ALMOST REDUCIBILITY OF QUASIPERIODIC SL(2, R)-COCYCLES IN ULTRADIFFERENTIABLE CLASSES

Maxime Chatal, Claire Chavaudret

Almost reducibility of quasiperiodic $SL(2,\mathbb{R})$-cocycles in ultradifferentiable classes

Maxime Chatal*
and
Claire Chavaudret†

May 24, 2022

Abstract: Given a quasiperiodic cocycle in $sl(2,\mathbb{R})$ sufficiently close to a constant, we prove that it is almost-reducible in ultradifferentiable class under an adapted arithmetic condition on the frequency vector. We also give a corollary on the Hölder regularity of the Lyapunov exponent.

1 Introduction

1.1 Presentation of the result

Let $d \geq 1$ and $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$ a rationally independent vector (meaning that no non trivial integer combination of the $(\omega_i)_{i=1,\dots,d}$ can vanish). We will assume that $\sup_i |\omega_i| \leq 1$. We will note $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ and $2\mathbb{T}^d := \mathbb{R}^d/2\mathbb{Z}^d$. Let $A : \mathbb{T}^d \to sl(2,\mathbb{R})$ be in a certain class of continuous matrix-valued functions. We call quasi-periodic cocycle the solution $X : \mathbb{T}^d \times \mathbb{R} \to SL(2,\mathbb{R})$ of the differential linear equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X^t(\theta) = A(\theta + t\omega)X^t(\theta) \\ X^0(\theta) = Id \end{cases} \tag{1}$$

One of the main motivations for studying quasi-periodic cocycles is the study of quasi-periodic Schrödinger equations

$$-y''(t) + q(\theta + t\omega)y(t) = Ey(t)$$

where $q : \mathbb{T}^d \to \mathbb{R}$ is called the potential, and $E \in \mathbb{R}$ the energy. It gives rise to a cocycle with values in $SL(2,\mathbb{R})$. The cocycle is said to be a constant cocycle if A is a constant matrix. A quasi-periodic cocycle as in (1) is said reducible if it can be conjugated by a quasi-periodic change of variable $Z : \mathbb{T}^d \to SL(2,\mathbb{R})$ to a constant cocycle, that is to say, if there exists $B \in sl(2,\mathbb{R})$ such that, for all $\theta \in 2\mathbb{T}^d$:

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B$$

In general, it is important to require the change of variables Z to be regular enough. Is this paper, we will be interested in the perturbative setting, that is to say, in quasi-periodic cocycles close to a constant :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X^t(\theta) = (A + F(\theta + t\omega))X^t(\theta) \\ X^0(\theta) = Id \end{cases} \tag{2}$$

where $A \in sl(2,\mathbb{R})$ and $F : \mathbb{T}^d \to sl(2,\mathbb{R})$ is of ultra-differentiable class and small enough, with a smallness condition depending on ω.

Reducibility is a strong property because it implies that the dynamics will be easily described by the constant equivalent of the system, in particular the Lyapunov exponents, the rotational properties of the solutions, the invariant subbundles etc. On the counterpart, reducibility results generally require many assumptions. Here we are interested in a weaker property which is almost reducibility. A cocycle like (2) is said almost-reducible if it can be conjugated by a sequence of quasi-periodic changes of variables to a cocycle of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X^t(\theta) = (\bar{A}_n(\theta + t\omega) + \bar{F}_n(\theta + t\omega))X^t(\theta) \\ X^0(\theta) = Id \end{cases}$$

*IMJ-PRG, Université Paris-Cité, chatal.maxime@gmail.com
†IMJ-PRG, Université Paris-Cité, chavaudr@math.univ-paris-diderot.fr

where $\bar{A}_n$ is reducible and $\bar{F}_n$ is arbitrarily small.
A quantitative version of almost reducibility, that is, almost reducibility together with estimates on the changes of variable, can have interesting corollaries such as approximate solutions, density of reducible cocycles, regularity of Lyapunov exponents.

Ultra-differentiability : To quantify the regularity of $F \in \mathcal{C}^\infty(\mathbb{T}^d, sl(2,\mathbb{R}))$ and the size of the sequence $(\bar{F}_n)$ above, we introduce the weight function $\Lambda : [0,+\infty[\to [0,+\infty[$ which we will assume to be increasing and differentiable. Expanding F in Fourier series $F(\theta) = \sum_{k\in\mathbb{Z}^d} \hat{F}(k) e^{2i\pi\langle k,\theta\rangle}$, we will say that F is Λ-ultra-differentiable if there exists $r > 0$ such that

$$|F|_r = |F|_{\Lambda,r} := \sum_{k\in\mathbb{Z}^d} \|\hat{F}(k)\| e^{2\pi\Lambda(|k|)r} < \infty$$

where $|k|$ is the sum of the absolute values of the components of k, and we will denote $F \in U_r(\mathbb{T}^d, sl(2,\mathbb{R})) = U_{\Lambda,r}(\mathbb{T}^d, sl(2,\mathbb{R}))$. To make this space a Banach algebra, we will require Λ to be subadditive :

$$\Lambda(x+y) \leq \Lambda(x) + \Lambda(y), \quad \forall x, y \geq 0$$

If $\Lambda \equiv id$, it is the analytic case.

Remark 1.1. The standard definition of ultra-differentiable functions involves Denjoy-Carleman sequences, that is, real sequences satisfying certain conditions which act as bounds on the successive derivatives of a given function. However, the above definition, introduced by Braun-Meise-Taylor ([8]), can be linked to Denjoy-Carleman classes (see [21], Theorem 11.6). Since Fourier series appear naturally in the problem considered here, we chose to use Braun-Meise-Taylor classes as a starting point.

Non-resonance condition on the frequency: An often studied situation is the case where the frequency vector ω if Diophantine (which we denote by $\omega \in DC(\kappa,\tau)$), for some $0 < \kappa < 1$ and $\tau \geq \max(1, d-1)$:

$$|\langle k,\omega\rangle| \geq \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d\backslash\{0\}$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product. It was proved by Eliasson [13] that in the analytic case, if $\omega \in DC(\kappa,\tau)$, and F is sufficiently small, Equation (2) is almost reducible. This result was improved by Chavaudret [11] who proved that the convergence occurs on analyticity strips of fixed width (whereas Eliasson's theorem gave the convergence on strips of width going to zero).

One of the aims of the present paper is to weaken this arithmetic condition by introducing the approximating function

$$\Psi : [0,+\infty[\to [0,+\infty[$$

with $\Psi \geq id$ (which is not restrictive since it is satisfied by the diophantine condition). We will assume Ψ to be increasing, differentiable and satisfying, for all $x, y \in [1,+\infty[$,

$$\Psi(x+y) \geq \Psi(x) + \Psi(y)$$

thus for all $n \in \mathbb{N}$, and for all $x \geq 1$, $\Psi(nx) \geq n\Psi(x)$. In our problem, ω will satisfy the following arithmetic condition for some $\kappa \in]0,1[$:

$$|\langle k,\omega\rangle| \geq \frac{\kappa}{\Psi(|k|)}, \quad \forall k \in \mathbb{Z}^d\backslash\{0\}$$

(notice that the case $\Psi(.) = |.|^\tau$, is the Diophantine case).

We will require the following condition:

$$\lim_{t\to+\infty} \frac{\log\Psi(t)}{\Lambda(t)} = 0.$$

and

$$\int_0^\infty \frac{\Lambda'(t)\ln\Psi(t)}{\Lambda(t)^2} dt < +\infty$$

This condition, known as the Λ-Brjuno-Rüssmann condition, will be denoted by $\omega \in BR(\kappa)$. This coincides with the well-known Brjuno condition if Λ is the identity.

If A is elliptic, an almost reducibility theorem was given in [5].

The purpose of this article is to show the following theorem:

Theorem 1.2. *Let $r_0 > 0$, $A_0 \in sl(2,\mathbb{R})$ and $F_0 \in U_{r_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$. Then, there exists ε_0 depending only on $A_0, \kappa, \Lambda, \Psi, r_0$ such that, if*

$$|F_0|_{r_0} \leq \varepsilon_0$$

then for all $\varepsilon \leq \varepsilon_0$, there exist

- $r_\varepsilon > 0$, $\zeta \in]0, \frac{1}{8}[$,
- $Z_\varepsilon \in U_{r_\varepsilon}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $A_\varepsilon \in sl(2,\mathbb{R})$,
- $\bar{A}_\varepsilon, \bar{F}_\varepsilon \in U_{r_\varepsilon}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $\psi_\varepsilon \in U_{r_\varepsilon}(2\mathbb{T}^d, SL(2,\mathbb{R}))$

such that

1. *$\bar{A}_\varepsilon$ is reducible to A_ε by ψ_ε, with $|\psi_\varepsilon|_{r_\varepsilon} \leq \varepsilon^{-\frac{1}{2}\zeta}$,*
2. *$|\bar{F}_\varepsilon|_{r_\varepsilon} \leq \varepsilon$,*
3. *$\lim_{\varepsilon \to 0} r_\varepsilon > 0$,*
4. *for all $\theta \in \mathbb{T}^d$,*
$$\partial_\omega Z_\varepsilon(\theta) = (A_0 + F_0(\theta))Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(\bar{A}_\varepsilon(\theta) + \bar{F}_\varepsilon(\theta))$$
5. $$|Z_\varepsilon^{\pm 1} - Id|_{r_\varepsilon} \leq \varepsilon_0^{\frac{9}{10}}$$

Moreover, either Ψ_ϵ becomes constant as $\epsilon \to 0$, or there exist arbitrarily small ε such that $||A_\varepsilon|| \leq \kappa\varepsilon^\zeta$.

This theorem states almost reducibility in an ultradifferentiable class with the same weight function as that of the initial system, but with a smaller parameter r_ε. Notice however that the parameter r_ε does not shrink to 0. In order to achieve this, the resonance cancellation technique is similar to the one in [11]. Notice that, for topological reasons, a period doubling is necessary in order to preserve the real structure. This phenomenon was already observed in [10].

1.2 Discussion

Almost reducibility in itself is an interesting property of quasi-periodic cocycles, in particular the quantitative version. A perturbative almost reducibility result in arbitrary dimension of space was given in [13], in the analytic framework, under a diophantine condition on the frequency vector. A quantitative version of perturbative almost reducibility, in a similar framework, was then proved in [11]. Using a technique developed in [1], in [16], Hou and You managed to remove the diophantine assumption on the frequency vector, in case it is 2-dimensional ($d = 2$); thus the result became non perturbative if not global (see also [20] for a non perturbative reducibility result). It is yet unknown whether arithmetical conditions can be removed in case $d > 2$. However in the analytic case, for any number of frequencies, it is known that the diophantine condition is not optimal for reducibility results and can be replaced by the Brjuno-Rüssmann condition (see [17], [12]). Here we give an almost reducibility result in which the arithmetical condition coincides with the Brjuno-Rüssmann condition in the analytic case.

Concerning the functional framework, a few results were known in the Gevrey class. The reference [11] contains almost reducibility in the Gevrey class as well, under a diophantine condition. The reference [15] gives a result on rigidity of reducibility in the Gevrey class, under a diophantine condition (see also [18] on Gevrey flows). But a simultaneous extension of Eliasson's reducibility result in [14] to more general ultradifferentiable classes and to a weaker arithmetical condition, which is linked to the considered class of functions, was given in [5] (see also [6] for a result in a hamiltonian setting). Here, we obtain this generalization for almost reducibility, the proof of which is more technical. The link between the arithmetical condition and the functional setting is similar to the one in [5], and also coincides with the Brjuno-Rüssmann condition in the analytic case.

1.3 Comments on the proof

The proof of the main result relies on the well-known KAM algorithm: a step of the algorithm will reduce the size of the perturbation to a power of it, by means of a change of variables which might be far from identity (if resonances have to be cancelled), but is still controlled by a small negative power of the size of the perturbation. The order at which one removes resonances to avoid small divisors has to be suitably chosen in order to decrease the perturbation sufficiently while having a sufficient control on the change of variables. One also has to shrink the parameter of the ultradifferentiable class at every step, and in order to have a strong almost reducibility result (i.e a sequence of parameters not shrinking to 0), the Λ-Brjuno-Rüssmann condition comes naturally.
If resonances are cancelled only finitely many times, then the change of variables remains close to identity at every step afterwards, which gives reducibility. Otherwise, the constant part of the system itself becomes small.

The main theorem, which is Theorem 12.1 below, is proved by iterating arbitrarily many times the Lemma 11.1 below; Lemma 11.1 gives a conjugation between to systems $\bar{A}+\bar{F}$ and $\bar{A}'+\bar{F}'$, where both $\bar{A}$ and $\bar{A}'$ are reducible maps and $\bar{F}'$ is smaller than $\bar{F}$, with a controlled loss of regularity.

The proof of Lemma 11.1 can be sketched by the following diagram:

$$\bar{A}+\bar{F} \xrightarrow{\psi} A+F \underset{Lemma\ 10.2}{\overset{\psi^{-1}\Phi^{-1}e^{X}\Phi}{\longrightarrow}} \bar{A}_1+\bar{F}_1 \xrightarrow{\Phi\psi} A_1+F_1 \underset{Lemma\ 9.3}{\overset{e^{X_1}}{\longrightarrow}} \dots \underset{Lemma\ 9.3}{\overset{e^{X_{l-1}}}{\longrightarrow}} A_l+F_l \overset{\psi^{-1}\Phi^{-1}}{\longrightarrow} \bar{A}_l+\bar{F}_l=\bar{A}'+\bar{F}'$$

where $A, A_1, \dots, A_l$ are constant matrices, $\bar{A}, \bar{A}_1, \dots, \bar{A}_l$ are reducible, and $\bar{F}, F, \bar{F}_i, F_i$ are small.

No non resonance condition is required on A, making it necessary to construct the change of variables Φ which will remove resonances, but may be far from the identity (Lemma 7.1). However, once this is done, the matrices $A_1, \dots, A_{l-1}$ remain non resonant enough in order to reduce the perturbation a lot without having to remove resonances again.

The superscripts on the arrows refer to the changes of variables. The changes of variables with an exponential expression are close to the identity, therefore the total conjugation, from $\bar{A}+\bar{F}$ to $\bar{A}'+\bar{F}'$, is close to identity, which makes it possible to obtain the density of reducible systems in the neighbourhood of a constant.

2 Notations

The notation $E(x)$ will refer to the integer part of a number x.

If $F \in L^2(2\mathbb{T}^d)$ and $N \in \mathbb{N}$, the truncation of F at order N (denoted F^N) is the function we obtain by cutting the Fourier series of F :

$$F^N(\theta) = \sum_{|m|\leq N} \hat{F}(m) e^{2i\pi\langle k,\theta\rangle}$$

In order to simplify the notation throughout this paper, we will write $\Psi(\cdot)$ for $\Psi(|\cdot|)$, and $\Lambda(\cdot)$ for $\Lambda(|\cdot|)$.
We will denote by $||\cdot||$ the norm of the greatest coefficient for matrices.

3 Decompositions, triviality

We take the following definitions from [11], describing decompositions of $\mathbb{R}^2$ and triviality, which will avoid to double the period more than once.

Definition 3.1 (Decomposition).

- If $A \in sl(2,\mathbb{R})$ has distinct eigenvalues, we call *A-decomposition* a decomposition of $\mathbb{R}^2$ as the direct sum of two eigenspaces of A. If L is an eigenspace of A, we write $\sigma(A_{|L})$ the spectrum of the restriction of A to the subspace L. We shall denote by $\mathcal{L}_A$ the decomposition of $\mathbb{R}^2$ into two distinct eigenspaces of A, if the related eigenvalues are distinct.
- If $\mathbb{R}^2 = L_1 \bigoplus L_2$, for all $u \in \mathbb{R}^2$, there exists a unique decomposition $u = u_1 + u_2, u_1 \in L_1, u_2 \in L_2$. For $i = 1, 2$, we call *projection on L_i with respect to $\mathcal{L} = \{L_1, L_2\}$*, and we write $P^{\mathcal{L}}_{L_i}$ the map defined by $P^{\mathcal{L}}_{L_i} u = u_i$.

Recall the following lemma on estimate of the projection (see [13]) :

Lemma 3.2 ([13]). *Let $\kappa' > 0$ and $A \in sl(2,\mathbb{R})$ with κ'-separated eigenvalues. There exists a constant $C_0 \geq 1$ such that, for any subspace $L \in \mathcal{L} = \mathcal{L}_A$,*

$$\|P_L^{\mathcal{L}}\| \leq C_0 \big(\frac{1}{\kappa'}\big)^6$$

Remark 3.3. The estimate given in [13] is more general since it also concerns matrices A with a nilpotent part. Here the setting in $sl(2,\mathbb{R})$ makes the estimate a little better.

Definition 3.4 (Triviality). Let $\mathcal{L} = \{L_1, L_2\}$ such that $L_1 \bigoplus L_2 = \mathbb{R}^2$. We say that a function $\psi \in \mathcal{C}^0(2\mathbb{T}^d, SL(2\mathbb{R}))$ is trivial with respect to $\mathcal{L}$ if there exists $m \in \frac{1}{2}\mathbb{Z}^d$ such that for all $\theta \in 2\mathbb{T}^d$,

$$\psi(\theta) = e^{2i\pi\langle m,\theta\rangle} P_{L_1}^{\mathcal{L}} + e^{-2i\pi\langle m,\theta\rangle} P_{L_2}^{\mathcal{L}}$$

If $|m| \leq N$, we say that ψ is trivial of order N.

Remark 3.5.

