeneous optimization problem given by f and K as in Example 2.1. One can see that $\mathcal{K}(K, f) = \{0\}$. According to Theorem 4.1, the solution set is nonempty and bounded. Meanwhile, we can see that $\text{Sol}(K, f) = \{(1, 0)\}$.

Remark 4.1 Our new notation (weakly homogeneous function) is stronger than that of [6] in sense that our kernel is a subset of $\text{Sol}(K^\infty, \hat{h})$, where \hat{h} is the asymptotically homogeneous function of f defined as in [6]. In Example 2.1, the asymptotic function in the sense of [6] is $\hat{h}(x_1, x_2) = x_1x_2$; hence, one has $\text{Sol}(K^\infty, \hat{h}) = \{(x_1, 0) : x_1 \geq 0\} \supset \{0\} = \mathcal{K}(K, f)$.

The following theorem considers the case that $\mathcal{K}(K, f)$ is non-trivial (i.e., unbounded) and the objective function f is pseudoconvex. A relationship between $\mathcal{K}(K, f)$ and $\nabla f(K)$ enables us to see the nonemptiness and boundedness of $\text{Sol}(K, f)$.

Theorem 4.2 *Assume that K is convex and $\mathcal{K}(K, f)$ is non-trivial. If f is pseudoconvex on an open set U containing K , then the following statements are equivalent:*

- (a) *For each $v \in \mathcal{K}(K, f) \setminus \{0\}$, there exists $x \in K$ such that $\langle \nabla f(x), v \rangle > 0$;*
- (b) *$\text{Sol}(K, f)$ is nonempty and bounded.*

Proof Suppose that f is pseudoconvex on an open set U containing K .

(a) \Rightarrow (b) Suppose that (a) holds. For each $k = 1, 2, \dots$, we denote $K_k = K \cap \mathbb{B}(0, k)$, where $\mathbb{B}(0, k)$ is the closed ball of radius k centered at the origin. Clearly, K_k is compact and convex. We can assume that K_k is nonempty. From the Bolzano-Weierstrass theorem, $\text{OP}(K_k, f)$ has a solution, say x^k .

We claim that the sequence $\{x^k\}$ is bounded. On the contrary, suppose that this sequence is unbounded and that $x^k \neq 0$ for all k , $\|x^k\|^{-1}x^k \rightarrow v$, where $v \in K^\infty$ with $\|v\| = 1$. For each k , we have

$$f(x) \geq f(x^k), \quad \forall x \in K_k. \quad (4.1)$$

By fixing $x \in K_1$, hence $x \in K_k$ for any k , dividing both sides of the inequality in (4.1) by $\|x^k\|^\alpha$, and letting $k \rightarrow +\infty$, we get $0 \geq f^\infty(v)$. This leads to $v \in \mathcal{K}(K, f) \setminus \{0\}$. For each k , since $x^k \in \text{Sol}(K_k, f)$ and f is pseudoconvex on the convex set K_k , from Remark 2.3, we have

$$\langle \nabla f(x), x - x^k \rangle \geq 0, \quad \forall x \in K_k. \quad (4.2)$$

Take an arbitrary $x \in K$. For k large enough, x belongs to K_k . Dividing both sides of the inequality in (4.2) by $\|x^k\|$ and letting $k \rightarrow +\infty$, we obtain $0 \geq \langle \nabla f(x), v \rangle$. This contradicts (a); hence $\{x^k\}$ is bounded.

We can assume that $x^k \rightarrow \bar{x}$. From (4.1), by the continuity of f , we see that \bar{x} solves $\text{OP}(K, f)$, so $\text{Sol}(K, f)$ is nonempty.

Let us give a brief proof for the boundedness of $\text{Sol}(K, f)$. We suppose on the contrary that there is an unbounded solution sequence $\{x^k\}$, $\|x^k\|^{-1}x^k \rightarrow v$, where $v \in K^\infty$ and $\|v\| = 1$. The inequalities in (4.1) and (4.2) hold for any

$x \in K_k$. By repeating the previous arguments, we can get a conclusion which contradicts (a). Hence, the first assertion is proved.

