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► **To cite this version:**

Kostia Chardonnet, Marc de Visme, Benoît Valiron, Renaud Vilmart. The Many-Worlds Calculus: Representing Quantum Control. 2022. hal-03654190v1

**HAL Id: hal-03654190**

**<https://hal.science/hal-03654190v1>**

Preprint submitted on 28 Apr 2022 (v1), last revised 3 Aug 2022 (v2)

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# The Many-Worlds Calculus: Representing Quantum Control

Kostia Chardonnet ✉ 🏠

Université Paris-Saclay, CNRS, ENS Paris-Saclay, LMF, 91190, Gif-sur-Yvette, France.

Université Paris Cité, CNRS, IRIF, F-75006, Paris, France

Marc de Visme ✉ 🏠

Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, LMF, 91190, Gif-sur-Yvette, France

Benoît Valiron ✉ 🏠 

Université Paris-Saclay, CNRS, CentraleSupélec, ENS Paris-Saclay, LMF, 91190, Gif-sur-Yvette, France

Renaud Vilmart ✉ 🏠 

Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, LMF, 91190, Gif-sur-Yvette, France

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## Abstract

We propose a new typed graphical language for quantum computation, based on compact categories with biproducts. Our language generalizes existing approaches such as ZX-calculus and quantum circuits, while offering a natural framework to support quantum control: it natively supports “quantum tests”.

The language comes equipped with a denotational semantics based on linear applications, and an equational theory. Through the use of normal forms for the diagrams, we prove the language to be universal, and the equational theory to be complete with respect to the semantics.

**2012 ACM Subject Classification** Theory of computation → Quantum computation theory; Theory of computation → Linear logic; Theory of computation → Equational logic and rewriting

**Keywords and phrases** Quantum Computation, Linear Logic, Graphical Languages, Geometry of Interaction

**Digital Object Identifier** 10.4230/LIPIcs.CVIT.2016.23

## 1 Introduction

Conventional wisdom has it that quantum computation is about *quantum data in superposition*. In the standard model, the memory holding quantum data is encapsulated inside a coprocessor accessed through a simple interface: The coprocessor holds individually addressable registers holding *quantum* bits, on which one can apply a fixed set of operations — *gates*— specified by the interface. If some of these gates can generate superposition of data, this is kept inside the coprocessor and opaque to the programmer. A typical interaction with the coprocessor is a purely classical sequence of elementary operations of the form “Apply gate X to register  $n$ ; apply gate Y to register  $m$ ; *etc*”. Such a sequence of instructions is usually represented as a *quantum circuit*. In this model, a quantum program is then a conventional program building a quantum circuit and sending it as a batch-job to the coprocessor.

From a semantical perspective, the *state* of a quantum memory consisting of  $n$  quantum bits is a vector in a  $2^n$ -dimensional Hilbert space. A quantum circuit is a linear, sequential description of elementary operations describing a *linear, unitary map* on the state space. The quantum coprocessor should come with a *universal* set of such operations, that is, such that any unitary operation on the state space can be realized using the given elementary gates.

If unitary maps form the original representation for (pure) quantum operations, this extensional presentation makes a very rough semantics for the interaction with the quantum memory. It is akin to say that one could abstract a conventional program by the plot of



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42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:31

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

45 the corresponding function: this hides useful information one might want to keep track of,  
 46 such as topological constraints, composition of subroutines and execution flow, required  
 47 resources or other costs, *etc* [33]. Most of the *structure* of the computation is lost in the  
 48 matrix representation.

49 Coming all the way from Feynman's diagrams [22], graphical languages for representing  
 50 quantum processes can be seen as an answer to the limitations of plain unitary matrices. In  
 51 recent year, in the field of quantum computation has blossomed a wide variety of proposals  
 52 to focus on specific aspects of the semantics of quantum programs [23, 14, 8].

53 Quantum circuits are an obvious candidate for a graphical language, and indeed, several  
 54 lines of research took them as their main object of study [24, 16, 30, 7]. Quantum circuits in  
 55 particular form a natural medium for describing the execution flow of a computation. Similar  
 56 to the intuition one can build for classical boolean circuits, one can make an operational  
 57 semantics for quantum circuits using a token-machine Geometry of Interaction. Each token  
 58 in the circuit carries a quantum bit, and they flow in the circuit from inputs to outputs  
 59 while getting modified as they pass through gates [16]. The main problem with the model of  
 60 quantum circuits is the lack of a satisfactory equational presentation. If several attempts have  
 61 been made for various subsets [12, 11, 26, 27], none of them provides a complete presentation.

62 A recent proposal answering the defects of quantum circuits as a (formal) language is the  
 63 ZX-calculus [14]. Rooted in category theory [1], it comes with a small set of generators and  
 64 a sound and complete equational theory [36]. Conservative extension of quantum circuits,  
 65 yet versatile, the ZX-calculus has shown its use in a wide variety of applications, ranging  
 66 from optimization [20, 3], verification [21, 19], and error-correction [17].

67 Despite its success, the ZX-calculus is nonetheless still tied to the quantum coprocessor  
 68 model. Indeed, it fails to account for one peculiar feature of quantum computation: non-  
 69 causal execution paths. Indeed, the Janus-faced quantum computational paradigm features  
 70 two seemingly distinct notions of control structure. On the one hand, a quantum program  
 71 follows *classical* control: it is hosted on the conventional computer governing the coprocessor,  
 72 and can therefore only enjoy loops, tests and other regular causally ordered sequences of  
 73 operations. On the other hand, the lab bench turns out to be more flexible than the rigid  
 74 coprocessor model, permitting more elaborate *purely quantum* computational constructs than  
 75 what quantum circuits or ZX-calculus allow.

76 The archetypal example of an quantum computational behavior hardly attainable within  
 77 ZX-calculus (or quantum circuits) is the *Quantum Switch*. Consider two quantum bits  $x$   
 78 and  $y$  and two unitary operations  $U$  and  $V$  acting on  $y$ . The problem consists in generating  
 79 the operation that performs  $UV$  on  $y$  if  $x$  is in state  $|0\rangle$  and  $VU$  if it is in state  $|1\rangle$ . As  
 80  $x$  can be in superposition, in general the operation is then sending  $(\alpha|0\rangle + \beta|1\rangle) \otimes |y\rangle$  to  
 81  $\alpha|0\rangle \otimes (UV|y\rangle)_x + \beta|1\rangle \otimes (VU|y\rangle)$ . It is a *purely quantum test*: not only can we have values  
 82 in superpositions (here,  $x$ ) but also *execution orders*. This is in sharp contrast with what  
 83 happens within the standard quantum coprocessor model.

84 Computational models supporting superpositions of execution orders have been studied  
 85 in the literature. One trend of research consists in proposing a suitable extension of  
 86 quantum circuits [9, 31, 35, 37]. These approaches typically aim at discussing the notion  
 87 of quantum channel from a quantum information theoretical standpoint. Another research  
 88 line is concerned with exhibiting the execution flow internal to ZX-terms in a distributed  
 89 manner [6]. Somewhat similarly to what has been proposed in [16], tokens are let to flow in  
 90 a ZX-term, realizing the computation, the difference being that the tokens can themselves  
 91 be in superposition. The limit is however that a ZX-term is inherently *causal*, therefore  
 92 restricting the possibilities for superposition of orders.

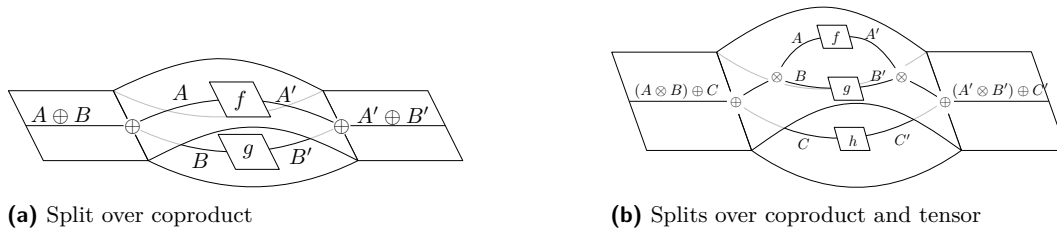


Figure 1 Examples of splits

93 One limitation of the existing extensions of quantum circuits or of graphical languages  
 94 based on ZX-calculus is the limited support for *typed quantum data*. In the context of  
 95 quantum circuits, the strategy have been to consider extended structures stemming from  
 96 proofs-nets and resource-sensitive logics such as linear logic. A proof-net is a structured graph  
 97 describing a proof of a linear logic formula. Through the Curry-Howard isomorphism, a proof  
 98 can be regarded as a program and a formula as a type specifying how the program handles  
 99 resources. This turns out to be particularly well-suited for quantum computation [18, 29, 16].

100 In the approaches merging quantum circuits with types based on linear logic, the *additive*  
 101 *types* such as the sum-type are *purely classical*:  $1 \oplus 1$  is always a (conventional) boolean. In  
 102 quantum computation, the sum-type  $1 \oplus 1$  can however be understood as a *sum of vector*  
 103 *spaces*, giving an alternative interpretation to  $1 \oplus 1$ : it can be regarded as the type of a quantum  
 104 bit, superposition of True and False. One should note that this appealing standpoint should  
 105 be taken cautiously: (Pure) quantum information imposes strong constraints on the structure  
 106 of the data in superposition: orthogonality and unit-norm have to be preserved [2, 32].

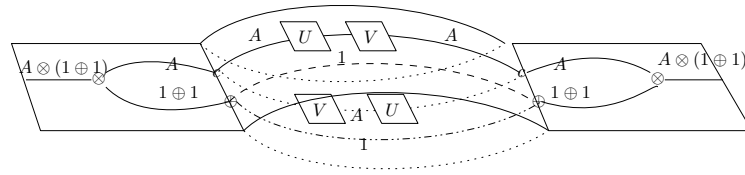
107 The standard categorical formalization underlying quantum computation, whether with  
 108 the coprocessor or with the purely quantum viewpoint, is symmetric monoidal categories,  
 109 and compact closed categories in particular. The monoidal structure stands for the tensor of  
 110 quantum data, while coproducts are used to represent “tests”, whether classical [16, 18] or  
 111 quantum [32].

112 Graphical languages for symmetric monoidal structure with coproducts usually rely on a  
 113 notion of *sheet*, or *worlds*, to handle tests and coproducts in general [18, 28]. Figure 1a shows  
 114 for instance how to represent the construction of the morphism  $f \oplus g : A \oplus A' \rightarrow B \oplus B'$   
 115 out of  $f : A \rightarrow B$  and  $g : A' \rightarrow B'$ . The symbol “ $\oplus$ ” stands for the “split” of worlds. Such a  
 116 graphical language therefore comes with two distinct “splits”: one for the monoidal structure  
 117 —leaving inside one specific world—, and one for the coproduct —splitting worlds—. They  
 118 can be intertwined, as shown in Figure 1b. Another approach followed by [15] externalizes the  
 119 two products (tensor product and coproduct) into the structure of the diagrams themselves,  
 120 at the price of a less intuitive tensor product and a form of synchronization constraint.

121 However, in the state of the art this “splitting-world” understanding has only been carried  
 122 for *classical* tests [16] and probabilistic branching stemming from measurements [18, 34],  
 123 and *not* for quantum superposition. The Quantum Switch can for instance be naturally  
 124 understood in this framework. Consider for instance Figure 2, read from left to right: as  
 125 input, a pair of an element of type  $A$  and a quantum bits. Based on the value of the qubit  
 126 (True or False), the wire  $A$  goes in the upper or the lower sheet, and is fed with  $U$  then  $V$  or  
 127  $V$  then  $U$ . Then everything is merged back together.

128 In this paper, we provide a rigorous interpretation for this type of (purely quantum)  
 129 behavior.

130 **Contributions** In this paper, we introduce a new graphical language for quantum



■ Figure 2 Quantum Switch with Worlds

131 computation, based on compact category with biproduct [?]. This language allows us to  
 132 express any quantum process, as we can encode the ZX-Calculus within it. We develop a  
 133 denotational semantic and an equational theory, and prove the soundness and completeness  
 134 of the semantics. As a case-study, we show how the Quantum Switch can naturally be  
 135 encoded in the language.  
 136 In the paper, the missing proofs can be found in Appendix.

## 2 The Many-Worlds Calculus

138 While the goal is to define a graphical language in which each wire can be enabled or disabled  
 139 depending on the world in which the computation takes place, we first define the category  
 140  $\mathbf{C}_D$  of diagrams without any "worlds", and will then add the world annotations.

### 2.1 A First Graphical Language

142 We define our graphical language within the paradigm of colored PROP [4, 25], meaning  
 143 that a diagram will be composed of nodes, or *generators*, linked to each others through *typed*  
 144 *wires*, wired that are allowed to cross each others. Additionally, we assume that our colored  
 145 PROP is compact closed and auto-dual, meaning that we allow to curve wires to obtain a  
 146 Cup or a Cap.

147 The generators of our language are described in Figure 3 and are respectively the Identity,  
 148 the Swap, the Cup, the Cap, the Plus, the Tensor, the Unit, the  $n$ -ary Contraction, and  
 149 the Scalar indexed with  $s$  ranging over  $\mathbb{C}$ . Mirrored versions of those generators are defined  
 150 as syntactic sugar through the compact closure, as shown for the mirrored Plus on the  
 151 right-hand-side of Figure 3. Diagrams are read top-to-bottom: the top-most wires are the  
 152 *input* wires and the bottom-most wires are the *output* wires.

153 The types of our wires are built from the syntax  $A, B ::= 1 \mid A \oplus B \mid A \otimes B$ . The objects  
 154 of our category are the set of wires as generated by the grammar  $\mathfrak{A}, \mathfrak{B} ::= \mathfrak{A} \square \mathfrak{B} \mid \diagup \mid A$ . The  
 155 choice of the notation  $\square$  for wires in parallel is uncommon, we use it to put an emphasis on  
 156 the fact that contrary to languages like the ZX-calculus, wires that are in parallel are not  
 157 necessarily "in tensor with one another". In fact,  $A \square B$  can be understood semantically as  
 158 "either  $A \otimes B$  or  $A \oplus B$ ".

