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Brief paper

Arbitrary order differentiator with varying homogeneity degree ☆

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ABSTRACT

In this paper, we propose an arbitrary order differentiator with time-varying homogeneity degree (exponent) and we establish its convergence properties and its robustness with respect to measurement noise, perturbations and the variation of the exponent. For this we use a unified Lyapunov function approach. We also propose two possible applications for such a differentiator. One provides a variation of the exponents that ensures a differentiator converging in *fixed and finite times*. The other that extends our previous results, aims at improving the behavior of the differentiator with respect to measurement noise, for which a noise-driven variation of the exponents is given. The results are illustrated using simulations for a second order differentiator.

1. Introduction

Noisy signal differentiation is a very important and challenging problem, having a large range of applications (electric machines, pneumatic cylinders, drones, etc.). Sensors of such applications are subject to electromagnetic compatibility, quantification problems and so on, then sensors measurements are very noisy in a large bandwidth. Many theoretical and applied papers are dedicated to this topic. Linear differentiators are the most common solutions employed so far, among them high-gain observers (HGO) have received recently a lot of attention (Prasov & Khalil, 2013; Vasiljevic & Khalil, 2008) (see also Khalil and Praly (2014)). Such differentiators are able to estimate asymptotically exactly only polynomial signals, i.e. those for which the nth derivative (for some $n \in \mathbb{N}$) vanishes, a rather thin class of signals. The discontinuous Levant's differentiator (Levant, 1998, 2003, 2005) is able to estimate exactly and in finite-time the derivatives of a signal with *bounded* nth derivative. Further developments of discontinuous differentiators are pursued by Bartolini et al. (2000), Bejarano and Fridman (2010), Floquet and Barbot (2007).

Recently, a whole family of arbitrary order homogeneous differentiators parametrized by its homogeneity degree d, and that includes both the HGO ($d = 0$) and Levant's ($d = -1$) differentiators as particular cases, has been proposed in Cruz-Zavala and Moreno (2016, 2019), Sanchez et al. (2018) (see also the alternative approach given in Lopez-Ramirez et al. (2018) for values of d close to $d = 0$). These encompass also previous algorithms as Andrieu et al. (2009), Efimov and Fridman (2011), Perruquetti et al. (2008). When $d \neq -1$ they are also able to estimate asymptotically exactly only polynomial signals.

Since no differentiator is able to estimate exactly the derivatives of a signal perturbed by arbitrary noise, the estimation error of the derivatives of any differentiator depends then on the size of the $n-$th derivative of the signal to be differentiated, on the size of the additive noise affecting it and the gains of the differentiator. For a given signal and noise these two effects depend (for fixed gains) on the homogeneity degree d of the differentiator. Since for the discontinuous differentiator the $n-$th derivative does not have any effect on the estimation, it is clear (by continuity reasons) that for slightly noisy signals it is better to maintain the homogeneity degree near $d = -1$ to minimize the differentiation error. For significantly noisy signals it may be better to select a different value of d, as e.g. $d = 0$. Motivated by this observation, recently in Ghanes et al. (2017a, 2017b, 2020) the authors proposed to modify on-line the homogeneity degree d ($d \in (-1, 0)$) of a second-order differentiator to improve the performance, i.e. to reduce the differentiation error of the first derivative of the signal, according to an estimation of the size of the noise. This approach is an alternative to many other attempts to reduce the effect of noise, as those proposed in Ball and Khalil (2011) for the HGO or in Levant and Yu (2018), Jbara et al. (2021) for the family of homogeneous differentiators, and allows to combine advantages of both HGO and homogeneous differentiators. In the absence of noise, fixed-time differentiators ($d > 0$) are also

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proposed to estimate exactly the derivatives of a signal in a finite-time that is independent of the initial condition (Angulo, Moreno, et al., 2013; Cruz-Zavala et al., 2011; Lopez-Ramirez et al., 2018; Polyakov, 2012).

In this paper, our main contributions in comparison to the existing results are to propose an arbitrary order homogeneous differentiator with time-varying homogeneity degree d in order:

- To consider a continuous change in d with positive ($d > 0$) and negative ($d \in (-1, 0)$) values. This allows integrating two differentiator's objectives: fixed-time convergence (independent of the initial condition) and less sensibility to noise by extending the idea of Ghanes et al. (2017a, 2017b, 2020) to an arbitrary order differentiator.
- To prove its convergence by analyzing the effects of noise, of the size of the nth derivative of the signal to be differentiated, and of the d variation in the asymptotic behavior of the estimation error. For this we use the Lyapunov framework developed in Cruz-Zavala and Moreno (2016, 2019), Sanchez et al. (2018) to analyze the effect of the time-varying homogeneity degree d.

The remainder of the paper is organized as follows. In Section 2 we propose an arbitrary order differentiator with varying homogeneity degree. In Section 3, we present the convergence and robustness properties of the proposed differentiator. Simulation results for a second order differentiator ($n = 3$) are given in Section 4. In Section 5 we provide all statements proofs. Section 6 concludes the paper and gives some perspectives.

2. Proposed arbitrary order differentiator with varying exponents

Let the input signal $f(t)$ to the differentiator be a Lebesgue-measurable function defined on $[0, \infty)$. The signal $f(t)$ is assumed to be decomposed as $f(t) = f_0(t) + \nu(t)$, where the first term is the n-times differentiable unknown base signal $f_0(t)$, to be differentiated, and the second term $\nu(t)$ corresponds to a uniformly bounded Lebesgue-measurable noise signal.

Assumption 1. There exist constants $\Delta > 0$ and $N > 0$ such that, $\forall\, t \geq 0$, $\left|f_0^{(n)}(t)\right| \leq \Delta$ and $|\nu(t)| \leq N$.

To estimate the time derivatives $f_0^{(i)}(t)$, $i = 1, \ldots, n-1$, of the base signal we consider the following family of arbitrary order homogeneous differentiators

$$\begin{aligned}
\dot{\hat{x}}_i &= -k_i L^i \left\lceil \hat{x}_1 - f \right\rfloor^{\frac{1-(n-i-1)d}{1-(n-1)d}} + \hat{x}_{i+1}\,,\ i = \{1, ., n-1\} \\
&\vdots \\
\dot{\hat{x}}_n &= -k_n L^n \left\lceil \hat{x}_1 - f \right\rfloor^{\frac{1+d}{1-(n-1)d}}\,,
\end{aligned} \tag{1}$$

where the function $\lceil x \rfloor^p$ corresponds to a signed power, i.e. $\lceil x \rfloor^p = |x|^p \operatorname{sign}(x)$. L and k_i are positive constants, selected to cancel exactly the perturbation for $d = -1$, i.e. $k_n L^n > \Delta$, and k_i are chosen such that the Laplace characteristic polynomial $s^n - \sum_{i=1}^{n} k_i s^{i-1} = 0$ is Hurwitz. d is the homogeneity degree of system (1) with the following assumption

Assumption 2. $d : \mathbb{R}_+ \to \left[-1, \frac{1}{n-1}\right)$ is a continuous function that is piecewise differentiable and there exists a constant $D > 0$ such that, $\forall\, t \geq 0$, $|\dot{d}(t)| \leq D$ in the meaning of Dini derivatives.