(b) \Rightarrow (a) Since K is convex, one has $K = K + K^\infty$. Suppose that $\text{Sol}(K, f)$ is nonempty and bounded, but on the contrary there exists $v \in \mathcal{K}(K, f) \setminus \{0\}$ such that $\langle \nabla f(x), v \rangle \leq 0$ for all $x \in K$. Let \bar{x} be a solution of $\text{OP}(K, f)$. For any $t \geq 0$, one has $\bar{x} + tv \in K$ and $\langle \nabla f(\bar{x} + tv), v \rangle \leq 0$. This implies that

$$\langle \nabla f(\bar{x} + tv), \bar{x} - (\bar{x} + tv) \rangle \geq 0.$$

The pseudoconvexity of f yields $f(\bar{x}) \geq f(\bar{x} + tv)$. Thus, $\bar{x} + tv \in \text{Sol}(K, f)$ for any $t \geq 0$. It follows that $\text{Sol}(K, f)$ is unbounded which contradicts our assumption. Hence, (a) holds. \square

To illustrate Theorem 4.2, we provide the following example.

Example 4.2 Consider the objective function $f(x_1, x_2) := \frac{1}{2}x_1^2 - x_1x_2 + x_2^3$ and the constraint set $K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 1, x_2 \geq 2\} \subset \mathbb{R}_+^2$. Clearly, $U := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 1\}$ is an open set containing K . The gradient and the Hessian matrix of f on U are determined respectively by

$$\nabla f = \begin{bmatrix} x_1 - x_2 \\ -x_1 + 3x_2^2 \end{bmatrix}, \quad H_f = \begin{bmatrix} 1 & -1 \\ -1 & 6x_2 \end{bmatrix}. \quad (4.3)$$

It is easy to check that K is convex and that H_f is positive semidefinite on the open set U ; hence f is convex on K . One has $K^\infty = \mathbb{R}_+^2$ and $f^\infty(x_1, x_2) = x_2^3$ with $\alpha = 3$. This yields

$$\mathcal{K}(K, f) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}. \quad (4.4)$$

For any v in $\mathcal{K}(K, f) \setminus \{0\}$, $v = (\beta, 0)$, where $\beta > 0$, by choosing $\bar{x} = (3, 2) \in K$, we get $\langle \nabla f(\bar{x}), v \rangle = \beta > 0$. According to Theorem 4.2, the solution set of $\text{OP}(K, f)$ is nonempty and bounded.

5 Linearly Parametric Optimization Problems

In the sequel, we assume that $\alpha > 1$. This section investigates the nonemptiness and boundedness of the solution sets of *linearly parametric* optimization problem $\text{OP}(K, f_u)$, where $u \in \mathbb{R}^n$ and

$$f_u(x) := f(x) + \langle u, x \rangle.$$

Theorem 5.1 *Assume that f is bounded from below on K and $\text{int}(\mathcal{K}(K, f)^*)$ is nonempty. If $u \in \text{int}(\mathcal{K}(K, f)^*)$, then $\text{Sol}(K, f_u)$ is nonempty and bounded.*

Proof Let γ be a lower bound of f on K . Suppose that u is an element of $\text{int}(\mathcal{K}(K, f)^*)$. We now prove the nonemptiness of $\text{Sol}(K, f_u)$. For each $k = 1, 2, \dots$, we define $K_k = K \cap \mathbb{B}(0, k)$. Clearly, K_k is compact. We can assume

that K_k is nonempty. By the Bolzano-Weierstrass theorem, $\text{OP}(K_k, f_u)$ has a solution, say x^k . One has

$$f(y) + \langle u, y \rangle \geq f(x^k) + \langle u, x^k \rangle \quad (5.1)$$

for all $y \in K_k$.