159 Diagrams are obtained from generators  
 160 by composing them in parallel (written  $\square$ ),  
 161 or sequentially (written  $\circ$ ) as follows. Se-  
 162 quential composition requires the type (and  
 163 number) of wires to match. We write  $\mathbf{C}_D$  for the category of diagrams we defined as such.

$$D_2 \circ D_1 := \begin{array}{c} \vdots \\ \boxed{D_1} \\ \vdots \\ \boxed{D_2} \\ \vdots \end{array} \quad D_1 \square D_2 := \begin{array}{c} \vdots \\ \boxed{D_1} \\ \vdots \end{array} \square \begin{array}{c} \vdots \\ \boxed{D_2} \\ \vdots \end{array}$$

164 ► **Example 1.** While our language lacks the worlds labeling, we can already illustrate it by  
 165 encoding some basic quantum primitive in it and show how they operate. In Figure 4 we  
 166 show the encoding of a quantum bit  $\alpha|0\rangle + \beta|1\rangle$  and the Hadamard unitary. In particular,

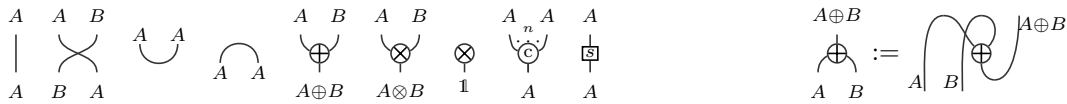


Figure 3 Generators of our First Graphical Language ( $n \geq 0, s \in \mathbb{C}$ )

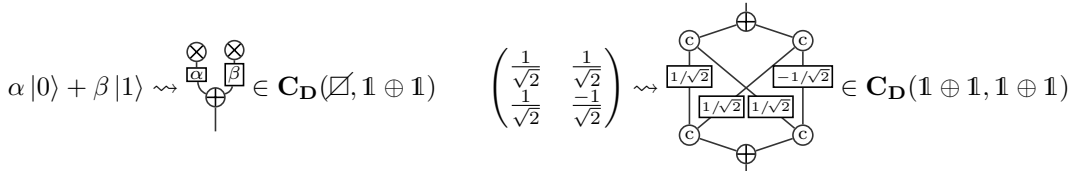
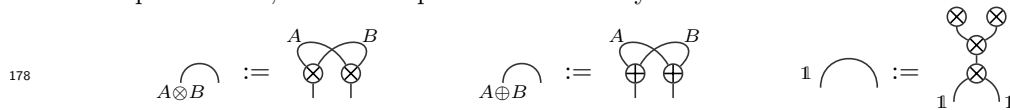


Figure 4 A Quantum Bit and the Hadamard Unitary

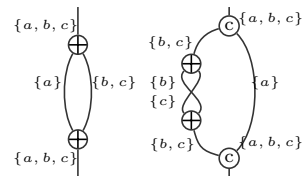
167 the Plus allows to "build" a new quantum bit from two scalars in parallel or to "open" a  
 168 quantum bit to recover its corresponding scalars, the left branch corresponding to  $|0\rangle$  and the  
 169 right branch to  $|1\rangle$ . The meaning of the Contraction is better seen when applying Hadamard  
 170 to a quantum bit as we show in Figure 5: it allows us to duplicate and sum scalars. The  
 171 rewriting sequence of Figure 5 is made using the equational theory defined in Section 4,  
 172 however to correctly define our equational theory, the worlds labeling are required. So while  
 173 this specific worlds-free rewriting sequence is sound, many other similar worlds-free rewriting  
 174 sequences are unsound.

175 ► Remark 2. Instead of having the Cup and the Cap as generators and defining the mirrored  
 176 version of each generator through them, one could proceed the other way around by defining  
 177 the Cap as follows, and the Cup in a mirrored way:



## 179 2.2 Adding Worlds Labeling

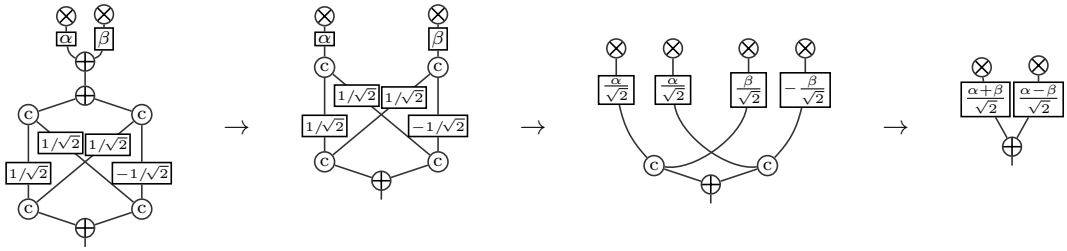
180 We now label wires of our diagram with worlds sets  $w \subseteq W$   
 181 for a given a set of worlds  $W$ . For each world  $a \in W$ ,  
 182 wires labeled by a set containing  $a$  are said to be "enabled  
 183 in  $a$ ", and the others are said "disabled in  $a$ ". This allows  
 184 us to correlate the enabling of wires. For example, the



185 "controlled not" can be represented by the diagram on the right, with worlds  $\{a, b, c, \star\}$ . The  
 186 left-hand-side of the diagram forces the world  $a$  to correspond to the case where the control  
 187 qubit is equal to  $|0\rangle$ , and the worlds  $b$  and  $c$  when it is equal to  $|1\rangle$ . The right-hand-side  
 188 of the diagram applies the identity in the world  $a$ , and a negation in the worlds  $b$  and  $c$ .  
 189 Lastly, the world  $\star$  appears nowhere in the labels, and corresponds to "we do not evaluate  
 190 this circuit at all". While not strictly necessary, it is often practical to have a world absent  
 191 from every wire.

192 ► Definition 3. Given a set of worlds  $W$ , we define the auto-dual compact closed colored  
 193 PROP ( $\mathbf{MW}_W, \square, \boxtimes$ ) of many-worlds calculus over  $W$  as follows:

194 Its colors are the pairs  $(A : w)$  of colors  $A$  of  $\mathbf{C}_D$  and subsets  $w \subseteq W$ . We write  $(\mathfrak{A}, \ell_{\mathfrak{A}})$  for  
 195 the objects, where  $\mathfrak{A}$  is an object of  $\mathbf{C}_D$  and  $\ell_{\mathfrak{A}}$  is a labeling function from the colors of  $\mathfrak{A}$  to  
 196 the subsets of  $W$ .



■ **Figure 5** Applying the Hadamard unitary to a quantum bit

197 *Its morphisms*  $f : (\mathfrak{A}, \ell_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \ell_{\mathfrak{B}})$  are pairs  $(\mathcal{D}_f, \ell_f)$  of a morphism  $\mathcal{D}_f \in \mathbf{C}_D(\mathfrak{A}, \mathfrak{B})$  and  
 198 a labeling function  $\ell_f$  from the wires of  $\mathcal{D}_f$  to the subsets of  $W$ , satisfying the following  
 199 constraints: The label on an input or output wire of color  $(A : w)$  must be equal to  $w$ , and

200 
$$|w \quad w \begin{array}{c} \diagup \\ \diagdown \end{array} v \quad \begin{array}{c} w \\ \frown \end{array} \quad \begin{array}{c} \smile \\ w \end{array} \quad \begin{array}{c} w \\ \boxed{\text{S}} \\ w \end{array} \quad \begin{array}{c} w \\ \oplus \\ w \sqcup v \end{array} \quad \begin{array}{c} w \\ \otimes \\ w \end{array} \quad \otimes_w \quad \forall n \geq 0, \quad \begin{array}{c} w_1 \dots w_n \\ \text{C} \\ w_1 \sqcup \dots \sqcup w_n \end{array}$$

201 where  $\sqcup$  denotes a set union which is disjoint. The constraints for the mirrored versions are  
 202 similar. The sequential composition  $\circ$  and the parallel composition  $\square$  preserve the labels.

203 Because of the first restriction on labels, given a morphism  $f$ , one can infer the labels on  
 204 its objects from the label  $\ell_f$ , hence we can write  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  instead of  $f : (\mathfrak{A}, \ell_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \ell_{\mathfrak{B}})$   
 205 unambiguously. It is discussed in Appendix B.1 how we can compose  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  and  $g : \mathfrak{B} \rightarrow \mathfrak{C}$   
 206 even when their induced labels do not match.

207 ► **Example 4 (The Quantum Switch).** Given two subdiagrams  $U$  and  $V$ , the Quantum  
 208 Switch of  $U; V$  and  $V; U$  presented in Figure 2 can be encoded as in ???.The figure is split  
 209 into two parts: the control part on the left-hand-side, and the computational part on the  
 210 right-hand-side. The idea is that the control part, that uses  $\oplus$ , will behave as an *if-then-else*  
 211 and will bind the world  $w$  to the case where the control quantum bit is  $|0\rangle$ , and the world  
 212  $v$  to the case where the control quantum bit is  $|1\rangle$ . On the computational part,  $U \circ V$  will  
 213 be applied within the world  $v$  while  $V \circ U$  will be applied within the world  $w$ . The “sheets”  
 214 presented in Figure 2 are here modeled with world labels.

215 For the sake of simplicity, we represented the Quantum Switch here with two copies of  $U$   
 216 and  $V$ , one for each branch. It is actually possible within our language to share  $U$  and  $V$   
 217 between the two worlds in order to only have one copy of each (which is the whole purpose  
 218 of the Quantum Switch). We describe in Example 8 this “true” Quantum Switch, and one  
 219 can show using the equational theory given in Section 4 that the two are equivalent.

### 2.3 Comparison with Other Graphical Languages

221 We draw some comparison between our language and two other graphical languages: the  
 222 ZX-Calculus [13], and Duncan’s Tensor-Sum Logic [18].

#### 2.3.1 ZX-Calculus

224 The first difference is the restrictions of the ZX-calculus to computations between qubits,  
 225 in other words linear map from  $\mathbb{C}^{2^n} \mapsto \mathbb{C}^{2^m}$ , while our language can encode any linear map  
 226 from  $\mathbb{C}^n \mapsto \mathbb{C}^m$ . The Tensor generator allowing the decomposition of  $\mathbb{C}^{2^n}$  into instances  
 227 of  $\mathbb{C}^2$  was already present in the *scalable* extension of the ZX-calculus [5], but the main  
 228 difference comes from the Plus (and the Contraction).

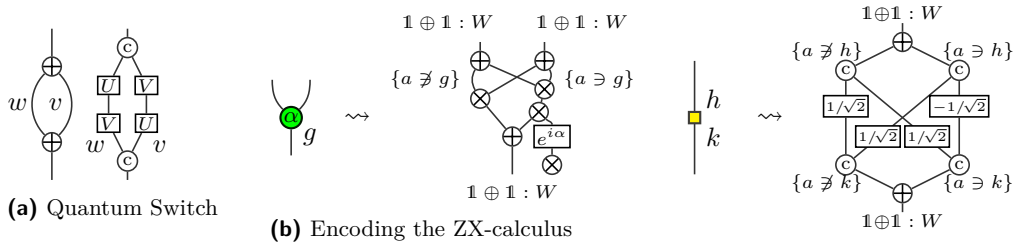
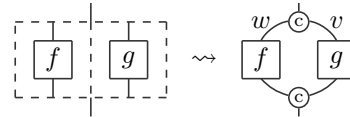


Figure 6

229 Additionally, every ZX-diagram can be encoded in our graphical language. The identity,  
 230 swap, cup and cap of the ZX calculus are encoded by the similar generators over the type  
 231  $\mathbb{1} \oplus \mathbb{1}$ , the green spider and Hadamard gate are encoded as described in ??, and an encoding  
 232 for the red spider can be deduced from those. The worlds set  $W$  is equal to  $\mathcal{P}(\text{Names})$  where  
 233 each green node of the ZX-diagram is given a name, and each input and each output of  
 234 an Hadamard gate is given a name too. For a name  $g$ , we write  $\{a \ni g\} \subseteq W$  for the set  
 235  $\{a \subseteq \text{Names} \mid g \in a\}$  and  $\{a \not\ni g\}$  for its complementary.

### 2.3.2 Tensor-Sum Logic

237 The core difference between their work and ours is the  
 238 presence of the contraction in our graphical language. They  
 239 instead rely on an enrichment of their category by a sum,  
 240 which they represent graphically with boxes. We show on  
 241 the right how the morphism  $f + g$  would be encoded in both their and our language. More  
 242 generally, their boxes correspond to uses of our contraction generator in a "well-bracketed"  
 243 way. Another point of difference is their approach to quantum computation, as we do  
 244 not assign the same semantics to those superpositions of morphisms. In their approach,  
 245 the superposition is a classical construction and corresponds to the measurement and the  
 246 classical control flow, while in our approach the superposition is a quantum construction and  
 247 corresponds to the quantum control.



## 3 Semantics of the Many-Worlds Calculus

249 Our calculus aims at pure quantum computations. Following the ZX approach, we relax the  
 250 condition on unitarity and define a semantics for our Many-World Calculus based on finite  
 251 dimensional Hilbert spaces and (general) linear operators. More precisely, we will define two  
 252 semantics, a world-dependent semantics  $\llbracket - \rrbracket_a$  for every world  $a$ , which will be a monoidal  
 253 functor from  $\mathbf{MW}_W$  to  $\mathbf{FdHilb}$ , and a worlds-agnostic semantics  $\llbracket - \rrbracket$  which will not be  
 254 functorial for the standard sequential composition and parallel composition, though we define  
 255 in Appendix B.1 the worlds-agnostic sequential composition and parallel composition for  
 256 which  $\llbracket - \rrbracket$  will be functorial.