Convergence of the differentiator means that the differentiation errors, defined as

$$e_i(t) \triangleq \hat{x}_i(t) - f_0^{(i-1)}(t)\,,\ i = 1, \ldots, n\,, \tag{2}$$

asymptotically converge to zero. Scaling these differentiation errors (2) as $z_1 = \frac{e_1}{1}, \ldots, z_i = \frac{e_i}{L^{i-1} k_{i-1}}$, we can write the differentiation error dynamics (2) as

$$\begin{aligned}
\dot{z}_i &= -L\tilde{k}_i \left(\lceil z_1 - \nu \rfloor^{\frac{r_{i+1}}{r_1}} - z_{i+1} \right)\,,\ i = 1, \ldots, n-1 \\
&\vdots \\
\dot{z}_n &= -L\tilde{k}_n \lceil z_1 - \nu \rfloor^{\frac{r_{n+1}}{r_1}} - \frac{f_0^{(n)}(t)}{k_{n-1} L^{n-1}}\,,
\end{aligned} \tag{3}$$

where for $i = 1, \ldots, n$, $\tilde{k}_i = \frac{k_i}{k_{i-1}}$, $k_0 = 1$. System (3), when $f_0^{(n)}(t) \equiv 0$ and $\nu = 0$, is $\mathbf{r}$-homogeneous (Bacciotti & Rosier, 2005; Bernuau et al., 2014; Bhat & Bernstein, 2005; Cruz-Zavala & Moreno, 2019; Levant, 2005; Levant & Livne, 2016; Lopez-Ramirez et al., 2018; Polyakov & Fridman, 2014; Sanchez et al., 2018) with homogeneity weights $r_i = 1-(n-i)\,d$, $i = 1, \ldots, n$, and homogeneity degree $-1 \leq d < \frac{1}{n-1}$. We also introduce $r_{n+1} = 1 + d$, and $r_0 = 1 - n\,d$.

Usually, the value of d is chosen to be a *constant* in the interval $d \in \left[-1, \frac{1}{n-1}\right)$. For $d = 0$ (1) corresponds to a linear (High-Gain) differentiator (Vasiljevic & Khalil, 2008). When $d = -1$ we obtain the arbitrary order Exact and Uniform Levant's Differentiator of arbitrary order (Levant, 2003, 2005). In this latter case system (1) has a discontinuous right-hand side, and their solutions are understood in the sense of Filippov (1988).

In this paper, we consider a continuous change of d (Assumption 2) with positive and negative values. There are several possible applications of varying d. Hereafter we explore two of them. When $d > 0$, we ensure a fixed-time differentiator convergence (see Section 2.1) while for $d \in (-1, 0)$, we provide an efficient balance between the arbitrary order Exact and Uniform Levant's Differentiator (Levant, 2003, 2005) and the linear (High-Gain) differentiator (Vasiljevic & Khalil, 2008) thanks to the noise driven d (see Section 2.2). Moreover, we establish the convergence properties of the differentiator (1) with varying exponent d.

2.1. Fixed-time differentiator

Let us consider $\left(\nu(t)\,,\, f_0^{(n)}(t)\right) \equiv 0$ for clarity in the presentation. For constant $d < 0$, it follows from (19) that $z(t)$ converges to zero in a finite-time, upper bounded by $T(z_0) = \frac{M}{L\kappa} \|z_0\|_{\mathbf{r}}^{-d}$. It is also clear from the previous expression that the convergence time $T(z_0)$ grows unboundedly with $\|z_0\|_{\mathbf{r}}$. On the other side, for constant $d > 0$, it follows from (20) that $z(t)$ converges to zero only asymptotically. However, from (20) we can obtain an upper bound $T_\epsilon(z_0)$ of the convergence time from an initial condition z_0 to the homogeneous ball $\mathcal{N}_\epsilon = \{z \in \mathbb{R}^n \mid \|z\|_{\mathbf{r}} \leq \epsilon\}$ centered at $z = 0$, given by $T_\epsilon(z_0) = \frac{\left(\frac{M}{\epsilon}\|z_0\|_{\mathbf{r}}\right)^d - 1}{L\kappa \|z_0\|_{\mathbf{r}}^d}$. Since $\lim_{\|z_0\| \to \infty} T_\epsilon(z_0) = \frac{M^d}{\epsilon^d L\kappa}$ is a constant, we conclude that convergence to $\mathcal{N}_\epsilon$ is in *fixed-time*, that is, any neighborhood of the origin will be attained in a finite-time independent of the initial condition. It is clear that combining these two properties, i.e. first using $d > 0$ and then $d < 0$, leads to a differentiator converging to $z = 0$ in fixed-time. The main result of Angulo, Fridman, et al. (2013) is precisely the use of a switching strategy between two values of d, first $d > 0$ and then $d = -1$, to build such a differentiator. Our results allow us to use a continuous change in d to obtain the same result, for arbitrary positive and negative homogeneity degrees. Although there are many possible strategies, as illustration we present just a simple one.

Proposition 1. *Select two values $0 < d_0 < \frac{1}{n-1}$ and $-1 \leq d_f < 0$, and a desired fixed-time $\mathcal{T}_f > 0$. Then, by considering the continuous signal*

$$d(t) = \begin{cases} d_0 & 0 \leq t \leq \frac{1}{3}\mathcal{T}_f \\ \frac{3(d_f - d_0)}{\mathcal{T}_f} t + 2d_0 - d_f & \frac{1}{3}\mathcal{T}_f \leq t \leq \frac{2}{3}\mathcal{T}_f \\ d_f & \frac{2}{3}\mathcal{T}_f \leq t \leq \mathcal{T}_f \,, \end{cases} \tag{4}$$

verifying Assumption 2, the differentiator (1) is a fixed-time differentiator for $N = 0$ and $\Delta = 0$. When $d_f = -1$ convergence to zero can be also achieved for $\Delta \neq 0$.

The proof is given in Section 5.

2.2. Noise driven time-varying d

In the works (Ghanes et al., 2017a, 2017b, 2020) a low-order differentiator with varying exponent has been shown to improve the behavior under noise and perturbation with respect to both, a linear and Levant's differentiator. Here, we propose an arbitrary order differentiator, with a time-varying exponent d in the interval $d \in [-1, 0]$. The value of d is determined on-line according to the magnitude of the high-frequency components of the noise $\nu(t)$, denoted by $\left|f_{\text{mhf}}(t)\right|$. $f_{\text{mhf}}(t)$ is determined by passing the signal f to be differentiated by a Butterworth fourth-order high-pass filter[1] as follows

$$F(s) = \mathfrak{L}(f(t)) \,, f_{\text{mhf}}(t) = \mathfrak{L}^{-1}\left(F_{\text{mhf}}(s)\right) \,, \tilde{s} = \frac{s}{\omega_{\text{cut}}}$$
$$F_{\text{mhf}}(s) = \frac{\tilde{s}^4}{\left(\tilde{s}^2 + 0.7654\tilde{s} + 1\right)\left(\tilde{s}^2 + 1.8478\tilde{s} + 1\right)} F(s) \,, \tag{5}$$

where $\mathfrak{L}$ represents the Laplace transform and $\omega_{\text{cut}} > 0$. The differentiator is given by (1), where the actual value of d is calculated as

$$\dot{\hat{x}}_{n+1}(t) = -\tau\left(\hat{x}_{n+1}(t) - \left|f_{\text{mhf}}(t)\right|\right) \tag{6}$$
$$d(t) = \frac{-1}{1 + \mu\left|\hat{x}_{n+1}(t)\right|} \,,$$