- If $\psi_1, \psi_2 : 2\mathbb{T}^d \to SL(2,\mathbb{R})$ are trivial with respect to $\mathcal{L}$, then the product $\psi_1\psi_2$ is also trivial with respect to $\mathcal{L}$ since, for all $L \neq L'$, $P_L^{\mathcal{L}} P_{L'}^{\mathcal{L}} = 0$.
- If ψ is trivial with respect to a a decomposition $\mathcal{L}$ of $\mathbb{R}^2$, then for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, we have $\psi G \psi^{-1} \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$. Indeed, notice that if $\psi = e^{2i\pi\langle m,\cdot\rangle} P_{L_1}^{\mathcal{L}} + e^{-2i\pi\langle m,\cdot\rangle} P_{L_2}^{\mathcal{L}}$ for some $m \in \frac{1}{2}\mathbb{Z}^d$, then

  $$\psi^{-1} = e^{-2i\pi\langle m,\cdot\rangle} P_{L_1}^{\mathcal{L}} + e^{2i\pi\langle m,\cdot\rangle} P_{L_2}^{\mathcal{L}}$$

  (it's a simple calculus to check that with this expression, $\psi\psi^{-1} = \psi^{-1}\psi \equiv I$.) Then,

  $$\begin{aligned}\psi G \psi^{-1} &= (e^{2i\pi\langle m,\cdot\rangle} P_{L_1}^{\mathcal{L}} + e^{-2i\pi\langle m,\cdot\rangle} P_{L_2}^{\mathcal{L}}) G (e^{-2i\pi\langle m,\cdot\rangle} P_{L_1}^{\mathcal{L}} + e^{2i\pi\langle m,\cdot\rangle} P_{L_2}^{\mathcal{L}}) \\ &= P_{L_1}^{\mathcal{L}} G P_{L_1}^{\mathcal{L}} + P_{L_2}^{\mathcal{L}} G P_{L_2}^{\mathcal{L}} + e^{2i\pi\langle 2m,\cdot\rangle} P_{L_1}^{\mathcal{L}} G P_{L_2}^{\mathcal{L}} + e^{-2i\pi\langle 2m,\cdot\rangle} P_{L_2}^{\mathcal{L}} G P_{L_1}^{\mathcal{L}}\end{aligned}$$

  which is well defined continuously on $\mathbb{T}^d$. Hence the function ψ will avoid a period doubling.

4 Choice of parameters

In this section, we define all the constants and parameters used in this paper.

$$\begin{cases} \delta = 100000 \\ \zeta = \frac{1}{1728} \end{cases}$$

Let for all $r, \varepsilon > 0$,

$$N(r,\varepsilon) = \Lambda^{-1}\big(\frac{50|\log\varepsilon|}{\pi r}\big)$$

$$R(r,\varepsilon) = \frac{1}{3N(r,\varepsilon)} \Psi^{-1}(\varepsilon^{-\zeta})$$

$$\kappa''(\varepsilon) = \kappa\varepsilon^{\zeta}$$

$$r'(r,\varepsilon) = r - \frac{50\delta|\log\varepsilon|}{\pi\Lambda(R(r,\varepsilon)N(r,\varepsilon))}$$

5 Smallness of the perturbation

Let $r_0 > 0, A_0 \in sl(2,\mathbb{R}), F_0 \in U_{r_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$.

Assumption 1. The functions Λ, Ψ satisfy

$$\lim_{t\to+\infty} \frac{\ln\Psi(t)}{\Lambda(t)} = 0$$

and ε_0 is small enough as to satisfy conditions of lemma 10.1 below and:

$$\frac{150\delta|\log\varepsilon_0|}{\pi\Lambda\circ\Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta}{\pi\zeta\log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty} \frac{\Lambda'(t)\ln\Psi(t)}{\Lambda(t)^2} dt < r_0$$

These conditions depend on r_0, Λ, Ψ.

We shall define the following sequences of parameters used throughout the iteration:

$$\varepsilon_k := \varepsilon_0^{(2\delta)^k}$$

$$\Lambda(N_k) := \Lambda(N(r_k, \varepsilon_k)) = \frac{50|\log \varepsilon_k|}{\pi r_k}$$

$$R_k := R(r_k, \varepsilon_k) = \frac{1}{3N(r_k, \varepsilon_k)} \Psi^{-1}(\varepsilon_k^{-\zeta})$$

and

$$r_k := r_0 - \sum_{i=0}^{k-1} \frac{50\delta|\log \varepsilon_i|}{\pi \Lambda(R(r_i, \varepsilon_i) N(r_i, \varepsilon_i))}$$

Lemma 5.1. *Under either the Assumption 1, the sequence r_k converges to a positive limit.*

Proof. Notice that, for all k, and since Λ is subadditive,

$$\Lambda(R_k N_k) \geq \frac{1}{3}\Lambda(3R_k N_k) \Rightarrow \frac{1}{\Lambda(R_k N_k)} \leq \frac{3}{\Lambda(3R_k N_k)}$$

Then

$$\begin{aligned}
\sum_{k\geq 0} \frac{50\delta|\log \varepsilon_k|}{\pi \Lambda(R_k N_k)} &\leq \sum_{k\geq 0} \frac{150\delta|\log \varepsilon_k|}{\pi \Lambda(3R_k N_k)} \\
&\leq \frac{150\delta}{\pi} \sum_{k\geq 0} \frac{(2\delta)^k|\log \varepsilon_0|}{\Lambda(3R_k N_k)} \\
&\leq \frac{150\delta}{\pi} \sum_{k\geq 0} \frac{(2\delta)^k|\log \varepsilon_0|}{\Lambda(\Psi^{-1}(\varepsilon_k^{-\zeta}))} \\
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta|\log \varepsilon_0|}{\pi} \int_0^{+\infty} \frac{(2\delta)^x}{\Lambda(\Psi^{-1}(\varepsilon_0^{-\zeta(2\delta)^x}))} dx
\end{aligned}$$

With the change of variable $t := (2\delta)^x$

$$\begin{aligned}
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta|\log \varepsilon_0|}{\pi} \int_1^{+\infty} \frac{t}{\Lambda(\Psi^{-1}(\varepsilon_0^{-\zeta t}))} \frac{1}{t\log(2\delta)} dt \\
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta|\log \varepsilon_0|}{\pi \log(2\delta)} \int_1^{+\infty} \frac{1}{\Lambda(\Psi^{-1}(\varepsilon_0^{-\zeta t}))} dt
\end{aligned}$$

With the change of variable $v := \Psi^{-1}(\varepsilon_0^{-\zeta t})$

$$\begin{aligned}
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta|\log \varepsilon_0|}{\pi \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty} \frac{1}{\Lambda(v)} \cdot \frac{\Psi'(v)}{-\zeta \log \varepsilon_0 \Psi(v)} dv \\
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta}{\pi\zeta \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty} \frac{\Psi'(v)}{\Lambda(v)\Psi(v)} dv
\end{aligned}$$

After integrating by parts,

$$\begin{aligned}
\sum_{k\geq 0} \frac{50\delta|\log \varepsilon_k|}{\pi \Lambda(R_k N_k)} &\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta}{\pi\zeta \log(2\delta)} [-\frac{\log(\varepsilon_0^{-\zeta})}{\Lambda(\Psi^{-1}(\varepsilon_0^{-\zeta}))} \\
&\quad + \int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty} \frac{\Lambda'(v)\log \Psi(v)}{\Lambda(v)^2} dv] \qquad (5.1) \\
&\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\epsilon_0^{-\zeta})} + \frac{150\delta}{\pi\zeta \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty} \frac{\Lambda'(v)\log \Psi(v)}{\Lambda(v)^2} dv
\end{aligned}$$

provided $\lim_{v\to+\infty} \frac{\log \Psi(v)}{\Lambda(v)} = 0$, thus the assumption 1 implies that (r_k) converges to a positive limit. □

Remark : We naturally find the Bruno-Rüssmann condition with respect to weight function Λ, which is the convergence of $\displaystyle\int \frac{\Lambda'(v)\log\Psi(v)}{\Lambda(v)^2}dv$.

6 Elimination of resonances

Given a matrix A, a useful technique in the KAM iteration will be to remove the resonances in the spectrum of the matrix A. To characterize the non-resonance of $z\in\mathbb{C}$ (depending on ω, a constant $\kappa'>0$ and on an order $N\in\mathbb{N}$) we will write $z\in BR^N_\omega(\kappa')$ if and only if :

$$\forall k\in\mathbb{Z}^d\backslash\{0\},\quad 0<|k|\leq N\Rightarrow|z-2i\pi\langle k,\omega\rangle|\geq\frac{\kappa'}{\Psi(k)}$$

Definition 6.1. We will say that A has $BR^N_\omega(\kappa')$ spectrum if

$$\sigma(A)=\{\alpha,\alpha'\}\Rightarrow\alpha-\alpha'\in BR^N_\omega(\kappa')$$

In particular, if the eigenvalues of A are $i\alpha$ and $-i\alpha$ with $\alpha\in\mathbb{R}$, A has if $BR^N_\omega(\kappa')$ spectrum if

$$2i\alpha\in BR^N_\omega(\kappa')$$

Remark 6.2. If A has real eigenvalues α,α', then for all $N\in\mathbb{N}$, A has $BR^N_\omega(\kappa)$-spectrum because $|\alpha-\alpha'-2i\pi\langle m,\omega\rangle|\geq|2i\pi\langle m,\omega\rangle|\geq\frac{\kappa}{\Psi(m)}$.

Lemma 6.3. *Let $\alpha\in\mathbb{R}$, $\tilde N\in\mathbb{N}^*$ and $\kappa'=\dfrac{\kappa}{\Psi(3\tilde N)}$. There exists $m\in\frac12\mathbb{Z}^d$, $|m|\leq\frac12\tilde N$ such that, if we denote $\alpha'=\alpha-2\pi\langle m,\omega\rangle$, then $2i\alpha'\in BR^{\tilde N}_\omega(\kappa')$ and if $m\neq0$ then $|\alpha'|\leq\frac{\kappa'}{2}$.*

Proof. We want to remove the resonances between $i\alpha$ and $-i\alpha$. If $2i\alpha\in BR^{\tilde N}_\omega(\kappa')$, let $m=0$ and we are done. Otherwise, if there exists $m'\in\frac12\mathbb{Z}^d$ with $|m'|\leq\tilde N$ such that

$$|2\alpha-2\pi\langle m',\omega\rangle|<\frac{\kappa}{\Psi(m')}$$

then let $m=\frac{m'}{2}$ and $2\alpha'=\alpha-2\pi\langle m,\omega\rangle$. It's a simple calculus to check that, in this case, $|2\alpha'|\leq\kappa'$, hence $\alpha'\leq\frac{\kappa'}{2}$. Now, for all $k\in\frac12\mathbb{Z}^d$, $k\leq\tilde N$,

$$|2i\alpha'-2i\pi\langle k,\omega\rangle|\geq\frac{\kappa}{\Psi(k)}-\kappa'\geq\frac{\kappa'}{\Psi(k)}$$

Then $2i\alpha'\in BR^{\tilde N}_\omega(\kappa')$. □

Lemma 6.4. *Let $\alpha\in\mathbb{R}$. For all $R\in\mathbb{R},N\in\mathbb{N}$, $N\geq1,R\geq2$, there exists $m\in\frac12\mathbb{Z}^d$, $|m|\leq\frac12N$ such that, if we denote $\kappa''=\dfrac{\kappa}{\Psi(3RN)}$ and $\alpha'=\alpha-2\pi\langle m,\omega\rangle$, then $2i\alpha'\in BR^{RN}_\omega(\kappa'')$ and if $m\neq0$ then $|\alpha'|\leq\frac{\kappa''}{2}$.*

Proof. If $\alpha\in BR^{RN}_\omega(\kappa'')$, then $m=0$. Otherwise, apply the previous Lemma with $\tilde N=N$ and $\kappa'=\kappa''$ to obtain $|m|\leq\frac12N$ such that $|\alpha-\langle m,\omega\rangle|\leq\frac{\kappa''}{2}$. Therefore for all $0\leq|k|\leq RN$,

$$|2\alpha'-2\pi\langle k,\omega\rangle|\geq|2\pi\langle k,\omega\rangle|-\kappa''\geq\frac{\kappa}{\Psi(k)}-\kappa''\geq\frac{\kappa''}{\Psi(k)}$$

and then $2i\alpha'=2i\alpha-2\pi\langle m,\omega\rangle\in BR^{RN}_\omega(\kappa'')$. □

7 Renormalization

We want to define a map Φ which conjugates A to a matrix with $BR_\omega^{RN}(\kappa'')$ spectrum.

Lemma 7.1. *Let $A \in sl(2,\mathbb{R}), R \geq 2, N \in \mathbb{N}\setminus\{0\}$. If $\kappa'' = \dfrac{\kappa}{\Psi(3RN)}$ and A has κ''-separated eigenvalues, then there exists a map $\Phi \in \mathcal{C}^0(2\mathbb{T}^d, SL(2,\mathbb{R}))$ which is trivial with respect to $\mathcal{L}_A$ (the decomposition into eigenspaces of A), and a constant $C_0 \geq 1$ such that,*

1. *For all $r' > 0$,*
$$|\Phi^{\pm 1}|_{r'} \leq 2C_0 e^{2\pi\Lambda(\frac{N}{2})r'}\big(\frac{1}{\kappa''}\big)^6$$
2. *If $\tilde{A}$ is defined by the following condition: for all $\theta \in 2\mathbb{T}^d$,*
$$\partial_\omega \Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A}$$
(note that $\tilde{A}$ actually does not depend on θ), then $\|\tilde{A} - A\| \leq \pi N$ and $\tilde{A}$ has $BR_\omega^{RN}(\kappa'')$ spectrum.
3. *For any function $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, we have $\Phi G \Phi^{-1} \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$.*
4. *If $\tilde{A} \neq A$ then $\|\tilde{A}\| \leq \frac{1}{2}\kappa''$.*

Proof. Let m given by Lemma 6.4 with α the imaginary part of an eigenvalue of matrix A. If the eigenvalues of A are in $\mathbb{R}$, then $m = 0$ and $\Phi \equiv I$.

Otherwise, let L_1 be the invariant subspace associated to $i\alpha$, L_2 associated to $-i\alpha$ and let for all $\theta \in 2\mathbb{T}^d$,

$$\Phi(\theta) = e^{2i\pi\langle m,\theta\rangle} P_{L_1}^{\mathcal{L}_A} + e^{-2i\pi\langle m,\theta\rangle} P_{L_2}^{\mathcal{L}_A}$$

For all θ, since the eigenvalues of $\Phi(\theta)$ are complex conjugate, we have $\Phi(\theta) \in SL(2,\mathbb{R})$, and from Lemma 6.4, $\tilde{A}$ has $BR_\omega^{RN}(\kappa'')$ spectrum (since the eigenvalues of $\tilde{A}$ are $\pm i\tilde{\alpha}$ obtained from lemma 6.4 where $\pm i\alpha$ are the eigenvalues of A). Moreover, the spectrum of $\tilde{A} - A$ is $\{\pm 2i\pi\langle m,\omega\rangle\}$ and $|2i\pi\langle m,\omega\rangle| \leq \pi N$ (remind that $|m| \leq \frac{1}{2}N$, and that we supposed $|\omega| \leq 1$) whence 2. Moreover, because $|m| \leq \frac{1}{2}N$, and from Lemma 3.2,

$$|\Phi|_{r'} \leq (\|P_{L_1}\| + \|P_{L_2}\|)e^{2\pi\Lambda(\frac{N}{2})r'} \leq 2C_0\big(\frac{1}{\kappa''}\big)^6 e^{2\pi\Lambda(\frac{N}{2})r'}$$

whence 1. The property 3 follows from the triviality of Φ (see the remark 3.5).

For the estimate in 4, notice that if $A \neq \tilde{A}$, that is to say if the spectrum of A was resonant, $\Phi \not\equiv I$ conjugates A to $\tilde{A}$. In particular, from lemma 6.4, the two eigenvalues $i\tilde{\alpha}$ and $-i\tilde{\alpha}$ of $\tilde{A}$ (which are the eigenvalues of A translated by $2i\pi\langle m,\omega\rangle$) satisfy $|i\tilde{\alpha} - (-i\tilde{\alpha})| \leq \kappa''$ and then $\|\tilde{A}\| \leq \frac{1}{2}\kappa''$. □

Definition 7.2. A function Φ satisfying conclusions of lemma 7.1 will be called *renormalization of A of order R, N*. Here the resonance is removed up to order RN whereas the estimate involves an exponential of $\Lambda(\frac{N}{2})$ and $\Psi(3RN)$.

8 Cohomological equation

In order to define a change of variables which will reduce the norm of the perturbation, we will first solve a linearized equation, which has the form:

$$\forall\theta \in \mathbb{T}^d, \partial_\omega \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + \tilde{F}^N - \hat{\tilde{F}}(0), \quad \hat{\tilde{X}}(0) = 0 \tag{8.1}$$

Here $\tilde{A} \in sl(2,\mathbb{R})$, therefore either it has real non zero eigenvalues, or it is the zero matrix, or it is nilpotent, or it has two eigenvalues $i\alpha, -i\alpha, \alpha \in \mathbb{R}^*$. Only in the latter case can $\tilde{A}$ be resonant.

Assume the eigenvalues are different (so, either they are distinct reals or they are complex conjugates). Let L_1, L_2 be the eigenspaces. For all $L, L' \in \{L_1, L_2\}$, define the following operator:

$$\mathcal{A}_{L,L'} : gl(2,\mathbb{R}) \to gl(2,\mathbb{R}), M \mapsto \mathcal{A}_{L,L'}M := \tilde{A}P_L M - MP_{L'}\tilde{A}$$

It will be necessary to compute the spectrum of every $\mathcal{A}_{L,L'}$ to estimate the solution of the linearized equation. This is done in the following lemma:

Lemma 8.1. *Let $L, L' \in \{L_1, L_2\}$, β the eigenvalue associated to L and γ the eigenvalue associated to L'. The spectrum of $\mathcal{A}_{L,L'}$ is $\{\beta, -\gamma, \beta-\gamma, 0\}$. Moreover, the operator $\mathcal{A}_{L,L'}$ is diagonalizable.*

Proof. Let $P \in GL(2,\mathbb{C})$ such that $P^{-1}\tilde{A}P = \begin{pmatrix} \bar{\beta} & 0 \\ 0 & \bar{\gamma} \end{pmatrix}$. Notice that $\{\beta,\gamma\} \subset \{\bar{\beta},\bar{\gamma}\}$. Denote by $E_{i,j}$ the elementary matrix which has 1 as the coefficient situated on line i and column j, and 0 elsewhere.

