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► **To cite this version:**

Arnaud Heibig, Pavel Krejčí, Adrien Petrov. Solvability for a dynamical beam problem on a frictionally damped foundation under a moving load. *Discrete and Continuous Dynamical Systems - Series B*, 2023, 28 (5), pp.3277-3293. 10.3934/dcdsb.2022217 . hal-03654473

HAL Id: hal-03654473

<https://hal.science/hal-03654473>

Submitted on 28 Apr 2022

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Solvability for a dynamical beam problem on a frictionally damped foundation under a moving load

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Abstract

This paper deals with a dynamic Euler-Bernoulli beam of infinite length subjected to a moving concentrated Dirac mass. The beam relies on a foundation composed of a continuous distribution of linear elastic springs associated in parallel with a uniform distribution of Coulomb friction elements and viscous dampers. The problem is stated in distributional form, and the existence and uniqueness results are established by means of a combination of L^∞ - L^2 - L^1 estimates together with a monotonicity argument. Traveling wave solutions are studied in detail in the case without Coulomb friction, and they are shown to be globally exponentially stable under positive viscous damping.

Keywords: Euler-Bernoulli beam equation, Coulomb friction law, existence and uniqueness results

2008 MSC: 35B45, 35D30, 49J40, 74H20, 74M10

1 Introduction

The increasing demand for sustainable development allowed a comeback of railway transport. This is a clear trend especially in Asia and Europe. The share in passenger and freight traffic increases during the last decades and it is accompanied with the development of high-speed railways. This

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Supported by GAČR Grant No. 20-14736S and by the European Regional Development Fund, Project No. CZ.02.1.01/0.0/0.0/16_019/0000778.

has in parallel stimulated mathematical models trying to describe the interactions between the rails and foundation which are in persistent frictional contact. The physical properties of the foundation, the so-called ballast composed of stones having different shapes and sizes, play a substantial role in the problem. The ballast track structures are successfully used nowadays all around the world in high-speed railway tracks. However, the track settlement is one of main causes of track deformations leading to a considerable maintenance effort. Thus a good understanding of this complex deterioration track mechanism is crucial to preserve the quality required by traffic safety and passenger comfort and to reduce track maintenance costs. The reliability of a railway track depends essentially on the limitation of the dynamic amplifications of the foundation behavior [17, 14]. It depends strongly on the ability of this structure to dissipate the energy transmitted by the transit of the moving loads. The train velocity, traffic loads, number of load cycles and other factors increase the energy that should be dissipated. For all these reasons, many studies are devoted to the foundation behavior (see [18, 14] and the references therein). It is also highlighted in [18, 9] that the properties of the foundation contribute substantially to possible deformations of the rail infrastructure, while the reduced scale experiments in [4] established that the sleepers subjected to sufficiently high accelerations increase strongly the settlement of the ballast.

Most mathematical models treat the rails as Euler-Bernoulli beams subjected to moving loads, and their properties are intensively studied. In the linear case, dynamic stresses are identified for beams under a moving load in [15, 19] and the transverse vibrations are investigated in [11]. Note that an analytical solution is exhibited for the free vibrations problem of a beam lying on an elastic foundation in [20], and [7, 8] contain some discrete approximations and numerical results for the problem of a vibrating beam between two rigid obstacles. However, the observed irreversible ballast accumulation in railway tracks experiments indicates that the main mechanism governing the interaction between the infrastructure constituents is driven by frictional contact. For this reason, it makes sense to study the dynamic rail behavior on a foundation including both frictional and visco-elastic dissipative mechanisms. Questions related to contact problems for Euler-Bernoulli beams have been studied in [2, 3]. More precisely, frictional contact at each end of the beam is treated in [3], while the distributed contact along the beam is investigated in [2].

We consider in this paper a straight infinite linear elastic Euler-Bernoulli beam with a constant cross sectional area A , central moment of inertia I ,

mass density ρ , and Young's modulus E , in persistent visco-elasto-frictional contact with a foundation. The mechanical properties of the foundation are represented by a continuous distribution of linear elastic springs with stiffness per unit length k associated in parallel with frictional and viscous dampers with yield limit $m \geq 0$ and damping coefficient $c \geq 0$, respectively. The present work is an extension of the results obtained in [13], where the force F depends on a time variable $t \in (0, T)$ and space variable $x \in \mathbb{R}$, and acts on the beam moving from left to right. Here the beam is activated by a concentrated force F moving at a constant velocity $v \in \mathbb{R}$ (see Figure 1).

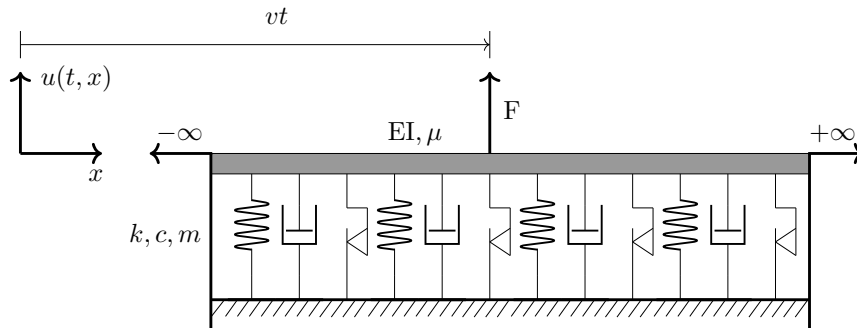


Figure 1: An infinite beam on a visco-elastic foundation subjected to frictional damping driven by a moving load.

The transverse displacement of the beam $u(t, x)$ for $(t, x) \in (0, T) \times \mathbb{R}$ is governed by a partial differential equation that can be formally written as

$$\mu u_{tt} + EI u_{xxxx} + m\xi + cu_t + ku = F\delta(x - vt), \quad \xi \in \text{sign}(u_t), \quad (1.1)$$

with positive constants μ, E, I, k, F . The constant μ is defined as the product $\mu \stackrel{\text{def}}{=} \rho A$ of the mass density ρ and the cross section area A and represents the mass of the beam per unit length. We denote by $(\cdot)_t \stackrel{\text{def}}{=} \frac{\partial(\cdot)}{\partial t}$ and $(\cdot)_x \stackrel{\text{def}}{=} \frac{\partial(\cdot)}{\partial x}$ the partial derivatives with respect to t and x , respectively. The symbol δ represents the Dirac measure on real line centered at 0, so that $\delta(x - vt)$ should be interpreted in distributional sense as

$$\int_0^T \int_{\mathbb{R}} \delta(x - vt) \phi(t, x) dx dt \stackrel{\text{def}}{=} \int_0^T \phi(t, vt) dt$$

for all $\phi \in L^1(0, T; C^0(\mathbb{R}))$. Last, as customary, “sign” denotes the multival-

ued sign function given by the formula

$$\text{sign}(z) = \begin{cases} \left\{ \frac{z}{|z|} \right\} & \text{for } z \neq 0, \\ [-1, 1] & \text{for } z = 0, \end{cases}$$

and we prescribe initial data

$$u(0, \cdot) = u^0 \quad \text{and} \quad u_t(0, \cdot) = u^1. \quad (1.2)$$

The paper is organized as follows. Section 2 is devoted to a rigorous formulation of (1.1) in distributional form in Problem 2.1 below, and to the statement of the main existence and uniqueness theorem. The first step in the existence argument consists in choosing in Section 3 an approximation parameter $\varepsilon > 0$, and replacing the singular terms in (1.1) by suitable regularizations. More specifically, we regularize the sign function by its Yosida approximation f^ε , and the Dirac measure by a smooth probability density ψ^ε with compact support on \mathbb{R} , with the intention to prove sufficient estimates independent of ε which will allow us to let ε tend to 0 in Section 4. The identification of the limit in Section 5 represents the most delicate step of the proof which consists in proving, using finer properties of the Yosida approximation, that the limit velocity u_t belongs to $L^1((0, T) \times \mathbb{R})$ although the sequence of its approximations u_t^ε is not bounded in this space. Uniqueness is proved in Section 6 by a standard monotonicity argument. Finally, Section 7 is devoted to an explicit computation of travelling wave solutions in the case without friction. We obtain not only complete agreement with previous numerical experiments (see [5], [6] and the references therein), but we prove in addition a global exponential stability result.