where $\tau > 0$, $\mu > 0$ are constants. In the absence of noise $\left|f_{\text{mhf}}(t)\right| = 0$, so that $\hat{x}_{n+1}(t) = 0$, d becomes $d(t) = -1$. In this case we obtain Levant's exact differentiator. In presence of noise, $\left|f_{\text{mhf}}(t)\right| \neq 0$ and $\hat{x}_{n+1}(t) \neq 0$, so that $d(t)$ moves towards 0. For a large value of $\left|f_{\text{mhf}}(t)\right|$ a differentiator close to the linear one is attained. The parameters of (1), the filter (5) and (6) are selected as follows: (i) for k_n and L, see just after (1); (ii) Since the dynamics of d is given by

$$\dot{d} = \tau \frac{\mu\left\lceil\hat{x}_{n+1}(t)\right\rfloor^0\left(\left|f_{\text{mhf}}(t)\right| - \hat{x}_{n+1}(t)\right)}{\left(1 + \mu\,\hat{x}_{n+1}\right)^2} \,, \tag{7}$$

and its bounds D must be largely smaller than a fixed L according to (22), consequently τ is chosen largely smaller than L, i.e., $\tau \lll L$. Thus, $\hat{x}_{n+1}$ behaves as a low pass filter of $\left|f_{\text{mhf}}(t)\right|$. (iii) μ is chosen large (resp. small) to shift the behavior of the differentiator towards the linear (respectively, the sliding mode) solution. (iv) The cutoff frequency ω_{cut} of the Butterworth fourth-order high-pass filter is chosen larger than the bandwidth of the signal $f_0(t)$, in order to achieve a separation between noise and signal. Note that in the implementations in Ghanes et al. (2017a, 2017b, 2020) a parameter different from d was used. The results are equivalent, since the two parameters are related by $\alpha = \frac{1-(n-2)d}{1-(n-1)d}$.

[1] A 4th-order Butterworth filter is chosen as a trade-off between ideal characteristics and implementation complexity.

3. Convergence properties of the proposed observer

The following Theorem is the main result of the paper. It shows that a slowly varying d does not destroy the (local) convergence of the differentiator, but also that the effect of a time-varying d is quite different for $d > 0$ and for $d < 0$.

Theorem 1. *Suppose that Assumptions 1 and 2 are satisfied. Then, for ϵ_1, ϵ_2, $\epsilon > 0$ as small as possible, the differentiator (1) with time-varying degree of homogeneity, has the following properties.*

(1) For $0 < \epsilon_1 \leq d(t) \leq \frac{1}{n-1} - \epsilon_2$, the error dynamics (3) is Globally Input-to-State Stable (GISS), i.e. for all initial condition $z_0 \in \mathbb{R}^n$, there exist $\mathcal{KL}_\infty$ function β and a $\mathcal{K}_\infty$ function γ such that the following holds

$$\|z(t)\|_{\mathbf{r}} \leq \beta\left(\|z_0\|_{\mathbf{r}}, t\right) + \gamma\left(\varrho\left(\Delta, N, D\right)\right) \,, \tag{8}$$

for some continuous function $\varrho : \mathbb{R}^3_+ \to \mathbb{R}_+$, $\varrho(0) = 0$.

(2) For $-1 < d(t) \leq -\epsilon < 0$, the error dynamics (3) is Locally Input-to-State Stable (LISS) i.e. for all initial conditions $\|z_0\|_{\mathbf{r}} \leq Z_0$, the following holds

$$\|z(t)\|_{\mathbf{r}} \leq \beta_D\left(\|z_0\|_{\mathbf{r}}, t\right) + \gamma_D\left(\varrho\left(\Delta, N\right)\right) \,, \tag{9}$$

for some $\mathcal{KL}$ function β_D and a $\mathcal{K}_\infty$ function γ_D and a continuous function $\varrho : \mathbb{R}^2_+ \to \mathbb{R}_+$, with $\varrho(0) = 0$. Moreover, $Z_0 \to \infty$ as $D \to 0$.

The proof is given in Section 5 using strong Lyapunov functions.[2] However, since the time varying homogeneity degree differentiator is not homogeneous, bounds for the trajectories, as given by (19)–(21) are, in general, no longer valid. Nevertheless, it is importance to mention two results:

- for $d < 0$, when $N = \Delta = 0$, the convergence to $z = 0$ can be bounded by (19) with $d = -\epsilon < 0$, when $d(t) \leq -\epsilon$ for all times (see the end of Section 5.1.6-b)). Moreover, convergence for time-varying $d < 0$ cannot be longer assured to be global, and the attraction region is smaller the larger is $\dot{d}$ (see (16) and Section 5.1.6-b)).
- for $d > 0$ the global boundedness of the trajectories is assured for any size of $\dot{d}$, but convergence to $z = 0$ is not guaranteed, even when $N = \Delta = 0$. For convergence to $z = 0$, it is necessary in this case that $d(t)$ converges to a constant value (see (16), (20) and Section 5.1.6-(a)).

Remark 1. a- For $D = 0$, $\Delta = 0$, $N = 0$, (8) tends to zero when $t \to +\infty$, asymptotically for $d \in \left(0, \frac{1}{n-1}\right)$ and in finite-time for $d \in (-1, 0)$.

b- For $D = 0$, $\Delta = 0$, $N = 0$ and $d = 0$ the convergence to zero of (8) is exponential (Khalil, 2002).

c- For $D = 0$, $\Delta \neq 0$, $N = 0$ and $d = -1$ the convergence to zero of (8) is in finite-time (Levant, 2003).

4. Simulation results

To show the effectiveness of the proposed differentiator with varying gains, a simulation study is performed. To estimate the first and second order derivatives of a noisy signal, three different second-order differentiators (1) (with $n = 3$) are designed for comparison: (i) a linear differentiator with constant $d = 0$; (ii) a discontinuous Levant's exact differentiator with constant $d = -1$; and (iii) the proposed varying exponent differentiator, with d. To illustrate in one simulation the two possible applications presented in Sections 2.1 and 2.2, during the first 6 seconds,

[2] Strong Lyapunov functions allow to prove asymptotic uniform convergences.

i.e. for $0 \leq t \leq 6$ of the simulation the differentiator with time-varying d given in Proposition 1, with $d_0 = \frac{1}{3}$ and $d_f = -1$ and ensuring a fixed-time convergence in $\mathcal{T}_f = 3s$, is implemented in the absence of noise. For $t \geq 6$ the varying differentiator of Section 2.2, taking values in the interval $d \in [-1, 0]$, and driven by noise according to (5)–(6), has been implemented.

