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Abstract

The Entringer's integers were counting the down-up (or up-down) permutations according to the first or the last term ; In this paper they are shown to enumerate the same permutations according to the absolute difference between the first and the last terms.

1.- An up-down (resp. down-up) permutation of order n is a permutation

$(a_1, a_2, ..., a_n)$ of $\{1, ..., n\}$ with

$a_{2p-1} < a_{2p}$, $a_{2p+1} < a_{2p}$ (resp. $a_{2p-1} > a_{2p}$, $a_{2p+1} > a_{2p}$)

for every $p \in \left\{1, ..., E\left[\frac{n}{2}\right]\right\}$.

An alternating permutation is an up-down or down-up permutation.

Entringer has studied the enumeration of up-down (or down-up) permutations according to the value of their first term.

For $n \geq 1$, $k \geq 1$ let ε_n^k be the number of up-down permutations of order $n+1$ starting with $n-k+1$; ε_n^k is also the number of down-up permutations of order $n+1$ starting with $k+1$.

We have the following recurrence relations :

$$\varepsilon_n^k = 0 \text{ for } n \geq 1 , k \geq n+1 ;$$

$$\varepsilon_1^1 = 0 ;$$

$$\varepsilon_n^k = \sum_{j=n-k}^{n-1} \varepsilon_{n-1}^j \text{ for } n \geq 2 , 1 \leq k \leq n-1 ;$$

$$\varepsilon_n^n = \varepsilon_n^{n-1} \text{ for } n \geq 2 .$$

We can write the first values of the coefficients ε_n^k

	k=1	2	3	4	5
n=1	1				
2	1	1			
3	1	2	2		
4	2	4	5	5	
5	5	10	14	16	16

Another way to get the Entringer numbers is to count the up-down or down-up permutations according to the value of their last term.

For instance, if n is even, there are ε_n^k up-down permutations of order $n+1$ ending with $n-k+1$.

Let $\mathcal{U}_n(\ell, k)$ (resp. $\mathcal{V}_n(\ell, k)$ and $\mathcal{S}_n(\ell, k)$) be the set of up-down (resp. dow-up and alternating) permutations of order n starting with ℓ and ending with k and let $u_n(\ell, k)$ (resp. $v_n(\ell, k)$ and $s_n(\ell, k)$) be its cardinality.

We also define, for instance, $u_n(\ell, .)$, the number of up-down permutations which start with ℓ.

For $d \in \mathbb{Z}$, $-n+1 \le d \le n-1$ et $d \neq 0$,

$\mathcal{C}_n(d) = \{(x, y) / 1 \le x \le n,\ 1 \le y \le n,\ x-y = d\}$,

we write $U_n(d) = \sum_{(i,j) \in \mathcal{C}_n(d)} u_n(i,j)$

and $V_n(d) = \sum_{(i,j) \in \mathcal{C}_n(d)} v_n(i,j)$

and $S_n(d) = \sum_{(i,j) \in \mathcal{C}_n(d)} s_n(i,j)$.

Let us consider the permutation $\{a_1, ..., a_n\}$. $a_1 - a_n$ will be its "deviation", and $|a_1 - a_n|$ its "separation".

We state the two following propositions :

for $1 \le d \le n-1$,

Proposition 1 : The numbers of alternating permutations of order n and deviation d is equal to the number of up-down (or down-up) permutations of order n and separation d.

Proposition 2 : the number of up-down (or down-up) permutations of order n and separation d is equal,

if n is even, to the number of up-down permutations of order n whose first term is d and if n is odd, to the multiplication of $n-d$ by the number of up-down permutations of order $n-1$ whose first term is d.

These two propositions may be reformulated as follows :

1) $S_n(d) = U_n(d) + U_n(-d) = V_n(d) + V_n(-d)$

2) $U_n(d) + U_n(-d) = V_n(d) + V_n(-d)$

$$= \begin{cases} u_n(d, .) & \text{if } n \text{ is even} \\ (n-d)\, u_{n-1}(d, .) & \text{if } n \text{ is odd} \end{cases}$$

We illustrate these propositions for $n = 6$ and $n = 7$ (cf p. 10, 11, 12).