2 The mathematical model

We present in this section a rigorous meaning to (1.1). To this aim, we fix a final time $T > 0$ and introduce the following spaces

- $\mathcal{W} \stackrel{\text{def}}{=} H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ endowed with the following norm:

$$\|w\|_{\mathcal{W}} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}} (|w(x)|^2 + |w_{xx}(x)|^2) dx \right)^{1/2} + \int_{\mathbb{R}} |w(x)| dx,$$

- $\mathcal{V} \stackrel{\text{def}}{=} L^1(0, T; \mathcal{W})$,
- $\mathcal{K} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} u \in C^0([0, T] \times \mathbb{R}) : u_t \in L^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; L^2(\mathbb{R})), \\ u, u_{xx} \in L^\infty(0, T; L^2(\mathbb{R})), u_{tt} \in L^\infty(0, T; \mathcal{W}') \end{array} \right\},$

where \mathcal{W}' is the dual space to \mathcal{W} . This non-standard functional framework allows us to reformulate (1.1) rigorously as follows:

Problem 2.1. *For given initial data $u^0 \in H^2(\mathbb{R})$ and $u^1 \in L^2(\mathbb{R})$ find $u \in \mathcal{K}$ and $\xi \in L^\infty((0, T) \times \mathbb{R})$ such that (1.2) holds, and for all $\tau \in (0, T)$ and for all $\phi \in \mathcal{V}$ we have*

$$\bullet \int_0^\tau \left(\langle \mu u_{tt}, \phi \rangle + \int_{\mathbb{R}} (E I u_{xx} \phi_{xx} + m \xi \phi + c u_t \phi + k u \phi) dx \right) dt \quad (2.1a)$$

$$= F \int_0^\tau \phi(t, vt) dt = F \left(\Phi(\tau, v\tau) - v \int_0^\tau \Phi_x(t, vt) dt \right),$$

$$\bullet \xi(t, x) \in \text{sign}(u_t(t, x)) \text{ a. e.}, \quad (2.1b)$$

where we put $\Phi(t, x) \stackrel{\text{def}}{=} \int_0^t \phi(s, x) ds$, and $\langle \cdot, \cdot \rangle$ denotes the duality between \mathcal{W} and \mathcal{W}' .

Notice that Φ and Φ_x are continuous functions, so that (2.1b) is meaningful. Indeed, by using Sobolev embeddings, we have, for any $x \in \mathbb{R}$ and almost every $s \in (0, T)$: $|\phi_x(s, x)| \leq C \|\phi(s, \cdot)\|_{H^2(\mathbb{R})}$. We thus have

$$\sup_{(t,x) \in (0,T) \times \mathbb{R}} |\Phi_x(t, x)| \leq C \int_0^T \|\phi(t, \cdot)\|_{\mathcal{W}} dt. \quad (2.2)$$

Furthermore, for $\tau > t$ and $x \in \mathbb{R}$ we have

$$|\Phi_x(\tau, x) - \Phi_x(t, x)| = \int_t^\tau |\phi_x(s, x)| ds \rightarrow 0$$

as $t \rightarrow \tau$ uniformly with respect to $x \in \mathbb{R}$. Similarly, we have also that

$$|\Phi_x(t, y) - \Phi_x(t, x)| \leq \int_0^t \int_x^y |\phi_{xx}(s, z)| dz ds \leq |x - y|^{1/2} \int_0^T \|\phi(t, \cdot)\|_{\mathcal{W}} dt$$

uniformly with respect to $t \in [0, T]$.

The positive constants μ, E, I, F, k in (1.1) play no role in the mathematical analysis carried out in the next sections. Thus without loss of generality, we set them equal to 1.

The main result of this paper can be stated as follows.

Theorem 2.2. *Let $v \in \mathbb{R}$, $m \geq 0$, and $c \geq 0$ be given constants and let $u^0 \in H^2(\mathbb{R})$ and $u^1 \in L^2(\mathbb{R})$ be given. Then there exists a unique $u \in \mathcal{K}$ and a not necessary unique $\xi \in L^\infty((0, T) \times \mathbb{R})$ solving Problem 2.1 with $\mu = E = I = F = k = 1$.*

The proof of Theorem 2.2 is given in the next sections. We first choose a small regularization parameter $\varepsilon > 0$ and solve a regularized problem. A priori estimates independent of ε and a compactness and monotonicity technique allow us to pass to the limit as $\varepsilon \rightarrow 0$ and prove that the limit is the desired solution of the problem. Uniqueness is obtained by a standard monotonicity argument.

3 The approximate problem

The starting point in the proof of existence result is to approximate (1.1)–(1.2) by using the Yosida regularization for the sign function and a convolution-type mollifier for the delta function. In other words, for $\varepsilon \in (0, 1)$, the sign function in (1.1) is replaced by a stiff response represented by the function

$$f^\varepsilon(u) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } u \geq \varepsilon, \\ -1 & \text{for } u \leq -\varepsilon, \\ \frac{u}{\varepsilon} & \text{for } u \in (-\varepsilon, \varepsilon). \end{cases} \quad (3.1)$$

The Dirac measure is approximated by a probability density ψ^ε of the form $\psi^\varepsilon(x) = \frac{1}{\varepsilon}\psi(x/\varepsilon)$, where $\psi : \mathbb{R} \rightarrow [0, 1]$ is an even function of class C^2 such that $\text{supp } \psi \subset [-1, 1]$ and $\int_{\mathbb{R}} \psi(x) dx = 1$. We then replace (1.1)–(1.2) with the equation for a new unknown function u^ε

$$\begin{aligned} & \int_0^\tau \int_{\mathbb{R}} (u_{tt}^\varepsilon \phi + u_{xx}^\varepsilon \phi_{xx} + m f^\varepsilon(u_t^\varepsilon) \phi + cu_t^\varepsilon \phi + u^\varepsilon \phi) dx dt \\ &= \frac{1}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) \phi(t, x) dx dt \end{aligned} \quad (3.2)$$

for every $\phi \in L^2(0, T; H^2(\mathbb{R}))$, coupled with initial conditions

$$u^\varepsilon(0, x) = u^{0,\varepsilon}(x) \quad \text{and} \quad u_t^\varepsilon(0, x) = u^{1,\varepsilon}(x), \quad (3.3)$$

where $u^{0,\varepsilon} \in H^4(\mathbb{R})$ and $u^{1,\varepsilon} \in H^2(\mathbb{R})$ satisfy the conditions

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} (|u^{0,\varepsilon} - u^0|^2 + |u_{xx}^{0,\varepsilon} - u_{xx}^0|^2)(x) dx = 0, \quad (3.4a)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |u^{1,\varepsilon} - u^1|^2(x) dx = 0. \quad (3.4b)$$

The existence and uniqueness of solutions to (3.2)–(3.3) have been established in [13, Thm 3.2] in a more general setting using the classical semigroup theory. For our purposes, we recall the result in the following special form.