The base signal to be differentiated, $f_0(t)$, is generated integrating the model $x_1 = f_0$, $\dot{x}_1 = x_2 = f_0^{(1)}$, $\dot{x}_2 = x_3 = f_0^{(2)}$, $\dot{x}_3 = f_0^{(3)} = x_1^{(3)}$ in Matlab/Simulink, using Euler's integration method, and the initial conditions $x(0) = [1, 2, 5] \times 10^{-2}$. The third-order derivative $f_0^{(3)} = x_1^{(3)}$ was chosen, as a piecewise bounded constant function, satisfying Assumption 1. It is given by

$$f_0^{(3)}(t) = \begin{cases} 0 & \text{for } t \in [0,\ 6.5] \cup [8.5,\ 11.5] \\ & \quad \cup [13.5,\ 16.5] \cup [18.5,\ 20] \\ 2 & \text{for } t \in [6.5,\ 8.5] \cup [11.5,\ 13.5] \\ & \quad \cup [16.5,\ 18.5]. \end{cases}$$

The additive noise $\nu(t)$ is white noise, with power 1×10^{-7}, sample time 1×10^{-4} [s] (fixed step-size), and starting seed for random number generator 23541. The noise is bounded, which satisfies Assumption 1, and it has been multiplied by the function

$$\sigma(t) = \begin{cases} 0 & \text{for } t \in [0,\ 6] \cup [7.25,\ 9] \cup [10.25,\ 12] \\ & \quad \cup [13.25,\ 15] \cup [16.25,\ 18] \\ & \quad \cup [19.25,\ 20] \\ 1 & \text{for } t \in [6,\ 7.25] \cup [9,\ 10.25] \cup [12,\ 13.25] \\ & \quad \cup [15,\ 16,\ 25] \cup [18,\ 19,\ 25]\ . \end{cases}$$

For all differentiators the gains have been chosen to be the same, with the values $k_1 = k_2 = k_3 = 5$, $L = 60$. For the noise-driven varying differentiator the following parameters were chosen for the filter: $\tau = 4$, $\omega_{\text{cut}} = 10$, $\mu = 300$. Two simulations are conducted with two different initial conditions, in order to illustrate the behavior of the proposed time-varying d according to the fixed-time and the noise-driven cases.

4.1. Estimation of the second derivative

4.1.1. Small initial estimation errors

For the three differentiators, the initial conditions are chosen null, i.e. $\hat{x}_i(0) = 0$, $i = 1 - 3$. For the system, they are selected near those of the differentiators, i.e. $f_0(0) = x_1(0) = 1$, $f_0^{(1)}(0) = x_2(0) = 2$, $f_0^{(2)}(0) = x_3(0) = 3$. Notice that for the sake of place, only the results for the estimation of the second derivative are presented.

Figs. 1 and 2 present the simulation results comparing the behavior of the three differentiators. The proposed exponent varying differentiator performs better than the linear and Levant's differentiator. This is particularly noticeable during the intervals of time when noise is present ($t \in [6,\ 7.25] \cup [9,\ 10.25] \cup [12,\ 13.25] \cup [15,\ 16,\ 25] \cup [18,\ 19,\ 25]$) and/or $f_0^{(3)}(t)$ is different from zero ($t \in [6.5,\ 8.5] \cup [11.5,\ 13.5] \cup [16.5,\ 18.5]$). Fig. 3 shows the changes in d for the proposed differentiator. Note that, the dynamics of d for $t \leq 3s$ (resp. for $t \geq 3s$) is given by Eq. (4) (resp. by Eq. (6)), verifying Assumption 2. Notice that during the intervals when noise is present, d is near to 0, so that a linear-like behavior is obtained. In the intervals with $f_0^{(3)}(t)$ different from zero and no noise d moves towards -1, so that a behavior close to Levant's differentiator is obtained. This may explain the improved performance of the proposed differentiator.

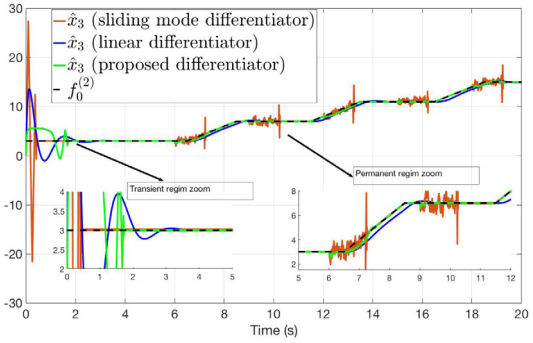

Fig. 1. State $f_0^{(2)} = x_3$ and its estimate $\hat{x}_3$.

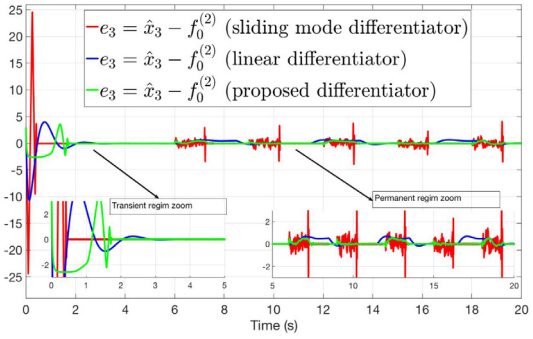

Fig. 2. Estimation errors ($e_3 = \hat{x}_3 - f_0^{(2)} = \hat{x}_3 - x_3$).

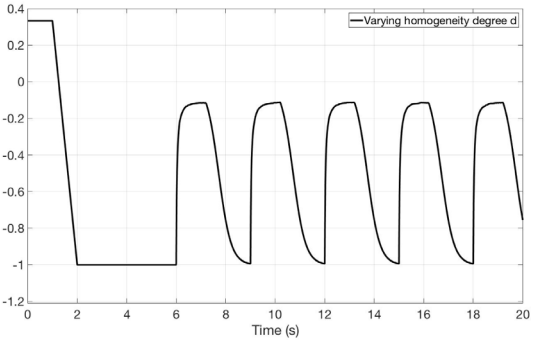

Fig. 3. Varying degree of homogeneity d.

4.1.2. Large initial estimation errors

The initial conditions for the three differentiators are chosen as zero. For the system they are selected far from those of the differentiators, i.e. $f_0 = 1000$, $f_0^{(1)} = 20000$, $f_0^{(2)} = 3000$. As expected, in Fig. 4 it can be seen that the proposed fixed-time differentiator with time-varying d converges before the fixed time $\mathcal{T}_f = 3s$ of Eq. (4), while the linear and the Levant's (sliding mode) differentiators have not converged before within this time.

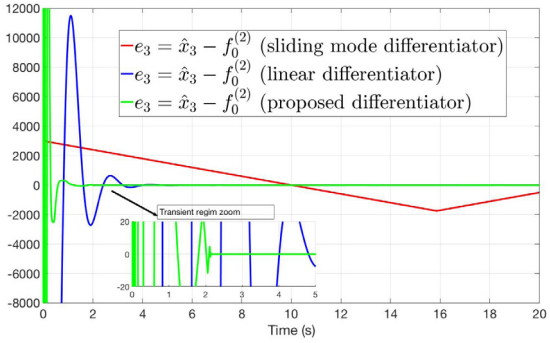

Fig. 4. Estimation errors ($e_3 = \hat{x}_3 - f_0^{(2)}$).

4.2. Estimation of the first derivative

For the sake of space, the estimation results of the first derivative $f_0^{(1)}(t) = x_1^{(1)}(t)$ are not displayed. They are in concordance to those of the second derivative, although, as expected, the first derivative is less prone to noise.