2. - First, we prove some relations concerning the numbers $u_n(\ell, k)$, $v_n(\ell, k)$ and $s_n(\ell, k)$.

1) The following symmetric relations go straight forward.

If n is odd,

$$u_n(\ell, k) = u_n(k, \ell) \qquad (1)$$

$$v_n(\ell, k) = v_n(k, \ell) \qquad (2)$$

For instance, to verify (1), we can use a bijection between $\mathcal{U}_n(\ell, k)$ and $\mathcal{U}_n(k, \ell)$: by reading from right to left, a permutation of $\mathcal{U}_n(\ell, k)$ becomes a permutation of $\mathcal{U}_n(k, \ell)$.

A similar argument can be applied to prove (2).

Likewise, it can be proved that, when n is even,

$$u_n(\ell, k) = v_n(k, \ell) \qquad (3)$$

It can also be easily proved that for all n, there is a bijection between $\mathcal{U}_n(\ell, k)$ and $\mathcal{V}_n(n+1-\ell, n+1-k)$:

we have only to transform the permutations of $\mathcal{U}_n(\ell, k)$ via complementation, i.e., to replace the term i of a permutation by $n+1-i$.

Therefore,

$$u_n(\ell, k) = v_n(n+1-\ell, n+1-k) \qquad (4)$$

2) Then, let us establish the following recurrence relations :

for $\ell > k$

$$u_n(\ell, k) = v_{n-1}(\ell, k) + u_n(\ell+1, k) \tag{5}$$

and for $\ell < k$

$$u_n(\ell, k) = v_{n-1}(\ell, k-1) + u_n(\ell+1, k) \quad \text{if } \ell+1 < k \tag{6}$$

and

$$u_n(\ell, k) = u_n(\ell+1, k-1) \quad \text{if } \ell+1 = k \tag{6'}$$

$$(v_{n-1}(\ell, k-1) = 0 \quad \text{if } \ell+1 = k)$$

Indeed, for $\ell > k$, we define a bijection between $\mathcal{U}_n(\ell, k)$ and $\bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i,k)$ as follows : given a permutation of $\mathcal{U}_n(\ell,k)$, we can define a permutation of $\bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i,k)$ by omitting its first term, by decreasing (substracting one unit) the terms of the permutation greater than ℓ and by keeping the others, i.e.,

$$\mathcal{U}_n(\ell, k) \rightarrow \bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i,k)$$

$$(a_1 = \ell, a_2, \ldots, a_n = k) \longmapsto (\tilde{a}_2, \ldots, \tilde{a}_n),$$

where, for $2 \le j \le n$

$$\tilde{a}_j = \begin{cases} a_j - 1 & \text{if } a_j > \ell \\ a_j & \text{otherwise} \end{cases}$$

$$(\tilde{a}_n = k)$$

Consequently

$$u_n(\ell, k) = \sum_{i=\ell}^{n-1} v_{n-1}(i, k) \quad .$$

We are now ready to prove (5).

Indeed, we can decompose $\bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i, k)$ in two sets $\mathcal{V}_{n-1}(\ell, k)$ and $\bigcup_{i=\ell+1}^{n-1} \mathcal{V}_{n-1}(i, k)$, and use the bijection between $\bigcup_{i=\ell+1}^{n-1} \mathcal{V}_{n-1}(i, k)$ and $\mathcal{U}_n(\ell+1, k)$.

For $\ell < k$, the sets $\mathcal{U}_n(\ell, k)$ and $\bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i, k-1)$ have the same cardinalities. For the proof, we can use the bijection described above (let us notice that $\tilde{a}_n = a_n - 1 = k-1$).

Thus for $\ell < k$

$$u_n(\ell, k) = \sum_{i=\ell}^{n-1} v_{n-1}(i, k-1) \quad .$$

To see why (6) and (6') hold, we can decompose $\bigcup_{i=\ell}^{n-1} \mathcal{V}_{n-1}(i, k-1)$ in two sets $\mathcal{V}_{n-1}(\ell, k-1)$ and $\bigcup_{i=\ell+1}^{n-1} \mathcal{V}_{n-1}(i, k-1)$, and use the bijection between $\bigcup_{i=\ell+1}^{n-1} \mathcal{V}_{n-1}(i, k-1)$

and $\mathcal{U}_n(\ell+1, k)$ if $\ell+1 < k$,

$\mathcal{U}_n(\ell+1, k-1)$ if $\ell+1 = k$.