Case 1: $L \neq L'$. Here $\{\beta,\gamma\} = \{\bar{\beta},\bar{\gamma}\}$. Without loss of generality, assume $\beta = \bar{\beta}, \gamma = \bar{\gamma}$, that is to say, $L = L_1, L' = L_2$. Then $P_L = PE_{1,1}P^{-1}$ and $P_{L'} = PE_{2,2}P^{-1}$. Thus $PE_{1,1}P^{-1}$ is an eigenvector associated to $\bar{\beta}$ and $PE_{2,2}P^{-1}$ is an eigenvector associated to $-\bar{\gamma}$. The matrix $PE_{1,2}P^{-1}$ is an eigenvector associated to $\bar{\beta}-\bar{\gamma}$ and $PE_{2,1}P^{-1}$is in the kernel.

Case 2: $L = L' = L_1$. Here $\beta = \gamma = \bar{\beta}$. Then $PE_{1,1}P^{-1}$ and $PE_{2,2}P^{-1}$ are in the kernel and $PE_{1,2}P^{-1}, PE_{2,1}P^{-1}$ are eigenvectors associated to $\bar{\beta}$ and $-\bar{\beta}$ respectively.

Case 3: $L = L' = L_2$. This case is very similar to the previous one. □

Now assume 0 is the only eigenvalue of $\tilde{A}$. If $\tilde{A}$ is the zero matrix, then $ad_{\tilde{A}} = 0$. Otherwise $\tilde{A}$ is nilpotent and in this case one has the following lemma:

Lemma 8.2. *Assume $\tilde{A}$ is nilpotent. Then the operator $ad_{\tilde{A}}$ has rank 2 and norm less than 1, and is nilpotent of order 3.*

Proof. Let P be such that $P^{-1}\tilde{A}P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $PE_{12}P^{-1}$ and $P(E_{11}+E_{22})P^{-1}$ are in the kernel. Moreover $ad_{\tilde{A}}PE_{11}P^{-1} = -PE_{12}P^{-1}$ and $ad_{\tilde{A}}PE_{21}P^{-1} = P(E_{11}-E_{22})P^{-1}$, therefore $ad_{\tilde{A}}$ has norm less than 1.

This also implies that $ad^2_{\tilde{A}}(PE_{1,1}P^{-1}) = 0$ and $ad^3_{\tilde{A}}(PE_{2,1}P^{-1}) = 0$. Finally $ad_{\tilde{A}}$ is nilpotent of order 3. □

Proposition 8.3. *Let $N \in \mathbb{N}, \kappa' \in]0,\kappa], r \in]0, r_0[$. Let $\tilde{A} \in sl(2,\mathbb{R})$ with $BR^N_\omega(\kappa')$ spectrum. Let $\tilde{F} \in U_r(\mathbb{T}^d, sl(2,\mathbb{R}))$. Then there exists a solution $\tilde{X} \in U_r(\mathbb{T}^d, sl(2,\mathbb{R}))$ of the equation*

$$\forall \theta \in \mathbb{T}^d, \partial_\omega \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + \tilde{F}^N - \hat{\tilde{F}}(0), \quad \hat{\tilde{X}}(0) = 0 \tag{8.2}$$

The truncation of $\tilde{X}$ at order N is unique.
Moreover,

1. *if $\tilde{A}$ is diagonalizable with distinct eigenvalues, let $\mathcal{L}_{\tilde{A}} = \{L_1, L_2\}$ (the decomposition into eigenspaces of $\tilde{A}$) and $\Phi = P^{\mathcal{L}}_{L_1} e^{2i\pi\langle m,\cdot\rangle} + P^{\mathcal{L}}_{L_2} e^{-2i\pi\langle m,\cdot\rangle}$ for some $m \in \frac{1}{2}\mathbb{Z}^d$, $|m| \leq N$, such that for $i \in \{1,2\}$, $\|P^{\mathcal{L}}_{L_i}\| \leq \frac{2C_0}{\kappa'^6}$, then*

$$|\Phi^{-1}\tilde{X}\Phi|_r \leq 4C_0^2\big(\frac{1}{\kappa'}\big)^{13}\Psi(N)|\Phi^{-1}\tilde{F}\Phi|_r$$

2. *if $\tilde{A}$ is nilpotent,*

$$|\tilde{X}|_r \leq \frac{3}{\kappa^3}\Psi(N)^3|\tilde{F}|_r$$

3. *if $ad_{\tilde{A}} = 0$, then*

$$|\tilde{X}|_r \leq \frac{1}{\kappa}\Psi(N)|\tilde{F}|_r$$

Proof. About existence, uniqueness and continuity of $\tilde{X} \in sl(2,\mathbb{R})$ on $\mathbb{T}^d$, the proof is the same as [11], proposition 3.2. We now have to show the estimate which also follows from [11] and we will adapt the proof to ultra-differentiable setting.

Case 1: $\tilde{A}$ has two κ'-separated eigenvalues. Let $\Phi = P_{L_1}e^{2i\pi\langle m_1,.\rangle} + P_{L_2}e^{-2i\pi\langle m_1,.\rangle}$ where L_1 and L_2 are the eigenspaces of $\tilde{A}$, and $|m_1| \leq N$. For all $L, L' \in \mathcal{L}_{\tilde{A}}$, let the linear operator $\mathcal{A}_{L,L'} : gl(2,\mathbb{R}) \to gl(2,\mathbb{R}), M \mapsto \mathcal{A}_{L,L'}M := \tilde{A}P_L M - MP_{L'}\tilde{A}$. We decompose (8.2) into blocks, and we get for all $L, L' \in \mathcal{L}_{\tilde{A}}$,

$$\partial_\omega(P_L\tilde{X}(\theta)P_{L'}) = \mathcal{A}_{L,L'}P_L\tilde{X}(\theta)P_{L'} + P_L(\tilde{F}^N - \hat{\tilde{F}}(0))P_{L'}$$

Then for all $m \in \frac{1}{2}\mathbb{Z}^d$, $0 < |m| \leq N$,

$$2i\pi\langle m,\omega\rangle(P_L\hat{\tilde{X}}(m)P_{L'}) = \mathcal{A}_{L,L'}(P_L\hat{\tilde{X}}(m)P_{L'}) + P_L\hat{\tilde{F}}(m)P_{L'}$$

Let

$$A_D := (2i\pi\langle m,\omega\rangle I - \mathcal{A}_{L,L'})$$

By Lemma 8.1, $\sigma(\mathcal{A}_{L,L'}) = \{\alpha - \alpha', \alpha, -\alpha', 0; \alpha \in \sigma(\tilde{A}_{|L}), \alpha' \in \sigma(\tilde{A}_{|L'})\}$, therefore $\sigma(\mathcal{A}_{L,L'} - 2i\pi\langle m,\omega\rangle I) = \{\alpha-\alpha'-2i\pi\langle m,\omega\rangle, \alpha-2i\pi\langle m,\omega\rangle, -\alpha'-2i\pi\langle m,\omega\rangle, -2i\pi\langle m,\omega\rangle; \alpha \in \sigma(\tilde{A}_{|L}), \alpha' \in \sigma(\tilde{A}_{|L'})\}$. Moreover $\mathcal{A}_{L,L'}$ is diagonalizable, therefore A_D as well, with non zero eigenvalues, and $\|A_D^{-1}\| = \max\{|\beta|, \beta \in \sigma(A_D^{-1})\} = \max\{|\gamma|^{-1}, \gamma \in \sigma(A_D)\}$.

Since $\forall \alpha \in \sigma(\tilde{A}_{|L}), \alpha' \in \sigma(\tilde{A}_{|L'})$, $|\alpha - \alpha' - 2i\pi\langle m,\omega\rangle| \geq \frac{\kappa'}{\Psi(m)}$ (for $m \in \mathbb{Z}^d$ if $L = L'$, $m \in \frac{1}{2}\mathbb{Z}^d$ if $L \neq L'$), then

$$\|(2i\pi\langle m,\omega\rangle - \mathcal{A}_{L,L'})^{-1}\| \leq \big(\frac{\Psi(m)}{\kappa'}\big)$$

Finally, for all $0 < |m| \leq N$,

$$\|P_L\hat{\tilde{X}}(m)P_{L'}\| = \|(2i\pi\langle m,\omega\rangle - \mathcal{A}_{L,L'})^{-1}P_L\hat{\tilde{F}}(m)P_{L'}\| \leq \big(\frac{\Psi(m)}{\kappa'}\big)\|P_L\hat{\tilde{F}}(m)P_{L'}\|$$

Denoting by m_L the vector appearing in Φ along the projection onto L, this estimate implies:

$$|P_L\tilde{X}e^{2i\pi\langle m_L - m_{L'},.\rangle}P_{L'}|_{r'} = \sum_{|m-m_L+m_{L'}|\leq N} \|P_L\hat{\tilde{X}}(m-m_L+m_{L'})P_{L'}\|e^{2\pi\Lambda(m)r'}$$

$$\leq \sum_{|m-m_L+m_{L'}|\leq N} \|P_L\hat{\tilde{F}}(m-m_L+m_{L'})P_{L'}\|e^{2\pi\Lambda(m)r'}\frac{\Psi(|m-m_L+m_{L'}|)}{\kappa'}$$

$$\leq \frac{\Psi(N)}{\kappa'}|P_L\tilde{F}e^{2i\pi\langle m_L-m_{L'},.\rangle}P_{L'}|_{r'} \tag{8.3}$$

We finally estimate $|\Phi^{-1}\tilde{X}\Phi|_{r'}$.

$$|\Phi^{-1}\tilde{X}\Phi|_{r'} = |\sum_{L,L'\in\mathcal{L}} P_L\Phi^{-1}\tilde{X}\Phi P_{L'}|_{r'} = |\sum_{L,L'\in\mathcal{L}} P_L\tilde{X}e^{2i\pi\langle m_L-m_{L'},.\rangle}P_{L'}|_{r'}$$

therefore, from (8.3),

$$|\Phi^{-1}\tilde{X}\Phi|_{r'} \leq \frac{\Psi(N)}{\kappa'}\sum_{L,L'\in\mathcal{L}}|P_L\tilde{F}e^{2i\pi\langle m_L-m_{L'},.\rangle}P_{L'}|_r = \frac{\Psi(N)}{\kappa'}\sum_{L,L'\in\mathcal{L}}|P_L\Phi^{-1}\tilde{F}\Phi P_{L'}|_r$$

therefore, since $\|P_L\| \leq \frac{2C_0}{\kappa'^6}$, we get the result

$$|\Phi^{-1}\tilde{X}\Phi|_{r'} \leq 4C_0^2\big(\frac{1}{\kappa'}\big)^{13}\Psi(N)|\Phi^{-1}\tilde{F}\Phi|_r.$$

Case 2: $\tilde{A}$ is nilpotent. One has to estimate the inverse of the operator $2i\pi\langle m,\omega\rangle I - ad_{\tilde{A}}$. By Lemma 8.2,

$$\begin{aligned}(2i\pi\langle m,\omega\rangle I - ad_{\tilde{A}})^{-1} = (2i\pi\langle m,\omega\rangle)^{-1}[I + (2i\pi\langle m,\omega\rangle)^{-1}ad_{\tilde{A}} \\ + (2i\pi\langle m,\omega\rangle)^{-2}ad_{\tilde{A}}^2]\end{aligned} \tag{8.4}$$

Therefore

$$\|(2i\pi\langle m,\omega\rangle I - ad_{\tilde{A}})^{-1}\| \leq 3|2i\pi\langle m,\omega\rangle|^{-3}$$

Finally, for all $0 < |m| \leq N$,

$$\|\hat{\tilde{X}}(m)\| = \|(2i\pi\langle m,\omega\rangle - \mathcal{A}_{L,L'})^{-1}\hat{\tilde{F}}(m)\| \leq 3\big(\frac{\Psi(m)}{\kappa}\big)^3\|\hat{\tilde{F}}(m)\|$$

Thus,

$$|\tilde{X}|_{r'} \leq \frac{3}{\kappa^3}\Psi(N)^3|\tilde{F}|_r$$

Case 3: The operator to invert is just $2i\pi\langle m,\omega\rangle I$, which makes the estimate much simpler.

□

9 Inductive lemma without renormalization

Before stating the inductive lemma, we will need this next result which will allow us to iterate the inductive lemma without needing a new renormalization map at each step.

Lemma 9.1. *Let $\kappa' \in]0,1[$, $\tilde{F} \in sl(2,\mathbb{R}), \tilde{\varepsilon} = \|\tilde{F}\|, \tilde{N} \in \mathbb{N}, \tilde{A} \in sl(2,\mathbb{R})$ with $BR_\omega^{\tilde{N}}(\kappa')$ spectrum.*
If

$$\tilde{\varepsilon} \leq \big(\frac{\kappa'}{32(1+\|\tilde{A}\|)}\big)^2\frac{1}{\Psi(\tilde{N})^2},$$

then $\tilde{A}+\tilde{F}$ has $BR_\omega^{\tilde{N}}(\frac{3\kappa'}{4})$ spectrum.

Proof. If $\tilde{\alpha} \in \sigma(\tilde{A}+\tilde{F})$, there exists $\alpha \in \sigma(\tilde{A})$ such that $|\alpha-\tilde{\alpha}| \leq 4(\|\tilde{A}\|+1)\tilde{\varepsilon}^{\frac{1}{2}}$ (see [11], lemma 4.1). Since $\tilde{A}$ has $BR_\omega^{\tilde{N}}(\kappa')$ spectrum, for all $\alpha,\alpha' \in \sigma(\tilde{A}+\tilde{F})$, for all $m \in \frac{1}{2}\mathbb{Z}^d$, $0<|m|\leq\tilde{N}$,

$$|\alpha-\alpha'-2i\pi\langle m,\omega\rangle| \geq \frac{\kappa'}{\Psi(m)} - 8(\|\tilde{A}\|+1)\tilde{\varepsilon}^{\frac{1}{2}}$$

We have to check that $8(\|\tilde{A}\|+1)\tilde{\varepsilon}^{\frac{1}{2}} \leq \frac{\kappa'}{4\Psi(m)}$, which is satisfied by assumption. □

Lemma 9.2. *Let $N \geq 1$. If $\mathcal{L} = \{L_1, L_2\}$ is a decomposition of $\mathbb{R}^2$ into supplementary subspaces, and Φ is trivial with respect to $\mathcal{L}$ of order N, then for all $0<r'<r$ and all $G \in U_r(\mathbb{T}^d, sl(2,\mathbb{R}))$,*

$$|\Phi^{-1}(G-G^{3N})\Phi|_{r'} \leq e^{-2\pi\Lambda(N)(r-r')}|\Phi^{-1}G\Phi|_r$$

Proof. Write $\Phi = P_{L_1}^{\mathcal{L}}e^{2i\pi\langle m_1,\cdot\rangle} + P_{L_2}^{\mathcal{L}}e^{2i\pi\langle m_1,\cdot\rangle}$, $m_1 \in \frac{1}{2}\mathbb{Z}^d$, then

$$|\Phi^{-1}(G-G^{3N})\Phi|_{r'} = |\sum_{L,L'\in\mathcal{L}} P_L^{\mathcal{L}}(G-G^{3N})e^{2i\pi\langle m_L-m_{L'},\cdot\rangle}P_{L'}^{\mathcal{L}}|_{r'}$$

$$= \sum_{k\in\mathbb{Z}^d}\|\sum_{L,L'} P_L^{\mathcal{L}}\widehat{(G-G^{3N})}(k-m_L+m_{L'})P_{L'}^{\mathcal{L}}\|e^{2\pi\Lambda(k)r'}$$

$$= \sum_{k\in\mathbb{Z}^d}\|\sum_{L,L'} P_L^{\mathcal{L}}\widehat{(G-G^{3N})}(k-m_L+m_{L'})P_{L'}^{\mathcal{L}}\|e^{2\pi\Lambda(k)r}e^{2\pi\Lambda(k)(r'-r)}$$

Now if $|k|\leq N$, then for all $L,L'\in\mathcal{L}$, $|k-m_L+m_{L'}|\leq 3N$, and $\sum_{L,L'\in\mathcal{L}} P_L^{\mathcal{L}}\widehat{(G-G^{3N})}(k-m_L+m_{L'})P_{L'}^{\mathcal{L}} = 0$, therefore

$$|\Phi^{-1}(G-G^{3N})\Phi|_{r'} \leq e^{2\pi\Lambda(N)(r'-r)}\sum_{|k|>N}\|\sum_{L,L'\in\mathcal{L}} P_L^{\mathcal{L}}\hat{G}(k-m_L+m_{L'})P_{L'}^{\mathcal{L}}\|e^{2\pi\Lambda(k)r}$$

$$\leq e^{2\pi\Lambda(N)(r'-r)}|\sum_{L,L'\in\mathcal{L}} P_L^{\mathcal{L}}Ge^{2i\pi\langle m_L-m_{L'},\cdot\rangle}P_{L'}^{\mathcal{L}}|_r = e^{2\pi\Lambda(N)(r'-r)}|\Phi^{-1}G\Phi|_r$$

□

We can now state the first inductive lemma, which does not require a renormalization map.

Lemma 9.3. *Let*

- $\tilde{\varepsilon} > 0, \tilde{r} > 0, 0 < \kappa' < 1, \tilde{N} \in \mathbb{N}^*, \tilde{r}' < \tilde{r}$,
- $\tilde{F} \in U_{\tilde{r}}(\mathbb{T}^d, sl(2,\mathbb{R})), \tilde{A} \in sl(2,\mathbb{R})$.