257 We start by defining those semantics on the objects. For every object  $\mathfrak{A}$  of  $\mathbf{C}_D$ , we define  
 258 its *enablings*  $\mathfrak{A}^\bullet$  as "replacing any number of wire type by  $\bullet$ ". For example  $(A \square B)^\bullet =$   
 259  $\{\bullet \square \bullet, \bullet \square B, A \square \bullet, A \square B\}$ . To each enabling  $\mathfrak{E} \in \mathfrak{A}^\bullet$  we associate a Hilbert space  $\mathcal{H}_{\mathfrak{E}}$  as  
 260 follows:  $\mathcal{H}_{\mathfrak{E} \square \mathfrak{F}} := \mathcal{H}_{\mathfrak{E}} \otimes \mathcal{H}_{\mathfrak{F}}$ ,  $\mathcal{H}_{\mathfrak{A}^\bullet} = \mathcal{H}_\bullet = \mathcal{H}_\mathbb{1} := \mathbb{C}$ ,  $\mathcal{H}_{A \otimes B} := \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\mathcal{H}_{A \oplus B} := \mathcal{H}_A \oplus \mathcal{H}_B$ .  
 261 Then, for any object  $(\mathfrak{A}, \ell_{\mathfrak{A}})$  of  $\mathbf{MW}_W$  and any world  $a \in W$ , we define  $(\mathfrak{A}, \ell_{\mathfrak{A}}) \Downarrow a$  to be  
 262 the enabling of  $\mathfrak{A}$  in which every  $(A : w)$  with  $a \in w$  is preserved and every  $(A : w)$   
 263 with  $a \notin w$  is replaced by  $\bullet$ . For example  $(A : \{a\} \square B : \{b\}) \Downarrow a = A \square \bullet$ . We can then



$$\begin{aligned}
 \llbracket \text{Cup} \rrbracket_a &= \begin{cases} \text{Id} & \in \mathbf{FdHilb}(\mathcal{H}_A, \mathcal{H}_A) & \text{if } a \in w \setminus v \\ \text{Id} & \in \mathbf{FdHilb}(\mathcal{H}_B, \mathcal{H}_B) & \text{if } a \in v \setminus w \\ h \otimes h' \mapsto h' \otimes h & \in \mathbf{FdHilb}(\mathcal{H}_{A \otimes B}, \mathcal{H}_{B \otimes A}) & \text{if } a \in w \cap v \\ (1) & \in \mathbf{FdHilb}(\mathbb{C}, \mathbb{C}) & \text{otherwise} \end{cases} \\
 \llbracket \text{Cup}^\dagger \rrbracket_a &= \begin{cases} h \otimes h' \mapsto \langle h|h' \rangle & \in \mathbf{FdHilb}(\mathcal{H}_{A \otimes A}, \mathbb{C}) & \text{if } a \in w \\ (1) & \in \mathbf{FdHilb}(\mathbb{C}, \mathbb{C}) & \text{otherwise} \end{cases} \quad \begin{array}{l} \text{and the (conjugate)} \\ \text{transposed operator} \\ \text{for the Cup} \end{array} \\
 \llbracket \text{S} \rrbracket_a &= \begin{cases} s \cdot \text{Id} & \in \mathbf{FdHilb}(\mathcal{H}_A, \mathcal{H}_A) & \text{if } a \in w \\ (1) & \in \mathbf{FdHilb}(\mathbb{C}, \mathbb{C}) & \text{otherwise} \end{cases} \\
 \llbracket \text{Plus} \rrbracket_a &= \begin{cases} \begin{pmatrix} \text{Id} \\ 0 \end{pmatrix} & \in \mathbf{FdHilb}(\mathcal{H}_{A \oplus B}, \mathcal{H}_A) & \text{if } a \in w \\ \begin{pmatrix} 0 \\ \text{Id} \end{pmatrix} & \in \mathbf{FdHilb}(\mathcal{H}_{A \oplus B}, \mathcal{H}_B) & \text{if } a \in v \\ (1) & \in \mathbf{FdHilb}(\mathbb{C}, \mathbb{C}) & \text{otherwise} \end{cases} \quad \begin{array}{l} \text{and the (conjugate)} \\ \text{transposed operator} \\ \text{for the mirrored} \\ \text{Plus} \end{array}
 \end{aligned}$$

■ **Figure 7** Semantics of the Generators of  $\mathbf{MW}_W$  in a World  $a \in W$ .

264 define the semantics  $\llbracket - \rrbracket_a : \mathbf{MW}_W \rightarrow \mathbf{FdHilb}$  and  $\llbracket - \rrbracket : \mathbf{MW}_W \rightarrow \mathbf{FdHilb}$  on objects  
 265 as  $\llbracket (\mathfrak{A}, \ell_{\mathfrak{A}}) \rrbracket_a := \mathcal{H}_{(\mathfrak{A}, \ell_{\mathfrak{A}}) \downarrow a}$  and  $\llbracket (\mathfrak{A}, \ell_{\mathfrak{A}}) \rrbracket := \bigoplus_{\mathfrak{C} \in \mathfrak{A}} \mathcal{H}_{\mathfrak{C}}$ . Then, for the morphisms, we  
 266 proceed by compositionality for  $\llbracket - \rrbracket_a$ , meaning that we define  $\llbracket - \rrbracket_a$  on every generator and  
 267 compute the semantics of a diagram by decomposing it with  $\llbracket g \circ f \rrbracket_a := \llbracket g \rrbracket_a \circ \llbracket f \rrbracket_a$  and  
 268  $\llbracket f \square g \rrbracket_a := \llbracket f \rrbracket_a \otimes \llbracket g \rrbracket_a$ . When looking at a generator  $gen \in \mathbf{MW}_W((\mathfrak{A}, \ell_{\mathfrak{A}}), (\mathfrak{B}, \ell_{\mathfrak{B}}))$ , in most  
 269 cases  $\llbracket (\mathfrak{A}, \ell_{\mathfrak{A}}) \rrbracket_a = \llbracket (\mathfrak{B}, \ell_{\mathfrak{B}}) \rrbracket_a = H$ : we then define  $\llbracket gen \rrbracket_a$  as the identity of  $\mathbf{Hilb}(H, H)$ .  
 270 The semantics of the generators are given in Figure 6, for all the other generators their  
 271 semantics is simply the identity.

272 The worlds-agnostic semantics is defined from the world-dependent semantics, as follows.  
 273 Consider  $f \in \mathbf{MW}_W((\mathfrak{A}, \ell_{\mathfrak{A}}), (\mathfrak{B}, \ell_{\mathfrak{B}}))$ . Then  $\llbracket f \rrbracket \in \mathbf{FdHilb}(\bigoplus_{\mathfrak{C} \in \mathfrak{A}} \mathcal{H}_{\mathfrak{C}}, \bigoplus_{\mathfrak{F} \in \mathfrak{B}} \mathcal{H}_{\mathfrak{F}})$  is  
 274 defined as  $\llbracket f \rrbracket := \left\{ \sum_{a \in W} \mathfrak{A} \mathfrak{B} \mathfrak{C} \mathfrak{F} \llbracket f \rrbracket_a \right\}_{\mathfrak{C} \in \mathfrak{A}, \mathfrak{F} \in \mathfrak{B}}$ , where  $W_{\mathfrak{A} \mathfrak{B} \mathfrak{C} \mathfrak{F}}$  is a shortcut notation  
 275 for the set  $\{a \in W \mid (\mathfrak{A}, \ell_{\mathfrak{A}}) \downarrow a = \mathfrak{C} \text{ and } (\mathfrak{B}, \ell_{\mathfrak{B}}) \downarrow a = \mathfrak{F}\}$ .

276 For example, the worlds-agnostic semantics of the Tensor and the Plus are simply the  
 277 collection of all their world-dependent semantics assembled into a single linear operator:

$$\begin{aligned}
 278 \quad \left[ \begin{array}{c} \text{World set: } \{a, \star\} \\ A : \{a\} \quad B : \{a\} \\ \text{Cup} \\ A \otimes B : \{a\} \end{array} \right] &= A \otimes B \begin{pmatrix} A \square B & A \square \bullet & \bullet \square B & \bullet \square \bullet \\ \text{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 279 \quad \left[ \begin{array}{c} \text{World set: } \{a, b, \star\} \\ A : \{a\} \quad B : \{b\} \\ \text{Plus} \\ A \oplus B : \{a, b\} \end{array} \right] &= A \oplus B \begin{pmatrix} A \square B & A \square \bullet & \bullet \square B & \bullet \square \bullet \\ 0 & \text{Id} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

281 The worlds-agnostic semantics is universal in the following sense:

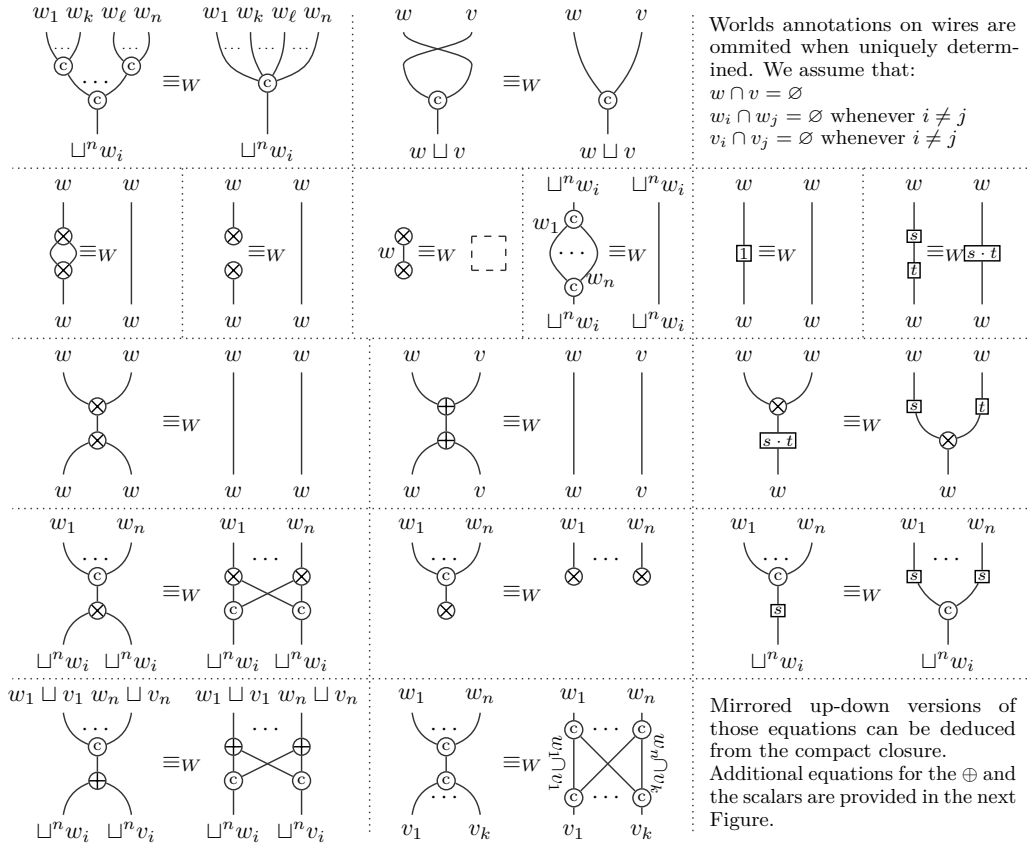


Figure 8 Equations with a Fixed Worlds Set  $W$

282 ▶ **Theorem 5 (Universality).** For every  $\mathfrak{A}, \mathfrak{B}$  objects of  $\mathbf{C}_D$ , and for every linear operator in  
 283  $U \in \mathbf{FdHilb} \left( \bigoplus_{\mathfrak{c} \in \mathfrak{A}} \mathcal{H}_{\mathfrak{c}}, \bigoplus_{\mathfrak{f} \in \mathfrak{B}} \mathcal{H}_{\mathfrak{f}} \right)$ , there exists a worlds set  $W$ , some labelings  $\ell_{\mathfrak{A}}, \ell_{\mathfrak{B}}$ ,  
 284 and a morphism  $f \in \mathbf{MW}_W((\mathfrak{A}, \ell_{\mathfrak{A}}), (\mathfrak{B}, \ell_{\mathfrak{B}}))$  such that  $\llbracket f \rrbracket = U$ . ◀

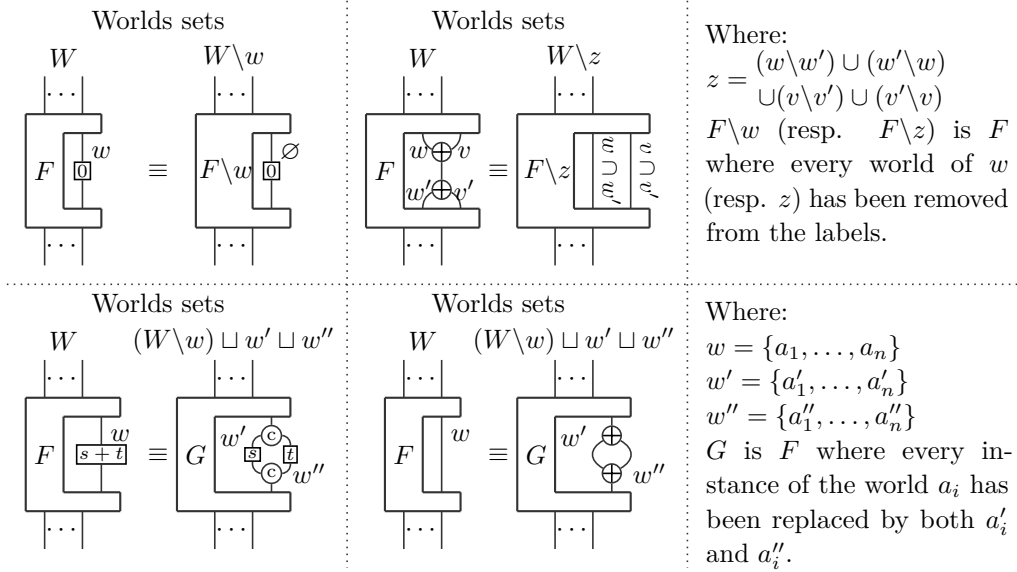
285 **4 The Equational Theory**

286 Similarly to how our semantics is defined in two steps, the equational theory is also defined  
 287 in two steps:

288 (1) A set of equations within  $\mathbf{MW}_W$  for a fixed set of worlds  $W$ , which will not be complete,  
 289 but will be sound for  $\llbracket - \rrbracket_a$  for every  $a \in W$ , hence sound for  $\llbracket - \rrbracket$  too. We write  $\equiv_W$  for the  
 290 induced congruence<sup>1</sup> over  $\mathbf{MW}_W$ . We list those equations in Figure 7. Quite notably, the  
 291 last two rows describe the fact that the contraction is a natural transformation.

292 (2) Five additional equations with side effects on the set of worlds, which will be sound and  
 293 complete for  $\llbracket - \rrbracket$ , but not for the  $\llbracket - \rrbracket_a$ . We write  $\equiv$  for the induced equivalence relation,  
 294 spanning over the morphisms of all the categories  $\mathbf{MW}_W$  for  $W$  any set of worlds. We have:  
 295 One equation that allows us to rename the worlds: for every morphism  $(\mathcal{D}_f, \ell_f)$  of  $\mathbf{MW}_W$ ,  
 296 and for every bijection  $i : W \rightarrow V$ , we have  $(\mathcal{D}_f, \ell_f) \equiv (\mathcal{D}_f, i \circ \ell_f) \in \mathbf{MW}_V$ ; Two equations

<sup>1</sup> In other words the smallest equivalence relation satisfying those equations and such that  $f \equiv_W f' \implies \forall g, h, l, g \circ (f \square h) \circ k \equiv_W g \circ (f' \square h) \circ k$ .



■ **Figure 9** Equations with Side-Effects on Worlds Sets

297 allowing the annihilation (or creation, when looking at them from right to left) of worlds  
 298 due to coproducts or scalars (first row of Figure 8); Two equations allowing the splitting (or  
 299 merging, when looking at them from right to left) of worlds due to coproducts or scalars  
 300 (second row of Figure 8).