Theorem 3.1. *Let $T > 0$. Let $\alpha \in C^1([0, T]; L^2(\mathbb{R}))$, $u^0 \in H^4(\mathbb{R})$, and $u^1 \in H^2(\mathbb{R})$ be given, and let $\beta : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded nondecreasing Lipschitz continuous function such that $\beta(0) = 0$. Then there exists a unique solution $u \in \cap_{i=0}^2 C^{2-i}([0, T]; H^{2i}(\mathbb{R}))$ to the system*

$$\begin{cases} u_{tt} + u_{xxxx} + \beta(u_t) + cu_t + u = \alpha, & (3.5a) \\ u(0, \cdot) = u^0 \quad \text{and} \quad u_t(0, \cdot) = u^1. & (3.5b) \end{cases}$$

Remark 3.2. *We can deduce from Theorem 3.1 that for the solution u of (3.5) we have $u \in C^0([0, T] \times \mathbb{R})$ and $u_x \in C^0([0, T] \times \mathbb{R})$.*

As an immediate consequence of Theorem 3.1 we see that Problem (3.2)–(3.3) has for every $\varepsilon \in (0, 1)$ a unique solution u^ε with the regularity as in Theorem 3.1. The convergence as $\varepsilon \rightarrow 0$ and the identification of the limit will be established in the next sections.

4 Limit as $\varepsilon \rightarrow 0$

We first derive some estimates independent of ε for solutions to (3.2)–(3.3) which will be subsequently used for passing to the limit in the approximate system and for proving that the limit is a solution to (2.1). Let us summarize the estimates in the following statement.

Lemma 4.1. *The solution u^ε to (3.2)–(3.3) satisfies the estimate*

$$\bullet \int_0^T \int_{\mathbb{R}} m f^\varepsilon(u_t^\varepsilon) u_t^\varepsilon dx dt \tag{4.1}$$

$$+ \sup_{\tau \in (0, T)} \int_{\mathbb{R}} (|u_t^\varepsilon|^2 + |u_{xx}^\varepsilon|^2 + |u^\varepsilon|^2)(\tau, x) dx \leq C,$$

$$\bullet \int_0^T \int_{\mathbb{R}} u_{tt}^\varepsilon(t, x) \phi(t, x) dx dt \leq C \int_0^T \|\phi(t, \cdot)\|_{\mathcal{W}} dt \tag{4.2}$$

with a constant $C > 0$ independent of ε .

Proof. We test (3.2) with $\phi(x, t) = u_t^\varepsilon(x, t)$. For every $\tau \in (0, T)$ we get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} (|u_t^\varepsilon|^2 + |u_{xx}^\varepsilon|^2 + |u^\varepsilon|^2)(\tau, x) dx \\ & - \frac{1}{2} \int_{\mathbb{R}} (|u^{1, \varepsilon}|^2 + |u_{xx}^{0, \varepsilon}|^2 + |u^{0, \varepsilon}|^2)(x) dx \\ & + \int_0^\tau \int_{\mathbb{R}} (m f^\varepsilon(u_t^\varepsilon) u_t^\varepsilon + c |u_t^\varepsilon|^2) dx dt = \frac{1}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x - vt}{\varepsilon}\right) u_t^\varepsilon dx dt. \end{aligned} \tag{4.3}$$

We rewrite the right-hand side of (4.3) as

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) u_t^\varepsilon(t, x) \, dx \, dt \\
&= \frac{1}{\varepsilon} \int_0^\tau \frac{d}{dt} \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) u^\varepsilon(t, x) \, dx \, dt \\
&\quad + \frac{v}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}} \psi'\left(\frac{x-vt}{\varepsilon}\right) u^\varepsilon(t, x) \, dx \, dt \\
&= \frac{1}{\varepsilon} \int_{\mathbb{R}} \left(\psi\left(\frac{x-v\tau}{\varepsilon}\right) u^\varepsilon(\tau, x) - \psi\left(\frac{x}{\varepsilon}\right) u^{0,\varepsilon}(x) \right) dx \\
&\quad - \frac{v}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) u_x^\varepsilon(t, x) \, dx \, dt,
\end{aligned} \tag{4.4}$$

which implies that there exists a constant $C > 0$ independent of ε such that

$$\begin{aligned}
& \left| \frac{1}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) u_t^\varepsilon(t, x) \, dx \, dt \right| \\
& \leq C \left(\sup_{x \in \mathbb{R}} |u^{0,\varepsilon}(x)| + \sup_{x \in \mathbb{R}} |u^\varepsilon(\tau, x)| + \int_0^\tau \sup_{x \in \mathbb{R}} |u_x^\varepsilon(t, x)| \, dt \right).
\end{aligned} \tag{4.5}$$

Using the L^1 - L^∞ convolution inequality and Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we get

$$\sup_{x \in \mathbb{R}} |u^{0,\varepsilon}(x)| \leq C \|u^0\|_{H^1(\mathbb{R})} \tag{4.6}$$

The same embedding provides

$$\sup_{x \in \mathbb{R}} (|u^\varepsilon(t, x)| + |u_x^\varepsilon(t, x)|) \leq C \|u^\varepsilon(t, \cdot)\|_{H^2(\mathbb{R})} \tag{4.7}$$

for almost every $t \in (0, T)$ and a constant $C > 0$ independent of ε . According to (4.4)–(4.7), we may deduce from (4.3) that

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} (|u_t^\varepsilon|^2 + |u_{xx}^\varepsilon|^2 + |u^\varepsilon|^2)(\tau, x) \, dx + \int_0^\tau \int_{\mathbb{R}} (mf^\varepsilon(u_t^\varepsilon)u_t^\varepsilon + c|u_t^\varepsilon|^2) \, dx \, dt \\
& \leq C \left(1 + \int_0^\tau \int_{\mathbb{R}} (|u_{xx}^\varepsilon|^2 + |u^\varepsilon|^2) \, dx \, dt \right)^{1/2}
\end{aligned}$$

with a constant $C > 0$ independent of ε . This is an inequality of the form

$$\dot{w}(\tau) + a(\tau) \leq C(1 + w(\tau))^{1/2}, \tag{4.8}$$

where (\cdot) denotes the derivative with respect to τ . Note that (4.8) can easily be integrated and we get $\sup_{\tau \in [0, T]} (w(\tau) + \dot{w}(\tau) + a(\tau)) \leq C$, which yields

in particular that

$$\begin{aligned} & \int_0^\tau \int_{\mathbb{R}} (mf^\varepsilon(u_t^\varepsilon)u_t^\varepsilon + c|u_t^\varepsilon|^2) dx dt \\ & + \sup_{\tau \in [0, T]} \int_{\mathbb{R}} (|u_t^\varepsilon|^2 + |u_{xx}^\varepsilon|^2 + |u^\varepsilon|^2) (\tau, x) dx \leq C, \end{aligned} \quad (4.9)$$

and (4.1) follows.

To prove the inequality (4.2), we first notice that (3.2) holds for all $\phi \in \mathcal{V}$ as a consequence of (4.1) and of the identity

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) \phi(t, x) dx dt = \frac{1}{\varepsilon} \int_{\mathbb{R}} \psi\left(\frac{x-v\tau}{\varepsilon}\right) \Phi(\tau, x) dx \\ & - \frac{v}{\varepsilon} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon}\right) \Phi_x(t, x) dx dt \end{aligned} \quad (4.10)$$

with Φ as in (2.1a). Inequality (4.2) now follows from (3.2), (2.2), (4.1), and (4.10) by comparison. \square

Due to the fact that the beam is assumed to be infinite, we cannot expect that the estimates in Lemma 4.1 would imply any compactness on the whole real axis. Instead, for any $R > 0$, we denote $I_R \stackrel{\text{def}}{=} [-R, R]$ and consider the restrictions of u^ε to $(0, T) \times I_R$. We first notice that $(u^\varepsilon)_{\varepsilon > 0}$ are bounded in $L^\infty(0, T; H^2(I_R))$ and $(u_t^\varepsilon)_{\varepsilon > 0}$ are bounded in $L^\infty(0, T; L^2(I_R))$. Using the Aubin-Lions lemma ([16, Theorem 5.1]), we can extract a subsequence $(\varepsilon_i)_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ and

$$\lim_{i \rightarrow \infty} \|u^{\varepsilon_i} - u\|_{L^\infty(0, T; L^2(I_R))} = 0 \quad (4.11)$$

for some $u \in L^\infty(0, T; H^2(\mathbb{R}))$ and for every $R > 0$. We need in the sequel the following stronger convergence result.