Similarly, we have for $\ell > k$,

$$v_n(\ell, k) = u_{n-1}(\ell-1, k) + v_n(\ell-1, k) \quad \text{if } \ell > k+1 \qquad (7)$$

and $v_n(\ell, k) = v_n(\ell-1, k+1)$ if $\ell = k+1$ (7')

$(u_{n-1}(\ell-1, k) = 0$ if $\ell = k+1)$

and for $\ell < k$,

$$v_n(\ell, k) = u_{n-1}(\ell-1, k-1) + v_n(\ell-1, k) \qquad (8)$$

3) If n is odd, we also establish a relation concerning the numbers $s(\ell, k)$.

More precisely, if n is odd and for $1 \leq \ell \leq n-1$, $1 \leq k \leq n-1$, $\ell \neq k$, the number of alternating permutations of order n starting with ℓ and ending with k is equal to the number of alternating permutations of order n starting with $\ell+1$ and ending with $k+1$.

In other words

$$s_n(\ell+1, k+1) = s_n(\ell, k) \qquad (9)$$

Indeed

for $\ell < k$,

$$s_n(\ell+1, k+1)$$

$$= u_n(\ell+1, k+1) + v_n(\ell+1, k+1)$$

$$= u_n(\ell+1, k+1) + u_{n-1}(\ell, k) + v_n(\ell, k+1) \qquad \text{(cf (8))}$$

$$= u_n(\ell+1, k+1) + u_{n-1}(\ell, k) + v_n(k+1, \ell) \qquad \text{(cf 2))}$$

$$= u_n(\ell+1, k+1) + u_{n-1}(\ell, k) + u_{n-1}(k, \ell) + v_n(k, \ell) \qquad \text{(cf (7))}$$

$$= u_n(\ell+1, k+1) + v_{n-1}(\ell, k) + v_{n-1}(k, \ell) + v_n(\ell, k) \qquad \text{(cf (2) et (3))}$$

$$= u_n(\ell, k+1) + v_{n-1}(k, \ell) + v_n(\ell, k) \qquad \text{(cf (6))}$$

$$= u_n(k+1, \ell) + v_{n-1}(k, \ell) + v_n(\ell, k) \qquad \text{(cf (1))}$$

$$= u_n(k, \ell) + v_n(\ell, k) \qquad \text{(cf (5))}$$

$$= u_n(\ell, k) + v_n(\ell, k) \qquad \text{(cf (1))}$$

$$= s_n(\ell, k)$$

For $\ell > k$, the proof is along the same lines as in the previous case.

3.- From these results, we can set relations between numbers $U_n(d)$, $V_n(d)$ and $S_n(d)$ and therefore we can prove Proposition 1 and Proposition 2 (cf. 2).

1) Proof of Proposition 1.

If n is even, we have

$$U_n(d) = V_n(-d) \text{ et } V_n(d) = U_n(-d) \qquad (10)$$

Indeed

$$U_n(d) = \sum_{(i,j)\in \mathfrak{T}_n(d)} u_n(i,j) = \sum_{(j,i)\in \mathfrak{T}_n(-d)} v_n(j,i) = V_n(-d)$$

Similarly, $V_n(d) = U_n(-d)$.

If n is odd, we have

$$U_n(d) = U_n(-d) = V_n(d) = V_n(-d) \qquad (11)$$

Indeed

$$U_n(d) = \sum_{(i,j)\in \mathfrak{T}_n(d)} u_n(i,j) = \sum_{(j,i)\in \mathfrak{T}_n(-d)} u_n(j,i) = U_n(-d)$$

likewise, $V_n(d) = V_n(-d)$.