5. Proof of the main results

5.1. Proof of Theorem 1

In the Cruz-Zavala and Moreno (2016), Sanchez et al. (2018) (see also Cruz-Zavala and Moreno (2019) for the discontinuous case in particular) it is shown that the function

$$V(z) = \sum_{j=1}^{n-1} \beta_j Z_j\left(z_j,\, z_{j+1}\right) + \beta_n \frac{1}{p}|z_n|^p \tag{10}$$

$$Z_i(z_i,\, z_{i+1}) = \frac{r_i}{p}|z_i|^{\frac{p}{r_i}} - z_i \lceil z_{i+1}\rfloor^{\frac{p-r_i}{r_{i+1}}} + \left(\frac{p-r_i}{p}\right)|z_{i+1}|^{\frac{p}{r_{i+1}}}, \tag{11}$$

is $\mathbf{r}-$homogeneous of degree p, positive definite and radially unbounded for any $\beta_i > 0$, $i = 1, \ldots, n$, i.e. there exist positive constants $0 < \alpha_1 \le \alpha_2$ for each value of d such that

$$\alpha_1 \|z\|_{\mathbf{r}}^p \le V(z) \le \alpha_2 \|z\|_{\mathbf{r}}^p . \tag{12}$$

V is differentiable for every $-1 \le d < \frac{1}{n-1}$ if $p \ge 2n-1$. Note that V is a function of d through the homogeneity weights r_i, appearing in the powers and in the coefficients. We assume that p is independent of d. The bounds α_1 and α_2 of (12) can be selected independent of d for any compact subinterval of $-1 \le d < \frac{1}{n-1}$. The same is valid for all bounding constants in the sequel. It is also shown in Cruz-Zavala and Moreno (2016), Sanchez et al. (2018) that the derivative of V along the trajectories of the error system (3) is given by

$$\dot{V} = -L W(z) - \partial_n V(z) \frac{f_0^{(n)}(t)}{k_{n-1} L^{n-1}} + L R(z,\, \nu) + \frac{\partial V(z)}{\partial d}\dot{d}, \tag{13}$$

$$W(z) = \sum_{j=1}^{n-1} \tilde{k}_j \partial_j V(\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - z_{j+1}) + \tilde{k}_n \partial_n V \lceil z_1\rfloor^{\frac{r_{n+1}}{r_1}},$$

$$R(z,\, \nu) = \sum_{j=1}^{n} \tilde{k}_j \partial_j V\left(\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - \lceil z_1 - \nu\rfloor^{\frac{r_{j+1}}{r_1}}\right), \tag{14}$$

where $\partial_j V$ stands for $\partial_j V = \frac{\partial V}{\partial z_j}$ and

$$\begin{aligned}
\partial_1 V &= \beta_1 \left(\lceil z_1\rfloor^{\frac{p-r_1}{r_1}} - \lceil z_2\rfloor^{\frac{p-r_1}{r_2}}\right),\\
\partial_i V &= -\beta_{i-1}\frac{p-r_{i-1}}{r_i}|z_i|^{\frac{p-r_{i-1}-r_i}{r_i}}\left(z_{i-1} - \lceil z_i\rfloor^{\frac{r_{i-1}}{r_i}}\right)\\
&\quad + \beta_i\left(\lceil z_i\rfloor^{\frac{p-r_i}{r_i}} - \lceil z_{i+1}\rfloor^{\frac{p-r_i}{r_{i+1}}}\right),\\
\partial_n V &= -\beta_{n-1}(p-r_{n-1})|z_n|^{p-r_{n-1}-1}\left(z_{n-1} - \lceil z_n\rfloor^{r_{n-1}}\right)\\
&\quad + \beta_n \lceil z_n\rfloor^{p-1}.
\end{aligned}$$

In (13) the first term corresponds to the stabilizing effect of the correction terms, the second and third terms are due to the nth derivative $f_0^{(n)}(t)$ and the noise ν, respectively. The last term is new in this paper, and it reflects the effect of the variation of d (the exponents) in the differentiator.

In Cruz-Zavala and Moreno (2016, 2019), Sanchez et al. (2018) it is shown that for each value of $-1 \le d \le 0$ there exist gains $\tilde{k}_i > 0$ such that $W(z) > 0$ (W is positive definite), so that in the absence of noise ($\nu \equiv 0$), for constant d and if $f_0^{(n)}(t) = 0$, the differentiator converges globally. The same procedure is valid for the whole range $-1 \le d < \frac{1}{n-1}$.

Since V is $\mathbf{r}$-homogeneous of degree p and W is $\mathbf{r}$-homogeneous of degree $p+d$, there exists a constant $\theta > 0$ such that $W(z) \ge \theta V^{\frac{p+d}{p}}(z)$. Since $\partial_j V$ is $\mathbf{r}$-homogeneous of degree $p - r_j$ there exist constants $w_j > 0$ such that $\left|\partial_j V(z)\right| \le w_j V^{\frac{p-r_j}{p}}(z)$.

5.1.1. The noise term $R(z,\, \nu)$

The effect of the noise ν on $\dot{V}$ is given by the term $\left(\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - \lceil z_1 - \nu\rfloor^{\frac{r_{j+1}}{r_1}}\right)$ in (14). When $d \in (-1,\, 0]$ functions $\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}}$ are globally Hölder (or Lipschitz for $d = 0$) continuous, and therefore there exist constants $h_j > 0$ such that $\left|\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - \lceil z_1 - \nu\rfloor^{\frac{r_{j+1}}{r_1}}\right| \le h_j |\nu|^{\frac{r_{j+1}}{r_1}}$. When $d \in \left(0,\, \frac{1}{n-1}\right)$ the ratios $\frac{r_{j+1}}{r_1} > 1$ and functions $\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}}$ are differentiable and locally (but not globally) Lipschitz continuous. Using the mean-value theorem of differential calculus we obtain $\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - \lceil z_1 - \nu\rfloor^{\frac{r_{j+1}}{r_1}} = -\frac{r_{j+1}}{r_1}|z_1 - \xi|^{\frac{r_{j+1}}{r_1}-1}\nu$, $\xi \in (0,\, \nu)$, and thus (using the classical generalized mean value inequality)

$$\left|\lceil z_1\rfloor^{\frac{r_{j+1}}{r_1}} - \lceil z_1 - \nu\rfloor^{\frac{r_{j+1}}{r_1}}\right| \le \frac{r_{j+1}}{r_1}\alpha_j\left(|\nu|^{\frac{r_{j+1}}{r_1}} + \tilde{h}_j V^{\frac{r_{j+1}-r_1}{p}}(z)\,|\nu|\right),$$

for some α_j and $\tilde{h}_j$. We conclude, that for some constants h_j and $\tilde{h}_j$ the following inequality is satisfied globally for any $-1 \le d < \frac{1}{n-1}$

$$R(z,\, \nu) \le \begin{cases} \sum_{j=1}^{n} \tilde{k}_j w_j h_j \left(V^{\frac{p-r_j}{p}}(z)\,|\nu|^{\frac{r_{j+1}}{r_1}} + \tilde{h}_j V^{\frac{p+r_{j+1}-r_1-r_j}{p}}(z)\,|\nu|\right), & \text{if } d \in \left(0,\, \frac{1}{n-1}\right)\\ \sum_{j=0}^{n} \tilde{k}_j w_j h_j V^{\frac{p-r_j}{p}}(z)\,|\nu|^{\frac{r_{j+1}}{r_1}}, & \text{if } d \le 0. \end{cases}$$

5.1.2. The term $\frac{\partial V(z)}{\partial d}\dot{d}$

To calculate $\frac{\partial V(z)}{\partial d}$ recall that $\frac{d}{dx}a^x = a^x \ln a$ if $a > 0$ $(a \neq 1)$ and $r_i = 1-(n-i)\,d$. A simple calculation shows that

$$\frac{\partial V(z)}{\partial d} = \mathcal{V}_1(z) + \mathcal{V}_2(z),$$

where

$$\mathcal{V}_1(z) = \sum_{j=1}^{n-1} \beta_j \frac{(n-j)}{p}\left(|z_{j+1}|^{\frac{p}{r_{j+1}}} - |z_j|^{\frac{p}{r_j}}\right)$$

$$\mathcal{V}_2(z) = \sum_{j=1}^{n-1} \beta_j \left(\frac{(n-j)}{r_j}|z_j|^{\frac{p}{r_j}} \ln|z_j| - \left(\frac{p(n-j-1)+1}{r_{j+1}^2}\right) z_j \lceil z_{j+1}\rfloor^{\frac{p-r_j}{r_{j+1}}} \ln|z_{j+1}| + \frac{(n-j-1)(p-r_j)}{r_{j+1}^2}|z_{j+1}|^{\frac{p}{r_{j+1}}} \ln|z_{j+1}|\right).$$

$\mathcal{V}_1(z)$ and $\mathcal{V}_2(z)$ are continuous functions and they vanish at the origin, i.e. $\mathcal{V}_1(0) = 0$ and $\mathcal{V}_2(0) = 0$ (see Lemma 1 for the terms containing the logarithms).