Lemma 4.2. *Let u and ε_i be as in (4.11), and let $R > 0$ be arbitrarily chosen. Then $(u^{\varepsilon_i})_{i \in \mathbb{N}}$ and $(u_x^{\varepsilon_i})_{i \in \mathbb{N}}$ converge strongly in $C^0([0, T] \times I_R)$ to u and u_x , respectively.*

Proof. Using the embedding $H^{5/3}(I_R) \hookrightarrow W^{1, \infty}(I_R)$ and the interpolation inequality, we deduce that there exists a constant $C > 0$ such that for all $i, j \in \mathbb{N}$ we have

$$\begin{aligned} & \|u^{\varepsilon_i} - u^{\varepsilon_j}\|_{L^\infty((0, T) \times I_R)} + \|u_x^{\varepsilon_i} - u_x^{\varepsilon_j}\|_{L^\infty((0, T) \times I_R)} \\ & \leq C \|u^{\varepsilon_i} - u^{\varepsilon_j}\|_{L^\infty(0, T; H^{5/3}(I_R))} \\ & \leq C \|u^{\varepsilon_i} - u^{\varepsilon_j}\|_{L^\infty(0, T; L^2(I_R))}^{1/6} \|u^{\varepsilon_i} - u^{\varepsilon_j}\|_{L^\infty(0, T; H^2(I_R))}^{5/6}. \end{aligned} \quad (4.12)$$

Since $(u^{\varepsilon_i} - u^{\varepsilon_j})_{(i,j) \in \mathbb{N}^2}$ are bounded in $L^\infty(0, T; H^2(I_R))$, we conclude using (4.11) that $(u^{\varepsilon_i})_{i \in \mathbb{N}}, (u_x^{\varepsilon_i})_{i \in \mathbb{N}}$ are Cauchy sequences in $C^0([0, T] \times I_R)$, and the assertion follows. \square

As a direct consequence of Lemma 4.1 and Lemma 4.2, we have the following result.

Lemma 4.3. *There exist $u \in \mathcal{K}$, $\xi \in L^\infty((0, T) \times \mathbb{R})$, and a sequence $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$ such that $u^{\varepsilon_i}, u_x^{\varepsilon_i}$ converge for $i \rightarrow \infty$ locally uniformly to u, u_x , respectively, and*

- $u_t^{\varepsilon_i} \xrightarrow{*} u_t$ in $L^\infty(0, T; L^2(\mathbb{R}))$,
- $u_{xx}^{\varepsilon_i} \xrightarrow{*} u_{xx}$ in $L^\infty(0, T; L^2(\mathbb{R}))$,
- $f^{\varepsilon_i}(u_t^{\varepsilon_i}) \xrightarrow{*} \xi$ in $L^\infty((0, T) \times \mathbb{R})$,

where the symbol $\xrightarrow{*}$ denotes the weak-star convergence.

Indeed, the identification of ξ represents a major problem. Note that the argument of [13] cannot be applied for lack of pointwise convergence, and we have to proceed instead in several consecutive steps. The first one is the L^1 bound for u_t established in Lemma 4.4 below.

Lemma 4.4. *Under the hypotheses of Lemma 4.3 we have*

$$\bullet u_{tt}^{\varepsilon_i} \xrightarrow{*} u_{tt} \text{ in } \mathcal{V}', \quad (4.13)$$

$$\bullet \int_0^T \int_{\mathbb{R}} |u_t| dx dt \leq \liminf_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}} f^{\varepsilon_i}(u_t^{\varepsilon_i}) u_t^{\varepsilon_i} dx dt \leq C, \quad (4.14)$$

$$\begin{aligned} \bullet & \frac{1}{2} \int_{\mathbb{R}} (|u_t|^2 + |u_{xx}|^2 + |u|^2)(\tau, x) dx + c \int_0^\tau \int_{\mathbb{R}} |u_t|^2 dx dt \\ & + \liminf_{\varepsilon \rightarrow 0} \int_0^\tau \int_{\mathbb{R}} m f^\varepsilon(u_t^\varepsilon) u_t^\varepsilon dx dt \leq u(\tau, v\tau) - u(0, 0) \\ & - v \int_0^\tau u_x(t, vt) dt + \frac{1}{2} \int_{\mathbb{R}} (|u^1|^2 + |u_{xx}^0|^2 + |u^0|^2)(x) dx. \end{aligned} \quad (4.15)$$

Proof. The convergence (4.13) follows from (4.2). To prove (4.14), we notice that for each $\eta \in (0, 1)$ and $R > 0$ we have

$$\begin{aligned} \int_0^T \int_{-R}^R f^\eta(u_t) u_t dx dt &= \lim_{i \rightarrow \infty} \int_0^T \int_{-R}^R f^\eta(u_t) u_t^{\varepsilon_i} dx dt \\ &\leq \liminf_{i \rightarrow \infty} \int_0^T \int_{-R}^R |u_t^{\varepsilon_i}| dx dt. \end{aligned}$$

We have

$$\int_0^T \int_{-R}^R |u_t^{\varepsilon_i}| dx dt \leq \int_0^T \int_{\mathbb{R}} f^{\varepsilon_i}(u_t^{\varepsilon_i}) u_t^{\varepsilon_i} dx dt + 2\varepsilon_i RT.$$

Hence, we find

$$\int_0^T \int_{-R}^R f^\eta(u_t) u_t dx dt \leq \liminf_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}} f^{\varepsilon_i}(u_t^{\varepsilon_i}) u_t^{\varepsilon_i} dx dt$$

for every $R > 0$ and $\eta > 0$. Passing to the limit as $\eta \rightarrow 0$, and by monotone convergence for $R \rightarrow \infty$ we obtain (4.14).

To prove (4.15), we use the weak-star convergences in Lemma 4.3 to get for all $\tau \in (0, T)$ that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} (|u_t|^2 + |u_{xx}|^2 + |u|^2)(\tau, x) dx + c \int_0^\tau \int_{\mathbb{R}} |u_t(t, x)|^2 dx dt \\ & \leq \liminf_{i \rightarrow \infty} \left(\frac{1}{2} \int_{\mathbb{R}} (|u_t^{\varepsilon_i}|^2 + |u_{xx}^{\varepsilon_i}|^2 + |u^{\varepsilon_i}|^2)(\tau, x) dx \right. \\ & \quad \left. + c \int_0^\tau \int_{\mathbb{R}} |u_t^{\varepsilon_i}(t, x)|^2 dx dt \right). \end{aligned} \quad (4.16)$$