301 ► **Proposition 6** (Soundness). *For  $f$  a morphism of  $\mathbf{MW}_W$  and  $g$  of  $\mathbf{MW}_W$ , whenever  $f \equiv g$   
 302 we have  $\llbracket f \rrbracket = \llbracket g \rrbracket$ . Additionally if  $W = V$ , whenever  $f \equiv_W g$  we have  $\forall a \in W, \llbracket f \rrbracket_a = \llbracket g \rrbracket_a$ .*

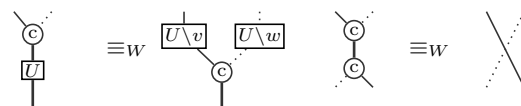
303 ► **Theorem 7** (Completeness). *For every  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  morphism of  $\mathbf{MW}_W$  and  $g : \mathfrak{A} \rightarrow \mathfrak{B}$  one  
 304 of  $\mathbf{MW}_V$ ,  $\llbracket f \rrbracket = \llbracket g \rrbracket \iff f \equiv g$  ◀*

305 The soundness can be proved by a case-by-case analysis on every equation. The complete-  
 306 ness theorem follows from the existence of a normal form for  $\equiv$ , as described in Section 5.1.

307 ► **Example 8** (The Quantum Switch). As claimed in Example 4, it is possible to represent  
 308 the Quantum Switch with only one copy of  $U$  and  $V$ , and one can rewrite it to the version  
 309 with two copies of each using the equational theory as shown in Figure 9.

310 In those diagrams, the worlds set is  $W = w \sqcup v$  and we rely on thick, thin and dotted wires  
 311 to indicate respectively worlds labels  $w \sqcup v$ ,  $w$  and  $v$ . Each figure has a control side which  
 312 operates on a quantum bit (type  $\mathbb{1} \oplus \mathbb{1}$ ) and binds the world  $w$  to  $|0\rangle$  and the world  $v$  to  
 313  $|1\rangle$ ; and a computational side which operates on some data of an arbitrary type  $A$ , on which  
 314 could be applied  $U$  and/or  $V$  which stand for two morphisms of  $\mathbf{MW}_W (A : W, A : W)$ .

315 The first rewriting step relies on the two lemmas on the right, both of which being deducible  
 from the equational theory (see Appendix A). The second rewriting step is simply using the  
 properties of a compact closed category.



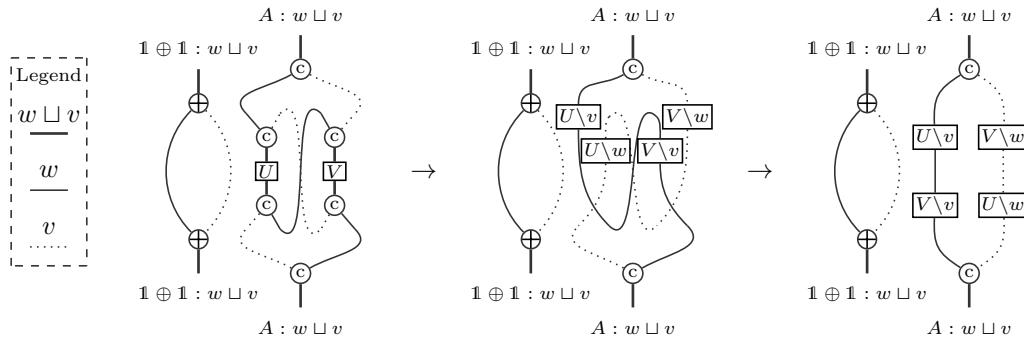


Figure 10 Rewriting the Quantum Switch

**5 Normal Form and Completeness**

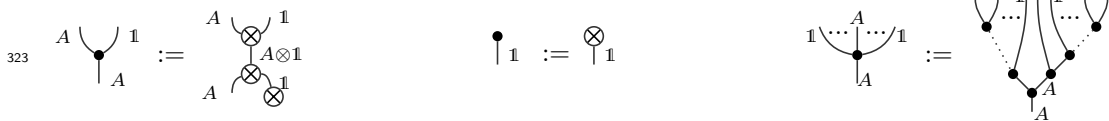
316

317 We can prove that the previous equational theory is complete, by defining a normal form on  
 318 the morphisms, and by showing that all diagrams can be put in this normal form, which is  
 319 unique.

**5.1 Normal Form**

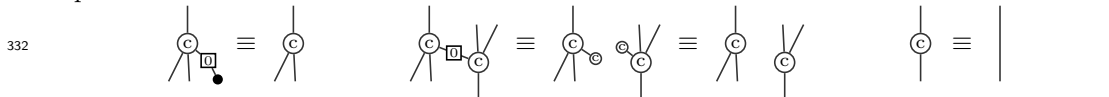
320

321 It will be practical for our normal forms to define the following syntactic sugar, which we  
 322 call the *unitor*, its *unit* and its generalized form:



323

324 We define the short-hand on the right with the  
 325 assumption that the worlds set is in bijection with the  
 326 set of scalars, in other words the scalars  $\lambda_{ij}$ ,  $\lambda'_i$ ,  $\lambda''_j$  and  
 327  $\lambda_0$  have for worlds label  $\{a_{ij}\}$ ,  $\{a'_i\}$ ,  $\{a''_j\}$  and  $\{a_0\}$   
 328 respectively. In particular, all the input (resp. output)  
 329 wires live in mutually exclusive worlds. An important observation is that any permutation of  
 330 wires (all mutually exclusive, and of type  $\mathbb{1}$ ) can easily be put in this form using the following  
 331 equations:



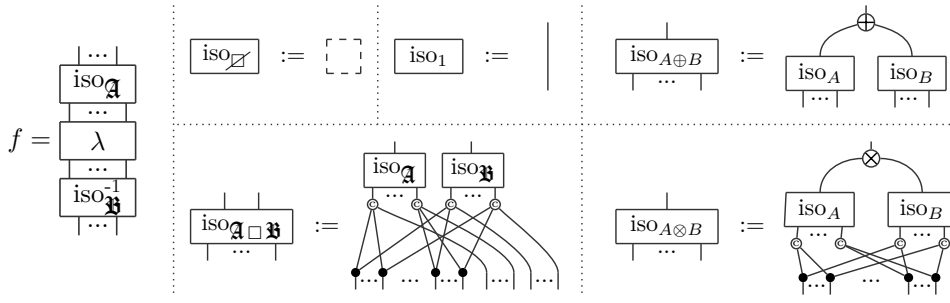
332

333 The normal form of a morphism  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  is defined as the form of the diagram on the left  
 334 of Figure 10, where the morphism  $\text{iso}_{\mathfrak{A}}$  is defined inductively right.

335 The output wires of  $\text{iso}_{\mathfrak{A}}$  for any  $\mathfrak{A}$  live in mutually exclusive worlds, but once again, we  
 336 don't overload the diagrams with unitors or world names encoding this information, although  
 337 it will be used in the following.

338 Notice that graphically, there is no difference between  $\mathfrak{A} \square (\mathfrak{B} \square \mathfrak{C})$  and  $(\mathfrak{A} \square \mathfrak{B}) \square \mathfrak{C}$ , in  
 339 other words  $\square$  is strictly associative. However  $\text{iso}_{\mathfrak{A} \square (\mathfrak{B} \square \mathfrak{C})}$  and  $\text{iso}_{(\mathfrak{A} \square \mathfrak{B}) \square \mathfrak{C}}$  are different,  
 340 but they are equivalent up to a rearranging of the output wires:

341 ► **Lemma 9.** *There exists a wire permutation  $\sigma$  such that  $\text{iso}_{\mathfrak{A} \square (\mathfrak{B} \square \mathfrak{C})} = \sigma \circ \text{iso}_{(\mathfrak{A} \square \mathfrak{B}) \square \mathfrak{C}}$ .* ◀



■ **Figure 11** The Normal Form (left) and the Inductive Definition of  $\text{iso}_{\mathfrak{a}}$  (right)

342 We hence have a choice to make here for canonicity, and choose  $\text{iso}_{A_0 \square A_1 \square A_2 \square \dots} :=$   
 343  $\text{iso}(\dots((A_0 \square A_1) \square A_2) \square \dots)$ .

344 We define  $\text{iso}_{\mathfrak{a}}^{-1}$  inductively in the same way, but upside-down. We note that  $\text{iso}_{\mathfrak{a}}^{-1} \circ \text{iso}_{\mathfrak{a}}$   
 345 is the normal form of  $\text{id}_{\mathfrak{a}}$ .

346 ► **Proposition 10.** *The normal form is unique.*

## 347 5.2 Completeness

348 We can now use this normal form to show that our equational theory is complete for arbitrary  
 349 morphisms. To do so, we need to show that all the generators can be put in normal form,  
 350 and then that any composition of morphisms in normal form can be put in normal form. We  
 351 do exactly so, and leave the details to the appendix.

352 ► **Proposition 11.** *The generators can be put in normal form.* ◀

353 ► **Proposition 12.** *Compositions of diagrams in normal form can be put in normal form.*

354 This allows us to prove the completeness theorem claimed above:

355 **Proof of Theorem 7.** The right-to-left direction of the equivalence can be directly checked  
 356 by verifying that all the axioms preserve the semantics.

357 Let  $f_1$  and  $f_2$  be two morphisms such that  $\llbracket f_1 \rrbracket = \llbracket f_2 \rrbracket$ . Both morphisms can be put in  
 358 normal form, resp.  $f_1^{NF}$  and  $f_2^{NF}$ , with  $f_i \equiv f_i^{NF}$  and thus  $\llbracket f_i^{NF} \rrbracket = \llbracket f_i \rrbracket$ . By uniqueness of  
 359 the normal form, and since  $\llbracket f_1^{NF} \rrbracket = \llbracket f_2^{NF} \rrbracket$ , we get  $f_1^{NF} \equiv f_2^{NF}$ , which ends the proof that  
 360  $f_1 \equiv f_2$ . ◀

## 361 6 Conclusion

362 We introduced a new sound and complete graphical language based on compact categories  
 363 with biproducts, along with a equational theory and a worlds system, helping us build a  
 364 denotational semantics of our language.

365 This language allows us to generalize already existing quantum graphical languages such  
 366 as the ZX-calculus with the additions of richer types than just the usual qubits and tensors  
 367 of qubits. In particular the biproduct allows us to reason about superposition of executions,  
 368 as shown by the encoding of the Quantum Switch in Example 8.

369 While our language allows for quantum control, it does not capture another language  
 370 that aims at formalizing quantum control, namely the PBS-Calculus [10]. How and in which  
 371 context could we capture the PBS-Calculus is left for future work.

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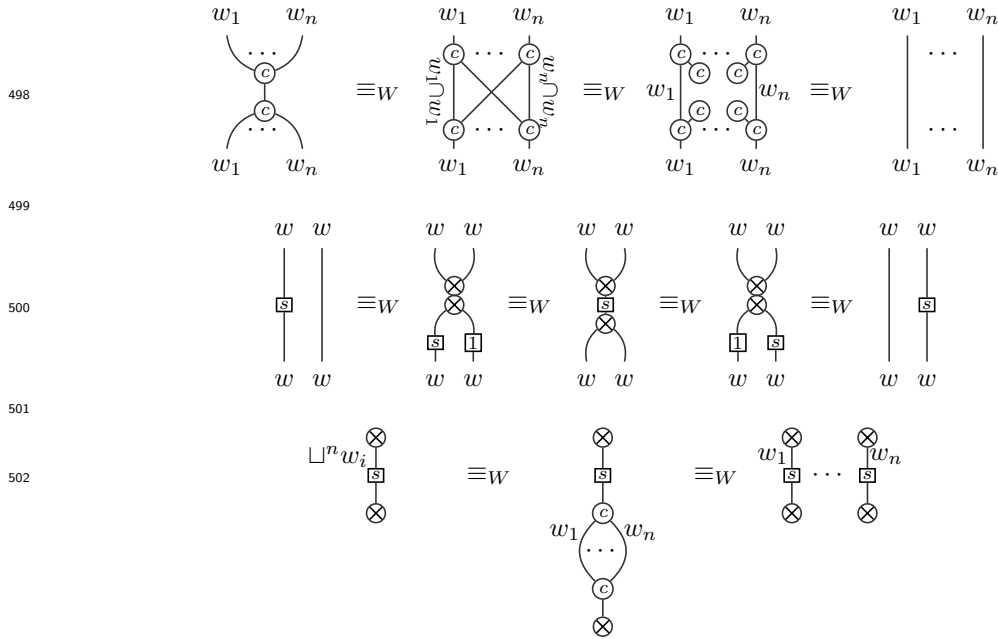
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494 **A** Induced Equations

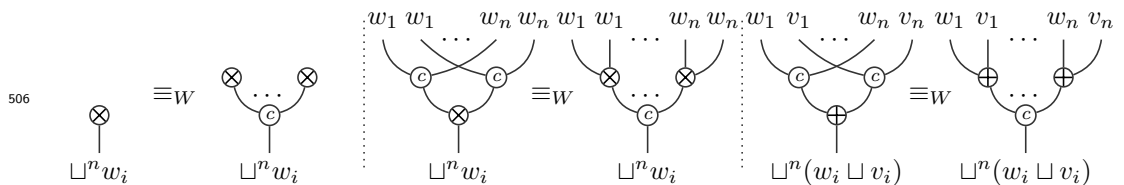
495 We include in this section a few practical lemmas that are provable from the equational  
 496 theory  $\equiv_W$ .

497 **► Lemma 13.** *Whenever  $w_i$  are disjoint sets of worlds, we have the following:*

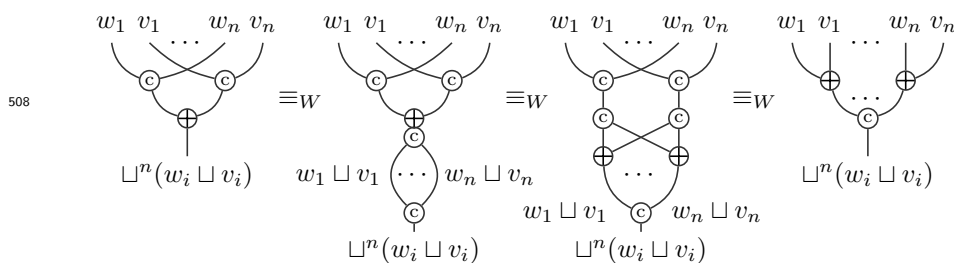


503 ◀

504 **► Lemma 14.** *Whenever  $w_i$  are disjoint sets of worlds, and that the  $v_i$  are disjoint set of  
 505 worlds too, we have the following:*

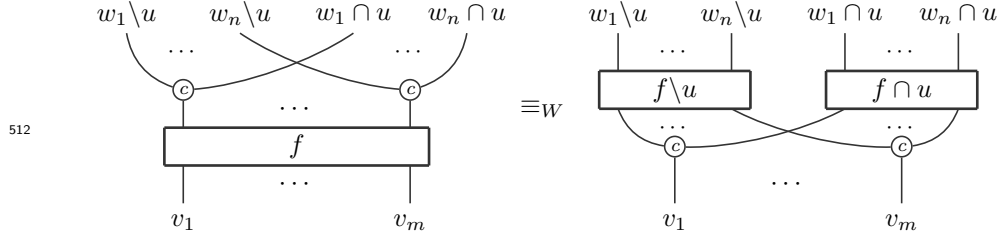


507 **Proof.** We provide a proof for the third equation, the first two are proven similarly.