And $$U_n(d) = \sum_{(i,j)\in \mathfrak{T}_n(d)} u_n(i,j) = \sum_{(i,j)\in \mathfrak{T}_n(d)} v_n(n+1-i, n+1-j)$$

$$= \sum_{(i,j')\in \mathfrak{T}_n(-d)} v_n(i',j') = V_n(-d) \quad .$$

Therefore, for all n

$$U_n(d) + U_n(-d) = V_n(d) + V_n(-d) = S_n(d) \quad .$$

Thus Proposition 1 is proved.

2) Proof of Proposition 2

For all n , we have the following relations between $U_n(d)$ et $V_n(d)$:

$$U_n(d) = V_{n+1}(d+1) - V_{n+1}(d) + u_{n+1}(1, d+1) \qquad (13)$$

and $$V_n(d) = U_{n+1}(d) - U_{n+1}(d+1) \qquad (14)$$

In order to prove (13), we use the relations (7) and (4)

$$V_{n+1}(d+1) - V_{n+1}(d) = \sum_{i=1}^{n-d} v_{n+1}(i+d+1, i) - \sum_{j=1}^{n-d+1} v_{n+1}(j+d, j)$$

$$= \sum_{m=1}^{n-d} u_n(m+d, m) - v_{n+1}(n+1, n-d+1)$$

$$= U_n(d) - u_{n+1}(1, d+1) \quad .$$

In the same way, let us prove (14) using (5)

If n is even, let us deduce considering (11), (13), (14) :

$$S_n(d) = U_n(d) + V_n(d) = u_{n+1}(1, d+1) \quad .$$

If n is odd, let us consider (9) and deduce :

$$s_n(d+j, j) = s_n(d+1, 1) \qquad \text{for } 1 \le j \le n-d \, .$$

But for each value of n , we state $s_n(d+1, 1) = s_n(1, d+1) = u_n(1, d+1)$, since there is an obvious bijection between $\mathcal{S}_n(\ell, k)$ and $\mathcal{S}_n(k, \ell)$, and $\mathcal{S}_n(1, m)$ and $\mathcal{U}_n(1, m)$ are identical.

Therefore when n is odd,

$$S_n(d) = \sum_{j=1}^{n-d} s_n(d+j, j) = (n-d)\, u_n(1, d+1) \quad .$$

Finally, we have

$$S_n(d) = U_n(d) + V_n(d) = U_n(d) + U_n(-d)$$

$$= V_n(d) + V_n(-d)$$

$$= u_{n+1}(1, d+1) \qquad \text{if n is even}$$

$$= (n-d)\, u_n(1, d+1) \qquad \text{if n is odd.}$$

Moreover, when n is odd, we can notice that $u_n(1, d+1)$ is the number of up-down permutations of order n–1 starting with d (denoted by $u_{n-1}(d, .)$), since there is a bijection between $\mathcal{U}_n(1, d+1)$ and the set of down-up permutations of order n–1 ending with d. This bijection can be seen as follows : given a permutation of $\mathcal{U}_n(1, d+1)$, we can define a down-up permutation of order n–1 ending with d by omitting its first term (equal to 1) and by decreasing (cf. [4]) the other terms. And since n is odd, it is easy to transform the down-up permutations of order n–1 ending with d in order to obtain the up-down permutations of order n–1 starting with d (by inversion of the terms).

Likewise, when n is even, $u_{n+1}(1, d+1)$ is equal to the number of up-down permutations of order n starting with d (denoted by $u_n(d, .)$).

This is the way we prove Proposition 2.

APPENDIX

For n=6 and n=7, using recurrence relations (5), (6), (7), (8), let us set the tables of numbers $u_n(\ell, k)$, $v_n(\ell, k)$ and $s_n(\ell, k)$.