If

1. *$\tilde{A}$ has $BR_\omega^{\tilde{N}}(\kappa')$ spectrum,*

2. $$\|\hat{\tilde{F}}(0)\| \leq \tilde{\varepsilon} \leq \big(\frac{\kappa'}{32(1+\|\tilde{A}\|)}\big)^2 \frac{1}{\Psi(\tilde{N})^2}$$

then there exist

- $X \in U_{\tilde{r}'}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $A' \in sl(2,\mathbb{R})$,

such that

1. *A' has $BR_\omega^{\tilde{N}}(\frac{3\kappa'}{4})$ spectrum,*

2. *$\|A' - \tilde{A}\| \leq \tilde{\varepsilon}$,*
   If $F' \in U_{\tilde{r}'}(\mathbb{T}^d, sl(2,\mathbb{R}))$ is defined by

$$\forall \theta \in \mathbb{T}^d, \partial_\omega e^{X(\theta)} = (\tilde{A} + \tilde{F}(\theta))e^{X(\theta)} - e^{X(\theta)}(A' + F'(\theta)), \tag{9.1}$$

   then we have the following estimates :

   If $\tilde{A}$ has two different eigenvalues, if Φ is of the form $\Phi = P_{L_1}e^{2i\pi\langle m,\cdot\rangle} + P_{L_2}e^{-2i\pi\langle m,\cdot\rangle}$ where L_1, L_2 are the two eigenspaces of $\tilde{A}$, $|m| \leq \tilde{N}$ and $\|P_{L_i}\| \leq \frac{2C_0}{\kappa'^6}$,

3. $$|\Phi^{-1}X\Phi|_{\tilde{r}'} \leq 4C_0^2\big(\frac{1}{\kappa'}\big)^{13}\Psi(3\tilde{N})|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}},$$

4. $$\begin{aligned}|\Phi^{-1}F'\Phi|_{\tilde{r}'} \leq 4C_0^2 e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}}\big(\frac{1}{\kappa'}\big)^{13}|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}\\ +|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}\Psi(3\tilde{N})(2e + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})\big].\end{aligned}$$

   If $\tilde{A}$ is nilpotent:

5. $$|X|_{\tilde{r}'} \leq \frac{3}{\kappa^3}\Psi(3\tilde{N})^3|\tilde{F}|_{\tilde{r}},$$

6. $$\begin{aligned}|F'|_{\tilde{r}'} \leq \frac{3}{\kappa^3}e^{|X|_{\tilde{r}'}}|\tilde{F}|_{\tilde{r}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}\\ +|\tilde{F}|_{\tilde{r}}\Psi(3\tilde{N})^3(2e + e^{|X|_{\tilde{r}'}})\big].\end{aligned}$$

   If $ad_{\tilde{A}} = 0$:

7. $$|X|_{\tilde{r}'} \leq \frac{1}{\kappa}\Psi(3\tilde{N})|\tilde{F}|_{\tilde{r}},$$

8. $$\begin{aligned}|F'|_{\tilde{r}'} \leq \frac{1}{\kappa}e^{|X|_{\tilde{r}'}}|\tilde{F}|_{\tilde{r}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}\\ +|\tilde{F}|_{\tilde{r}}\Psi(3\tilde{N})(2e + e^{|X|_{\Lambda,\tilde{r}'}})\big].\end{aligned}$$

   In any case, there is the estimate

9.

$$|\partial_\omega X|_{\tilde{r}'} \leq 2\|\tilde{A}\|\ |X|_{\tilde{r}'} + |\tilde{F}|_{\tilde{r}'}.$$

Proof. By assumption, $\tilde{A}$ has $BR_\omega^{\tilde{N}}(\kappa')$ spectrum, so we apply Proposition 8.3 with $N = 3\tilde{N}$. Let $X \in U_{\tilde{r}}(\mathbb{T}^d, sl(2,\mathbb{R}))$ a solution of

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega X(\theta) = [\tilde{A}, X(\theta)] + \tilde{F}^{3\tilde{N}}(\theta) - \hat{\tilde{F}}(0)$$

satisfying the conclusion of Proposition 8.3. This obviously implies the property 9.

Let $A' := \tilde{A} + \hat{\tilde{F}}(0)$. We have $A' \in sl(2,\mathbb{R})$ and $\|\tilde{A} - A'\| = \|\hat{\tilde{F}}(0)\|$, and then property 2. With assumption (2) we can apply lemma 9.1 to deduce that A' has $BR_\omega^{\tilde{N}}(\frac{3\kappa'}{4})$ spectrum, and then property 1.

If F' is defined in equation (9.1),

$$F' = e^{-X}(\tilde{F} - \tilde{F}^{3\tilde{N}}) + e^{-X}\tilde{F}(e^X - Id) + (e^{-X} - Id)\hat{\tilde{F}}(0) - e^{-X}\sum_{k\geq 2}\frac{1}{k!}\sum_{l=0}^{k-1} X^l(\tilde{F}^{3\tilde{N}} - \hat{\tilde{F}}(0))X^{k-1-l} \tag{9.2}$$

- Case 1: $\tilde{A}$ has two different eigenvalues : Let Φ be as required, then

$$|\Phi^{-1}F'\Phi|_{\tilde{r}'} \leq e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}}[|\Phi^{-1}(\tilde{F} - \tilde{F}^{3\tilde{N}})\Phi|_{\tilde{r}'} + |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}|\Phi^{-1}X\Phi|_{\tilde{r}'}(2e + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})]$$

From proposition 8.3, estimate 1,

$$|\Phi^{-1}X\Phi|_{\tilde{r}'} \leq |\Phi^{-1}X\Phi|_{\tilde{r}} \leq 4C_0^2\big(\frac{1}{\kappa'}\big)^{13}\Psi(3\tilde{N})|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}$$

whence (3); and from lemma 9.2, since $\tilde{r}' < \tilde{r}$,

$$|\Phi^{-1}(\tilde{F} - \tilde{F}^{3\tilde{N}})\Phi|_{\tilde{r}'} \leq e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}$$

which finally gives

$$\begin{aligned}|\Phi^{-1}F'\Phi|_{\tilde{r}'} &\leq e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}} \\ &\qquad + |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}4C_0^2\big(\frac{1}{\kappa'}\big)^{13}\Psi(3\tilde{N})|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}(2e + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})\big] \\ &\leq 4C_0^2 e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}}\big(\frac{1}{\kappa'}\big)^{13}|\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')} + |\Phi^{-1}\tilde{F}\Phi|_{\tilde{r}}\Psi(3\tilde{N})(2e + e^{|\Phi^{-1}X\Phi|_{\tilde{r}'}})\big]\end{aligned}$$

hence 4 holds.

- Case 2: $\tilde{A}$ is nilpotent : (9.2) implies

$$|F'|_{\tilde{r}'} \leq e^{|X|_{\tilde{r}'}}[|\tilde{F} - \tilde{F}^{3\tilde{N}}|_{\tilde{r}'} + |\tilde{F}|_{\tilde{r}}|X|_{\tilde{r}'}(2e + e^{|X|_{\tilde{r}'}})]$$

From proposition 8.3, estimate 2,

$$|X|_{\tilde{r}'} \leq |X|_{\tilde{r}} \leq \frac{3}{\kappa^3}\Psi(3\tilde{N})^3|\tilde{F}|_{\tilde{r}}$$

which is estimate 5. Moreover, from Lemma 9.2,

$$|\tilde{F} - \tilde{F}^{3\tilde{N}}|_{\tilde{r}'} \leq e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}|\tilde{F}|_{\tilde{r}}$$

Therefore, similarly to the previous case, we get

$$\begin{aligned}|F'|_{\tilde{r}'} \leq \frac{3}{\kappa^3}e^{|X|_{\tilde{r}'}}|\tilde{F}|_{\tilde{r}}\big[&e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')} \\ &+ |\tilde{F}|_{\tilde{r}}\Psi(3\tilde{N})^3(2e + e^{|X|_{\tilde{r}'}})\big]\end{aligned}$$

which is estimate 6.

- Case 3: $ad_{\tilde{A}} = 0$: From proposition 8.3, estimate 3

$$|\tilde{X}|_{\tilde{r}} \leq \frac{1}{\kappa}\Psi(3\tilde{N})|\tilde{F}|_{\tilde{r}}$$

which is estimate (7), and similarly to the two previous cases, we get the estimate (8):

$$|F'|_{\tilde{r}'} \leq \frac{1}{\kappa}e^{|X|_{\tilde{r}'}}|\tilde{F}|_{\tilde{r}}\big[e^{-2\pi\Lambda(\tilde{N})(\tilde{r}-\tilde{r}')}$$
$$+|\tilde{F}|_{\tilde{r}}\Psi(3\tilde{N})(2e+e^{|X|_{\tilde{r}'}})\big].$$

□

10 Inductive lemma with renormalization

The following Lemma is used to define the smallness assumption on ϵ_0 mentioned in section 5. This smallness assumption shall be sufficient for Lemmas 10.2 and 11.1.

Lemma 10.1. *Let $l = 56$. There exists $\varepsilon_0 > 0$ depending on C_0, C' κ, b_0 and D_5, such that, for all $\varepsilon \in]0, \varepsilon_0]$, the following inequalities hold for all $2 \leq j \leq l$:*
In lemma 10.2

$$\frac{1}{2}\kappa\varepsilon^{\frac{1}{1728}} + \varepsilon^{\frac{845}{864}} \leq \frac{3}{4}\kappa\varepsilon^{\frac{1}{1728}} \tag{10.1}$$

$$\frac{4C_0^2}{\kappa^{13}}\varepsilon^{-13\zeta}\varepsilon^{-3\zeta}\varepsilon^{1-2\zeta} \leq \varepsilon^{\frac{7}{8}} \tag{10.2}$$

$$8C_0^2\varepsilon^{1-2\zeta-\frac{1}{96}}(\varepsilon^{100\delta} + 3\varepsilon^{1-6\zeta}) \leq \varepsilon^{\frac{3}{2}-4\zeta-\frac{1}{96}} \tag{10.3}$$

In lemma 11.1

$$\varepsilon^{1-576\zeta} \leq (2C_0)^{-96}\Big(\frac{\kappa}{32(\varepsilon^{-\frac{\zeta}{2}}+1)}\Big)^{576} \tag{10.4}$$

$$\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \Big(\frac{\frac{3}{4}\frac{\kappa}{C_0}\varepsilon^{\zeta}}{32(1+(1+\pi)\varepsilon^{-\frac{\zeta}{2}}+\varepsilon^{\frac{23}{24}})}\Big)^2\varepsilon^{2\zeta} \tag{10.5}$$

$$\varepsilon^{(\frac{5}{4})^j-\frac{1}{48}} \leq \Big(\frac{(\frac{3}{4})^j\frac{\kappa}{C_0}\varepsilon^{\zeta}}{32(1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\frac{\zeta}{2}}+\sum_{i=1}^{j-1}\varepsilon^{(\frac{5}{4})^i-\frac{1}{96}})}\Big)^2\varepsilon^{2\zeta} \tag{10.6}$$

$$256C_0^2\varepsilon^{-14\zeta}\Big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa}{C_0}}\Big)^{13}\varepsilon^{(\frac{5}{4})^{j-1}}(\varepsilon^{\frac{50\delta}{l}}+\varepsilon^{(\frac{5}{4})^{j-1}}) \leq \varepsilon^{(\frac{5}{4})^j} \tag{10.7}$$

$$\varepsilon^{\frac{23}{24}} + \pi\varepsilon^{-\frac{\zeta}{2}} + \sum_{i=1}^{l}\varepsilon^{(\frac{5}{4})^i-\frac{1}{48}} \leq \varepsilon^{-\zeta} \tag{10.8}$$

$$\frac{1}{2}\kappa\varepsilon^{\zeta} + 2\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \kappa\varepsilon^{\zeta} \tag{10.9}$$

$$\varepsilon^{-\frac{\zeta}{2}} + \varepsilon^{\frac{23}{24}} + \pi\varepsilon^{-\zeta} \leq \varepsilon^{-2\zeta} \tag{10.10}$$

$$4\varepsilon^{-2\zeta+\frac{59}{48}} + 2\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \varepsilon \tag{10.11}$$

$$2\varepsilon^{\frac{1}{2}} + 2\varepsilon^{\frac{7}{8}} \leq \varepsilon^{\frac{1}{4}} \tag{10.12}$$

Proof. **Equations in lemma 10.2**

Equation (10.1) holds for

$$\varepsilon \leq (\frac{1}{4}\kappa)^{\frac{1728}{2123}}.$$

Equation (10.2) holds for

$$\varepsilon \leq \big(\frac{4C_0^2}{\kappa^{13}}\big)^{-\frac{96}{11}}.$$

Equation (10.3) holds if

$$8C_0^2\varepsilon^{\frac{427}{432}}(\varepsilon^{100\delta}+3\varepsilon^{\frac{287}{288}}) \leq \varepsilon^{\frac{1285}{864}} \Leftrightarrow 8C_0^2(\varepsilon^{100\delta}+3\varepsilon^{\frac{287}{288}}) \leq \varepsilon^{\frac{431}{864}}$$

therefore, if we have

$$\begin{cases} 8C_0^2\varepsilon^{100\delta} \leq \frac{1}{2}\varepsilon^{\frac{431}{864}} \\ 24C_0^2\varepsilon^{\frac{287}{288}} \leq \frac{1}{2}\varepsilon^{\frac{431}{864}} \end{cases}$$

which is satisfied if

$$\varepsilon \leq (48C_0^2)^{-\frac{432}{215}}$$

then inequality (10.3) holds.

Equations in lemma 11.1

Equation (10.4) holds if

$$\varepsilon^{\frac{1}{1152}}+\varepsilon^{\frac{1}{864}} \leq (2C_0)^{-\frac{1}{6}}\frac{\kappa}{32}$$

which is satisfied if

$$\begin{cases} \varepsilon^{\frac{1}{1152}} \leq \frac{\kappa}{64}(2C_0)^{-\frac{1}{6}} \\ \varepsilon^{\frac{1}{864}} \leq \frac{\kappa}{64}(2C_0)^{-\frac{1}{6}} \end{cases} \Leftrightarrow \begin{cases} \varepsilon \leq (\frac{\kappa}{64})^{1152}(2C_0)^{-192} \\ \varepsilon \leq (\frac{\kappa}{64})^{864}(2C_0)^{-144} \end{cases}$$

Equation (10.5) is satisfied if

$$\varepsilon^{\frac{5}{4}-\frac{1}{48}-4\zeta}(1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\frac{\zeta}{2}})^2 \leq (\frac{3\kappa}{128C_0})^2.$$

For $\varepsilon \leq 1$, we have

$$1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\frac{\zeta}{2}} \leq 4\pi\varepsilon^{-\frac{\zeta}{2}}$$

then we need

$$16\pi^2\varepsilon^{\frac{5}{4}-\frac{1}{48}-5\zeta} \leq (\frac{3\kappa}{128C_0})^2$$

which is satisfied if

$$\varepsilon \leq (\frac{3\kappa}{512\pi C_0})^{\frac{2119}{1728}}$$

Equation (10.6) is satisfied if

$$\varepsilon^{(\frac{5}{4})^{j-1}-\frac{1}{96}-4\zeta}(1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\frac{\zeta}{2}}+2\varepsilon^{\frac{5}{4}-\frac{1}{48}})^2 \leq (\frac{3}{4})^{2j}(\frac{\kappa}{32C_0})^2.$$

If $\varepsilon \leq 1$, then

$$1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\frac{\zeta}{2}}+2\varepsilon^{\frac{5}{4}-\frac{1}{96}} \leq 4\pi\varepsilon^{-\frac{\zeta}{2}}$$

then it's enough to have

$$16\pi^2\varepsilon^{\frac{5}{4}-\frac{1}{48}-4\zeta-\zeta} \leq (\frac{3}{4})^{2\cdot 104}(\frac{\kappa}{32C_0})^2$$

which is satisfied if

$$\varepsilon \leq \big((\frac{3}{4})^{208}(\frac{\kappa}{128\pi C_0})^2)\big)^{\frac{3456}{2119}}$$

Equation (10.7) holds if

$$256C_0^2(\frac{4}{3})^{13(j-1)}(\frac{C_0}{\kappa})^{13}(\varepsilon^{\frac{50\delta}{l}}+\varepsilon^{(\frac{5}{4})^{j-1}}) \leq \varepsilon^{14\zeta+\frac{1}{4}(\frac{5}{4})^{j-1}}.$$

We will first show that, for all $j \in [\![2, l]\!]$, and for ε small enough,

$$\varepsilon^{\frac{50\delta}{l}} + \varepsilon^{(\frac{5}{4})^{j-1}} \leq \varepsilon^{\frac{1}{3}(\frac{5}{4})^{j-1}}.$$

Since $l = 56$, this condition is satisfied if for all $j \in [\![2, l]\!]$ if

$$\begin{cases} 2 \leq \varepsilon^{\frac{1}{3}(\frac{5}{4})^{j-1} - \frac{50\delta}{l}} \\ 2 \leq \varepsilon^{-\frac{1}{3}(\frac{5}{4})^{j-1}} \end{cases}$$

which holds if

$$\begin{cases} \varepsilon \leq 2^{\frac{1}{\frac{1}{2}(\frac{5}{4})^{55} - \frac{50\delta}{56}}} \\ \varepsilon \leq 2^{-\frac{12}{25}} \end{cases}$$

then equation (10.7) is satisfied if

$$256 C_0^2 (\frac{4}{3})^{13(j-1)} (\frac{C_0}{\kappa})^{13} \leq \varepsilon^{14\zeta - \frac{1}{12}(\frac{5}{4})^{j-1}} \Leftrightarrow \varepsilon \leq \big(256 C_0^{15} (\frac{4}{3})^{13(j-1)} (\frac{1}{\kappa})^{13}\big)^{\frac{1}{14\zeta - \frac{1}{12}(\frac{5}{4})^{j-1}}}.$$

Now, as $C_0 \geq 1$ and $0 < \kappa < 1$, since $\varepsilon \leq 1$,

$$(\frac{24 C_0^{15}}{\kappa^{13}})^{\frac{1}{5\zeta - \frac{1}{12}(\frac{5}{4})^{j-1}}} \geq (\frac{24 C_0^{15}}{\kappa^{13}})^{\frac{1}{14\zeta - \frac{1}{12}(\frac{5}{4})}} = (\frac{24 C_0^{15}}{\kappa^{13}})^{-\frac{864}{83}}$$

and

$$(\frac{4}{3})^{\frac{13(j-1)}{14\zeta - \frac{1}{12}(\frac{5}{4})^{j-1}}} \geq (\frac{4}{3})^{\frac{13 \cdot 4}{14\zeta - \frac{1}{12}(\frac{5}{4})^{4}}} = (\frac{4}{3})^{-\frac{1437696}{5401}}$$

Finally, equation (10.7) is satisfied with

$$\varepsilon \leq (\frac{24 C_0^{15}}{\kappa^{13}})^{-\frac{1728}{178}} (\frac{4}{3})^{-\frac{1437696}{5545}}$$

Equation (10.8) is satisfied if

$$\varepsilon^{\frac{23}{25}} + \pi \varepsilon^{-\frac{\zeta}{2}} + 2\varepsilon^{\frac{5}{4} - \frac{1}{48}} \leq \varepsilon^{-\zeta}.$$

So if we have

$$\begin{cases} \varepsilon^{\frac{23}{24}} \leq \frac{1}{10}\varepsilon^{-\zeta} \\ \pi \varepsilon^{-\frac{\zeta}{2}} \leq \frac{4}{5}\varepsilon^{-\zeta} \\ 2\varepsilon^{\frac{59}{48}} \leq \frac{1}{10}\varepsilon^{-\zeta} \end{cases} \Leftrightarrow \begin{cases} \varepsilon \leq (\frac{1}{10})^{\frac{1728}{1628}} \\ \varepsilon \leq (\frac{4\pi}{5})^{3456} \\ \varepsilon \leq (\frac{1}{20})^{\frac{1728}{1837}} \end{cases}$$

then equation (10.8) holds.