509 ◀

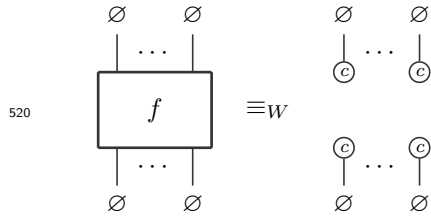
510 ► **Corollary 15.** For every  $f : \Box^n(A_i : w_i) \rightarrow \Box^m(B_j : v_j)$  with worlds set  $W$  and every  
 511  $u \subseteq W$ , we have



513 where  $f \setminus u : \Box^n(A_i : w_i \setminus u) \rightarrow \Box^m(B_j : v_j \setminus u)$  is equal to  $f$  where every worlds label  $w$  has  
 514 been replaced by  $w \setminus u$ , and similarly for  $f \cap u$ .

515 This is simply proven by induction over  $f$ . All the generator cases (including Cup and  
 516 Cap) follow directly from the equations given in Figure 7 and Lemma 14 together with the  
 517 properties of a compact close category.

518 ► **Lemma 16.** For every  $f : \Box^n(A_i : \emptyset) \rightarrow \Box^m(B_j : \emptyset)$  with worlds set  $W$  but such that  
 519 every worlds label of  $f$  is  $\emptyset$ , we have



521 This is simply proven by replacing every wire by two contractions of arity zero (sixth  
 522 axiom of Figure 7 with  $n = 0$ ), and then using the naturality of the contraction of arity zero  
 523 (last two lines of Figure 7 with  $n = 0$ ) to consume every generator.

## 524 **B Dealing with Worlds Labeling**

525 The worlds labeling can appear to be a very strict structure making it unpractical to  
 526 manipulate diagrams. In this appendix, we show a number of tools that allow to manipulate  
 527 worlds labeling in a more flexible way.

### 528 **B.1 Worlds-Agnostic Category**

529 In this section, we define the category  $\mathbf{MW}_V$  which has the same objects as  $\mathbf{C}_D$ , so no worlds  
 530 labeling on the objects, and has for morphisms all the morphisms of the categories  $\mathbf{MW}_W$   
 531 for every worlds set  $W$ . In other words, we explain how to make sequential composition and  
 532 parallel composition of morphisms that have different worlds set.

533 We write  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$  for a morphism  $(\mathcal{D}_f, \ell_f) \in \mathbf{MW}_W((\mathfrak{A}, \ell_f^{\mathfrak{A}}), (\mathfrak{B}, \ell_f^{\mathfrak{B}}))$ , where  $\ell_f^{\mathfrak{A}}$  and  
 534  $\ell_f^{\mathfrak{B}}$  are the labeling induced on  $\mathfrak{A}$  and  $\mathfrak{B}$  by  $\ell_f$ .

535 The goal of this section is to allow to compose  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$  and  $[g]_V : \mathfrak{B} \rightarrow \mathfrak{C}$  even when  
 536 the inferred labels on  $\mathfrak{B}$  do not match and when  $W$  and  $V$  are different.

$$\begin{array}{c}
 W = \{a, \star\} \quad V = \{b, \star\} \quad W \times V = \begin{array}{l} \{(a,b), (a,\star), \\ (\star,b), (\star,\star)\} \end{array} \\
 \left. \{a\} \right| \quad \square \quad \left. \{b\} \right| \quad = \quad \left. \begin{array}{l} \{(a,b), \\ (a,\star)\} \end{array} \right| \quad \left. \begin{array}{l} \{(a,b), \\ (\star,b)\} \end{array} \right|
 \end{array}$$

■ **Figure 12** Example of Worlds-Agnostic Parallel Composition

### 537 B.1.1 Extending the Worlds Set

538 Before tackling the sequential composition, we start by defining the parallel composition  $\square$   
 539 between morphisms with different world sets. For  $W, V$  two sets of worlds, and for  $\ell : S \rightarrow W$ ,  
 540 we define  $\ell^{-\times V} : S \rightarrow W \times V$  as  $\ell^{-\times V}(s) = \{(a, b) \mid a \in \ell(s), b \in V\}$ . Similarly, for  
 541  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$ , we define  $[f^{-\times V}]_{W \times V}$  as the morphism  $(\mathcal{D}_f, \ell_f^{-\times V})$ . We define  $\ell^{V \times -}$  and  
 542  $f^{V \times -}$  symmetrically.

543 ► **Definition 17** (Worlds-Agnostic Parallel Composition). For  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$  and  $[g]_V : \mathfrak{C} \rightarrow \mathfrak{D}$   
 544 two morphisms, we define

$$545 \quad [f]_W \square [g]_V := [f^{-\times V} \square g^{W \times -}]_{W \times V} : (\mathfrak{A} \square \mathfrak{B}) \rightarrow (\mathfrak{C} \square \mathfrak{D})$$

546 In Figure 11, we show the result of this parallel composition when  $[f]_{\{a, \star\}} = \mathbf{id}_{A: \{a\}}$  and  
 547  $[g]_{\{b, \star\}} = \mathbf{id}_{A: \{b\}}$  for a color  $A$ .

### 548 B.1.2 Restricting the Worlds Set

549 We now tackle the sequential composition. For  $W$  a set of worlds,  $w \subseteq W$ , we define the  
 550 restriction of  $\ell : S \rightarrow W$  to  $W \setminus w$  written  $\ell^{\setminus w} : S \rightarrow W \setminus w$  as  $\ell^{\setminus w}(s) = \{a \in \ell(s) \mid a \notin w\}$ .  
 551 For  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$  we define  $[f^{\setminus w}]_{W \setminus w} : \mathfrak{A} \rightarrow \mathfrak{B}$  as the morphism  $(\mathcal{D}_f, \ell_f^{\setminus w})$ .

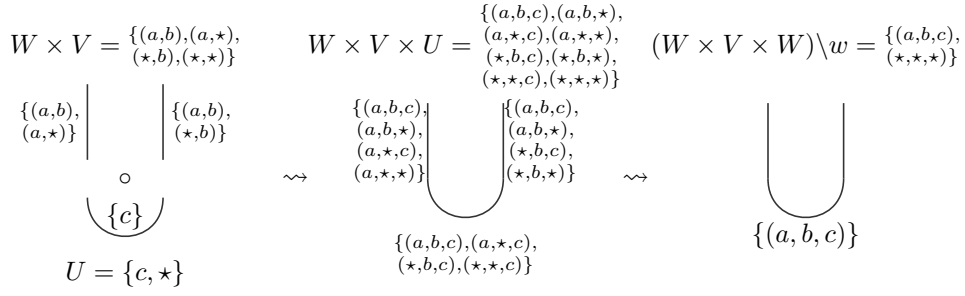
552 ► **Definition 18.** Given two morphisms  $[f]_W : (\mathfrak{A}, \ell_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \ell_{\mathfrak{B}})$  and  $[g]_V : (\mathfrak{B}, \ell_{\mathfrak{B}}) \rightarrow (\mathfrak{C}, \ell_{\mathfrak{C}})$ ,  
 553 we write  $w$  for the smallest subset of  $W \times V$  such that  $(f^{-\times V})^{\setminus w}$  induces the same worlds  
 554 labeling on  $\mathfrak{B}$  as  $(g^{W \times -})^{\setminus w}$ . We then define

$$555 \quad [g]_V \circ [f]_W := [(g^{W \times -})^{\setminus w} \circ (f^{-\times V})^{\setminus w}]_{(W \times V) \setminus w}$$

556 We continue the previous example in Figure 12 by composing the result with the Cup over  
 557  $A : \{c, \star\}$ . We proceed in two steps: first we handle the situation as if it was a parallel  
 558 composition, leading to a diagram labeled over  $W \times V \times U$ , but with multiple contradictory  
 559 labels on the wire. Then, we eliminate as few worlds as possible to make those labels  
 560 compatible:

- 561 ■ We eliminate  $(a, b, \star)$  and  $(a, \star, \star)$  which are on the *left* label but not on the *bottom* one.
- 562 ■ We eliminate  $(\star, b, c)$  and  $(\star, \star, c)$  which are on the *bottom* label but not on the *left* one.
- 563 ■ We eliminate  $(\star, b, \star)$  which is on the *right* label but not on the *bottom* one. We would  
 564 also eliminate  $(a, b, \star)$  if we had not done so already.
- 565 ■ We eliminate  $(a, \star, c)$  which is on the *bottom* label but not on the *right* one. We would  
 566 also eliminate  $(\star, \star, c)$  if we had not done so already.

567 The eliminated worlds are  $w = \{(a, b, \star), (a, \star, c), (\star, b, c), (a, \star, \star), (\star, b, \star), (\star, \star, c)\}$ , and what  
 568 remains is  $\{(a, b, c), (\star, \star, \star)\}$ .



■ **Figure 13** Example of Worlds-Agnostic Sequential Composition

### B.1.3 The Worlds-Agnostic Category and its Semantics

569

570  $\mathbf{MW}_\forall$  with the sequential and parallel composition as described above forms a category (up  
571 to renaming of worlds), and is in fact an auto-dual compact closed colored PROP (up to  
572 renaming of worlds).

573 ► **Proposition 19.** *The worlds-agnostic semantics  $\llbracket - \rrbracket$  defined in Section 3 is a monoidal  
574 functor from  $\mathbf{MW}_\forall$  to  $\mathbf{FdHilb}$ .*

575 **Proof.** We recall here the definition of the worlds-agnostic semantics of  $[f]_W : \mathfrak{A} \rightarrow \mathfrak{B}$ :

$$576 \quad \llbracket [f]_W \rrbracket := \left\{ \sum_{\substack{a \in W \\ (\mathfrak{A}, \ell_a^f) \Downarrow a = \mathfrak{A}' \\ (\mathfrak{B}, \ell_a^f) \Downarrow a = \mathfrak{B}'}} \llbracket [f]_a \rrbracket \right\}_{\mathfrak{A}' \in \mathfrak{A}^*, \mathfrak{B}' \in \mathfrak{B}^*}$$

577 From the definition of the worlds-agnostic compositions, we directly have:

$$578 \quad \llbracket [f]_W \square [g]_V \rrbracket_{(a,b)} = \llbracket [f]_W \rrbracket_a \otimes \llbracket [g]_V \rrbracket_b \quad \llbracket [g]_V \circ [f]_W \rrbracket_{(a,b)} = \llbracket [g]_V \rrbracket_b \circ \llbracket [f]_W \rrbracket_a$$

579 The functoriality with respect to the parallel composition is then immediate:

$$580 \quad \llbracket [f]_W \square [g]_V \rrbracket = \left\{ \sum \llbracket [f]_W \square [g]_V \rrbracket_{(a,b)} \right\} = \left\{ \sum \llbracket [f]_W \rrbracket_a \right\} \otimes \left\{ \sum \llbracket [g]_V \rrbracket_b \right\} = \llbracket [f]_W \rrbracket \otimes \llbracket [g]_V \rrbracket$$

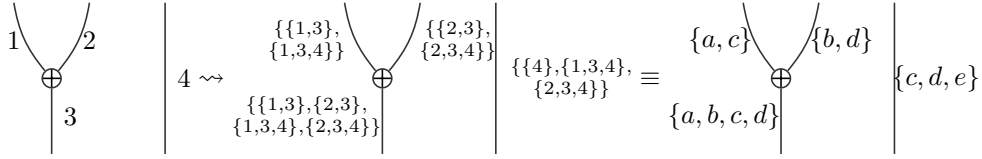
581 The functoriality with respect to the sequential composition is more subtle, as one must  
582 carefully manipulate the indices of the sum and remark that the set of worlds  $w$  eliminated  
583 by the worlds-agnostic composition satisfies the following:

$$584 \quad (a, b) \notin w \iff (\mathfrak{B}, \ell_a^f) \Downarrow a = (\mathfrak{B}, \ell_b^g) \Downarrow b \text{ where } \begin{array}{l} f : (\mathfrak{A}, \ell_a^f) \rightarrow (\mathfrak{B}, \ell_b^f) \\ g : (\mathfrak{B}, \ell_b^g) \rightarrow (\mathfrak{C}, \ell_c^g) \end{array}$$

585 Then, we have

$$586 \quad \llbracket [g]_V \circ [f]_W \rrbracket = \left\{ \sum \llbracket [g]_V \circ [f]_W \rrbracket_{(a,b)} \right\} = \left\{ \sum \llbracket [g]_V \rrbracket_b \right\} \circ \left\{ \sum \llbracket [f]_W \rrbracket_a \right\} = \llbracket [g]_V \rrbracket \circ \llbracket [f]_W \rrbracket$$

587



■ **Figure 14** A diagram, its canonical labeling, and its equivalent with worlds set  $\{a, b, c, d, e, \star\}$

## 588 B.2 Keeping the Worlds Labeling Implicit

589 In Figure 5, we provide an example where one can use the equational theory without the  
 590 worlds. In the previous section, we provided a worlds-agnostic semantics to our diagrams. It  
 591 is reasonable to wonder how much can be done within  $\mathbf{C}_D$ , in other words without using the  
 592 worlds labeling. In the following, we show how to provide a canonical labeling for diagrams  
 593 of  $\mathbf{C}_D$  and deduce a denotational semantics for  $\mathbf{C}_D$ .

594 We consider a diagram  $\mathcal{D}$  of  $\mathbf{C}_D$  and we want to define a canonical worlds labeling over  
 595 its set of wires  $\mathbb{W}$ . The core idea is that a world will be a set of wires that can cohabit with  
 596 each others while satisfying the constraints of the nodes Plus, Tensor, etc. For example in  
 597 Figure 13, there are four wires with wire 1 and 2 being incompatible due to the Plus, but  
 598 also inseparable from wire 3, hence the canonical worlds set would be:

$$599 \quad \{\emptyset, \{1, 3\}, \{2, 3\}, \{4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

600 More formally, a subset of wires  $S \subseteq \mathbb{W}$  is said *valid* if the labeling  $\ell_S : x \mapsto \{\star\}$  if  $x \in S$   
 601 and  $\emptyset$  otherwise is a valid labeling for  $\mathcal{D}$ , in other words if  $(\mathcal{D}, \ell_S)$  is a morphism of  $\mathbf{MW}_{\{\star\}}$ .  
 602 We will take for worlds set  $W_{\mathcal{D}} \subseteq \mathcal{P}(\mathbb{W})$  the sets of valid subsets of wires. The labeling will  
 603 simply be:  $\ell_{\mathcal{D}} : x \mapsto \{S \in W_{\mathcal{D}} \mid x \in S\}$ .