One claims that $\{x^k\}$ is a bounded sequence. Indeed, on the contrary, suppose that the sequence is unbounded. We can assume that $x^k \neq 0$ for all k , $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ with $\bar{x} \in K^\infty$ and $\|\bar{x}\| = 1$. Let y in K_1 be fixed. By dividing the inequality (5.1) by $\|x^k\|^\alpha$ and letting $k \rightarrow +\infty$, since $\alpha > 1$, we obtain $0 \geq f^\infty(\bar{x})$. By the boundedness from below of f on K , one can see that $f^\infty(\bar{x}) \geq 0$. It follows that $f^\infty(\bar{x}) = 0$; hence $\bar{x} \in \mathcal{K}(K, f)$. From (5.1), we can see that

$$f(y) + \langle u, y \rangle \geq \gamma + \langle u, x^k \rangle.$$

This leads to $0 \geq \langle u, \bar{x} \rangle$ which contradicts our assumption $u \in \text{int}(\mathcal{K}(K, f)^*)$. Thus, the sequence $\{x^k\}$ is bounded.

We can suppose that $x^k \rightarrow z$. Let $y \in K$ be given, then $y \in K_k$ for k large enough and (5.1) holds. By the continuity of f and (5.1), we have $f(y) + \langle u, y \rangle \geq f(z) + \langle u, z \rangle$. This conclusion holds for all $y \in K$; hence z solves $\text{OP}(K, f_u)$, and the nonemptiness of $\text{Sol}(K, f_u)$ follows.

To prove the boundedness of $\text{Sol}(K, f_u)$, on the contrary, assume that there exists an unbounded sequence $\{x^k\} \subset \text{Sol}(K, f_u)$, $x^k \neq 0$ for all k , and $\|x^k\|^{-1}x^k \rightarrow \bar{x}$. Repeating the previous arguments for $\{x^k\}$, we also obtain the facts that $\bar{x} \in \mathcal{K}(K, f)$ and $0 \geq \langle u, \bar{x} \rangle$. This implies $u \notin \text{int}(\mathcal{K}(K, f)^*)$ which contradicts our assumption. \square

We denote by $\mathcal{R}(K, f)$ the set of all $u \in \mathbb{R}^n$ such that $\text{OP}(K, f_u)$ has a solution, i.e.,

$$\mathcal{R}(K, f) := \{u \in \mathbb{R}^n : \text{Sol}(K, f_u) \neq \emptyset\}.$$

Corollary 5.1 *Assume that K is a pointed cone and f is bounded from below on K . Then, $\mathcal{R}(K, f)$ is nonempty.*

Proof By Theorem 5.1, one has $\text{int}(\mathcal{K}(K, f)^*) \subset \mathcal{R}(K, f)$. Because K is a cone, we have $\mathcal{K}(K, f) \subset K$ and $K^* \subset \mathcal{K}(K, f)^*$. Since the cone K is pointed, K^* has a nonempty interior; hence $\text{int}(\mathcal{K}(K, f)^*)$ is nonempty. This implies the nonemptiness of $\mathcal{R}(K, f)$. \square

Example 5.1 Let us consider the polynomial optimization problem given by $f := x_2(x_1x_2 - 1)$ and $K := \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 1, x_1 - x_2 \geq 0, x_2 \geq 0\}$. It is clear that f is bounded from below on K . One has $f^\infty(x_1, x_2) = x_1x_2^2$ with $\alpha = 3$ and $K^\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \geq 0, x_2 \geq 0\}$. Therefore, $\mathcal{K}(K, f) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$. The interior of the dual cone of the kernel is determined by

$$\text{int}(\mathcal{K}(K, f)^*) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > 0\}.$$

According to Theorem 5.1, the solution set of $\text{OP}(K, f_u)$ is nonempty and bounded for any $u = (u_1, u_2)$ with $u_1 > 0$. To see the (un)boundedness of

$\text{OP}(K, f_u)$ when $u \notin \text{int}(\mathcal{K}(K, f)^*)$, we consider the two following cases where u belongs to the boundary of $\mathcal{K}(K, f)^*$ as follows:

- First, we take $u = (0, 1)$, then $f_u = x_1x_2^2$. It is not difficult to show that f_u is coercive on K , i.e., $f_u(x^k) \rightarrow +\infty$ as $\|x^k\| \rightarrow +\infty$. Hence, the solution set of $\text{OP}(K, f_u)$ is nonempty and bounded.
- Second, we choose $u = (0, 0)$, then $f_u = x_2(x_1x_2 - 1)$. This function is non-negative on K and vanishes on $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 = 1, x_1 \geq 1\}$. Hence, S is the solution set of $\text{OP}(K, f_u)$ that is unbounded.