The next goal is to pass to the limit in (4.3) as $\varepsilon = \varepsilon_i \rightarrow 0$. It remains to prove convergence of the last term on the right-hand side of (4.3) rewritten in the form (4.4). Note that the support of ψ is contained in $[-1, 1]$. Hence, assuming that $\varepsilon_i < 1$ for all $i \in \mathbb{N}$ we have

$$\begin{aligned} & \frac{v}{\varepsilon_i} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon_i}\right) u_x^{\varepsilon_i}(t, x) dx dt \\ & = \frac{v}{\varepsilon_i} \int_0^\tau \int_{-|v|T-1}^{|v|T+1} \psi\left(\frac{x-vt}{\varepsilon_i}\right) u_x^{\varepsilon_i}(t, x) dx dt \\ & = \frac{v}{\varepsilon_i} \int_0^\tau \int_{-|v|T-1}^{|v|T+1} \psi\left(\frac{x-vt}{\varepsilon_i}\right) (u_x^{\varepsilon_i} - u_x)(t, x) dx dt \\ & \quad + \frac{v}{\varepsilon_i} \int_0^\tau \int_{-|v|T-1}^{|v|T+1} \psi\left(\frac{x-vt}{\varepsilon_i}\right) u_x(t, x) dx dt. \end{aligned} \quad (4.17)$$

Let us estimate the right-hand side of (4.17). First, by using Lemma 4.2 we observe that

$$\begin{aligned} & \left| \frac{v}{\varepsilon_i} \int_0^\tau \int_{-|v|T-1}^{|v|T+1} \psi\left(\frac{x-vt}{\varepsilon_i}\right) (u_x^{\varepsilon_i} - u_x)(t, x) dx dt \right| \\ & \leq C \|u_x^{\varepsilon_i} - u_x\|_{L^\infty((0, T) \times (-|v|T-1, |v|T+1))} \rightarrow 0 \end{aligned} \quad (4.18)$$

as $i \rightarrow \infty$. The fact that $u_{xx}^{\varepsilon_i}$ are bounded in $L^\infty(0, T; L^2(\mathbb{R}))$ yields that there exists a constant $C > 0$ such that

$$|u_x(t, x) - u_x(t, vt)| = \left| \int_{vt}^x u_{xx}(t, y) dy \right| \leq C|x - vt|^{1/2}, \quad (4.19)$$

which implies that

$$\begin{aligned} & \lim_{i \rightarrow +\infty} \frac{v}{\varepsilon_i} \int_0^\tau \int_{-|v|T-1}^{|v|T+1} \psi\left(\frac{x-vt}{\varepsilon_i}\right) u_x(t, x) dx dt \\ &= v \int_0^\tau u_x(t, vt) dt. \end{aligned} \quad (4.20)$$

By using (4.18) and (4.20), we may conclude that

$$\begin{aligned} & \lim_{i \rightarrow +\infty} \frac{1}{\varepsilon_i} \int_0^\tau \int_{\mathbb{R}} \psi\left(\frac{x-vt}{\varepsilon_i}\right) u_t^{\varepsilon_i}(t, x) dx dt \\ &= u(\tau, v\tau) - u(0, 0) - v \int_0^\tau u_x(t, vt) dt. \end{aligned} \quad (4.21)$$

Combining (4.3) with (3.4), (4.16), and (4.21), we obtain (4.15). \square

5 Identification of the limit

There is nothing to prove if $m = 0$. Indeed, using the convergences in Lemmas 4.3 and 4.4 we can directly pass to the limit as $\varepsilon \rightarrow 0$ in (3.2) and (4.10) and check that the limit is a solution to (2.1a). Hence, assume in the sequel that $m > 0$. We can still pass to the limit as $\varepsilon \rightarrow 0$ in (3.2) and (4.10) and obtain (2.1a). However, (2.1b) is not a direct consequence of the weak convergence results, and a monotonicity argument will be necessary.

A natural idea would be to choose $\phi = u_t$ in (2.1a). This is, however, not possible, as u_t is not sufficiently regular. Instead, we take the function ψ as in (3.2) and for $\alpha > 0$ put

$$u^\alpha(t, x) = \frac{1}{\alpha^2} \iint_{\mathbb{R}^2} \psi\left(\frac{x-y}{\alpha}\right) \psi\left(\frac{z-y}{\alpha}\right) u(t, z) dz dy, \quad (5.1a)$$

$$\widehat{u}^\alpha(t, x) = \frac{1}{\alpha} \int_{\mathbb{R}} \psi\left(\frac{x-y}{\alpha}\right) u(t, y) dy, \quad (5.1b)$$

and choose $\phi = u_t^\alpha$ in (2.1a). We claim that we obtain

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} (|\widehat{u}_t^\alpha|^2 + |\widehat{u}_{xx}^\alpha|^2 + |\widehat{u}^\alpha|^2) (\tau, x) dx \\
& + \int_0^\tau \int_{\mathbb{R}} (m\xi u_t^\alpha + c|\widehat{u}_t^\alpha|^2) (t, x) dx dt \\
& = \frac{1}{2} \int_{\mathbb{R}} (|\widehat{u}_t^\alpha|^2 + |\widehat{u}_{xx}^\alpha|^2 + |\widehat{u}^\alpha|^2) (0, x) dx + \int_0^\tau u_t^\alpha(t, vt) dt.
\end{aligned} \tag{5.2}$$

The only term which may perhaps need some explanation is the identity

$$\int_0^\tau \langle u_{tt}, u_t^\alpha \rangle (t) = \int_0^\tau \int_{\mathbb{R}} (\widehat{u}_{tt}^\alpha \widehat{u}_t^\alpha)(t, x) dx dt.$$

This follows from the general identity

$$\langle w, \psi * (\psi * \phi) \rangle = \int_{\mathbb{R}} (\psi * w)(x) (\psi * \phi)(x) dx, \tag{5.3}$$

where $\psi * \phi$ is the convolution product $(\psi * \phi)(x) = \int_{\mathbb{R}} \psi(x-y)\phi(y) dy$. Formula (5.3) holds for all $w, \phi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and by density of $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in \mathcal{W}' it can be extended to all elements $w \in \mathcal{W}'$.

In the identity (5.2), the convergences $\widehat{u}_t^\alpha(t, \cdot) \rightarrow u_t(t, \cdot)$, $\widehat{u}_{xx}^\alpha(t, \cdot) \rightarrow u_{xx}(t, \cdot)$, $\widehat{u}^\alpha(t, \cdot) \rightarrow u(t, \cdot)$, $u_t^\alpha(t, \cdot) \rightarrow u_t(t, \cdot)$ as α tends to 0 are strong in $L^2(\mathbb{R})$ for almost all $t \in (0, T)$, and $u_t^\alpha \rightarrow u_t$ strongly in $L^1((0, T) \times \mathbb{R})$. The right-hand side of (5.2) can be rewritten similarly as in (4.4), more specifically,

$$\begin{aligned}
& \int_0^\tau u_t^\alpha(t, vt) dt \\
& = u^\alpha(\tau, v\tau) - u^\alpha(0, 0) - \frac{v}{\alpha^3} \int_0^\tau \iint_{\mathbb{R}^2} \psi' \left(\frac{vt-y}{\alpha} \right) \psi \left(\frac{z-y}{\alpha} \right) u(t, z) dz dy dt \\
& = u^\alpha(\tau, v\tau) - u^\alpha(0, 0) + \frac{v}{\alpha^3} \int_0^\tau \iint_{\mathbb{R}^2} \psi \left(\frac{vt-y}{\alpha} \right) \psi' \left(\frac{z-y}{\alpha} \right) u(t, z) dz dy dt \\
& = u^\alpha(\tau, v\tau) - u^\alpha(0, 0) - \frac{v}{\alpha^2} \int_0^\tau \iint_{\mathbb{R}^2} \psi \left(\frac{vt-y}{\alpha} \right) \psi \left(\frac{z-y}{\alpha} \right) u_x(t, z) dz dy dt.
\end{aligned}$$

We now argue as in Section 4 and pass to the limit in (5.2) as α tends to 0

to get the identity

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} (|u_t|^2 + |u_{xx}|^2 + |u|^2)(\tau, x) dx + \int_0^\tau \int_{\mathbb{R}} (m\xi u_t + c|u_t|^2) dx dt \\
&= u(\tau, v\tau) - u(0, 0) - v \int_0^\tau u_x(t, vt) dt \\
&+ \frac{1}{2} \int_{\mathbb{R}} (|u^1|^2 + |u_{xx}^0|^2 + |u^0|^2)(x) dx.
\end{aligned} \tag{5.4}$$