Equation (10.9) holds for

$$\varepsilon \leq (\frac{\kappa}{4})^{\frac{1728}{2141}}.$$

Equation (10.10) holds if

$$\begin{cases} \varepsilon^{-\frac{\zeta}{2}} \leq \frac{1}{3}\varepsilon^{-2\zeta} \\ \varepsilon^{\frac{23}{24}} \leq \frac{1}{3}\varepsilon^{-2\zeta} \\ \pi \varepsilon^{-\zeta} \leq \frac{1}{3}\varepsilon^{-2\zeta} \end{cases} \Leftrightarrow \begin{cases} \varepsilon \leq (\frac{1}{3})^{1152} \\ \varepsilon \leq (\frac{1}{3})^{\frac{864}{829}} \\ \varepsilon \leq (\frac{1}{3\pi})^{1728} \end{cases}$$

Equation (10.11) holds if

$$\begin{cases} 4\varepsilon^{\frac{1061}{864}} \leq \frac{1}{2}\varepsilon \\ 2\varepsilon^{\frac{59}{48}} \leq \frac{1}{2}\varepsilon \end{cases} \Leftrightarrow \begin{cases} \varepsilon \leq (\frac{1}{8})^{\frac{864}{197}} \\ \varepsilon \leq (\frac{1}{4})^{\frac{48}{11}} \end{cases}$$

and then equation (10.11) holds.

Since, for $\varepsilon \leq 1$ we have $\varepsilon^{\frac{7}{8}} \leq \varepsilon^{\frac{1}{2}}$, equation (10.12) holds if

$$4\varepsilon^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{4}}$$

that's it to say, if

$$\varepsilon \leq \frac{1}{256}.$$

Now define ε_0 in order to satisfy conditions (10.1) to (10.12).

□

Lemma 10.2 (Inductive lemma with renormalization)**.** *Let*

- $A \in sl(2,\mathbb{R})$,
- $r > 0$,
- $\bar{A}, \bar{F} \in U_r(\mathbb{T}^d, sl(2,\mathbb{R})), \psi \in U_r(2\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $|\bar{F}|_r = \varepsilon$,
- $$N = \Lambda^{-1}\big(\frac{50|\log\varepsilon|}{\pi r}\big)$$
- $$R = \frac{1}{3N}\Psi^{-1}(\varepsilon^{-\zeta})$$
- $$r' = r - \frac{50\delta|\log\varepsilon|}{\pi\Lambda(RN)}$$

Assume $r' > 0$. Let $\kappa'' = \dfrac{\kappa}{\Psi(3RN)} = \kappa\varepsilon^{\zeta}$. Suppose that $\varepsilon \le \varepsilon_0$ which was defined in Lemma 10.2 and

1. $$\varepsilon \le (2C_0)^{-96}\big(\frac{\kappa''}{32(\|A\|+1)}\big)^{576}$$
2. *$\bar{A}$ is reducible to A by ψ,*
3. *$\|A\| \le \varepsilon^{-\frac{\zeta}{2}}$,*
4. *for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi^{-1}G\psi \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,*
5. *$|\psi^{\pm 1}|_r \le \varepsilon^{-\zeta}$,*

then there exist

- $Z' \in U_{r'}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $\bar{A}', \bar{F}' \in U_{r'}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $\psi' \in U_r(2\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $A' \in sl(2,\mathbb{R})$

satisfying the following properties :

1. *$\bar{A}'$ is reducible by ψ' to A',*
2. *for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi'^{-1}G\psi' \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,*
3. *A' has $BR_{\omega}^{RN}(\frac{3}{4C_0}\kappa'')$ spectrum, where C_0 was defined in Lemma 7.1,*
4. $$\partial_\omega Z' = (\bar{A}+\bar{F})Z' - Z'(\bar{A}'+\bar{F}')$$
5. *$\|A'\| \le \|A\| + \varepsilon^{\frac{23}{24}} + \pi N$,*
6. $$|Z'^{\pm 1} - Id|_{r'} \le \varepsilon^{\frac{8}{9}}$$
7. *for all $s' > 0$,*
$$|\psi'^{-1}\psi|_{s'} \le 2C_0\big(\frac{1}{\kappa''}\big)^6 e^{2\pi\Lambda(\frac{N}{2})s'}$$
$$|\psi^{-1}\psi'|_{s'} \le 2C_0\big(\frac{1}{\kappa''}\big)^6 e^{2\pi\Lambda(\frac{N}{2})s'},$$
8. *$|\psi'^{\pm 1}|_r \le \varepsilon^{-\zeta-\frac{1}{96}} e^{2\pi\Lambda(\frac{N}{2})r}$,*
9. *$|\psi^{-1}\bar{F}'\psi|_{r'} \le \varepsilon^{\frac{5}{4}}$.*

10. If moreover the spectrum of A is not $BR_\omega^{RN}(\kappa'')$, then $\|A'\| \leq \frac{3}{4}\kappa''$,

11. If the spectrum of A is $BR_\omega^{RN}(\kappa'')$, we have $\Phi \equiv I$, then $\psi' = \psi$ and $\tilde{A} = A$, and then

$$|\psi^{-1} Z'^{\pm 1} \psi|_{r'} \leq e^{\varepsilon^{\frac{7}{8}}},$$

$$|\partial_\omega(\psi^{-1} Z'^{\pm 1} \psi)|_{r'} \leq \varepsilon^{\frac{1}{2}},$$

Proof. **Algebraic aspects**

If A has a double eigenvalue or κ''-close eigenvalues, let Φ be defined on $2\mathbb{T}^d$ as constantly equal to I and let $\tilde{A} = A$. Otherwise, let Φ a renormalization of A of order R, N given by lemma 7.1. Let $\tilde{A} \in sl(2,\mathbb{R})$ such that

$$\forall \theta \in 2\mathbb{T}^d, \partial_\omega \Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A}$$

so $||A - \tilde{A}|| \leq \pi N$ and $\tilde{A}$ has $BR_\omega^{RN}(\kappa'')$ spectrum. Notice that in this case, $\tilde{A}$ is not nilpotent. Let $\psi' = \psi\Phi$, and

$$\tilde{F} := \psi'^{-1} \bar{F} \psi'$$

Moreover, Φ is trivial with respect to $\mathcal{L}_A$:

$$\Phi = P_{L_1}^{\mathcal{L}_A} e^{2i\pi\langle m,\cdot\rangle} + P_{L_2}^{\mathcal{L}_A} e^{-2i\pi\langle m,\cdot\rangle} \tag{10.13}$$

with $|m| \leq N$ and $||P_{L_i}|| \leq \frac{C_0}{\kappa''^6}$. Since Φ is trivial with respect to $\mathcal{L}_A$, for all $s' \geq 0$, Lemma 7.1 implies

$$|\Phi^{\pm 1}|_{s'} \leq 2C_0 \Big(\frac{1}{\kappa''}\Big)^6 e^{2\pi\Lambda(\frac{N}{2})s'} \tag{10.14}$$

which gives property 7.

Let $\psi' = \psi\Phi$. Let $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, then by triviality of Φ, $\Phi^{-1} G \Phi \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, and by the assumption 4, $\psi'^{-1} G \psi' \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$. Therefore the property 2 on ψ' holds.

Moreover,

$$\|\hat{\tilde{F}}(0)\| \leq |\tilde{F}|_0 \leq |\Phi|_0 |\Phi^{-1}|_0 |\psi|_0 |\psi^{-1}|_0 |\bar{F}|_0$$

Therefore by (10.14) and by assumption 5,

$$\|\hat{\tilde{F}}(0)\| \leq \varepsilon^{1-2\zeta} (2C_0)^2 \big(\frac{1}{\kappa''}\big)^{12}.$$

Since $\varepsilon \leq (2C_0)^{-96} \big(\frac{\kappa''}{32(\|A\|+1)}\big)^{576} \leq (2C_0)^{-96} \kappa''^{576}$, we get

$$\|\hat{\tilde{F}}(0)\| \leq \varepsilon^{1-2\zeta-\frac{1}{48}}.$$

Since $\tilde{A}$ has a $BR_\omega^{RN}(\kappa'')$ spectrum, we want to apply lemma 9.3 with

$$\tilde{\varepsilon} = \varepsilon^{1-2\zeta-\frac{1}{48}}, \tilde{r} = r, \tilde{r}' = r', \kappa' = \frac{\kappa''}{C_0}, \tilde{N} = RN,$$

then we need

$$\varepsilon^{1-2\zeta-\frac{1}{48}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(1+\|\tilde{A}\|)} \varepsilon^\zeta\right)^2$$

or sufficiently

$$\varepsilon^{1-2\zeta-\frac{1}{48}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(1+\|A\|+\pi N)} \varepsilon^\zeta\right)^2$$

which holds true if

$$\varepsilon^{1-2\zeta-\frac{1}{48}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(2+\pi)} \varepsilon^{2\zeta}\right)^2$$

(where we have used the assumption that $\Psi \geq id$), which holds true by assumption 1. Therefore we can apply lemma 9.3 to get the maps $X \in U_{r'}(\mathbb{T}^d, sl(2,\mathbb{R}))$, $F' \in U_{r'}(\mathbb{T}^d, sl(2,\mathbb{R}))$, and a matrix $A' \in sl(2,\mathbb{R})$ such that

- A' has $BR^{RN}_{\omega}(\frac{3\kappa''}{4C_0})$ spectrum,
- $\|A'-\tilde{A}\| \leq \tilde{\varepsilon} \leq \varepsilon^{\frac{23}{24}}$ (because $1-2\zeta-\frac{1}{48} \geq \frac{23}{24}$), which implies that

$$\|A'-A\| \leq \|A'-\tilde{A}\| + \|A-\tilde{A}\| \leq \varepsilon^{\frac{23}{24}} + \pi N$$

and thus

$$\|A'\| \leq \|A\| + \varepsilon^{\frac{23}{24}} + \pi N$$

which is property 5,
- $\partial_\omega e^X = (\tilde{A}+\tilde{F})e^X - e^X(A'+F')$.
Let $\bar{F}' = \psi' F'(\psi')^{-1} \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$ and $\bar{A}' \in U_r(2\mathbb{T}^d, sl(2,\mathbb{R}))$ such that

$$\partial_\omega \psi' = \bar{A}'\psi' - \psi' A'$$

(which means that $\bar{A}'$ is reducible to A', hence Property 1 with $\psi' := \psi\Phi$). Then the function $Z' := \psi' e^X (\psi')^{-1} \in \mathcal{C}^0(\mathbb{T}^d, SL(2,\mathbb{R}))$ is solution of

$$\partial_\omega Z' = (\bar{A}+\bar{F})Z' - Z'(\bar{A}'+\bar{F}')$$

hence Property 4. This conjugation also implies that $\bar{A}' \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$.
- if $\tilde{A}$ has two different eigenvalues, since Φ is trivial with respect to $\mathcal{L}_A$ which is identical to $\mathcal{L}_{\tilde{A}}$, by Lemma 9.3 and the expression (10.13),

$$|\Phi X \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} \Psi(3RN) |\Phi \tilde{F} \Phi^{-1}|_r$$

and

$$|\Phi F' \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} e^{|\Phi X \Phi^{-1}|_{r'}} \big(|\Phi \tilde{F} \Phi^{-1}|_r \big[e^{-2\pi\Lambda(RN)(r-r')} + |\Phi \tilde{F} \Phi^{-1}|_r \Psi(3RN)(2e + e^{|\Phi X \Phi^{-1}|_{r'}}) \big]$$

otherwise if $\tilde{A}$ is nilpotent,

$$|X|_{r'} \leq \frac{3}{\kappa^3} \Psi(3RN)^3 |\tilde{F}|_r$$

and

$$|F'|_{r'} \leq \frac{3}{\kappa^3} e^{|X|_{r'}} |\tilde{F}|_r \big[e^{-2\pi\Lambda(RN)(r-r')} + |\tilde{F}|_r \Psi(3RN)^3 (2e + e^{|X|_{r'}}) \big]$$

and if $ad_{\tilde{A}} = 0$,

$$|X|_{r'} \leq \frac{1}{\kappa} \Psi(3RN) |\tilde{F}|_r$$

and

$$|F'|_{r'} \leq \frac{1}{\kappa} e^{|X|_{r'}} |\tilde{F}|_r \big[e^{-2\pi\Lambda(RN)(r-r')} + |\tilde{F}|_r \Psi(3RN)(2e + e^{|X|_{r'}}) \big].$$

Notice that in any case (since $\Phi \equiv I$ if $\tilde{A}$ is nilpotent or $\mathrm{ad}_{\tilde{A}} = 0$), we have

$$|\Phi X \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} \Psi(3RN)^3 |\Phi \tilde{F} \Phi^{-1}|_r \tag{10.15}$$

and

$$|\Phi F' \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} e^{|\Phi X \Phi^{-1}|_{r'}} |\Phi \tilde{F} \Phi^{-1}|_r \big[e^{-2\pi\Lambda(RN)(r-r')} + |\Phi \tilde{F} \Phi^{-1}|_r \Psi(3RN)^3 (2e + e^{|\Phi X \Phi^{-1}|_{r'}}) \big] \tag{10.16}$$

Estimates

Estimate of Ψ', Ψ'^{-1}, A'

With the assumption $\varepsilon \leq (2C_0)^{-96}\big(\frac{\kappa''}{\|A\|+1}\big)^{576}$, we have

$$|\Phi|_r \leq \varepsilon^{-\frac{1}{96}} e^{2\pi\Lambda(\frac{N}{2})r}$$

and similarly for Φ^{-1}. Moreover since $|\psi|_r \leq \varepsilon^{-\zeta}$, we get property 8:

$$|\psi'|_r = |\psi\Phi|_r \leq |\psi|_r|\Phi|_r \leq \varepsilon^{-\zeta-\frac{1}{96}} e^{2\pi\Lambda(\frac{N}{2})r}$$

and similarly for ψ'^{-1}. Notice that this inequality remains true if $\Phi \equiv id$.

Notice that if $\Phi \not\equiv I$ (that is to say if the spectrum of A is resonant), then from lemma 7.1 we get $\|\tilde{A}\| \leq \frac{1}{2}\kappa''$ and then for $\varepsilon \leq \varepsilon_0$ defined in lemma 10.1 (see equation (10.1)),

$$\begin{aligned}
\|A'\| &\leq \|\tilde{A}\| + \|\hat{\tilde{F}}(0)\| \\
&\leq \frac{1}{2}\kappa\varepsilon^{\zeta} + \varepsilon^{1-2\zeta-\frac{1}{48}} \\
&\leq \frac{1}{2}\kappa\varepsilon^{\frac{1}{1728}} + \varepsilon^{\frac{845}{864}} \\
&\leq \frac{3}{4}\kappa\varepsilon^{\frac{1}{1728}} = \frac{3}{4}\kappa''
\end{aligned}$$

and property 10 is satisfied.

Estimate of $Z'^{\pm 1} - I, \psi^{-1}(Z'^{\pm 1})\psi$ and its derivative

Since $\tilde{F} = (\psi\Phi)^{-1}\bar{F}\psi\Phi$, then

$$|\Phi\tilde{F}\Phi^{-1}|_r = |\psi^{-1}\bar{F}\psi|_{r'} \leq |\bar{F}|_{r'}\varepsilon^{-2\zeta} = \varepsilon^{1-2\zeta} \tag{10.17}$$

Recall the estimate (10.15):

$$|\Phi X\Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}}\Psi(3RN)^3|\Phi\tilde{F}\Phi^{-1}|_r \tag{10.18}$$

therefore by (10.17), and for $\varepsilon \leq \varepsilon_0$ defined in lemma 10.1 ((see equation (10.2)),

$$|\Phi X\Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa^{13}}\varepsilon^{-16\zeta}\varepsilon^{1-2\zeta} \leq \varepsilon^{\frac{7}{8}} \tag{10.19}$$

then

$$e^{|\Phi X\Phi^{-1}|_{r'}} \leq e^{\varepsilon^{\frac{7}{8}}} \leq 2$$

We now estimate $|Z' - I|_{r'} = |\psi\Phi(e^X - I)(\psi\Phi)^{-1}|_{r'}$. From (10.19),

$$|\Phi e^X\Phi^{-1} - Id|_{r'} \leq e|\Phi X\Phi^{-1}|_{r'} \leq e\varepsilon^{\frac{7}{8}}$$

Then

$$|Z' - I|_{r'} = |\psi\Phi e^X(\psi\Phi)^{-1} - Id|_{r'} \leq |\psi|_{r'}|\Phi e^X\Phi^{-1} - Id|_{r'}|\psi^{-1}|_{r'} \leq e\varepsilon^{\frac{7}{8}-2\zeta}$$

hence property 6 is satisfied. If $\Phi \equiv I$, we have

$$\psi^{-1}Z'\psi = \psi^{-1}\psi\Phi e^X(\psi\Phi)^{-1}\psi = e^X,$$

therefore

$$|\psi^{-1}Z'\psi|_{r'} \leq |e^X|_{r'} \leq e^{\varepsilon^{\frac{7}{8}}}$$

which is the first part of the property 11. Now Lemma 9.3 also states that if $\Phi \equiv I$ (that is, $\tilde{A} = A$),

$$|\partial_\omega X|_{r'} \leq 2\|A\| \ |X|_{r'} + |\tilde{F}|_{r'}$$

which implies that

$$|\partial_\omega X|_{r'} \leq 2\|A\|\varepsilon^{\frac{7}{8}} + \varepsilon^{1-2\zeta} \leq \varepsilon^{\frac{7}{8}}(2\|A\| + 1)$$

and by the assumption 1,

$$|\partial_\omega X|_{r'} \leq \varepsilon^{\frac{4}{5}}.$$

Therefore,

$$|\partial_\omega(\psi^{-1}Z'\psi)|_{r'} = |\partial_\omega(X)e^X|_{r'} \leq e^{\varepsilon^{\frac{7}{8}}}\varepsilon^{\frac{4}{5}} \leq \varepsilon^{\frac{1}{2}}$$

hence property 11.