Since $\mathcal{V}_1(z)$ is $\mathbf{r}$-homogeneous of the same degree as $V(z)$, there exists $\theta_1 > 0$ (independent of L) such that

$$\mathcal{V}_1(z) \leq \theta_1 V(z), \ \forall z.$$

The following result, useful in the sequel, allows us to conclude that x^α dominates $x^\beta \ln x$ near $x = 0$ if $\alpha < \beta$, and near $x \to \infty$ if $\alpha > \beta$. Its proof is elementary.

Lemma 1. *Supose that* $\alpha > 0$, $\beta > 0$ *and* $x > 0$. *Then function* $r(x) \triangleq \frac{x^\beta \ln x}{x^\alpha}$ *has the following limits*

$$\lim_{x\to 0^+} r(x) = \lim_{x\to 0^+} x^{\beta-\alpha}\ln x = \begin{cases} 0 & \text{if } \beta-\alpha > 0 \\ -\infty & \text{if } \beta-\alpha \leq 0 \end{cases},$$

$$\lim_{x\to \infty} r(x) = \lim_{x\to \infty} x^{\beta-\alpha}\ln x = \begin{cases} 0 & \text{if } \beta-\alpha < 0 \\ \infty & \text{if } \beta-\alpha \geq 0 \end{cases}.$$

5.1.3. Global bound of $\mathcal{V}_2(z)$

Although $\mathcal{V}_2(z)$ is not homogeneous, using Lemma 1, we can obtain a global bound with two homogeneous functions.

Consider first the function $|z_j|^{\frac{p}{r_j}} \ln|z_j|$. It is possible to upper-bound the function globally by

$$\left||z_j|^{\frac{p}{r_j}} \ln|z_j|\right| \leq \gamma_1 |z_j|^{\frac{p-\delta_1}{r_j}} + \gamma_2 |z_j|^{\frac{p+\delta_2}{r_j}}, \quad \forall z_j \in \mathbb{R},\ \delta_1 > 0,\ \delta_2 > 0,$$

where δ_1 and δ_2 can be selected arbitrarily. The first term on the righthand side of the inequality is homogeneous of degree $p-\delta_1$ while the second term is homogeneous of degree $p+\delta_2$. Moreover, using also Young's inequality, we obtain the following inequality $\forall (z_j, z_{j+1}) \in \mathbb{R}^2$

$$z_j \lceil z_{j+1}\rfloor^{\frac{p-r_j}{r_{j+1}}} \ln|z_{j+1}| \leq \gamma_1\left(\frac{r_j}{p-\delta_1}|z_j|^{\frac{p-\delta_1}{r_j}} + \frac{p-r_j-\delta_1}{p-\delta_1}|z_{j+1}|^{\frac{p-\delta_1}{r_{j+1}}}\right) + \gamma_2\left(\frac{r_j}{p+\delta_2}|z_j|^{\frac{p+\delta_2}{r_j}} + \frac{p-r_j+\delta_2}{p+\delta_2}|z_{j+1}|^{\frac{p+\delta_2}{r_{j+1}}}\right), \forall p > \delta_1 > 0,\ \delta_2 > 0.$$

It is important to note that γ_1 and γ_2 are independent of L and can also be selected independent of d and j. This implies that there exist constants $\theta_2 > 0$, $\theta_3 > 0$ (independent of L) such that $\forall z \in \mathbb{R}^n$, $p > \delta_1 > 0$, $\delta_2 > 0$, we have

$$|\mathcal{V}_2(z)| \leq \theta_2 V^{\frac{p-\delta_1}{p}}(z) + \theta_3 V^{\frac{p+\delta_2}{p}}(z).$$

5.1.4. Global inequality for $\dot{V}$

Using all previous bounds with Assumption 1 in the expression (13), we obtain, when $-1 < d < \frac{1}{n-1}$, the following globally valid inequality for $\dot{V}$

$$\dot{V} \leq -\theta L V^{\frac{p+d}{p}}(z) + \vartheta \frac{\Delta}{L^{n-1}} V^{\frac{p-1}{p}}(z) + \left(\theta_1 V(z) + \theta_2 V^{\frac{p-\delta_1}{p}}(z) + \theta_3 V^{\frac{p+\delta_2}{p}}(z)\right)|\dot{d}(t)| + L\sum_{j=0}^{n} \vartheta_j V^{\frac{p-1+(n-j)d}{p}}(z) N^{\frac{1-(n-j-1)d}{1-(n-1)d}}, \tag{15}$$

$$\vartheta_0 = 0 \text{ if } d \notin \left(0, \frac{1}{n-1}\right),$$

Since $p-1 \geq p-1+(n-j)\,d$ for $d \leq 0$ and $p-1+(n-j)\,d \geq p-1$ for $d > 0$, for large values of z, i.e. $V(z) > 1$, $V^{\frac{p-1+(n-j)d}{p}}(z) \leq V^{\frac{p+\mu}{p}}(z)$ for any $j = 0, \ldots, n$ and

$$\mu = \begin{cases} -1 & -1 < d \leq 0 \\ -1+n\,d & d > 0 \end{cases}.$$

Similarly, since $\delta_2 > 0 > -\delta_1$, for $V(z) > 1$ the term $V^{\frac{p+\delta_2}{p}}$ dominates. And therefore the inequality (15) becomes, for $V(z) > 1$ and any $0 < r < 1$,

$$\dot{V} \leq -r\theta L V^{\frac{p+d}{p}}(z) - (1-r)\,\theta L V^{\frac{p+d}{p}}(z) + c_P V^{\frac{p+\mu}{p}}(z) + c_d V^{\frac{p+\delta_2}{p}}(z)|\dot{d}(t)|, \tag{16}$$

where $c_P(\Delta, N, L) = \vartheta \frac{\Delta}{L^{n-1}} + L\sum_{j=0}^{n} \vartheta_j N^{\frac{1-(n-j-1)d}{1-(n-1)d}}$, $\vartheta_0 = 0$ if $d \leq 0$, $c_d = \theta_1+\theta_2+\theta_3$. Note that $\frac{p+d}{p} > \mu$ for all $-1 < d < \frac{1}{n-1}$, and therefore the term (with coefficient c_P) due to noise and perturbation, is dominating near the origin, but is dominated by the negative term far from it. Furthermore, and since δ_1 and δ_2 can be selected arbitrarily close to zero, the term due to $\dot{d}$ is dominated by the negative term far from the origin for $d > 0$ and near the origin for $d < 0$.