604 ► **Proposition 20.** *For every diagram  $\mathcal{D}$  of  $\mathbf{C}_D$ ,  $(\mathcal{D}, \ell_{\mathcal{D}})$  is a morphism of  $\mathbf{MW}_{W_{\mathcal{D}}}$ . Up to  
 605 renaming of the worlds, this construction is a monoidal functor from  $\mathbf{C}_D$  to  $\mathbf{MW}_{\vee}$ .*

606 It follows that if we define  $\llbracket \mathcal{D} \rrbracket$  as  $\llbracket (\mathcal{D}, \ell_{\mathcal{D}}) \rrbracket$ , this semantics is a monoidal functor from  
 607  $\mathbf{C}_D$  to  $\mathbf{FdHilb}$ . We note that it is enough to compute the semantics of the generators of  
 608  $\mathbf{C}_D$  (see FIGURE) to obtain the semantics for every diagram using  $\llbracket \mathcal{D} \circ \mathcal{D}' \rrbracket = \llbracket \mathcal{D} \rrbracket \circ \llbracket \mathcal{D}' \rrbracket$   
 609 and  $\llbracket \mathcal{D} \square \mathcal{D}' \rrbracket = \llbracket \mathcal{D} \rrbracket \otimes \llbracket \mathcal{D}' \rrbracket$ .

## C Proofs of Universality and Uniqueness of Normal Form

### Universality

In this section, we show that for every  $\mathfrak{A}, \mathfrak{B}$  objects of  $\mathbf{C}_D$ , and for every linear operator

$$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{n1} & \lambda'_1 \\ \vdots & & \vdots & \vdots \\ \lambda_{1m} & \cdots & \lambda_{nm} & \lambda''_m \\ \lambda'_1 & \cdots & \lambda'_n & \lambda_0 \end{pmatrix} \in \mathbf{FdHilb} \left( \bigoplus_{\mathfrak{C} \in \mathfrak{A}^\bullet} \mathcal{H}_{\mathfrak{C}}, \bigoplus_{\mathfrak{F} \in \mathfrak{B}^\bullet} \mathcal{H}_{\mathfrak{F}} \right)$$

then the morphism  $f = \text{iso}_{\mathfrak{B}}^{-1} \circ \lambda \circ \text{iso}_{\mathfrak{A}}$  defined in Figure 10 satisfies  $\llbracket f \rrbracket = \Lambda$ .

As stated in the corresponding section, there is one world for each scalar in  $\lambda$ , so we write  $a_{ij}$  the world associated to  $\lambda_{ij}$  and  $a'_i, a''_j, a_0$  similarly, and  $W$  the set of all those worlds. Writing  $\bullet^k$  for  $\square \dots \square \bullet$  (with  $k$  elements) and  $\mathbb{1}_i^k$  for  $\bullet^k$  where the  $i$ -th  $\bullet$  has been replaced by  $\mathbb{1}$ , we have the following:

$$\begin{cases} \llbracket \lambda \rrbracket_{a_{ij}} : \mathbb{1}_i^n \rightarrow \mathbb{1}_j^m : x \mapsto \lambda_{ij} \cdot x \\ \llbracket \lambda \rrbracket_{a'_i} : \mathbb{1}_i^n \rightarrow \bullet^m : x \mapsto \lambda'_i \cdot x \\ \llbracket \lambda \rrbracket_{a''_j} : \bullet^n \rightarrow \mathbb{1}_j^m : x \mapsto \lambda''_j \cdot x \\ \llbracket \lambda \rrbracket_{a_0} : \bullet^n \rightarrow \bullet^m : x \mapsto \lambda_0 \cdot x \end{cases} \text{ where } \boxed{\lambda} := \begin{array}{c} \begin{array}{c} \dots \\ \lambda'_1 \\ \dots \end{array} \\ \begin{array}{c} \begin{array}{c} \lambda_{11} \\ \lambda_{12} \\ \dots \end{array} \\ \begin{array}{c} \lambda_{nm} \\ \lambda''_m \\ \dots \end{array} \\ \begin{array}{c} \lambda_0 \\ \dots \end{array} \end{array}$$

Additionally, one can show by induction that  $\llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_{ij}}$  and  $\llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a'_i}$  (resp.  $\llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a''_j}$  and  $\llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_0}$ ) are simply the projection on the  $i$ -th (resp.  $(n+1)$ -th) element of the canonical basis, and  $\llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_{ij}}$  and  $\llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a'_i}$  (resp.  $\llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a''_j}$  and  $\llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_0}$ ) are simply the injection on the  $j$ -th (resp.  $(m+1)$ -th) element of the canonical basis. Since we have  $\llbracket f \rrbracket_a = \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_a \circ \llbracket \lambda \rrbracket_a \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_a$  for every  $a \in W$ , and  $\llbracket f \rrbracket$  being the collection of all the  $\llbracket f \rrbracket$ , we obtain that  $\llbracket f \rrbracket = \Lambda$ .

### Uniqueness of Normal Form

Let  $f$  and  $g$  be two diagrams in normal form (with respectively  $\lambda$  and  $\mu$  as inner block), such that  $\llbracket f \rrbracket = \llbracket g \rrbracket$  (the naming of the worlds is taken to be the same in both diagrams, and is the same as in the previous proof). By the definition of  $\llbracket \cdot \rrbracket$ , we have  $\llbracket f \rrbracket_a = \llbracket g \rrbracket_a$  for every  $a \in W$ . We hence have  $\llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_a \circ \llbracket \lambda \rrbracket_a \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_a = \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_a \circ \llbracket \mu \rrbracket_a \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_a$ .

Denoting  $e_i^{\mathfrak{A}}$  (resp.  $e_i^{\mathfrak{B}}$ ) the  $i$ -th element of the basis of  $\mathfrak{A}$  (resp.  $\mathfrak{B}$ ), we have:

$$\begin{cases} \lambda_{ij} = e_j^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_{ij}} \circ \llbracket \lambda \rrbracket_{a_{ij}} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_{ij}} e_i^{\mathfrak{A}} = e_j^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_{ij}} \circ \llbracket \mu \rrbracket_{a_{ij}} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_{ij}} e_i^{\mathfrak{A}} = \mu_{ij} \\ \lambda'_i = \bullet^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a'_i} \circ \llbracket \lambda \rrbracket_{a'_i} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a'_i} e_i^{\mathfrak{A}} = \bullet^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a'_i} \circ \llbracket \mu \rrbracket_{a'_i} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a'_i} e_i^{\mathfrak{A}} = \mu'_i \\ \lambda''_j = e_j^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a''_j} \circ \llbracket \lambda \rrbracket_{a''_j} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a''_j} \bullet^{\mathfrak{A}} = e_j^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a''_j} \circ \llbracket \mu \rrbracket_{a''_j} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a''_j} \bullet^{\mathfrak{A}} = \mu''_j \\ \lambda_0 = \bullet^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_0} \circ \llbracket \lambda \rrbracket_{a_0} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_0} \bullet^{\mathfrak{A}} = \bullet^{\mathfrak{B}^\dagger} \llbracket \text{iso}_{\mathfrak{B}}^{-1} \rrbracket_{a_0} \circ \llbracket \mu \rrbracket_{a_0} \circ \llbracket \text{iso}_{\mathfrak{A}} \rrbracket_{a_0} \bullet^{\mathfrak{A}} = \mu_0 \end{cases}$$

where  $|\mathfrak{B}|$  denotes the number of wires in  $\mathfrak{B}$ .

Hence, all coefficients in the scalars of  $f$  and  $g$  are the same. Since the structure is otherwise the same for  $f$  and  $g$ , they are the same diagram.

636 **D Proof of Soundness**

637 Given that most of the time,  $\llbracket \text{gen} \rrbracket_a$  is the identity, the equations defining  $\equiv_W$  are quite  
 638 straightforward to verify. We immediately have that  $\equiv_W$  is sound with respect to  $\llbracket - \rrbracket_a$  for  
 639 every  $a \in W$ . Since  $\llbracket - \rrbracket$  is defined from  $\llbracket - \rrbracket_a$ , soundness with respect to  $\llbracket - \rrbracket$  is also correct.  
 640 We then handle the five additional equations of  $\equiv$ .

641 **Renaming** Applying a bijection to the worlds set  $W$  does not change the result computed  
 642 by  $\sum_{a \in W} \dots$ , hence this equation is sound with respect to  $\llbracket - \rrbracket$ .

643 **Annihilation due to Scalars** This equation simply removes elements equal to zero from the  
 644 sum  $\sum_{a \in W} \dots$ , hence it is sound with respect to  $\llbracket - \rrbracket$ .

645 **Annihilation due to Plus** Since  $\oplus$  is a biproduct in  $\mathbf{FdHilb}$ , we have  $\text{proj}_H^{H \oplus K} \circ \text{inj}_H^{H \oplus K} =$   
 646  $\mathbf{id}_H$ ,  $\text{proj}_K^{H \oplus K} \circ \text{inj}_K^{H \oplus K} = \mathbf{id}_K$ ,  $\text{proj}_K^{H \oplus K} \circ \text{inj}_H^{H \oplus K} = 0$  and  $\text{proj}_H^{H \oplus K} \circ \text{inj}_K^{H \oplus K} = 0$ . One  
 647 can then simply remove from the  $\sum_{a \in W} \dots$  the elements equal to zero, which proves  
 648 that *Annihilation due to Plus* is sound with respect to  $\llbracket - \rrbracket$ .

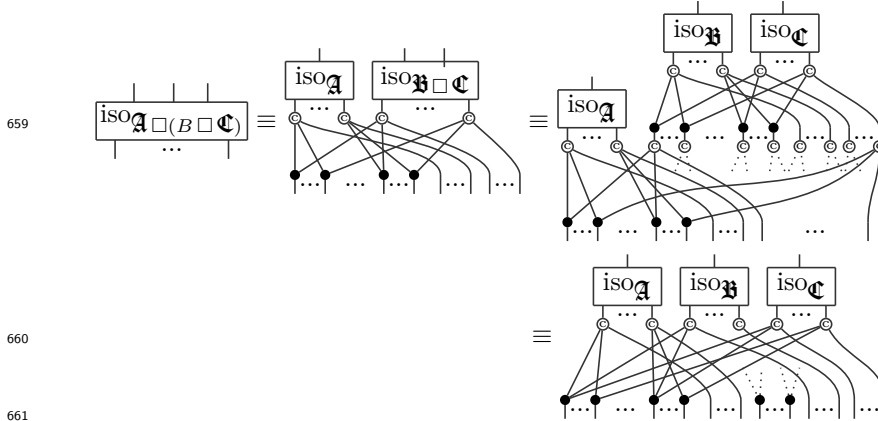
649 **Splitting due to Scalars** Since  $\mathbf{FdHilb}$  is a vector space, we have  $(s + t) \cdot f = s \cdot f + t \cdot f$ ,  
 650 which is exactly the property required for this equation to be sound for  $\llbracket - \rrbracket$ .

651 **Splitting due to Plus** Similarly, we have in  $\mathbf{FdHilb}$  the property that  $\mathbf{id}_{H \oplus K} = \text{inj}_H^{H \oplus K} \circ$   
 652  $\text{proj}_H^{H \oplus K} + \text{inj}_K^{H \oplus K} \circ \text{proj}_K^{H \oplus K}$ , which is the property required for this equation to be  
 653 sound for  $\llbracket - \rrbracket$ .

**E Proofs for Completeness**

**E.1 The Normal Form**

**Proof of Lemma 9.** First notice that in both  $\text{iso}_{\mathcal{A} \square (\mathcal{B} \square \mathcal{C})}$  and  $\text{iso}_{(\mathcal{A} \square \mathcal{B}) \square \mathcal{C}}$ , we can use the bialgebra between contractions and unitors, followed by their respective fusions in the following way:



and similarly for  $\text{iso}_{(\mathcal{A} \square \mathcal{B}) \square \mathcal{C}}$ . It then suffices to check which contractions the bottom unitors are linked to. Naming the  $i$ th contraction existing  $\text{iso}_A$  as  $a_i$ , and similarly for  $B$  and  $C$ , we can see that for each triple  $(a_i, b_j, c_k)$  there is exactly one unitor connected to precisely contraction  $a_i, b_j$  and  $c_k$ , in both diagrams. The same is true for every pair  $(a_i, b_j)$ ,  $(a_i, c_k)$  and  $(b_j, c_k)$ , as well as for every 1-tuple  $(a_i, \cdot)$ ,  $(b_j, \cdot)$  and  $(c_k, \cdot)$ . This shows that both diagrams are equal up to rearranging of the outputs. ◀

► **Lemma 21.** Notice that  $\text{iso}_{A \oplus B} \equiv \text{iso}_A \square B$  and  $\text{iso}_{A \otimes B} \equiv \text{iso}_A \square B$ .

► **Lemma 22.** We have the following identities:

$\text{iso}_A \square \text{iso}_A^{-1} \equiv \mathcal{A}$  and  $\text{iso}_A^{-1} \square \text{iso}_A \equiv \text{c}$  where  $\ell(s_i) \cap \ell(s_j) = \emptyset$  when  $i \neq j$ .