Suppose that f is differentiable on an open set U containing K , we denote

$$\mathcal{D}(K, f) := (K^\infty)^* - \nabla f(K).$$

Remark 5.1 It is not difficult to see that $\mathcal{K}(K, f) = \mathcal{K}(K, f_u)$, for any $u \in \mathbb{R}^n$. Furthermore, if $\mathcal{K}(K, f_u)$ is nonempty, then $\mathcal{D}(K, f) \subset \mathcal{K}(K, f_u)^* - \nabla f(K)$ because of $\mathcal{K}(K, f_u) \subset K^\infty$.

The next theorem provides another sufficient condition concerning $\mathcal{D}(K, f)$ for the nonemptiness and boundedness of solutions to linearly parametric optimization problems.

Theorem 5.2 *Let K be a convex set. Assume that f is differentiable, convex, and bounded from below on K . If $u \in \text{int}(\mathcal{D}(K, f)) \neq \emptyset$, then $\text{Sol}(K, f_u)$ is nonempty and bounded.*

Proof Because f is bounded from below on K , from Proposition 3.3, $\mathcal{K}(K, f)$ is nonempty. Suppose that u belongs to $\text{int}(\mathcal{D}(K, f)) \neq \emptyset$. It follows from Remark 5.1 that $u \in \text{int}(\mathcal{K}(K, f_u)^* - \nabla f(K))$. Applying Lemma 2.1 to the case $V = \mathcal{K}(K, f_u)$, we conclude that, for each $v \in \mathcal{K}(K, f_u) \setminus \{0\}$, there is $x \in K$ such that $\langle \nabla f(x) + u, v \rangle > 0$. Note that $f + \langle u, \cdot \rangle$ is convex on K .

Because $\mathcal{K}(K, f)$ is nonempty, we have the following two cases: If $\mathcal{K}(K, f)$ is the trivial cone, then the nonemptiness and boundedness of $\text{Sol}(K, f_u)$ follows Theorem 4.1. If $\mathcal{K}(K, f)$ is non-trivial then, by applying Theorem 4.2 to the convex problem $\text{OP}(K, f_u)$, one asserts that $\text{Sol}(K, f_u)$ is nonempty and bounded. \square

For unconstrained optimization problems, we have the following property:

Corollary 5.2 *Assume that f is differentiable, convex on \mathbb{R}^n , and $\text{int}(\nabla f(\mathbb{R}^n))$ is nonempty. If $u \in -\text{int}(\nabla f(\mathbb{R}^n))$, then $\text{Sol}(\mathbb{R}^n, f_u)$ is nonempty and bounded.*

Proof The constraint set in the problem is $K = \mathbb{R}^n$. Because of $(K^\infty)^* = \{0\}$, one has $\mathcal{D}(\mathbb{R}^n, f) = -\nabla f(\mathbb{R}^n)$. The conclusion in the corollary is straightforward from Theorem 5.2.

Consider a quadratic optimization problem $\text{OP}(A, b, D, c)$ given by $f(x) := \frac{1}{2}x^T D x + c^T x$, where $D \in \mathbb{R}^{n \times n}$ is symmetric, $c \in \mathbb{R}^n$, and $K := \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. We denote by $\text{Sol}(A, b, D, c)$ the solution

set of the problem. Assume that K is unbounded and $h(x) := \frac{1}{2}x^T Dx$ is an asymptotically homogeneous function of degree $\alpha = 2$ of f relative to K . Clearly, one has $\nabla f(x) = Dx$ and $K^\infty = \{x \in \mathbb{R}^n : Ax \leq 0\}$.

The following corollary follows directly from Theorem 5.2.

Corollary 5.3 *Assume that D is positive semidefinite on K . Then, the set $\text{Sol}(A, b, D, c + u)$ is nonempty and bounded if $u \in \text{int}((K^\infty)^* - D(K)) \neq \emptyset$.*

To illustrate Corollary 5.3, we consider the following example.