From (4.15), (5.4) it follows that

$$\liminf_{\varepsilon \rightarrow 0} \int_0^\tau \int_{\mathbb{R}} f^\varepsilon(u_t^\varepsilon) u_t^\varepsilon dx dt \leq \int_0^\tau \int_{\mathbb{R}} \xi u_t dx dt \tag{5.5}$$

for all $\tau \in (0, T)$. Hence, by (4.14) and (5.5) we have the inequality

$$\int_0^T \int_{\mathbb{R}} |u_t| dx dt \leq \liminf_{i \rightarrow \infty} \int_0^T \int_{\mathbb{R}} f^{\varepsilon_i}(u_t^{\varepsilon_i}) u_t^{\varepsilon_i} dx dt \leq \int_0^T \int_{\mathbb{R}} \xi u_t dx dt,$$

which implies that $\xi(x, t) \in \text{sign}(u_t(x, t))$ almost everywhere. We have thus proved the following existence statement of Theorem 2.2.

Proposition 5.1. *Under the hypotheses of Theorem 2.2, there exists a solution to Problem 2.1.*

We conclude this section by proving the following higher regularity result.

Proposition 5.2. *Under the additional hypothesis*

$$u^0 \in H^4(\mathbb{R}) \quad \text{and} \quad u^1 \in H^2(\mathbb{R})$$

the solution to Problem 2.1 has the regularity $u_{tt} \in L^\infty(0, T, L^2(\mathbb{R}))$, and (2.1a) can be stated in the form

$$\int_0^\tau \int_{\mathbb{R}} (u_{tt}\phi + u_{xx}\phi_{xx} + m\xi\phi + cu_t\phi + u\phi) dx dt = \int_0^\tau \phi(t, vt) dt \tag{5.6}$$

for all $\phi \in \mathcal{V}$ and $\tau \in (0, T)$.

Proof of Proposition 5.2. We consider (3.2) in the form (3.5a), that is,

$$u_{tt}^\varepsilon - u_{xxxx}^\varepsilon + m f^\varepsilon(u_t^\varepsilon) + cu_t^\varepsilon + u^\varepsilon = \frac{1}{\varepsilon} \psi\left(\frac{x - vt}{\varepsilon}\right) \tag{5.7}$$

with initial conditions $u^\varepsilon(0, x) = u^0(x)$, $u_t^\varepsilon(0, x) = u^1(x)$. By virtue of Theorem 3.1, the identity (5.7) is satisfied in the sense of $L^2((0, T) \times \mathbb{R})$. For $(t, x) \in (0, T) \times \mathbb{R}$ and $\sigma > 0$ sufficiently small we introduce the functions

$$u^{\varepsilon, \sigma}(t, x) \stackrel{\text{def}}{=} \frac{1}{\sigma} (u^\varepsilon(t + \sigma, x + v\sigma) - u^\varepsilon(t, x)). \quad (5.8)$$

It follows from (5.7) that the functions $u^{\varepsilon, \sigma}$ satisfy the equation

$$u_{tt}^{\varepsilon, \sigma} - u_{xxxx}^{\varepsilon, \sigma} + \frac{m}{\sigma} (f^\varepsilon(u_t^\varepsilon(t + \sigma, x + v\sigma)) - f^\varepsilon(u_t^\varepsilon(t, x))) + cu_t^{\varepsilon, \sigma} + u^{\varepsilon, \sigma} = 0 \quad (5.9)$$

in the sense of $L^2((0, T) \times \mathbb{R})$. We now multiply (5.9) by $u_t^{\varepsilon, \sigma}$, integrate over $(0, t) \times \mathbb{R}$ for an arbitrary $t \in (0, T)$, and using integration by parts and the monotonicity of f^ε we obtain

$$\begin{aligned} & \int_{\mathbb{R}} (|u_t^{\varepsilon, \sigma}|^2 + |u_{xx}^{\varepsilon, \sigma}|^2 + |u^{\varepsilon, \sigma}|^2)(t, x) dx \\ & \leq \int_{\mathbb{R}} (|u_t^{\varepsilon, \sigma}|^2 + |u_{xx}^{\varepsilon, \sigma}|^2 + |u^{\varepsilon, \sigma}|^2)(0, x) dx. \end{aligned} \quad (5.10)$$

Still keeping $\varepsilon > 0$ fixed, we pass to the limit as $\sigma \rightarrow 0$ using the regularity of u^ε , which yields

$$\begin{aligned} & \int_{\mathbb{R}} (|u_{tt}^\varepsilon + vu_{xt}^\varepsilon|^2 + |u_{xxt}^\varepsilon + vu_{xxx}^\varepsilon|^2 + |u_t^\varepsilon + vu_x^\varepsilon|^2)(t, x) dx \\ & \leq \int_{\mathbb{R}} (|u_{tt}^\varepsilon + vu_{xt}^\varepsilon|^2 + |u_{xxt}^\varepsilon + vu_{xxx}^\varepsilon|^2 + |u_t^\varepsilon + vu_x^\varepsilon|^2)(0, x) dx \\ & \leq C (\|u^0\|_{H^4(\mathbb{R})} + \|u^1\|_{H^2(\mathbb{R})})^2 \end{aligned} \quad (5.11)$$

with a constant $C > 0$ independent of ε . Taking into account the fact that

$$\int_{\mathbb{R}} |u_{xt}^\varepsilon + vu_{xx}^\varepsilon|^2(t, x) dx \leq \int_{\mathbb{R}} (|u_{xxt}^\varepsilon + vu_{xxx}^\varepsilon|^2 + |u_t^\varepsilon + vu_x^\varepsilon|^2)(t, x) dx$$

and (4.1) we obtain that

$$\sup_{t \in (0, T)} \int_{\mathbb{R}} (|u_{tt}^\varepsilon + vu_{xt}^\varepsilon|^2 + |u_{xt}^\varepsilon + vu_{xx}^\varepsilon|^2 + |u_{xx}^\varepsilon|^2)(t, x) dx \leq C$$

with a constant $C > 0$ depending only on the initial data, hence,

$$\sup_{t \in (0, T)} \int_{\mathbb{R}} (|u_{tt}^\varepsilon|^2 + |u_{xt}^\varepsilon|^2 + |u_{xx}^\varepsilon|^2)(t, x) dx \leq C. \quad (5.12)$$

The regularity (5.12) is preserved in the limit as $\varepsilon \rightarrow 0$, which proves that $u_{tt} \in L^\infty(0, T, L^2(\mathbb{R}))$. Notice also that this implies compactness of u_t^ε in $C^0([0, T] \times [-R, R])$ for every $R > 0$, which would thus simplify the convergence argument. \square

6 Uniqueness

We establish in this section the uniqueness result by using a classical approach. Namely, let u_1 and u_2 be two solutions of Problem 2.1, and let $\bar{u} \stackrel{\text{def}}{=} u_1 - u_2$ and $\bar{\xi} \stackrel{\text{def}}{=} \xi_1 - \xi_2$. Then for every test function ϕ as in Theorem 2.2 the function \bar{u} satisfies the identity

$$\int_0^\tau \left(\langle \bar{u}_{tt}, \phi \rangle + \int_{\mathbb{R}} \bar{u}_{xx} \phi_{xx} + m \bar{\xi} \phi + c \bar{u}_t \phi + \bar{u} \phi \, dx \right) dt = 0. \quad (6.1)$$