5.1.5. Constant d

In this case function $f(V) = -(1-r)\,\theta L V^{\frac{p+d}{p}} + c_P V^{\frac{p+\mu}{p}}$ has a root at $V = \bar{V}_1 = 0$ and a positive root given by $V = \bar{V}_2 = \left(\frac{c_P}{(1-r)\theta L}\right)^{\frac{p}{d-\mu}}$. In the set $\Xi = \left\{z \in \mathbb{R}^n \mid \bar{V}_2 \leq V(z)\right\}$ the differential inequality (16) becomes

$$\dot{V} \leq -r\theta L V^{\frac{p+d}{p}}(z). \tag{17}$$

Trajectories starting in Ξ converge in finite-time to the sublevel set $\Omega_{\bar{V}_2} = \left\{z \in \mathbb{R}^n \mid V(z) \leq \bar{V}_2\right\}$, which is positively invariant, and remain there for all future times. Since trajectories remain in $\Omega_{\bar{V}_2}$, from (12) we conclude that $z(t)$ is ultimately bounded by

$$\|z(t)\|_{\mathbf{r}} \leq \left(\frac{\bar{V}_2}{\alpha_1}\right)^{\frac{1}{p}} = \frac{1}{\alpha_1^{\frac{1}{p}}}\left(\frac{c_P}{(1-r)\,\theta L}\right)^{\frac{1}{d-\mu}}. \tag{18}$$

To describe the evolution of the trajectories in Ξ, note that the solution of (17) for $d \neq 0$ is given by (we use the comparison lemma (Khalil, 2002))

$$\frac{p}{-d} V^{-\frac{d}{p}}(z(t)) \leq \frac{p}{-d} V^{-\frac{d}{p}}(z_0) - r\theta L t.$$

For $d<0$, $V(z(t)) \leq \left(V^{-\frac{d}{p}}(z_0) - \frac{(-d)}{p} r\theta L t\right)^{\frac{p}{(-d)}}$. Using (12) and (18) we arrive at

$$\|z(t)\|_{\mathbf{r}} \leq \begin{cases} \left(M\|z_0\|_{\mathbf{r}}^{-d} - L\kappa t\right)^{\frac{1}{(-d)}} \text{ if } & 0 \leq t \leq \frac{M\|z_0\|_{\mathbf{r}}^{-d}}{L\kappa} \\ 0 & \text{if } t \geq \frac{M\|z_0\|_{\mathbf{r}}^{-d}}{L\kappa} \end{cases} + \left(\varrho_\Delta \frac{\Delta}{L^n} + \varrho_N \sum_{j=1}^{n} N^{\frac{1-(n-j-1)d}{1-(n-1)d}}\right)^{\frac{1}{1+d}}, \tag{19}$$

with $\kappa = \frac{(-d)}{p\alpha_1^{-\frac{d}{p}}} r\theta$, $M = \left(\frac{\alpha_2}{\alpha_1}\right)^{-\frac{d}{p}}$. For $d>0$, we get

$$V(z(t)) \leq \frac{V(z_0)}{\left(1+\frac{d}{p} r\theta L V^{\frac{d}{p}}(z_0)\, t\right)^{\frac{p}{d}}},$$

and the use of (12) and (18) leads to

$$\|z(t)\|_{\mathbf{r}} \leq \frac{M\|z_0\|_{\mathbf{r}}}{\left(1+L\kappa\|z_0\|_{\mathbf{r}}^{d} t\right)^{\frac{1}{d}}} + \left(\varrho_\Delta \frac{\Delta}{L^n} + \varrho_N \sum_{j=0}^{n} N^{\frac{1-(n-j-1)d}{1-(n-1)d}}\right)^{\frac{1}{1-(n-1)d}}, \tag{20}$$

with $\kappa = \frac{d}{p} r\theta \alpha_1^{\frac{d}{p}}$, $M = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p}}$. Similarly, for $d=0$, solution of (17) is given by

$$V(z(t)) \leq V(z_0)\exp(-r\theta L t),$$

and this results in

$$\|z(t)\|_{\mathbf{r}} \leq M\|z_0\|_{\mathbf{r}} \exp(-L\kappa t) + \left(\varrho_\Delta \frac{\Delta}{L^n} + \varrho_N N\right), \tag{21}$$

with $\kappa = \frac{r\theta}{p}$, $M = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p}}$. The results in (19), (20) and (21) are inspired from those in Levant (2003, 2005) for $d=-1$ using homogeneity properties, and for $d\in[-1,\,0]$ in Cruz-Zavala and Moreno (2016, 2019), Sanchez et al. (2018) by means of Lyapunov functions. The proof in Cruz-Zavala and Moreno (2016), Sanchez et al. (2018) is also applied for $d\in\left(0,\,\frac{1}{n-1}\right)$ with slight modifications.

5.1.6. Time-varying d

We note that all constants α_1, α_2 and those in (15), θ, ϑ, θ_i, ϑ_j, depend (continuously) on d. And they can be selected independent of d in any compact subinterval of $-1 \leq d < \frac{1}{n-1}$. In this subsection we consider that such a uniform (in d) bound is used.

Since the effect of $\dot{d}$ in V, as seen in (16), is different for $d>0$ than for $d<0$, we consider here two scenarios:

(a) *High Degree* $d(t)>0$

We consider the case that $\frac{1}{n-1} > \epsilon_2 \geq d(t) \geq \epsilon_1 > 0$, with constant ϵ_i. Important here is that the negative term is dominating for large values of z. Select $\delta_2 = \epsilon_1/2$ and define $\bar{\mu} = \max\{\delta_2,\, -1+nd\}$. Then $p+d > p+\bar{\mu} \geq p-1+(n-j)d \geq p-1$, and for large values of z, i.e. $V(z)>1$,

$$\dot{V} \leq -r\theta L V^{\frac{p+d}{p}}(z) - (1-r)\theta L V^{\frac{p+d}{p}}(z) + cV^{\frac{p+\bar{\mu}}{p}}(z),$$

where $c = c_P + c_d D$, according to Assumption 2. The same analysis as in the previous section shows that the root $\bar{V}_2$ of $f(V) = -(1-r)\theta L V^{\frac{p+d}{p}} + cV^{\frac{p+\bar{\mu}}{p}}$ is given by

$$V = \bar{V}_2 = \left(\frac{c}{(1-r)\theta L}\right)^{\frac{p}{d-\bar{\mu}}}.$$

Since $\bar{V}_2$ depends on d, we obtain $\bar{V}_{2,\max} = \max_d \bar{V}_2$. Trajectories starting in the set

$$\Xi_{\max} = \left\{z\in\mathbb{R}^n \mid \bar{V}_{2,\max} \leq V(z)\right\},$$

converge in finite-time to the positively invariant sublevel set $\Omega_{\bar{V}_{2,\max}}$, and remain there for all future times, i.e. $\|z(t)\|_{\mathbf{r}} \leq \left(\frac{\bar{V}_{2,\max}}{\alpha_1}\right)^{\frac{1}{p}}$. In the set $\Xi_{\max}$ the differential inequality becomes $\dot{V} \leq -r\theta L V^{\frac{p+d}{p}}$. Since d is time dependent, it is not possible to solve this inequality by separation of variables. However, in case $V>1$, since $d \geq \epsilon_1 > 0$ then $V^{\frac{p+d}{p}} > V^{\frac{p+\epsilon_1}{p}}$, and thus $\dot{V} \leq -r\theta L V^{\frac{p+\epsilon_1}{p}}$ is valid. In this case the evolution of trajectories is thus given by (20), with $d=\epsilon_1$. In case $V<1$, since $0<d\leq\epsilon_2<\frac{1}{n-1}$, then $V^{\frac{p+d}{p}} > V^{\frac{p+\epsilon_2}{p}}$, and thus $\dot{V} \leq -r\theta L V^{\frac{p+\epsilon_2}{p}}$ is valid. In this case the evolution of trajectories is thus given by (20), with $d=\epsilon_2$.