Example 5.2 Consider the convex quadratic optimization problem given by $f(x_1, x_2) := (x_1 - 2x_2 + 1)^2$ and $K := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$. We have $(K^\infty)^* = \{(0, \mu) : \mu \in \mathbb{R}\}$ and $\nabla f(K) = \{\gamma(1, -2) : \gamma \in \mathbb{R}\}$. Besides, $\mathcal{D}(K, f) = \{(-\gamma, \mu + 2\gamma) : \mu, \gamma \in \mathbb{R}\} = \mathbb{R}^2$. From Theorem 5.2, $\text{Sol}(K, f_u)$ is nonempty and bounded, for any u in \mathbb{R}^2 .

6 Upper Semicontinuity of the Solution Map

This section is devoted to investigate the upper semicontinuity of the solution map

$$\mathcal{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \quad \mathcal{S}(u) = \text{Sol}(K, f_u).$$

The two sets $\mathcal{K}(K, f)$ and $\mathcal{D}(K, f)$ play vital roles in this investigation.

Recall that a set-valued map $\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *locally bounded* at \bar{x} if there exists an open neighborhood U of \bar{x} such that $\cup_{x \in U} \Phi(x)$ is bounded. The map Φ is *upper semicontinuous* at $x \in T$, where T is a subset of \mathbb{R}^n , if and only if for any open set $V \subset \mathbb{R}^n$ such that $\Phi(x) \subset V$ there exists a neighborhood U of x such that $\Phi(x') \subset V$ for all $x' \in U$. If Φ is upper semicontinuous at every $x \in T \subset \mathbb{R}^n$, then Φ is said that to be upper semicontinuous on T .

Remark 6.1 If the set-valued map Φ is closed, i.e., its graph

$$\text{gph}(\Phi) := \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n : v \in \Phi(u)\}$$

is closed in $\mathbb{R}^m \times \mathbb{R}^n$, and locally bounded at x , then Φ is upper semicontinuous at x [12, Theorem 5.19].

Remark 6.2 The solution map \mathcal{S} is closed. Indeed, we take a sequence $\{(u^k, x^k)\}$ in $\text{gph}(\mathcal{S})$ with $(u^k, x^k) \rightarrow (\bar{u}, \bar{x})$. It follows that $u^k \rightarrow \bar{u}$ and $x^k \rightarrow \bar{x}$. Let $y \in K$ be arbitrary. By definition, one has $f(y) + \langle u^k, y \rangle \geq f(x^k) + \langle u^k, x^k \rangle$ for any k . Taking $k \rightarrow +\infty$, by the continuity of f , we get $f(y) + \langle \bar{u}, y \rangle \geq f(\bar{x}) + \langle \bar{u}, \bar{x} \rangle$. Thus, $\bar{x} \in \text{Sol}(K, f_{\bar{u}})$; hence \mathcal{S} is closed.

Theorem 6.1 *Assume that f is bounded from below on K . Then, \mathcal{S} is upper semicontinuous on $\text{int}(\mathcal{K}(K, f)^*)$.*

Proof From Proposition 3.3, $\mathcal{K}(K, f)$ is nonempty. By Remarks 6.1 and 6.2, we need only to prove that \mathcal{S} is locally bounded at u , for $u \in \text{int}(\mathcal{K}(K, f)^*)$. Let $\varepsilon > 0$ be given, $\mathbb{B}(u, \varepsilon)$ and $\overline{\mathbb{B}}(u, \varepsilon)$ be the open ball and the closed ball, respectively, centered at u of radius ε . Consider the following sets:

$$L := \bigcup_{u \in \mathbb{B}(u, \varepsilon)} \mathcal{S}(u) \subset \bigcup_{u \in \overline{\mathbb{B}}(u, \varepsilon)} \mathcal{S}(u) =: R. \quad (6.1)$$

We will show that R is bounded. Suppose on the contrary that R is unbounded. Then, there exist an unbounded sequence $\{x^k\}$ and a bounded sequence $\{u^k\} \subset \overline{\mathbb{B}}(u, \varepsilon)$ such that x^k solves $\text{OP}(K, f_{u^k})$ with $x^k \neq 0$ for every k , $\|x^k\| \rightarrow +\infty$, and $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ with $\|\bar{x}\| = 1$. By the compactness of $\overline{\mathbb{B}}(u, \varepsilon)$, we can assume that $u^k \rightarrow \bar{u}$ with $\bar{u} \in \overline{\mathbb{B}}(u, \varepsilon)$.