We define for $\alpha > 0$ similarly as in (5.1)

$$\bar{u}^\alpha(t, x) \stackrel{\text{def}}{=} \frac{1}{\alpha^2} \iint_{\mathbb{R}^2} \psi\left(\frac{x-y}{\alpha}\right) \psi\left(\frac{z-y}{\alpha}\right) \bar{u}(t, z) \, dz \, dy, \quad (6.2a)$$

$$\tilde{u}^\alpha(t, x) \stackrel{\text{def}}{=} \frac{1}{\alpha} \int_{\mathbb{R}} \psi\left(\frac{x-y}{\alpha}\right) \bar{u}(t, y) \, dy, \quad (6.2b)$$

and choose $\phi = \bar{u}^\alpha$ in (6.1). Repeating the argument of (5.2)–(5.4) we pass to the limit as α tends to 0 and get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} (|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + |\bar{u}|^2) (\tau, x) \, dx \\ & + \int_0^\tau \int_{\mathbb{R}} (m \bar{\xi} \bar{u}_t + c |\bar{u}_t|^2) \, dx \, dt = 0. \end{aligned} \quad (6.3)$$

We have indeed $\bar{\xi}(t, x) \bar{u}_t(t, x) \geq 0$ a. e., hence $\bar{u} = 0$ and $u_1 = u_2$.

More generally, we can consider different initial conditions u_i^0, u_i^1 for $i = 1, 2$. Putting $\bar{u}^0 = u_1^0 - u_2^0$, $\bar{u}^1 = u_1^1 - u_2^1$ we obtain by the same argument the inequality

$$\int_{\mathbb{R}} (|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + |\bar{u}|^2) (\tau, x) \, dx \leq \int_{\mathbb{R}} (|\bar{u}^1|^2 + |\bar{u}_{xx}^0|^2 + |\bar{u}^0|^2) (x) \, dx, \quad (6.4)$$

which implies continuous dependence of the solution on the initial data.

7 Traveling wave solutions

In order to make a link between the theory developed in the previous sections and the classical approach to (2.1a), we first notice that if the nonlinearity ξ does not appear in (2.1a), then for some special initial conditions there may exist traveling wave solutions of the form $u(t, x) = w(x - vt)$. Let us first consider one special case of (1.1) with $m = c = 0$, that is,

$$\mu u_{tt} + E I u_{xxxx} + k u = \delta(x - vt). \quad (7.1)$$

In terms of the new unknown function w , this equation reads

$$\mu v^2 w^{ii}(z) + \text{EI} w^{iv}(z) + kw(z) = \delta(z), \quad (7.2)$$

where w^{ii} and w^{iv} denote the second and fourth derivatives with respect to z . The solution can be constructed in such a way that we solve the linear autonomous equations

$$\mu v^2 w_-^{ii}(z) + \text{EI} w_-^{iv}(z) + kw_-(z) = 0 \quad \text{for } z < 0, \quad (7.3)$$

$$\mu v^2 w_+^{ii}(z) + \text{EI} w_+^{iv}(z) + kw_+(z) = 0 \quad \text{for } z > 0, \quad (7.4)$$

and prescribe at $z = 0$ the transition conditions

$$\begin{aligned} w_+(0+) &= w_-(0-), \\ w_+^i(0+) &= w_-^i(0-), \\ w_+^{ii}(0+) &= w_-^{ii}(0-), \\ w_+^{iii}(0+) &= w_-^{iii}(0-) + 1, \end{aligned} \quad (7.5)$$

where w^i and w^{iii} denote the first and third derivatives with respect to z . The characteristic equation of both (7.3) and (7.4) reads

$$\text{EI}\lambda^4 + \mu v^2 \lambda^2 + k = 0, \quad (7.6)$$

hence

$$\lambda^2 = \frac{-\mu v^2 \pm \sqrt{\mu^2 v^4 - 4\text{EI}k}}{2\text{EI}}. \quad (7.7)$$

The condition that $u(t, \cdot)$ belongs to $L^2(\mathbb{R})$ requires that $w_-(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $w_+(x) \rightarrow 0$ as $x \rightarrow +\infty$. If $\mu^2 v^4 \geq 4\text{EI}k$, then the right-hand side of (7.7) is real and negative for both choices of the sign, so that the characteristic equation (7.6) has four roots on the imaginary axis, and no solution of (7.3)–(7.4) can asymptotically vanish at $\pm\infty$. The situation is different if $\mu^2 v^4 < 4\text{EI}k$. Then (7.7) is of the form $\lambda^2 = -p \pm ri$ with $p > 0$, $r > 0$, so that $\lambda = \pm a \pm bi$ with

$$a = \sqrt{\frac{1}{2}(\sqrt{p^2 + r^2} - p)} \quad \text{and} \quad b = \frac{r}{2a}.$$

The condition that w_- has to vanish at $-\infty$ and w_+ has to vanish at $+\infty$ implies that we search for solutions of the form

$$w_-(z) \stackrel{\text{def}}{=} e^{az}(\alpha \cos(bz) + \beta \sin(bz)), \quad (7.8a)$$

$$w_+(z) \stackrel{\text{def}}{=} e^{-az}(\hat{\alpha} \cos(bz) + \hat{\beta} \sin(bz)), \quad (7.8b)$$

where $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}) \in \mathbb{R}^4$. Conditions (7.5) read

$$\widehat{\alpha} = \alpha, \quad (7.9a)$$

$$-a\widehat{\alpha} + b\widehat{\beta} = a\alpha + b\beta, \quad (7.9b)$$

$$(a^2 - b^2)\widehat{\alpha} - 2ab\widehat{\beta} = (a^2 - b^2)\alpha + 2ab\beta, \quad (7.9c)$$

$$(-a^3 + 3ab^2)\widehat{\alpha} + (3a^2b - b^3)\widehat{\beta} = (a^3 - 3ab^2)\alpha + (3a^2b - b^3)\beta + 1, \quad (7.9d)$$

which is equivalent to

$$\begin{aligned} \widehat{\alpha} &= \alpha = \frac{1}{4a(a^2 + b^2)}, \\ \widehat{\beta} &= -\beta = \frac{1}{4b(a^2 + b^2)}. \end{aligned} \quad (7.10)$$

The above computations show that a traveling wave solution in $L^2(\mathbb{R})$ exists if and only if $|v|$ stays below the *critical velocity*

$$v_c \stackrel{\text{def}}{=} (4EI k / \mu^2)^{1/4}. \quad (7.11)$$

A similar formula to (7.11) for the critical velocity of load travelling on finite and infinite beams was exhibited in [6] and it is in accordance with experimental and numerical results (see, for instance, [4, 5]). The critical velocity is defined as the load velocity inducing the highest upward and/or downward deflections of the rail. In particular, upward displacements of the rail may have a negative impact on the stability and the lifetime of the system. In practice, however, the theoretical critical speed is still far from standard train velocities.

Note also that the nonexistence of traveling wave solutions for supercritical velocities is not in contradiction with the existence statement in Theorem 2.2. The point is that initial conditions of the traveling wave solution in the “bad case” $\mu^2 v^4 \geq 4EI k$ do not belong to $L^2(\mathbb{R})$, so that we can only conclude that the solution constructed in Theorem 2.2 does not have the form of traveling wave. On the other hand, in the “good case” $\mu^2 v^4 < 4EI k$, a traveling wave solution exists in $L^2(\mathbb{R})$ only for one particular choice of the initial condition, namely

$$u^0(x) \stackrel{\text{def}}{=} \begin{cases} w_-(x) & \text{for } x < 0 \\ w_+(x) & \text{for } x > 0 \end{cases} \quad \text{and} \quad u^1(x) \stackrel{\text{def}}{=} \begin{cases} w_-^i(x) & \text{for } x < 0 \\ w_+^i(x) & \text{for } x > 0 \end{cases},$$

which satisfies the hypotheses of Theorem 2.2.