Estimate of $\psi^{-1}\bar{F}'\psi = \Phi F'\Phi^{-1}$

From Equation (10.16),

$$|\Phi F'\Phi^{-1}|_{r'} \leq 4C_0^2 e^{|\Phi X\Phi^{-1}|_{r'}}\big(\frac{C_0}{\kappa''}\big)^{13}|\Phi\tilde{F}\Phi^{-1}|_r\big[e^{-2\pi\Lambda(RN)(r-r')} + |\Phi\tilde{F}\Phi^{-1}|_r\Psi(3RN)^3(2e + e^{|\Phi X\Phi^{-1}|_{r'}})\big]$$

Moreover, by definition, we have $\Lambda(RN) = \frac{50\delta|\log\varepsilon|}{\pi(r-r')}$, thus

$$e^{-2\pi\Lambda(RN)(r-r')} = \varepsilon^{100\delta}$$

and then, because we assumed $\varepsilon \leq (2C_0)^{-96}\big(\frac{\kappa''}{\|A\|+1}\big)^{576}$ and $\Psi(3RN) = \varepsilon^{-\zeta}$,

$$|\Phi F'\Phi^{-1}|_{r'} \leq 8C_0^2\varepsilon^{-\frac{1}{96}}\varepsilon^{1-2\zeta}(\varepsilon^{100\delta} + 8\Psi(3RN)^3\varepsilon^{1-2\zeta}).$$

Thus

$$|\Phi F'\Phi^{-1}|_{r'} \leq 8C_0^2\varepsilon^{1-2\zeta-\frac{1}{96}}(\varepsilon^{100\delta} + \varepsilon^{1-6\zeta}) \leq \varepsilon^{\frac{3}{2}}.$$

Hence property 9 holds for $\varepsilon \leq \varepsilon_0$ as defined in lemma 10.1 (see equation (10.3)).

□

11 Inductive step

Let's define the following functions which will be used for the complete iterative step :

$$\begin{cases} \kappa''(\varepsilon) = \kappa\varepsilon^\zeta \\ N(r,\varepsilon) = \Lambda^{-1}\big(\frac{50|\log\varepsilon|}{\pi r}\big) \\ R(r,\varepsilon) = \dfrac{1}{3N(r,\varepsilon)}\Psi^{-1}(\varepsilon^{-\zeta}) \\ r''(r,\varepsilon) = r - \dfrac{50\delta|\log\varepsilon|}{\pi\Lambda(R(r,\varepsilon)N(r,\varepsilon))} \end{cases} \tag{P}$$

Note that these definitions match with lemma 10.2.

Lemma 11.1. *Let*

- $A \in sl(2,\mathbb{R})$,
- $r > 0$,
- $\bar{A}, \bar{F} \in U_r(\mathbb{T}^d, sl(2,\mathbb{R})), \psi \in U_r(2\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $|\bar{F}|_r = \varepsilon$.

Suppose that

1. $\varepsilon \leq \varepsilon_0$, *where* ε_0 *is defined in Lemma 10.1,*
2. $r'' > 0$,
3. $\bar{A}$ *is reducible to* A *by* ψ,
4. *for all* $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi^{-1}G\psi \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
5. $|\psi^{\pm 1}|_r \leq \varepsilon^{-\zeta}$,
6. $\|A\| \leq \varepsilon^{-\frac{\zeta}{2}}$,

then, there exist

- $Z' \in U_{r''}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $\bar{A}', \bar{F}' \in U_{r''}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $\psi' \in U_r(2\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $A' \in sl(2,\mathbb{R})$

satisfying the following properties:

1. $\bar{A}'$ *is reducible to* A' *by* ψ',
2. *for all* $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi^{-1}G\psi \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
3. $|\bar{F}'|_{r''} \leq \varepsilon^{2\delta}$,
4. $|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-2\delta\zeta}$,
5. $\|A'\| \leq \|A\| + \varepsilon^{-\zeta} \leq \varepsilon^{-(2\delta)\frac{\zeta}{2}}$,
6. $$\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}'),$$
7. $$|Z'^{\pm 1} - Id|_{r''} \leq \varepsilon^{\frac{9}{10}}.$$
8. *If moreover the spectrum of* A *was not* $BR_\omega^{R(r,\varepsilon)N(r,\varepsilon)}(\kappa''(\varepsilon))$, *we actually have* $\|A'\| \leq \kappa''(\varepsilon)$;
9. *If the spectrum of* A *was* $BR_\omega^{R(r,\varepsilon)N(r,\varepsilon)}(\kappa''(\varepsilon))$, *we actually have* $\psi' = \psi$ *and then*
$$|\psi^{-1}Z'^{\pm 1}\psi|_{r''} \leq (1+2\varepsilon)e^{2\varepsilon} \tag{11.1}$$
and
$$|\partial_\omega(\psi^{-1}Z'^{\pm 1}\psi)|_{r''} \leq \varepsilon^{\frac{1}{4}}.$$

Proof. **Removing the resonances and first step**

Let $R = R(r,\varepsilon)$, $N = N(r,\varepsilon)$, $\kappa'' = \kappa''(r,\varepsilon)$, $r'' = r''(r,\varepsilon)$. Since $\kappa'' = \kappa\varepsilon^\zeta$, $\|A\| \leq \varepsilon^{-\frac{\zeta}{2}}$ and $\varepsilon \leq \varepsilon_0$ as defined in Lemma 10.1 (see equation (10.4)),

$$\varepsilon^{1-576\zeta} \leq (2C_0)^{-96}\big(\frac{\kappa}{32(\|A\|+1)}\big)^{576}$$

therefore

$$\varepsilon \leq (2C_0)^{-96}\Big(\frac{\kappa''}{32(\|A\|+1)}\Big)^{576}$$

and the assumption of lemma 10.2 is satisfied. We can apply lemma 10.2 to get:

- $Z_1 \in U_{\frac{r+r''}{2}}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $\psi' \in U_{\frac{r+r''}{2}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $A_1 \in sl(2,\mathbb{R})$,
- $\bar{A}_1 \in U_{\frac{r+r''}{2}}(\mathbb{T}^d, sl(2,\mathbb{R}))$,

- and $F_1 = (\psi')^{-1}\bar{F}_1\psi'$, with $\bar{F}_1 \in U_{\frac{r+r''}{2}}(\mathbb{T}^d, sl(2,\mathbb{R}))$

such that

1. $\bar{A}_1$ is reducible to A_1 by ψ',
2. for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi'^{-1}G\psi' \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, which implies that $F_1 \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
3. A_1 has $BR^{RN}_\omega(\frac{3}{4}\frac{\kappa''}{C_0})$ spectrum,
4. $$\partial_\omega Z_1 = (\bar{A}+\bar{F})Z_1 - Z_1(\bar{A}_1+\bar{F}_1),$$
5. $$\|A_1\| \leq \|A\| + \varepsilon^{\frac{23}{24}} + \pi N, \tag{11.2}$$
6. $$|Z_1^{\pm 1} - Id|_{r''} \leq \varepsilon^{\frac{8}{9}}, \tag{11.3}$$
7. for all $s' > 0$,
$$|\psi'^{-1}\psi|_{s'} \leq 2C_0\big(\frac{1}{\kappa''}\big)^6 e^{2\pi\Lambda(\frac{N}{2})s'}$$
$$|\psi^{-1}\psi'|_{s'} \leq 2C_0\big(\frac{1}{\kappa''}\big)^6 e^{2\pi\Lambda(\frac{N}{2})s'}$$
8. $|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-\zeta-\frac{1}{96}} e^{2\pi\Lambda(\frac{N}{2})r}$,
9. $$|\psi^{-1}\bar{F}_1\psi|_{r''} \leq \varepsilon^{\frac{5}{4}},$$
10. If the spectrum of A was not $BR^{RN}_\omega(\kappa'')$, $\|A_1\| \leq \frac{1}{2}\kappa''$;
11. If the spectrum of A was $BR^{RN}_\omega(\kappa'')$, we actually have $\psi' = \psi$ and then
$$|\psi^{-1}Z_1^{\pm 1}\psi|_{r''} \leq e^{\varepsilon^{\frac{7}{8}}}$$
$$|\partial_\omega(\psi^{-1}Z_1^{\pm 1}\psi)|_{r''} \leq \varepsilon^{\frac{1}{2}}. \tag{11.4}$$

Second step : iteration without resonances

We will now iterate lemma 9.3 a certain number of times, without renormalization.
Let $l = E(\frac{\log(100\delta)}{\log(\frac{4}{3})}) = 56$ which satisfies

$$\varepsilon^{(\frac{4}{3})^{l+1}} \leq e^{-2\pi\Lambda(RN)(r-r'')} = \varepsilon^{100\delta} \leq \varepsilon^{(\frac{4}{3})^l}.$$

Define for all $j \geq 0$, the sequences $\varepsilon'_j = \varepsilon^{(\frac{5}{4})^j}\varepsilon^{-\frac{1}{48}}$ and $r'_j = \frac{r+r''}{2} - j\frac{r-r''}{2l}$. Thus $r'_0 = \frac{r+r''}{2}$ and $r'_l = r'' < r$.
We want to iterate $l-1$ times lemma 9.3, from $j = 2$, with

- $\tilde{\varepsilon} = \varepsilon'_{j-1}$,
- $\tilde{r} = r'_{j-2}$,
- $\tilde{r}' = r'_{j-1}$,
- $\kappa' = (\frac{3}{4})^{j-1}\frac{\kappa''}{C_0}$,
- $\tilde{N} = RN$,
- $\tilde{F} = F_{j-1}$,
- $\tilde{A} = A_{j-1}$,
- $\Phi = \psi^{-1}\psi'$,

First iterate of lemma 9.3: From

$$|\psi^{-1}\bar{F}_1\psi|_0 \leq \varepsilon^{\frac{5}{4}}$$

and

$$|\psi'^{-1}\psi|_0 \leq 2C_0\big(\frac{1}{\kappa''}\big)^6 \leq \varepsilon^{-\frac{1}{96}},\ |\psi^{-1}\psi'|_0 \leq 2C_0\big(\frac{1}{\kappa''}\big)^6 \leq \varepsilon^{-\frac{1}{96}},$$

then

$$\|\hat{F}_1(0)\| \leq |\psi'^{-1}\psi|_0|\psi^{-1}\psi'|_0|\psi^{-1}\bar{F}_1\psi|_0 \leq \varepsilon^{\frac{5}{4}-\frac{1}{48}}.$$

As A_1 has $BR^{RN}_\omega(\frac{3}{4}\frac{\kappa''}{C_0})$ spectrum, to apply lemma 9.3 we need

$$\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \big(\frac{(\frac{3}{4})\frac{\kappa''}{C_0}}{32(1+\|A_1\|)}\big)^2\frac{1}{\Psi(RN)^2}$$

and since

$$\|A_1\| \leq \|A\| + \varepsilon^{\frac{23}{24}} + \pi N \leq \varepsilon^{-\frac{\zeta}{2}} + \varepsilon^{\frac{23}{24}} + \pi\varepsilon^{-\zeta}$$

(this last inequality comes from the fact that $\Psi \geq id$, which implies that $N = \frac{1}{3R}\Psi^{-1}(\varepsilon^{-\zeta}) \leq \varepsilon^{-\zeta}$), this remains true if

$$\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \big(\frac{(\frac{3}{4})\frac{\kappa''}{C_0}}{32(1+(\pi+1)\varepsilon^{-\zeta}+\varepsilon^{\frac{23}{24}})}\big)^2\frac{1}{\Psi(RN)^2}$$

which holds for $\varepsilon \leq \varepsilon_0$ as in lemma 10.1 (see equation (10.5)).

Iteration of lemma 9.3
If for some $j \geq 2$

$$\varepsilon^{(\frac{5}{4})^j-\frac{1}{48}} \leq \Big(\frac{(\frac{3}{4})^j\frac{\kappa''}{C_0}}{32(1+\varepsilon^{\frac{23}{24}}+(1+\pi)\varepsilon^{-\zeta}+2\varepsilon^{\frac{5}{4}-\frac{1}{48}})}\Big)^2\frac{1}{\Psi(RN)^2},$$

which holds true for $\varepsilon \leq \varepsilon_0$ as in lemma 10.1 (see equation (10.6)), then

$$\varepsilon'_j \leq \Big(\frac{(\frac{3}{4})^j\frac{\kappa''}{C_0}}{32(1+\|A_1\|+\sum_{i=1}^{j-1}\varepsilon_i)}\Big)^2\frac{1}{\Psi(RN)^2}.$$

Let $j \geq 2$ and assume that A_{j-1} has $BR^{RN}_\omega((\frac{3}{4})^{j-1}\frac{\kappa''}{C_0})$ spectrum, $F_{j-1} \in U_{r'_{j-2}}(\mathbb{T}^d, sl(2,\mathbb{R}))$, and

$$\|\hat{F}_{j-1}(0)\| \leq \varepsilon'_{j-1};\ |\Psi^{-1}\Psi'F_{j-1}\Psi'^{-1}\Psi|_{r'_{j-2}} \leq \varepsilon^{(\frac{5}{4})^{j-1}}.$$

We obtain via lemma 9.3 functions F_j, $X_j \in U_{r_{j-1}}(\mathbb{T}^d, sl(2,\mathbb{R}))$ and a matrix $A_j \in sl(2,\mathbb{R})$ such that

1. A_j has $BR^{RN}_\omega((\frac{3}{4})^j\frac{\kappa''}{C_0})$ spectrum,
2. $\|A_j\| \leq \|A_{j-1}\| + \varepsilon'_{j-1}$,
3. $$\partial_\omega e^{X_j} = (A_{j-1}+F_{j-1})e^{X_j} - e^{X_j}(A_j+F_j),$$
4. the following estimates hold:
   - if A_{j-1} has two different eigenvalues:

$$|\psi^{-1}\psi'X_j\psi'^{-1}\psi|_{r'_{j-1}} \leq 4C_0^2\big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa''}{C_0}}\big)^{13}\Psi(3RN)|\psi^{-1}\psi'F_j\psi'^{-1}\psi|_{r'_{j-1}},$$

$$|\psi^{-1}\psi'F_j\psi'^{-1}\psi|_{r'_{j-1}} \leq 4C_0^2\big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa''}{C_0}}\big)^{13}e^{|\psi^{-1}\psi'X_{j-1}\psi'^{-1}\psi|_{r'_{j-1}}}|\psi^{-1}\psi'F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}$$
$$\big[e^{-2\pi\Lambda(RN)(r'_{j-2}-r'_{j-1})} + |\psi^{-1}\psi'F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}\Psi(3RN)(2e+e^{|\psi^{-1}\psi'X_{j-1}\psi'^{-1}\psi|_{r'_{j-1}}})\big];$$

- if A_{j-1} is nilpotent:

$$|X_j|_{r'_{j-1}} \leq \frac{3}{\kappa^3}\Psi(3RN)^3|F_{j-1}|_{r'_{j-1}},$$

$$|F_j|_{r'_{j-1}} \leq \frac{3}{\kappa^3}e^{|X_{j-1}|_{r'_{j-1}}}|F_{j-1}|_{r'_{j-2}}\big[e^{-2\pi\Lambda(RN)(r'_{j-2}-r'_{j-1})} + |F_{j-1}|_{r'_{j-2}}\Psi(3RN)^3(2e+e^{|X_{j-1}|_{r'_{j-1}}})\big].$$

- if $ad_{A_{j-1}} = 0$:

$$|X_j|_{r'_{j-1}} \leq \frac{1}{\kappa}\Psi(3RN)|F_{j-1}|_{r'_{j-1}},$$

$$|F_j|_{r'_{j-1}} \leq \frac{1}{\kappa}e^{|X_{j-1}|_{r'_{j-1}}}|F_{j-1}|_{r'_{j-2}}\big[e^{-2\pi\Lambda(RN)(r'_{j-2}-r'_{j-1})} + |F_{j-1}|_{r'_{j-2}}\Psi(3RN)(2e+e^{|X_{j-1}|_{r'_{j-1}}})\big].$$

Notice that in any case we have

$$|\psi^{-1}\psi' X_j \psi'^{-1}\psi|_{r'_{j-1}} \leq 4C_0^2\big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa''}{C_0}}\big)^{13}\Psi(3RN)|\psi^{-1}\psi' F_j \psi'^{-1}\psi|_{r'_{j-1}}, \tag{11.5}$$

$$\begin{aligned}|\psi^{-1}\psi' F_j \psi'^{-1}\psi|_{r'_{j-1}} \leq 4C_0^2\big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa''}{C_0}}\big)^{13}e^{|\psi^{-1}\psi' X_{j-1}\psi'^{-1}\psi|_{r'_{j-1}}}|\psi^{-1}\psi' F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}\\ \big[e^{-2\pi\Lambda(RN)(r'_{j-2}-r'_{j-1})} + 8|\psi^{-1}\psi' F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}\Psi(3RN)\big],\end{aligned} \tag{11.6}$$

so we will use these estimates to iterate lemma 9.3.