By assumptions, for every k , one has

$$f(y) + \langle u^k, y \rangle \geq f(x^k) + \langle u^k, x^k \rangle \quad (6.2)$$

for all $y \in K$. Fix $\bar{y} \in K$. Then (6.2) yields $f(\bar{y}) + \langle u^k, \bar{y} \rangle \geq f(x^k) + \langle u^k, x^k \rangle$. Dividing this inequality by $\|x^k\|^\alpha$ and letting $k \rightarrow +\infty$, we get $0 \geq f^\infty(\bar{x})$. It follows from the nonemptiness of $\mathcal{K}(K, f)$ and Remark 2.2 that $\bar{x} \in \mathcal{K}(K, f)$. Furthermore, since f is bounded from below on K by γ , from (6.2) we have

$$f(\bar{y}) + \langle u^k, \bar{y} \rangle - \gamma \geq \langle u^k, x^k \rangle.$$

Dividing this inequality by $\|x^k\|$ and letting $k \rightarrow +\infty$, we get $\langle \bar{u}, \bar{x} \rangle \leq 0$. Since $\bar{u} \in \text{int}(\mathcal{K}(K, f)^*)$ and $\bar{x} \in \mathcal{K}(K, f)$, we have $\langle \bar{u}, \bar{x} \rangle > 0$. Hence, we have a contradiction. \square

When the kernel is trivial, we get the upcoming corollary of Theorem 6.1

Corollary 6.1 *If $\mathcal{K}(K, f) = \{0\}$, then \mathcal{S} is upper semicontinuous on \mathbb{R}^n .*

Proof The proof is straightforward from Theorem 6.1 with the note that the dual cone of $\{0\}$ is the whole space \mathbb{R}^n . \square

Example 6.1 The solution map of the optimization problem in Example 5.1 is upper semicontinuous on the open half-plane $\{(u_1, u_2) \in \mathbb{R}^2 : u_1 > 0\}$ by Theorem 6.1.

Theorem 6.2 *Assume that K is convex, f is differentiable, convex, and bounded from below on K . Then, \mathcal{S} is upper semicontinuous on $\text{int}(\mathcal{D}(K, f))$.*

Proof From Proposition 3.3, $\mathcal{K}(K, f)$ is nonempty. Let $u \in \text{int}(\mathcal{D}(K, f))$ and take $\varepsilon > 0$ such that $\overline{\mathbb{B}}(u, \varepsilon) \subset \text{int}(\mathcal{D}(K, f))$. We need to prove that \mathcal{S} is locally bounded at u . By repeating the argument and the notations from the proof of Theorem 6.1 with two sequences $\{x^k\}$, $\{u^k\}$ such that x^k solves $\text{OP}(K, f_{u^k})$, $\{x^k\}$ is unbounded, with $\|x^k\|^{-1}x^k \rightarrow \bar{x}$ and $u^k \rightarrow \bar{u}$, we can prove that $\bar{x} \in \mathcal{K}(K, f)$. Clearly, f_{u^k} is convex on K . By Remark 2.3, one has $\langle \nabla f(x) + u^k, x - x^k \rangle \geq 0$, for all $x \in K$. We fix $x \in K$, divide the last inequality by $\|x^k\|$, and let $k \rightarrow +\infty$; then we get $\langle \nabla f(x) + \bar{u}, \bar{x} \rangle \leq 0$. From Lemma 2.1, we conclude that $u \notin \text{int}(\mathcal{D}(K, f))$. This contradicts our assumption. \square

The following example illustrates the previous theorem.

Example 6.2 Consider the convex optimization problem given as in Example 5.2. As $\mathcal{D}(K, f) = \mathbb{R}^2 = \mathcal{R}(K, f)$, from Theorem 6.2, \mathcal{S} is upper semicontinuous on \mathbb{R}^2 .

Acknowledgments

The author would like to express his deep gratitude to the two referees for their corrections and valuable comments which helped to improve the manuscript. This work has been supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

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