(b) *Low Degree* $d(t)<0$

We consider the case $-1 < d(t) \leq -\epsilon < 0$ for some ϵ. Distinctive in this case is that the (destabilizing) effect of $\dot{d}$ is dominating for large values of z, leading to a restricted attraction region. Select $\delta_1 = \epsilon/2$ and $\delta_2 > 0$. Since $\delta_2 > 0 > -\delta_1$, and $p+\delta_2 > p+d > p-1 \geq p-1+(n-j)d$ for $d\leq 0$, for large values of z ($V(z)>1$) (16) is not satisfied outside the restricted attractive region, where the latter is the largest neighborhood of the origin for which (16) is satisfied.

Consider the function in (16) using Assumption 2

$$f(V) = -(1-r)\theta L V^{\frac{p+d}{p}} + c_P V^{\frac{p-1}{p}} + c_d D V^{\frac{p+\delta_2}{p}}. \tag{22}$$

We have seen that for $D = |\dot{d}| = 0$ this function has two real roots: $\bar{V}_1 = 0 < \bar{V}_2 = \left(\frac{c_P}{(1-r)\theta L}\right)^{\frac{p}{d-\mu}}$. Since the roots of $f(V)$ change continuously with its coefficients, for small values of D function $f(V)$ has three positive real roots, $V = \left\{\bar{V}_1, \bar{V}_2, \bar{V}_3\right\}$, with $\bar{V}_1 = 0 < \bar{V}_2 < \bar{V}_3$. If $c_P = 0$ then $\bar{V}_1 = \bar{V}_2 = 0 < \bar{V}_3$. Moreover, $f(V) > 0$ for $0 = \bar{V}_1 < V < \bar{V}_2$, and $\bar{V}_3 < V$, while $f(V) < 0$ for $\bar{V}_2 < V < \bar{V}_3$. The roots $\bar{V}_2$ and $\bar{V}_3$ change continuously with D, and $\bar{V}_3 \to \infty$ when $D \to 0$. Since the roots depend on d we calculate $\bar{V}_{2,\max} = \max_d \bar{V}_2$ and $\bar{V}_{3,\min} = \min_d \bar{V}_3$. Selecting D sufficiently small it is possible to achieve that $\bar{V}_1 = 0 < \bar{V}_{2,\max} < \bar{V}_{3,min}$. This leads to $D \ll L$ with a fixed L, which implies that $\tau \ll L$ for (7).

Consider the set

$$\Xi = \left\{z\in\mathbb{R}^n \mid \bar{V}_{2,\max} < \ell_1 \leq V(z) \leq \ell_2 < \bar{V}_{3,\min}\right\},$$

for some ℓ_1 and ℓ_2. The sets $\Omega_{\ell_1} = \{z\in\mathbb{R}^n \mid V(z)\leq\ell_1\}$ and $\Omega_{\ell_2} = \{z\in\mathbb{R}^n \mid V(z)\leq\ell_2\}$ are positively invariant, and any trajectory starting in Ξ will converge in finite-time to Ω_{ℓ_1}, and will remain there for all future times. Note that $z\in\Omega_{\ell_2}$ implies

$$V(z) \leq \ell_2 \Rightarrow \alpha_1\|z\|_{\mathbf{r},p}^{p} \leq \ell_2 \Leftrightarrow \|z\|_{\mathbf{r}} \leq \left(\frac{\ell_2}{\alpha_1}\right)^{\frac{1}{p}}.$$

In the set Ξ the differential inequality becomes $\dot{V} \leq -r\theta L V^{\frac{p+d}{p}}$. Similarly to previous paragraph, for large V we obtain $\dot{V} \leq -r\theta L V^{\frac{p-1}{p}}$ and trajectories in Ξ evolve as (19) with $d=-1$. For small V we get $\dot{V} \leq -r\theta L V^{\frac{p-\epsilon}{p}}$, and trajectories in Ξ evolve as (19) with $d=-\epsilon$.

5.2. Proof of Proposition 1

During the intervals with constant d the reaching times are given by: (i) $T(z_0) = \frac{M}{L\kappa}\|z_0\|_{\mathbf{r}}^{-d}$, for $d<0$ to reach $z=0$ from the initial condition z_0. (ii) $T_\epsilon = \frac{M^d}{\epsilon^d L\kappa}$ for $d>0$ to reach the

ball $\mathcal{N}_\epsilon = \{z \in \mathbb{R}^n \mid \|z\|_{\mathbf{r}} \leq \epsilon\}$ from any initial state. Note that these times can be made arbitrarily small by increasing the gain L. During the interval where d changes we have just to show that V remains bounded. Under Assumption 2, with $c_p = 0$, it follows from (16), that $\dot{V} \leq -\theta L V^{\frac{p+d}{p}} + c_d V^{\frac{p+\delta_2}{p}} D$. In the worst possible case, $\dot{V} \leq c_d V^{\frac{p+\delta_2}{p}} D$, i.e. solving the inequality

$$V(t) \leq \frac{V_0}{\left[1 - \frac{\delta_2}{p} c_d D V_0^{\frac{\delta_2}{p}} t\right]^{\frac{p}{\delta_2}}}.$$

The time to escape to infinity is given by $T_e = \frac{p}{\delta_2 c_d D V_0^{\frac{\delta_2}{p}}}$. We need to assure that $\frac{1}{3}\mathcal{T}_f < T_e$, i.e. $\mathcal{T}_f < \frac{3p}{\delta_2 c_d D V^{\frac{\delta_2}{p}}(t_0)}$. This can always be reached by selecting $V_0 = V\left(\frac{1}{3}\mathcal{T}_f\right)$, the value of V reached in the first interval $t \in \left[0, \frac{1}{3}\mathcal{T}_f\right]$, and/or δ_2 sufficiently small. When $d_f = -1$ and $\Delta \neq 0$, the same argument applies, since in that case the inequality becomes $\dot{V} \leq -\left(\theta L - \vartheta \frac{\Delta}{L^{n-1}}\right) V^{\frac{p-1}{p}} + c_d V^{\frac{p+\delta_2}{p}} D$.

6. Conclusion

In this paper, we proposed an arbitrary order differentiator with continuous time-varying homogeneity degree (d exponent) including positive and negative degrees. By doing so, the benefit is to have, by varying continuously only one parameter, both fixed time ($d > 0$) and finite-time ($d < 0$) convergence properties. Using a strong Lyapunov function, we showed that for $d(t) < 0$, only a local finite-time convergence is ensured and for $d(t) > 0$ a global fixed-time convergence is established. Following this result, we gave two applications. The first proposed variation law of $d(t)$ ensures the continuity between fixed and finite times convergences while the second one allows the differentiator to be less-sensible to noise. In the future, we will firstly focus on testing the proposed differentiator within real applications and extending the idea of the fixed-time convergence (Proposition 1) to the control law design. Then, we will investigate the differentiator based control with varying homogeneity degree. Moreover, we will study the case of hybrid dynamical systems where the time-varying homogeneity degree can be written in function of the discrete state even if its variation is continuous.

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