**Proof of Lemma 22.** We will use the following identities:

$\text{iso}_A \square \text{iso}_A^{-1} \equiv \mathcal{A}$  when all the wires  $a_{ij}, b_i, c_j$  are mutually exclusive (1)

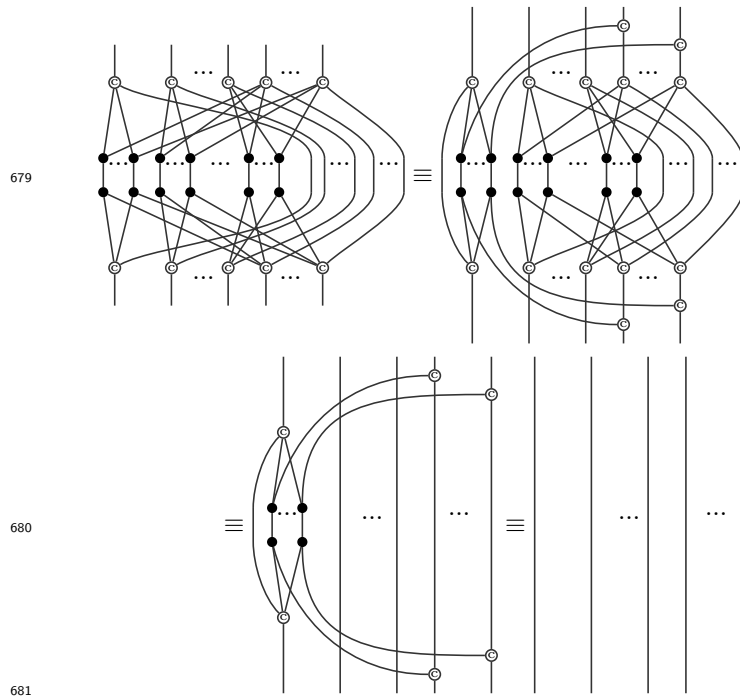
We can show this result by induction on  $n$  and  $m$ . Case  $(0, m)$  is obvious. Case  $(1, 1)$  can be proven easily using worlds sets:

$\text{iso}_A \square \text{iso}_A^{-1} \equiv \mathcal{A}$

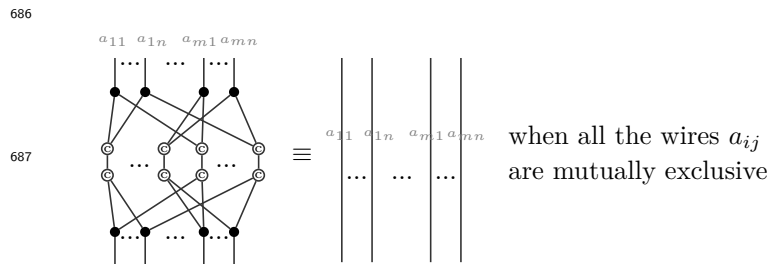
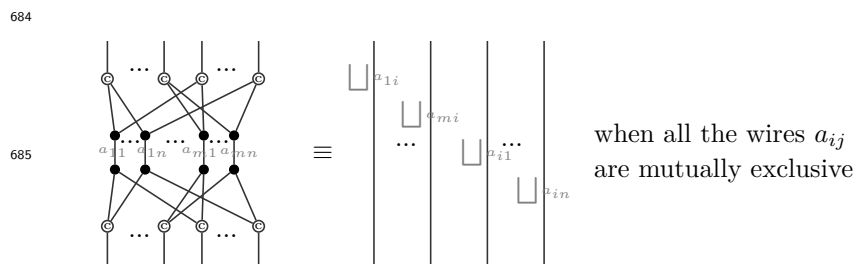
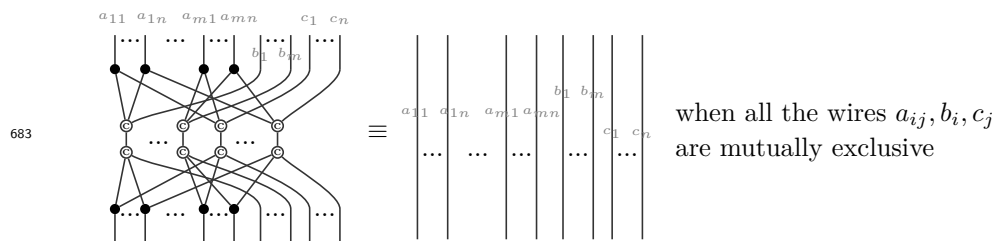


## 23:24 The Many-Worlds Calculus: Representing Quantum Control

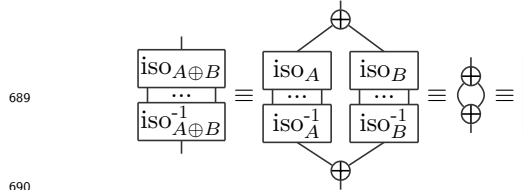
677 For any  $n$  and  $m$ , we can then prove the case  $(n + 1, m)$  using the cases  $(n, m)$  and  $(1, m)$   
 678 (the case  $(n, m + 1)$  is completely symmetric):



682 In a similar way, it is possible to show the following three identities:



688 ■  $[\text{iso}_A^{-1} \circ \text{iso}_A]$ : The result is obvious in cases  $\mathbb{1}$  and  $\mathbb{Z}$ . For  $A \oplus B$ :



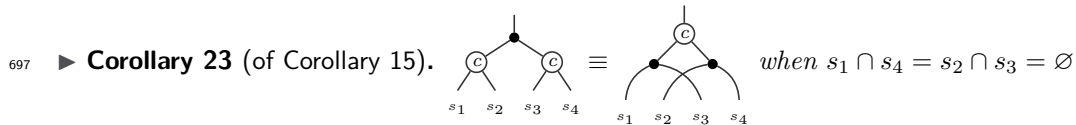
691 The proof is similar for  $\otimes$  and  $\square$  using the previous identities.

692 ■  $[\text{iso}_A \circ \text{iso}_A^{-1}]$ : The result is again obvious for  $\mathbb{1}$  and  $\mathbb{Z}$ . The general result is easy to prove  
 693 by induction using the above identities.

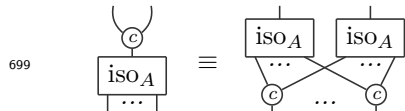
694

### 695 E.2 The Completeness

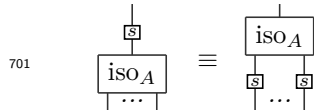
696 We want to show here the results of Section 5.2. To do so we will first derive a few lemmas:



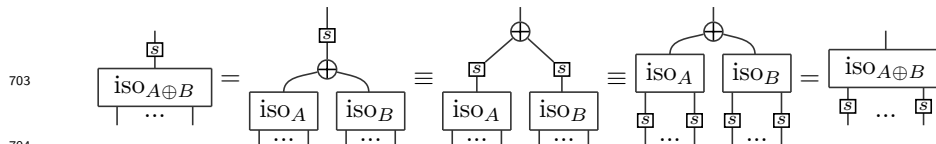
698 ► **Corollary 24** (of Corollary 15). *Single-colored isos distribute over the contraction:*



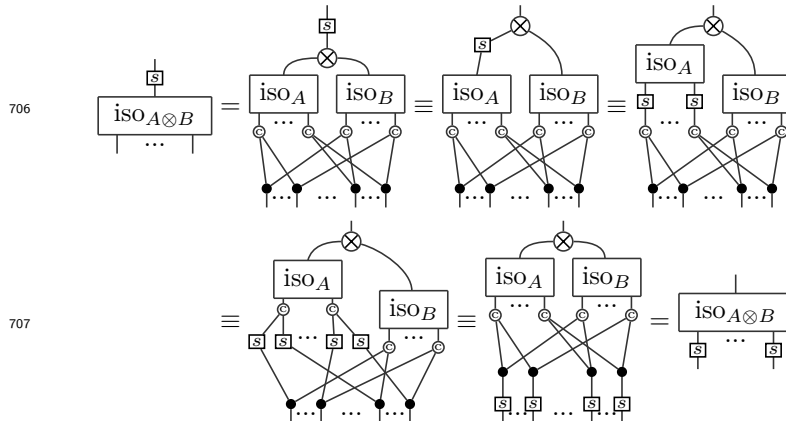
700 ► **Lemma 25.** *Scalars distribute over single-colored isos:*



702 **Proof.** The result is obvious for  $\mathbb{1}$  and  $\mathbb{Z}$ . For  $A \oplus B$ :



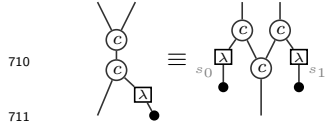
705 For  $A \otimes B$ :



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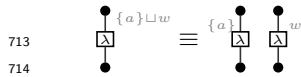
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► **Corollary 26** (of Corollary 15).

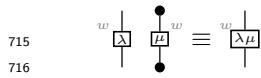


710  
711  
712 with  $s_0 \cap s_1 = \emptyset$ .

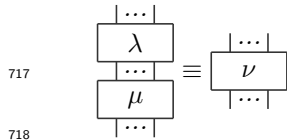
► **Corollary 27** (of Lemma 13).



713  
714  
► **Corollary 28** (of Lemma 13).

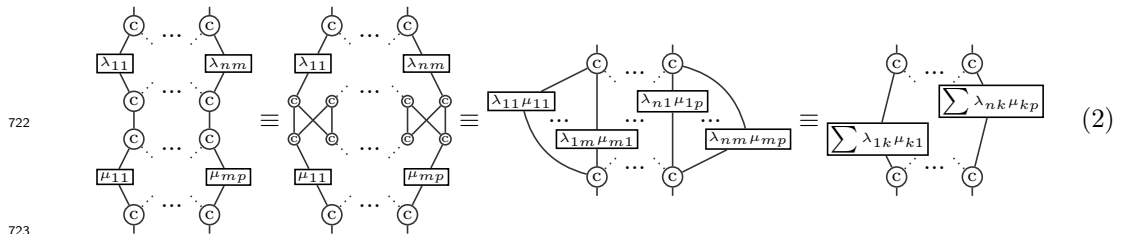


715  
716  
► **Lemma 29.**



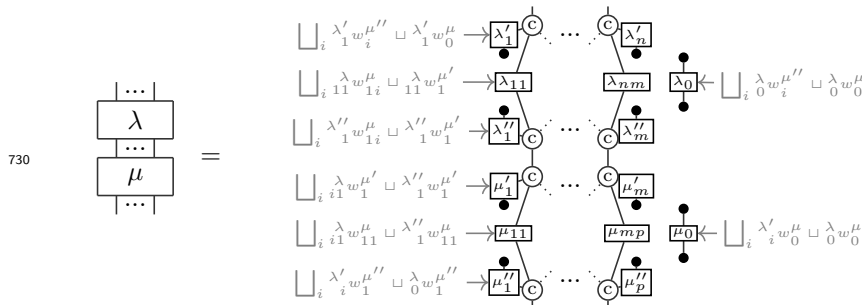
717  
718  
719 with  $\nu_{ij} = \sum_k \lambda_{ik} \mu_{kj} + \lambda'_i \mu''_j$ ,  $\nu'_i = \sum_k \lambda_{ik} \mu''_k + \mu_0 \lambda'_i$ ,  $\nu''_i = \sum_k \lambda''_k \mu_{ki} + \lambda_0 \mu''_i$  and  $\nu_0 =$   
720  $\lambda_0 \mu_0 + \sum_k \lambda''_k \mu'_k$ .

721 **Proof.** Consider first the following simpler case:



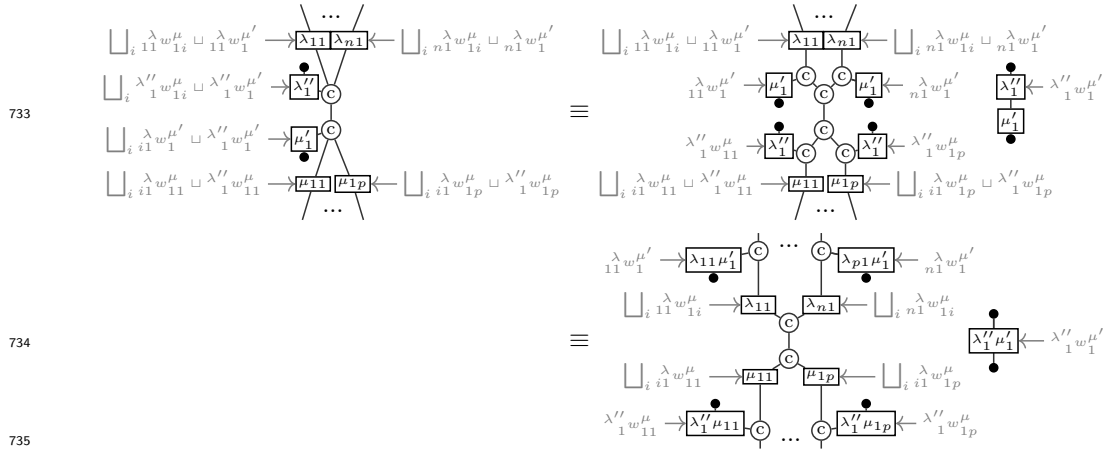
724 Importantly, in the last diagram, we have that each internal wire lives in a set of worlds  
725 that has empty intersection with any of the others. It is hence possible to rename this set of  
726 worlds to a singleton, and to do it for each internal wire.

727 For the general case, it is important to properly look at the worlds. Denoting  $w_{ij}^\lambda$  the  
728 (singleton) world that bears  $\lambda_{ij}$ ,  $w_i^{\lambda'}$  the one that bears  $\lambda'_i$ ,  $w_i^{\lambda''}$  the one that bears  $\lambda''_i$  and  
729  $w_0^\lambda$  the one that bears  $\lambda_0$  in matrix block  $\lambda$ , and similarly in matrix block  $\mu$ , we get:



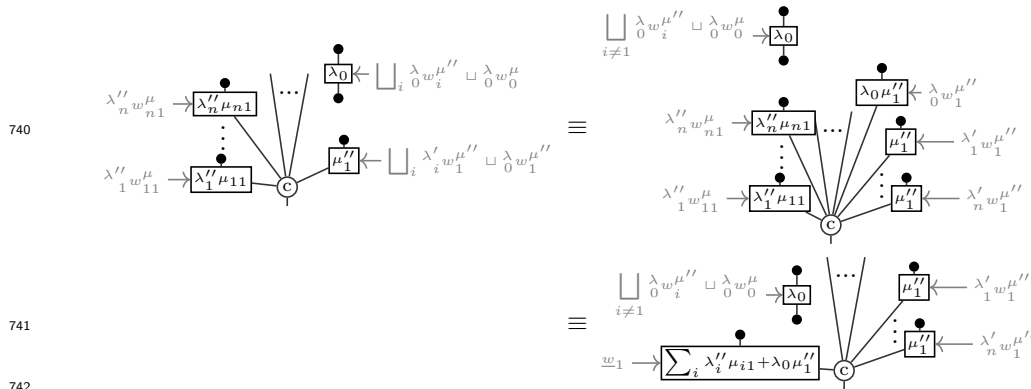
730

731 where  $a_x w_y^b = \{(z_1, z_2) \mid z_1 \in w_x^a, z_2 \in w_y^b\}$ . Let us now look at how to push the middle  
 732 dangling wires through. At the first junction, we have:



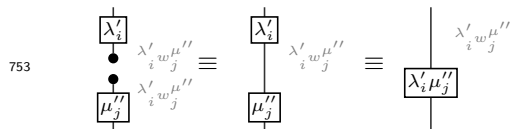
736 Doing this at every junction wire, we are able to push all the middle dangling wires to the  
 737 boundaries, and where the middle is in the shape of the previous simple case (2).

738 Let us now look at the boundaries, for instance the first output. After pushing all dangling  
 739 wires, and taking into account the scalar diagram  $\lambda_0$ , we have:



743 where on the right hand side of the diagram, we decomposed the dangling wire by its different  
 744 elementary worlds. One of them was shared with the  $\lambda_0$  scalar, and using Lemmas 27 and  
 745 28, we could bring them together. Then on the left hand side, we used the fact that none of  
 746 the  $\lambda'_i w_{i1}^{\mu''}$  nor the  $\lambda_0 w_1^{\mu''}$  could be found elsewhere in the diagram, so we could sum all the  
 747 scalars without altering the rest of the diagram.