If the damping coefficient $c > 0$ is present in the equation, the counterpart of the equation (7.2) for the traveling wave solution w is of the form

$$\mu v^2 w^{ii}(z) + EI w^{iv}(z) - cv w^i(z) + kw(z) = \delta(z). \quad (7.12)$$

As a first observation, we see that only the sign of the product cv has an influence in the solution, that is, we get the same solution for negative damping (which physically makes no sense) when the train goes backward in space.

The characteristic equation has the form

$$EI\lambda^4 + \mu v^2 \lambda^2 - cv\lambda + k = 0. \quad (7.13)$$

Consider for example the case $cv > 0$. The intersection of the straight line $\omega = cv\lambda$ with the “parabola” $\omega = EI\lambda^4 + \mu v^2 \lambda^2 + k$ for $\lambda \in \mathbb{R}$ is either empty, or consists of one or two points with positive values of λ . Moreover, the sum of the four complex roots is zero. Hence, for the roots of (7.13), only one of the following three cases can occur:

- (1) complex roots $a \pm bi$, $p \pm ri$, $p > 0$, $a + p = 0$;
- (2) complex roots $a \pm bi$, two real roots $0 < p_1 < p_2$, $2a + p_1 + p_2 = 0$;
- (3) complex roots $a \pm bi$, one double real root $p > 0$, $a + p = 0$.

Then (note that $a < 0$) the counterpart of (7.8) reads

$$\begin{aligned} w_+(z) &= e^{az}(\alpha \cos(bz) + \beta \sin(bz)), \\ w_-(z) &= \begin{cases} e^{pz}(\widehat{\alpha} \cos(rz) + \widehat{\beta} \sin(rz)) & \text{in case (1),} \\ \widehat{\alpha} e^{p_1 z} + \widehat{\beta} e^{p_2 z} & \text{in case (2),} \\ e^{pz}(\widehat{\alpha} + \widehat{\beta} z) & \text{in case (3).} \end{cases} \end{aligned} \quad (7.14)$$

System (7.5) is a linear system of four equations for four unknowns $\alpha, \beta, \widehat{\alpha}, \widehat{\beta}$, and the matrix of the system is nonsingular, hence it has a unique solution. The same conclusion holds if $cv < 0$ with the difference that p and p_i are negative and a is positive, so that we can only simply interchange w_+ and w_- . We conclude that even negative dissipation may give a “good” traveling wave solution. The difference is that while for $c \geq 0$ the traveling wave solution is stable with respect to perturbations of the initial condition by virtue of (6.4), for $c < 0$ the traveling wave solution is unstable. In fact, in the case $c > 0$, the traveling wave solution is *globally exponentially stable*, and the result can be stated as follows.

Proposition 7.1. *There exist constants $\kappa > 0$ and $M > 0$ such that if $u_1, u_2 \in \mathcal{K}$ are two solutions to (2.1) with $\mu = \mathbf{E} = \mathbf{I} = \mathbf{F} = k = 1$, $m = 0$, and $c > 0$, corresponding to initial conditions u_i^0, u_i^1 for $i = 1, 2$, then the function*

$$\mathcal{E}(t) = \int_{\mathbb{R}} (|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + |\bar{u}|^2)(t, x) dx, \quad (7.15)$$

where $\bar{u} = u_1 - u_2$, satisfies the estimate

$$\mathcal{E}(t) \leq M e^{-\kappa t}. \quad (7.16)$$

Proof. The argument of Section 6 yields for every $\tau > t \geq 0$ the following counterpart of (6.3) with $m = 0$

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} (|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + |\bar{u}|^2)(\tau, x) dx + c \int_t^\tau \int_{\mathbb{R}} |\bar{u}_t|^2(s, x) dx ds \\ &= \frac{1}{2} \int_{\mathbb{R}} (|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + |\bar{u}|^2)(t, x) dx. \end{aligned} \quad (7.17)$$

Furthermore, we can test (6.1) with $m = 0$ by $\phi = \bar{u}$, which is an admissible test function, and obtain

$$\begin{aligned} & \frac{d}{ds} \int_{\mathbb{R}} \left(\frac{c}{2} |\bar{u}|^2 + \bar{u}_t \bar{u} \right) (s, x) dx + \int_{\mathbb{R}} (|\bar{u}_{xx}|^2 + |\bar{u}|^2) (s, x) dx \\ &= \int_{\mathbb{R}} |\bar{u}_t|^2(s, x) dx. \end{aligned} \quad (7.18)$$

Integrating (7.18) from t to τ yields that

$$\begin{aligned} & \int_{\mathbb{R}} \left(\frac{c}{2} |\bar{u}|^2 + \bar{u}_t \bar{u} \right) (\tau, x) dx + \int_t^\tau \int_{\mathbb{R}} (|\bar{u}_{xx}|^2 + |\bar{u}|^2) (s, x) dx \\ &= \int_t^\tau \int_{\mathbb{R}} |\bar{u}_t|^2(s, x) dx + \int_{\mathbb{R}} \left(\frac{c}{2} |\bar{u}|^2 + \bar{u}_t \bar{u} \right) (t, x) dx. \end{aligned} \quad (7.19)$$

We now multiply (7.19) by $c/2$, add the result to (7.17), denote

$$\mathcal{E}_0(t) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \left(|\bar{u}_t|^2 + |\bar{u}_{xx}|^2 + \left(1 + \frac{c^2}{2}\right) |\bar{u}|^2 + c \bar{u}_t \bar{u} \right) (t, x) dx,$$

and obtain the inequality

$$\mathcal{E}_0(\tau) - \mathcal{E}_0(t) + \frac{c}{2} \int_t^\tau \mathcal{E}(s) ds \leq 0 \quad (7.20)$$

for all $\tau > t \geq 0$. There exists a constant $\eta \in (0, 1)$ such that

$$\eta \mathcal{E}_0(t) \leq \mathcal{E}(t) \leq \frac{1}{\eta} \mathcal{E}_0(t) \quad (7.21)$$

for all $t \geq 0$. Hence, it follows from (7.20) that

$$\mathcal{E}_0(\tau) - \mathcal{E}_0(t) + \frac{c\eta}{2} \int_t^\tau \mathcal{E}_0(s) ds \leq 0 \quad (7.22)$$

for all $\tau > t \geq 0$. The function

$$G(t) \stackrel{\text{def}}{=} \mathcal{E}_0(t) + \frac{c\eta}{2} \int_0^t \mathcal{E}_0(s) ds$$

is therefore nonincreasing in $(0, +\infty)$, hence the Stieltjes integral inequality

$$0 \geq \int_0^T q(t) dG(t) = [q(t)G(t)]_0^T - \int_0^T q'(t)G(t) dt \quad (7.23)$$

holds for every positive absolutely continuous function q . Choosing in particular

$$q(t) = e^{(c\eta/2)t}$$

we have

$$q'(t)G(t) = \frac{c\eta}{2} \frac{d}{dt} \left(e^{(c\eta/2)t} \int_0^t \mathcal{E}_0(s) ds \right),$$

and we obtain from (7.23) that

$$e^{(c\eta/2)T} \left(\mathcal{E}_0(T) + \frac{c\eta}{2} \int_0^T \mathcal{E}_0(s) ds \right) - \mathcal{E}_0(0) \leq \frac{c\eta}{2} e^{(c\eta/2)T} \int_0^T \mathcal{E}_0(s) ds.$$

We conclude that the inequality

$$\mathcal{E}_0(T) \leq e^{-(c\eta/2)T} \mathcal{E}_0(0)$$

holds for all $T > 0$, and the assertion follows from (7.21). \square

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