Estimates $\varepsilon \leq (2C_0)^{-96}\Big(\frac{\kappa''}{32(\|A\|+1)}\Big)^{576}$ and $|\psi^{-1}\psi' F_{j-1}\psi'^{-1}\psi|_{r_{j-2}} \leq \varepsilon^{(\frac{5}{4})^{j-1}}$ give

$$e^{|\psi^{-1}\psi' X_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}} \leq 2$$

Hence from (11.6)

$$\begin{aligned}|\psi^{-1}\psi' F_j \psi'^{-1}\psi|_{r'_{j-1}} &\leq 64\cdot 4C_0^2\Psi(3RN)\big(\frac{\Psi(3RN)}{(\frac{3}{4})^{j-1}\frac{\kappa}{C_0}}\big)^{13}|\psi^{-1}\psi' F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}\big[e^{-2\pi\Lambda(RN)(r'_{j-2}-r'_{j-1})}\\ &\quad +|\psi^{-1}\psi' F_{j-1}\psi'^{-1}\psi|_{r'_{j-2}}\big]\\ &\leq 256C_0^2\Psi(3RN)^{14}\big(\frac{1}{(\frac{3}{4})^{j-1}\frac{\kappa}{C_0}}\big)^{13}\varepsilon^{(\frac{5}{4})^{j-1}}(\varepsilon^{\frac{50\delta}{l}} + \varepsilon^{(\frac{5}{4})^{j-1}})\end{aligned}$$

Since $\Psi(3RN) = \varepsilon^{-\zeta}$ and $\varepsilon \leq \varepsilon_0$ defined in lemma 10.1 ((see equation (10.7)),

$$|\psi^{-1}\psi' F_j \psi'^{-1}\psi|_{r'_{j-1}} \leq \varepsilon^{(\frac{5}{4})^j} \tag{11.7}$$

We will now estimate $\|\hat{F}_j(0)\|$ to iterate lemma 9.3:

$$\|\hat{F}_j(0)\| \leq |F_j|_0 = |(\psi'^{-1}\psi)\psi^{-1}\psi' F_j \psi'^{-1}\psi(\psi^{-1}\psi')|_0 \leq |\psi^{-1}\psi'|_0|\psi'^{-1}\psi|_0|\psi^{-1}\psi' F_j\psi'^{-1}\psi|_{r_{j-1}},$$

therefore

$$\|\hat{F}_j(0)\| \leq |\psi^{-1}\psi'|_0|\psi'^{-1}\psi|_0\varepsilon^{(\frac{5}{4})^j} \leq \varepsilon^{(\frac{5}{4})^j}\varepsilon^{-\frac{1}{48}} = \varepsilon'_j,$$

and we can iterate lemma 9.3, $l-1$ times.

Equations (11.7) and (11.5) imply that

$$|\psi^{-1}\psi' X_j \psi'^{-1}\psi|_{r_{j-1}} \leq \varepsilon'_j \tag{11.8}$$

and

$$e^{|\psi^{-1}\psi' X_j\psi'^{-1}\psi|_{r_{j-1}}} \leq 2 \qquad .$$

Conclusion :

Let $Z = e^{X_2}...e^{X_{l+1}} \in U_{r''}(\mathbb{T}^d, SL(2,\mathbb{R}))$.
Let $Z' = Z_1\psi' Z\psi'^{-1}$, $A' = A_{l+1}$, $F' = F_{l+1}$, $\bar{F}' = \psi' F' \psi'^{-1}$ (hence Property 2) and $\bar{A}'$ such that

$$\partial_\omega \psi' = \bar{A}'\psi' - \psi' A',$$

then

$$\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}'),$$

hence the properties 1 and 6 hold.

We have

$$\partial_\omega Z = (A_1 + F_1)Z - Z(A_{l+1} + F_{l+1})$$

and since for all $j \geq 2$, we have

$$\|A_j\| \leq \|A_{j-1}\| + \varepsilon^{(\frac{5}{4})^{j-1}}\varepsilon^{-\frac{1}{48}} \leq \|A_1\| + \sum_{i=1}^{j-1} \varepsilon^{(\frac{5}{4})^i}\varepsilon^{-\frac{1}{48}} = \|A_1\| + \sum_{i=1}^{j-1} \varepsilon_i', \tag{11.9}$$

then

$$\|A'\| \leq \|A_1\| + \sum_{i=1}^{l} \varepsilon_i \leq \|A\| + \varepsilon^{\frac{23}{24}} + \pi N + \sum_{i=1}^{l} \varepsilon^{(\frac{5}{4})^i - \frac{1}{48}}.$$

Remind that $\Psi \geq id$ implies

$$N \leq \varepsilon^{-\frac{\zeta}{2}}$$

and then, since $\|A\| \leq \varepsilon^{\frac{-\zeta}{2}}$,

$$\|A'\| \leq \|A\| + \varepsilon^{-\zeta} \leq 2\varepsilon^{-\zeta} \leq \varepsilon^{-\delta\zeta}$$

thus the property 5 holds if $\varepsilon \leq \varepsilon_0$ as defined in lemma 10.1 (see equation (10.8)).

Moreover,

$$|\psi^{-1}\psi' F_{l+1} \psi'^{-1}\psi|_{r_l'} \leq \varepsilon^{(\frac{5}{4})^{l+1}} \tag{11.10}$$

and since $l = 56$, one has

$$|\psi' F_{l+1}\psi'^{-1}|_{r_l'} \leq |\psi|_{r_l'}|\psi^{-1}|_{r_l'}\varepsilon^{(\frac{5}{4})^{l+1}} \leq \varepsilon^{(\frac{5}{4})^{57}-2\zeta} \leq \varepsilon^{2\delta}$$

thus the property 3 holds. In the case the spectrum of A was resonant, the function Φ used in lemma 10.2 is not the identity and we have

$$\|A'\| \leq \|A_1\| + \sum_{i=1}^{l} \varepsilon_i' \leq \frac{1}{2}\kappa''(r,\varepsilon) + 2\varepsilon_1' \leq \frac{1}{2}\kappa\varepsilon^{\zeta} + 2\varepsilon^{\frac{5}{4}-\frac{1}{48}} \leq \kappa\varepsilon^{\zeta} = \kappa''(r,\varepsilon)$$

since $\varepsilon \leq \varepsilon_0$ as defined in lemma 10.1 (see equation (10.9)), whence the property 8.

Estimates

Now we will show property 4 : $|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-2\delta\zeta}$.
We know that $|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-\zeta-\frac{1}{96}} e^{2\pi\Lambda(\frac{N}{2})r}$. But, by definition of $\Lambda(N) = \frac{50|\log\varepsilon|}{\pi r}$,

$$e^{2\pi\Lambda(\frac{N}{2})r} \leq e^{2\pi\Lambda(N)r} \leq e^{100|\log\varepsilon|} = \varepsilon^{-100}$$

therefore

$$|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-\zeta-\frac{1}{96}-100} \leq \varepsilon^{-2\zeta\delta}.$$

which is Property 4. Now we will show the property 3. One has

$$|\bar{F}'|_{r''} = |\psi' F' \psi'^{-1}|_{r''} = |\psi\psi^{-1}\psi' F' \psi'^{-1}\psi\psi^{-1}|_{r''} \leq |\psi|_r|\psi^{-1}|_r|\psi^{-1}\psi' F' \psi'^{-1}\psi|_{r''} \leq \varepsilon^{-2\zeta}\varepsilon^{(\frac{5}{4})^l} \leq \varepsilon^{2\delta}$$

where the last inequality uses equation (11.10), which gives property 3.

According to the estimate

$$|Z_1^{\pm 1} - Id|_{r''} \leq \varepsilon^{\frac{8}{9}}$$

obtained in (11.3), we get

$$|Z' - Id|_{r''} \leq |Z_1 - Id|_{r'_1} + |Z_1|_{r'_1}|\psi|_r|\psi^{-1}|_r \sum_{j=2}^{l+1} |\psi^{-1}\psi' X_j \psi'^{-1}\psi|_{r'_j}$$

$$\leq \varepsilon^{\frac{8}{9}} + (\varepsilon^{\frac{8}{9}} + 1)\varepsilon^{-2\zeta} \sum_{j=2}^{l+1} |\psi^{-1}\psi' X_j \psi'^{-1}\psi|_{r'_j}$$

Therefore, by the estimate (11.8) and by definition of $l = E(\frac{\log(100\delta)}{\log(\frac{7}{6})})$,

$$|Z' - Id|_{r''} \leq \varepsilon^{\frac{8}{9}} + (\varepsilon^{\frac{8}{9}} + 1)2\varepsilon'_1\varepsilon^{-2\zeta} \leq \varepsilon^{\frac{9}{10}}$$

hence 7.

Proof of the property 9

We now have to estimate $\psi^{-1}Z'\psi$ and its directional derivative in the case A has a $BR_\omega^{R(r,\varepsilon)N(r,\varepsilon)}(\kappa''(\varepsilon))$ spectrum. In this case $\Phi \equiv I$ and $\psi = \psi'$, therefore $\psi^{-1}Z'\psi = \psi^{-1}Z_1\psi Z$. Therefore

$$|\psi^{-1}Z'\psi|_{r''} \leq |\psi^{-1}Z_1\psi|_{r''}|Z|_{r''} \leq (1 + 2\varepsilon)|Z|_{r''}.$$

(where the last inequality comes from (11.3)). Moreover,

$$|Z|_{r''} = |\Pi_{k=2}^{l+1} e^{X_k}|_{r''}.$$

Now for all $k \in [\![2, l]\!]$, we have seen in (11.8) that $|X_k|_{r''} \leq \varepsilon'_k$.
Therefore

$$|Z|_{r''} \leq |\Pi_{k=2}^{l+1} e^{X_k}|_{r''} \leq e^{\sum_{k=2}^{l+1} \varepsilon'_k} \leq e^{2\varepsilon}$$

and finally,

$$|\psi^{-1}Z'\psi|_{r''} \leq (1 + 2\varepsilon)e^{2\varepsilon}.$$

The estimate of $\psi^{-1}Z'^{-1}\psi$ is obtained in a similar way. This gives the property (11.1).

Moreover,

$$|\partial_\omega(\psi^{-1}Z'\psi)|_{r''} \leq |\partial_\omega(\psi^{-1}Z_1\psi)Z|_{r''} + |\psi^{-1}Z_1\psi\partial_\omega(Z)|_{r''} \leq \varepsilon^{\frac{1}{2}}|Z|_{r''} + (1 + 2\varepsilon)|\partial_\omega(Z)|_{r''}$$

where the last inequality comes from (11.4). Now

$$|\partial_\omega(Z)|_{r''} \leq \sum_{k=2}^{l+1} |\partial_\omega X_k|_{r''} \prod_{j=2}^{l+1} e^{|X_j|_{r''}}.$$

For all $k \in [\![2, l+1]\!]$, by construction of X_k,

$$|\partial_\omega X_k|_{r''} \leq 2\|A_k\||X_k|_{r''} + |F_k|_{r''} \leq 2\|A_k\||X_k|_{r''} + \varepsilon^{(\frac{5}{4})^k - \frac{1}{48}}.$$

Now for all $k \in [\![2, l+1]\!]$, by the estimate (11.9), the estimate (11.2) and condition 3 of this lemma, for $\varepsilon \leq \varepsilon_0$ given by lemma 10.1 (see equation (10.10)),

$$\|A_k\| \leq \varepsilon^{-\frac{\zeta}{2}} + \varepsilon^{\frac{23}{24}} + \pi\varepsilon^{-\zeta} \leq \varepsilon^{-2\zeta}$$

therefore for $\varepsilon \leq \varepsilon_0$ given by lemma 10.1 (see equation (10.11)),

$$\sum_{k=2}^{l=1} |\partial_\omega X_k|_{r''} \leq 2\sum_{k=2}^{l+1} \|A_k\||X_k|_{r''} + \sum_{k=2}^{l+1} |F_k|_{r''}$$
$$\leq 2\sum_{k=2}^{l+1} \varepsilon^{-2\zeta}\varepsilon'_k + \sum_{k=2}^{l+1} \varepsilon'_k$$
$$\leq \varepsilon$$

and finally, for $\varepsilon \leq \varepsilon_0$ like in lemma 10.1 (see equation (10.12)),

$$|\partial_\omega(\psi^{-1}Z'\psi)|_{r''} \leq 2\varepsilon^{\frac{1}{2}} + 2\varepsilon^{\frac{7}{8}} \leq \varepsilon^{\frac{1}{4}}.$$

The estimate of $\partial_\omega(\psi^{-1}Z'^{-1}\psi)$ is similar, which gives property 9. □

12 Almost reducibility

Here we complete the proof of the main theorem.

Theorem 12.1. *Let $r_0 > 0$, $A \in sl(2,\mathbb{R})$ and $F \in U_{r_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$. Then, if*

$$|F|_r \leq \varepsilon_0$$

where ε_0 satisfies the assumptions above, and

$$\|A\| \leq \varepsilon_0^{-\frac{\zeta}{2}},$$

then for all $\varepsilon \leq \varepsilon_0$, there exist

- $r_\varepsilon > 0$, $k_\varepsilon \in \mathbb{N}$,
- $Z_\varepsilon \in U_{r_\varepsilon}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $A_\varepsilon \in sl(2,\mathbb{R})$,
- $\bar{A}_\varepsilon, \bar{F}_\varepsilon \in U_{r_\varepsilon}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $\psi_\varepsilon \in U_{r_\varepsilon}(2\mathbb{T}^d, SL(2,\mathbb{R}))$,

such that

1. *$\bar{A}_\varepsilon$ is reducible to A_ε by ψ_ε, with $|\psi_{k_\varepsilon}|_{r_\varepsilon} \leq \varepsilon^{-\zeta}$,*
2. *$|\bar{F}_\varepsilon|_{r_\varepsilon} \leq \varepsilon$,*
3. *for all $\theta \in \mathbb{T}^d$,*
$$\partial_\omega Z_\varepsilon(\theta) = (A + F(\theta))Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(\bar{A}_\varepsilon(\theta) + \bar{F}_\varepsilon(\theta))$$
4. $$|Z_\varepsilon^{\pm 1} - Id|_{r_\varepsilon} \leq \varepsilon_0^{\frac{9}{10}}.$$

Moreover, either $|\partial_\omega Z_\varepsilon|_{r_\varepsilon}$ is bounded as $\varepsilon \to 0$ and $A + F$ is a reducible cocycle in $U_{r_\infty}(\mathbb{T}^d, sl(2,\mathbb{R}))$ for some $r_\infty > 0$, or for all $\varepsilon \leq \varepsilon_0$ there exists $\varepsilon' \leq \varepsilon$ such that

$$\|A_{\varepsilon'}\| \leq \kappa\varepsilon'^{\zeta}.$$

Proof. Remind parameters (P) defined in section 11 and define, for all $k \in \mathbb{N}, k \geq 1$,

$$\varepsilon_k := \varepsilon_0^{(2\delta)^k}; \quad r_k := r_0 - \sum_{i=0}^{k-1} \frac{50\delta|\log\varepsilon_i|}{\pi\Lambda(R(r_i,\varepsilon_i)N(r_i,\varepsilon_i))}.$$

Notice that, by Lemma 5.1, under assumption 1, for all $k \in \mathbb{N}$, $r_k > 0$.

We can apply a first time lemma 11.1. There exist

- $Z_1 \in U_{r_1}(\mathbb{T}^d, SL(2,\mathbb{R}))$,

- $\bar{A}_1$, $\bar{F}_1 \in U_{r_1}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $A_1 \in sl(2,\mathbb{R})$,
- $\psi_0 \in U_{r_1}(2\mathbb{T}^d, SL(2,\mathbb{R}))$,

such that

- $\bar{A}_1$ is reducible to A_1 by ψ_0,
- for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi_0^{-1} G \psi_0 \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $|\bar{F}_1|_{r_1} \leq \varepsilon_1$,
- $|\psi_0^{\pm 1}|_{r_1} \leq \varepsilon_1^{-\zeta}$,
- $\|A_1\| \leq \varepsilon_1^{-\frac{\zeta}{2}}$,
- for all $\theta \in \mathbb{T}^d$

$$\partial_\omega Z_1(\theta) = (A + F(\theta)) Z_1(\theta) - Z_1(\theta)(\bar{A}_1(\theta) + \bar{F}_1(\theta)),$$

- $$|Z_1^{\pm 1} - Id|_{r_1} \leq \varepsilon_0^{\frac{9}{10}},$$
- if moreover A had a $BR_\omega^{R(r_0,\varepsilon_0)N(r_0,\varepsilon_0)}(\kappa''(\varepsilon_0))$ spectrum,

$$|\psi_0^{-1} Z_1 \psi_0|_{r_1} \leq (1 + 2\varepsilon_0) e^{2\varepsilon_0} \tag{12.1}$$

and if not,

$$\|A_1\| \leq \kappa''(\varepsilon_0)$$

and

$$|\partial_\omega(\psi_0^{-1} Z_1 \psi_0)|_{r_1} \leq \varepsilon_0^{\frac{1}{4}}. \tag{12.2}$$

Iterative step : let $k \geq 1$ and

- $\bar{A}_k, \bar{F}_k \in U_{r_k}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $A_k \in sl(2,\mathbb{R})$,
- $\psi_{k-1} \in U_{r_k}(2\mathbb{T}^d, SL(2,\mathbb{R}))$,

such that

- $\bar{A}_k$ is reducible to A_k by ψ_{k-1},
- for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi_{k-1}^{-1} G \psi_{k-1} \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $|\bar{F}_k|_{r_k} \leq \varepsilon_k$,
- $|\psi_{k-1}^{\pm 1}|_{r_k} \leq \varepsilon_k^{-\zeta}$,
- $\|A_k\| \leq \varepsilon_k^{-\frac{\zeta}{2}}$.