For given n , in the table of numbers $u_n(\ell, k)$ (resp. $v_n(\ell, k)$, $s_n(\ell, k)$), we find the numbers $U_n(d)$ and $U_n(-d)$ (resp. $V_n(d)$ and $V_n(-d)$, $S_n(d)$ and $S_n(-d)$) in the last column and in the last row.

for n=6

$S_6(1) = U_6(1) + U_6(-1) = V_6(1) + V_6(-1) = 8 + 8 = 16 = u_6(1, .)$

$S_6(2) = U_6(2) + U_6(-2) = V_6(2) + V_6(-2) = 12 + 4 = 16 = u_6(2, .)$

$S_6(3) = U_6(3) + U_6(-3) = V_6(3) + V_6(-3) = 13 + 1 = 14 = u_6(3, .)$

$S_6(4) = U_6(4) + U_6(-4) = V_6(4) + V_6(-4) = 10 + 0 = 10 = u_6(4, .)$

$S_6(5) = U_6(5) + U_6(-5) = V_6(5) + V_6(-5) = 5 + 0 = 5 = u_6(5, .)$

and for n=7

$S_7(1) = U_7(1) + U_7(-1) = V_7(1) + V_7(-1) = 48 + 48 = 96 = 6 \times 16 = 6\, u_6(1, .)$

$S_7(2) = U_7(2) + U_7(-2) = V_7(2) + V_7(-2) = 40 + 40 = 80 = 5 \times 16 = 5\, u_6(2, .)$

$S_7(3) = U_7(3) + U_7(-3) = V_7(3) + V_7(-3) = 28 + 28 = 56 = 4 \times 14 = 4\, u_6(3, .)$

$S_7(4) = U_7(4) + U_7(-4) = V_7(4) + V_7(-4) = 15 + 15 = 30 = 3 \times 10 = 3\, u_6(4, .)$

$S_7(5) = U_7(5) + U_7(-5) = V_7(5) + V_7(-5) = 5 + 5 = 10 = 2 \times 5 = 2\, u_6(5, .)$

n=6

$\ell \backslash k$	1	2	3	4	5	6	
1	0	0	2	4	5	5	
2	0	0	2	4	5	5	5
3	0	2	0	4	4	4	10
4	0	2	4	0	2	2	13
5	0	1	2	2	0	0	12
6	0	0	0	0	0	0	8
		0	0	1	4	8	

table of numbers
$u_6(\ell,k)$

$\ell \backslash k$	1	2	3	4	5	6	
1	0	0	0	0	0	0	
2	0	0	2	2	1	0	0
3	2	2	0	4	2	0	0
4	4	4	4	0	2	0	1
5	5	5	4	2	0	0	4
6	5	5	4	2	0	0	8
		5	10	13	12	8	0

table of numbers
$v_6(\ell,k)$

$\ell \backslash k$	1	2	3	4	5	6	
1	0	0	2	4	5	5	
2	0	0	4	6	6	5	5
3	2	4	0	8	6	4	10
4	4	6	8	0	4	2	14
5	5	6	6	4	0	0	16
6	5	5	4	2	0	0	16
		5	10	14	16	16	0

table of numbers
$s_6(\ell,k)$

n=7

$\ell \backslash k$	1	2	3	4	5	6	7	
1	0	16	16	14	10	5	0	
2	16	0	16	14	10	5	0	0
3	16	16	0	12	8	4	0	5
4	14	14	12	0	4	2	0	15
5	10	10	8	4	0	0	0	28
6	5	5	4	2	0	0	0	40
7	0	0	0	0	0	0	0	48
		0	5	15	28	40	48	0

table of numbe
$u_7(\ell,k)$

$\ell \backslash k$	1	2	3	4	5	6	7	
1	0	0	0	0	0	0	0	
2	0	0	0	2	4	5	5	0
3	0	0	0	4	8	10	10	5
4	0	2	4	0	12	14	14	15
5	0	4	8	12	0	16	16	28
6	0	5	10	14	16	0	16	40
7	0	5	10	14	16	16	0	48
		0	5	15	28	40	48	0

table of numbe
$v_7(\ell,k)$

$\ell \backslash k$	1	2	3	4	5	6	7	
1	0	16	16	14	10	5	0	
2	16	0	16	16	14	10	5	0
3	16	16	0	16	16	14	10	10
4	14	16	16	0	16	16	14	30
5	10	14	16	16	0	16	16	56
6	5	10	14	16	16	0	16	80
7	0	5	10	14	16	16	0	96
		0	10	30	56	80	96	

table of numbe
$s_7(\ell,k)$

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