748 Doing so for all boundaries, scalars  $\lambda_0$  and  $\mu_0$  are only left on world  $\lambda_0 w_0^{\mu''}$ . We can then use  
 749 Lemma 28 to bring them together. All the worlds on right hand side of the last diagram  
 750 above  $\lambda'_i w_1^{\mu''}$  can be found on the boundaries at the top (i.e. for each input  $i$  and output  $j$   
 751 there is world  $\lambda'_i w_j^{\mu''}$  shared between the two). Using the small following derivation, we can  
 752 bring them together:

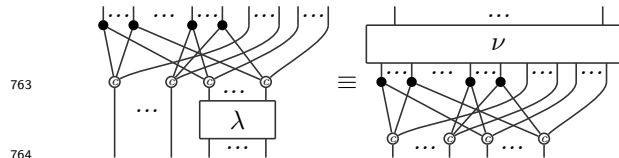


754 Now this world can be found nowhere else in the diagram, it is hence possible to sum the  
 755 scalar with the other that has the same neighbourhood, obtained from (2).

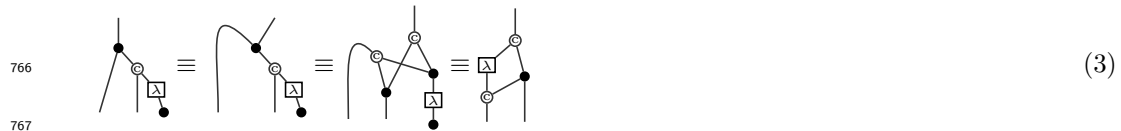
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756 It remains to deal with the scalars. We have  $\lambda_0 \mu_0$  on world  $\lambda_0 w_0^\mu$  from what we just  
 757 discussed. We also have scalar  $\lambda_i' \mu_i'$  on world  $\lambda_i' w_i^{\mu'}$  from the dangling wires at the junction.  
 758 All these worlds are again mutually exclusive and found nowhere else in the diagram. We  
 759 can hence sum them and get  $\lambda_0 \mu_0 + \sum_i \lambda_i' \mu_i'$  as the scalar on the side. After all this, the  
 760 diagram is in the appropriate form (up to renaming of worlds), with all worlds in the internal  
 761 wires being mutually exclusive. ◀

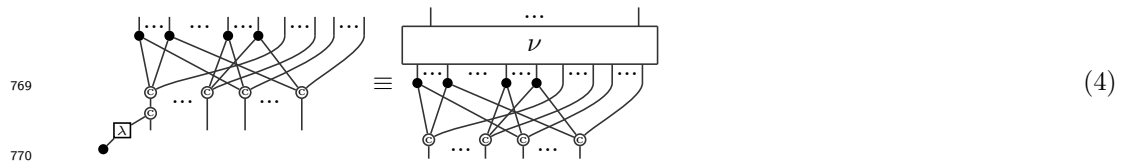
762 ▶ **Lemma 30.** For any "matrix block"  $\lambda$ , there exists a "matrix block"  $\nu$  such that:



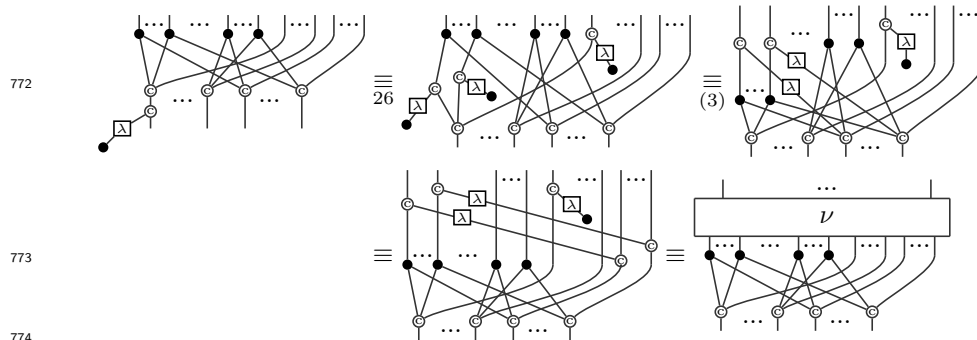
765 **Proof.** We show the result through several steps. First we show the following:



768 Then, we deal with the top dangling scalars and prove that:

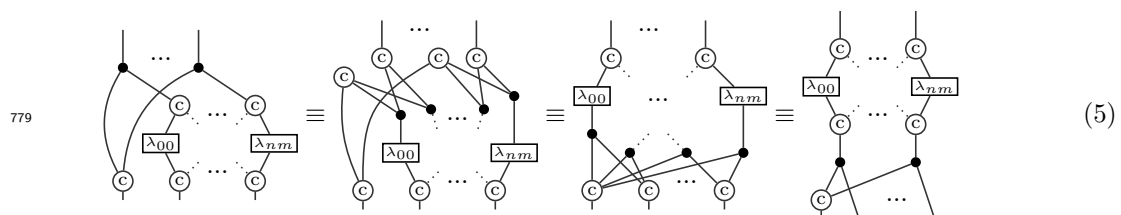


771 Indeed:



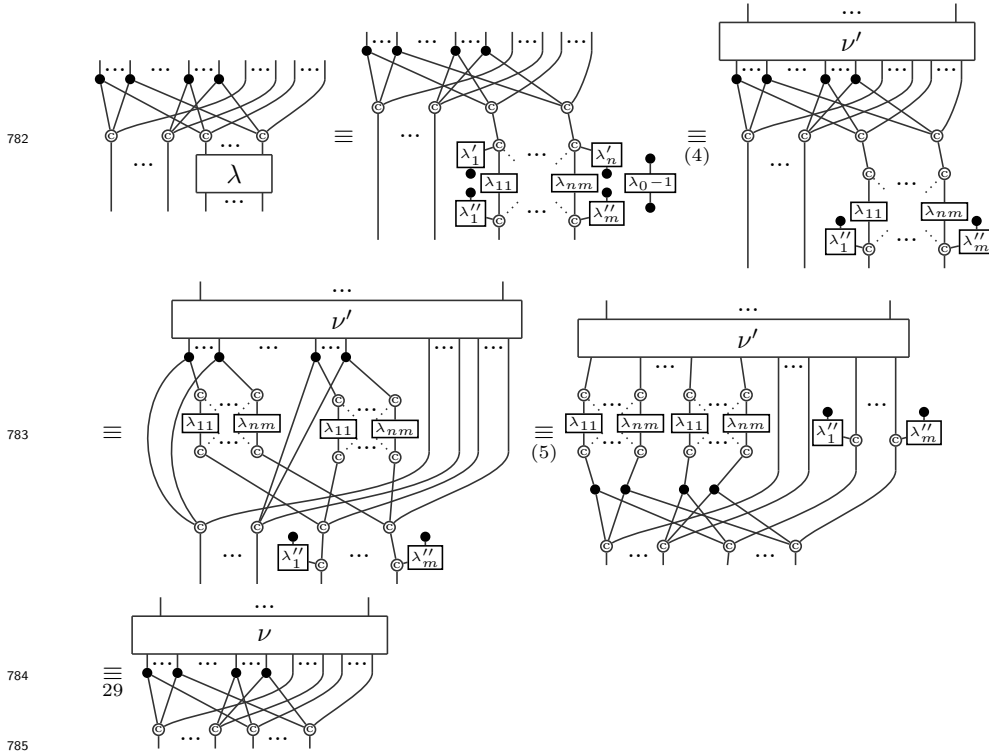
775 This case generalizes to any bottom wire the dangling scalar is applied on. When we have  
 776 several of them, we can make them go through the top part in turns, then aggregate them  
 777 under a single "matrix block" using Lemma 29.

778 We then derive the following equation:



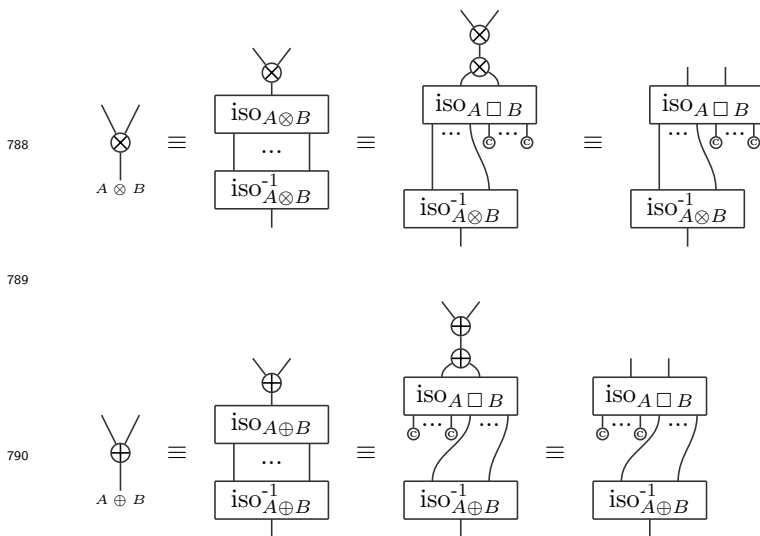
780

781 We can finally prove the lemma:



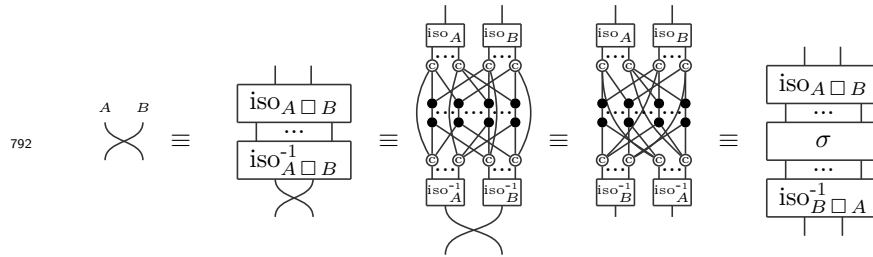
We can now move on to show that generators can be put in normal form:

**Proof of Proposition 11.**

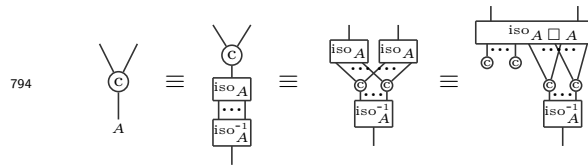


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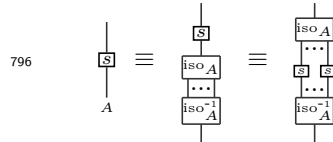
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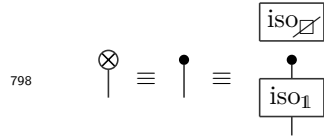
793 with  $\sigma$  a simple permutation of wires.



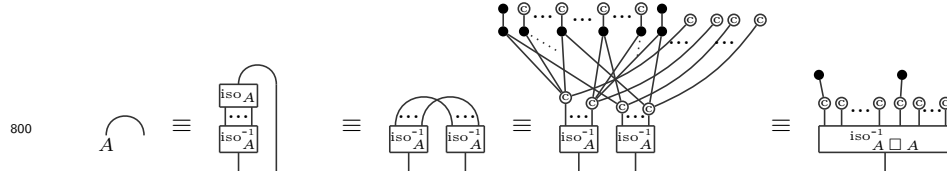
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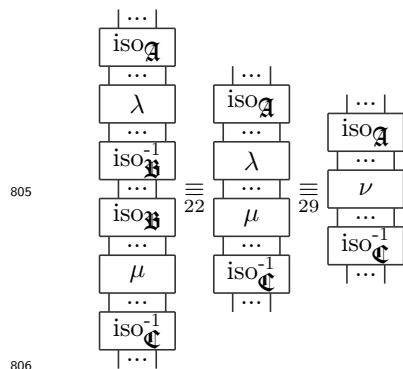
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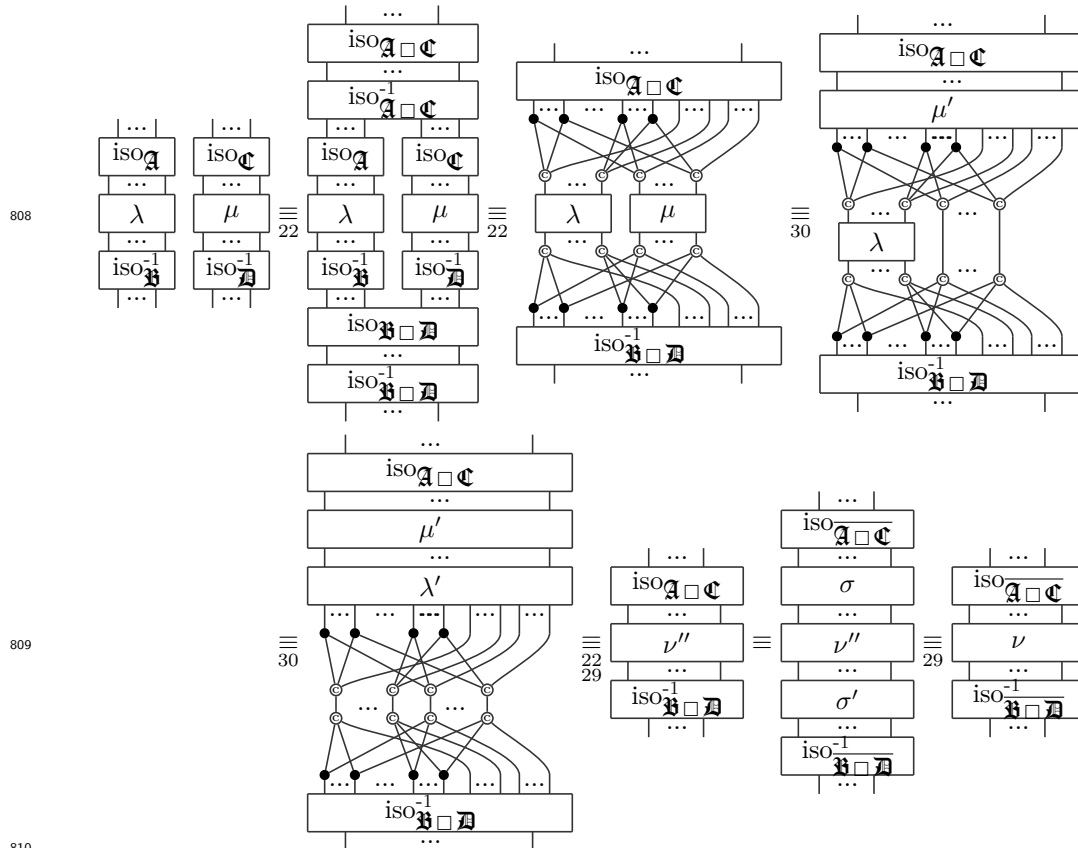
801 The upside-down versions of the generators are provided in exactly the same way (but  
802 upside-down). ◀

803 And then that compositions of diagrams in normal form can be put in normal form:

804 **Proof of Proposition 12.** In the case of sequential composition:



807 In the case of parallel composition:



811 where  $\overline{a \square c}$  represents the canonical choice of composition with  $\square$  (and similarly for  $\overline{b \square d}$ ).

812