We can one again apply lemma 11.1 to get

- $Z_{k+1} \in U_{r_{k+1}}(\mathbb{T}^d, SL(2,\mathbb{R}))$,
- $\bar{A}_{k+1}$, $\bar{F}_{k+1} \in U_{r_{k+1}}(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $A_{k+1} \in sl(2,\mathbb{R})$,
- $\psi_k \in U_{r_{k+1}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$,

such that

- $\bar{A}_{k+1}$ is reducible to A_{k+1} by ψ_k,
- for all $G \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$, $\psi_k^{-1} G \psi_k \in \mathcal{C}^0(\mathbb{T}^d, sl(2,\mathbb{R}))$,
- $|\bar{F}_{k+1}|_{r_{k+1}} \leq \varepsilon_{k+1}$,
- $|\psi_k^{\pm 1}|_{r_{k+1}} \leq \varepsilon_{k+1}^{-\zeta}$,
- $\|A_{k+1}\| \leq \varepsilon_{k+1}^{-\frac{\zeta}{2}}$,
- for all $\theta \in \mathbb{T}^d$,

$$\partial_\omega Z_{k+1}(\theta) = (\bar{A}_k(\theta) + \bar{F}_k(\theta)) Z_1(\theta) - Z_{k+1}(\theta)(\bar{A}_{k+1}(\theta) + \bar{F}_{k+1}(\theta)),$$

- $$|Z_{k+1}^{\pm 1} - Id|_{r_{k+1}} \leq \varepsilon_k^{\frac{9}{10}}$$
- if moreover A_k had a $BR_\omega^{R(r_k,\varepsilon_k)N(r_k,\varepsilon_k)}(\kappa''(\varepsilon_k))$ spectrum,

$$|\psi_{k+1}^{-1} Z_{k+1} \psi_{k+1}|_{r_{k+1}} \leq (1 + 2\varepsilon_k) e^{2\varepsilon_k} \tag{12.3}$$

and if not,

$$||A_{k+1}|| \leq \kappa''(\varepsilon_k)$$

and

$$|\partial_\omega(\psi_{k+1}^{-1} Z_{k+1} \psi_{k+1})|_{r_{k+1}} \leq \varepsilon_k^{\frac{1}{4}}. \tag{12.4}$$

Result: Let $\varepsilon \leq \varepsilon_0$ and $k_\varepsilon \in \mathbb{N}$ such that $|F|_r^{(2\delta)^{k_\varepsilon}} \leq \varepsilon$. Let

$$\begin{cases} Z_\varepsilon = Z_1 \cdots Z_{k_\varepsilon} \\ \bar{A}_\varepsilon = \bar{A}_{k_\varepsilon} \\ \bar{F}_\varepsilon = \bar{F}_{k_\varepsilon} \\ \psi_\varepsilon = \psi_{k_\varepsilon} \\ r_\varepsilon = r_{k_\varepsilon} \end{cases}$$

then the properties 1 and 2 hold. Moreover for all $\theta \in \mathbb{T}^d$,

$$\partial_\omega Z_\varepsilon(\theta) = (A + F(\theta)) Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(\bar{A}_\varepsilon(\theta) + \bar{F}_\varepsilon(\theta))$$

and the property 3 holds. Notice that, for all $k' > \kappa_\varepsilon$, if $\|A_{k_\varepsilon}\| \leq \kappa \varepsilon^\zeta$ (which is satisfied if, for example, the matrix $A_{k_\varepsilon - 1}$ was resonant), then

$$\|A_{k'}\| \leq \|A_{k_\varepsilon}\| + \sum_{i=k_\varepsilon+1}^{k'} \varepsilon_i \leq \kappa''(\varepsilon) + 2\varepsilon \leq 2\kappa''(\varepsilon).$$

We also have

$$|Z_1^{\pm 1} - Id|_{r_\varepsilon} \leq \varepsilon_0^{\frac{9}{10}} \Rightarrow |Z_1|_{r_\varepsilon} \leq 1 + \varepsilon_0^{\frac{9}{10}}$$

Let $k \in \mathbb{N}$ and suppose that for all $j \leq k-1$,

$$|Z_1 \dots Z_j|_{r_\varepsilon} \leq 2$$

then

$$\begin{aligned} c_k := |Z_1 \dots Z_k - Id|_{r_\varepsilon} &\leq |Z_{k-1} - Id|_{r_\varepsilon} |Z_1 \dots Z_{k-1}|_{r_\varepsilon} + |Z_1 \dots Z_{k-1} - Id|_{r_\varepsilon} \\ &\leq 2\varepsilon_{k-2}^{\frac{9}{10}} + c_{k-1} \end{aligned}$$

which implies

$$c_k \leq 2 \sum_{i=0}^{k-2} \varepsilon_i^{\frac{9}{10}} v \leq 4\varepsilon_0^{\frac{9}{10}}.$$

Finally

$$|(Z_1 \dots Z_k)^{\pm 1} - I|_{r_\varepsilon} \leq \varepsilon_0^{\frac{9}{10}}$$

hence the property 4 holds. **Reducible case**

Suppose that there exists $\bar{k}$ such that for all $k' \geq \bar{k}$, $\psi_{k'} \equiv \psi_{\bar{k}}$ (which means that for all $k' \geq \bar{k}$, $A_{k'}$ has a $BR_\omega^{N(r_{k'},\varepsilon_{k'})R(r_{k'},\varepsilon_{k'})}(\kappa''(\varepsilon_{k'}))$ spectrum). Then

$$
\begin{aligned}
\partial_\omega Z_\varepsilon &= \partial_\omega(\prod_{i=1}^{k_\varepsilon} Z_i) \\
&= \partial_\omega(\prod_{i=1}^{\bar{k}-1} Z_i)(\prod_{j=\bar{k}}^{k_\varepsilon} Z_j) + (\prod_{i=1}^{\bar{k}-1} Z_i)\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} Z_j) \\
&= \partial_\omega(\prod_{i=1}^{\bar{k}-1} Z_i)(\prod_{j=\bar{k}}^{k_\varepsilon} Z_j) + (\prod_{i=1}^{\bar{k}-1} Z_i)\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_{\bar{k}}\psi_j^{-1} Z_j \psi_j \psi_{\bar{k}}^{-1}) \\
&= \partial_\omega(\prod_{i=1}^{\bar{k}-1} Z_i)(\prod_{j=\bar{k}}^{k_\varepsilon} Z_j) + (\prod_{i=1}^{\bar{k}-1} Z_i)[\partial_\omega(\psi_{\bar{k}}) \prod_{j=\bar{k}-1}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j \psi_{\bar{k}}^{-1} + \\
&\qquad \psi_{\bar{k}}\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)\psi_{\bar{k}}^{-1} + \psi_{\bar{k}} \prod_{j=\bar{k}}^{k_\varepsilon} (\psi_j^{-1} Z_j \psi_j)\partial_\omega(\psi_{\bar{k}}^{-1})],
\end{aligned}
\tag{12.5}
$$

thus

$$
\begin{aligned}
|\partial_\omega Z_\varepsilon|_{r_\varepsilon} \leq\ & |\partial_\omega(\prod_{i=1}^{\bar{k}-1} Z_i)|_{r_\varepsilon} |\prod_{j=\bar{k}}^{k_\varepsilon} Z_j|_{r_\varepsilon} + |\prod_{i=1}^{\bar{k}-1} Z_i|_{r_\varepsilon} |\psi_{\bar{k}}|_{r_\varepsilon} |\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} |\psi_{\bar{k}}^{-1}|_{r_\varepsilon} \\
&+ |\prod_{i=1}^{\bar{k}-1} Z_i|_{r_\varepsilon} |(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} |\psi_{\bar{k}}^{-1}|_{r_\varepsilon} |\partial_\omega \psi_{\bar{k}}|_{r_\varepsilon} \\
&+ |\prod_{i=1}^{\bar{k}-1} Z_i|_{r_\varepsilon} |(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} |\partial_\omega \psi_{\bar{k}}^{-1}|_{r_\varepsilon} |\psi_{\bar{k}}|_{r_\varepsilon}.
\end{aligned}
$$

Since the factors $|\prod_{i=1}^{\bar{k}-1} Z_i|_{r_\varepsilon}, |\partial_\omega \prod_{i=1}^{\bar{k}-1} Z_i|_{r_\varepsilon}, |(\prod_{j=\bar{k}}^{k_\varepsilon} Z_j)|_{r_\varepsilon}, |\psi_{\bar{k}}|_{r_\varepsilon}, |\psi_{\bar{k}}^{-1}|_{r_\varepsilon}, |\partial_\omega \psi_{\bar{k}}|_{r_\varepsilon}, |\partial_\omega \psi_{\bar{k}}^{-1}|_{r_\varepsilon}$ are bounded uniformly in ε (here we use (12.3)), there exist $K_1, K_2 \geq 0$ independent of ε such that

$$
|\partial_\omega Z_\varepsilon|_{r_\varepsilon} \leq K_1 + K_2 |\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon}.
$$

Moreover, by (12.4) and (12.3),

$$
\begin{aligned}
|\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} &\leq \sum_{j=\bar{k}}^{k_\varepsilon} |\partial_\omega(\psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} \prod_{\substack{\bar{k} \leq i \leq k_\varepsilon \\ i \neq j}} |\psi_i^{-1} Z_i \psi_i|_{r_\varepsilon} \\
&\leq \sum_{j=\bar{k}}^{k_\varepsilon} \varepsilon_j^{\frac{1}{4}} \prod_{\substack{\bar{k} \leq i \leq k_\varepsilon \\ i \neq j}} (1 + 2e^{\varepsilon_i}) e^{2\varepsilon_i}
\end{aligned}
$$

therefore

$$
\begin{aligned}
|\partial_\omega(\prod_{j=\bar{k}}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j)|_{r_\varepsilon} &\leq 2 \sum_{j=\bar{k}}^{k_\varepsilon} \varepsilon_j^{\frac{1}{4}} e^{2\varepsilon_{\bar{k}}} \\
&\leq 8 \varepsilon_{\bar{k}}^{\frac{1}{4}} e^{2\varepsilon_{\bar{k}}} \\
&\leq 16 \varepsilon_{\bar{k}}^{\frac{1}{4}}
\end{aligned}
$$

and finally, $|\partial_\omega Z_\varepsilon|_{r_\varepsilon}$ is bounded as $\varepsilon \to 0$. In this case, Z_ε and $\partial_\omega Z_\varepsilon$ have adherent values; let Z_∞ be an adherent value of Z_ε. Since

$$\partial_\omega(Z_\varepsilon\psi_{\bar{k}}) = (A+F)Z_\varepsilon\psi_{\bar{k}} - Z_\varepsilon\psi_{\bar{k}}(A_\varepsilon + \psi_{\bar{k}}^{-1}\bar{F}_\varepsilon\psi_{\bar{k}})$$

(where $A_\varepsilon \in sl(2,\mathbb{R})$), and since all factors except A_ε are known to converge in a subsequence, then there exists a constant $A_\infty \in sl(2,\mathbb{R})$ such that

$$\partial_\omega(Z_\infty\psi_{\bar{k}}) = (A+F)Z_\infty\psi_{\bar{k}} - Z_\infty\psi_{\bar{k}}A_\infty$$

and thus $A+F$ is actually is a reducible cocycle in $U_{r_\infty}(\mathbb{T}^d, sl(2,\mathbb{R}))$ for some $r_\infty > 0$.

Non reducible case

If the system $A+F$ is not reducible, then for all $k \geq 1$, there exists $k' \geq k$ such that $A_{k'}$ does not have a $BR_\omega^{R_{k'}N_{k'}}(\kappa''(\varepsilon_{k'}))$ spectrum. In this case, $||A_{k'+1}|| \leq \kappa''(\varepsilon_{k'}) = \kappa\varepsilon_{k'}^{\zeta}$.

□

We will now show a density corollary.

Corollary 12.1 (Density of reducible cocycles close to a constant cocycle)**.** *Let $r_0 > 0$, $A \in sl(2,\mathbb{R})$, and $G \in U_{r_0}(2\mathbb{T}^d, sl(2,\mathbb{R}))$ such that $|G-A|_{r_0} \leq \varepsilon_0$ and $\|A\| \leq \varepsilon_0^{-\frac{\zeta}{2}}$ with ε_0 as in 10.1, and satisfying the assumption 1. Denote*

$$\rho = r_0 - \frac{150\delta|\log\varepsilon_0|}{\pi\Lambda\circ\Psi^{-1}(\epsilon_0^{-\zeta})} - \frac{150\delta}{\pi\zeta\log(2\delta)}\int_{\Psi^{-1}(\varepsilon_0^{-\zeta})}^{+\infty}\frac{\Lambda'(t)\ln\Psi(t)}{\Lambda(t)^2}dt.$$

Then for all $\varepsilon > 0$ there exists $H \in U_\rho(2\mathbb{T}^d, sl(2,\mathbb{R}))$ such that $|G-H|_\rho \leq \varepsilon$ and H is reducible.

Proof. Apply theorem 12.1 with $F = G-A$. Since $\rho \leq r_\varepsilon$, we in particular get matrices $Z_\varepsilon \in U_\rho(\mathbb{T}^d, SL(2,\mathbb{R}))$, $\bar{A}_\varepsilon, \bar{F}_\varepsilon \in U_\rho(\mathbb{T}^d, sl(2,\mathbb{R}))$ and $A_\varepsilon \in sl(2,\mathbb{R})$ such that

- $\bar{A}_\varepsilon$ is reducible to A_ε,
- $\partial_\omega Z_\varepsilon = (A+(G-A))Z_\varepsilon - Z_\varepsilon(\bar{A}_\varepsilon + \bar{F}_\varepsilon) = GZ_\varepsilon - Z_\varepsilon(\bar{A}_\varepsilon + \bar{F}_\varepsilon)$,
- $|Z_\varepsilon^{\pm 1}|_\rho \leq 1 + \varepsilon_0^{\frac{9}{10}} \leq 2$,
- $|\bar{F}_\varepsilon|_\rho \leq \frac{\varepsilon}{4}$

Let $H := G - Z_\varepsilon\bar{F}_\varepsilon Z_\varepsilon^{-1}$. We have

$$\partial_\omega Z_\varepsilon = HZ_\varepsilon - Z_\varepsilon\bar{A}_\varepsilon$$

and then H is reducible to A_ε (as $\bar{A}_\varepsilon$ is). Moreover, H satisfies

$$|H-G|_\rho = |Z_\varepsilon^{-1}\bar{F}_\varepsilon Z_\varepsilon|_\rho \leq 4|\bar{F}_\varepsilon|_\rho \leq \varepsilon.$$

□

References

[1] A.Avila, B.Fayad, R.Krikorian. A KAM scheme for SL(2,R) cocycles with Liouvillean frequencies. Geom. Funct. Anal. 21 (2011),1001-1019

[2] A.Avila, S.Jitomirskaya. Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators Commun. Math. Phys. 301 (2011), 563-581

[3] A.Avila, S.Jitomirskaya. Almost localization and almost reducibility. J.Eur.Math.Soc. 12 (2010), 93-131

[4] Sana Ben Hadj Amor. Hölder continuity of the rotation number for the quasi-periodic cocycles in SL(2,R). Commun. Math. Phys. 287, 565-588 (2009)

[5] Abed Bounemoura, Claire Chavaudret, Shuqing Liang. Reducibility of ultra-differentiable quasi-periodic cocycles under an adapted arithmetic condition, Proc. Amer. Math. Soc. 149 (2021), no. 7, 2999–3012.

[6] A. Bounemoura and J. Féjoz. KAM, α-Gevrey regularity and the α-Brjuno-Rüssmann condition. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze, XIX (2019), 1-55

[7] Jean Bourgain. Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime. Lett. Math. Phys. 51 (2000), no. 2, 83-118.

[8] R.W.Braun, R.Meise and B.A.Taylor. Ultradifferentiable functions and Fourier analysis Results Math. 17 no 3-4 (1990), 206-237

[9] Ao Cai, Claire Chavaudret, Jiangong You and Qi Zhou. Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles, Math. Z. 291 (2019), 931-958

[10] Claire Chavaudret. Reducibility of Quasi-Periodic Cocycles in Linear Lie Groups. Ergodic Theory Dyn. Syst., 31 (2010), 741-769

[11] Claire Chavaudret. Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles. Bull. Soc. Math. Fr. 141 n.1 (2013), 47-106

[12] C.Chavaudret, S.Marmi. Reducibility of quasiperiodic cocycles under a Brjuno-Rüssmann arithmetical condition. J. Mod. Dyn. 6 n.1 (2012), 59-78

[13] Lars-Hakan Eliasson. Almost reducibility of linear quasi-periodic systems, Proc. Symp. Pure Math. 69, 2001, 679-705.

[14] Lars-Hakan Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. Commun. Math. Phys. 146 (1992), 447-482

[15] X.Hou, G.Popov. Rigidity of the Reducibility of Gevrey Quasi-periodic Cocycles on U(n). Bull. Soc. Math. Fr. 144 n.1 (2016), 1-52

[16] X.Hou, J.You. Almost reducibility and non perturbative reducibility of quasi-periodic linear systems. Invent. math. 190 (2012), 209-260

[17] J.Lopes Dias. A normal form theorem for Brjuno skew-systems through renormalization. J. Differential Equations 230, 1-23 (2006).

[18] J. Lopes Dias and J. P. Gaivao. Linearization of Gevrey flows on $\mathbb{T}^d$ with a Brjuno type arithmetical condition. J. Differ. Equations, 267 (2019), 7167-7212.

[19] P.Duarte, S.Klein, M.Poletti. Hölder continuity of the Lyapunov exponents of linear cocycles over hyperbolic maps. arXiv:2110.10265

[20] J.Puig. A nonperturbative Eliasson's reducibility theorem. Nonlinearity 19 (2006), 355-376

[21] Armin Rainer. Ultradifferentiable extension theorems: a survey, ArXiv:2107.01061v1, 2021

[22] David Ruelle. Analyticity properties of the characteristic exponents of random matrix products, Adv. in Math. 32, 1979, 68-80.