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Introduction to quiver spaces

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Abstract

This article introduces quiver spaces, a class of structures that generalize metric spaces and multiplicative metric spaces. They provide a representation framework that is both faithful and general for the principle of the identity of indiscernibles, avoiding the limitations of the classical metric and emphasizing a purely relational and algebraic perspective. Several topologies naturally associated with quiver spaces are also introduced and studied.

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1 Introduction

The principle of the identity of indiscernibles asserts that two entities cannot be distinct if no property allows one to distinguish them. In classical physics, distance plays precisely this role: two objects are distinct because they occupy different positions in space. In quantum mechanics, and in particular in the case of entanglement [2], this idea becomes fragile: entangled particles do not possess well-defined individual spatial properties, and the distance between them may lose its fundamental physical meaning. Indiscernibility then becomes structural, related to the global state of the system rather than to a measurable spatial separation. In modern physics the concept of the distance was formalized in a subtle way using *Von Neumann* algebras where the algebra of coordinates is considered *non-commutative*, and the synthesis was done by *Alain Connes*[1]. The first abstract formulation of the concept of distance was introduced by *Maurice Fréchet*[6]. This concept, later formalized as a *metric space* by *Felix Hausdorff*, is based on a function d

that assigns a nonnegative real number $d(x, y)$ to each pair of points (x, y) in a set X . The function d is required to satisfy the following three properties, for all $x, y, z \in X$,

1. *Identity of indiscernibles:* $d(x, y) = 0$ if and only if $x = y$.
2. *Symmetry:* $d(x, y) = d(y, x)$.
3. *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$.

Together with the set X , the pair (X, d) defines a *metric space*. The purpose of this paper is to introduce the *quiver spaces*. This framework provides a faithful and general representation of the identity of indiscernibles without relying on the traditional notion of a *metric*. Several topologies naturally associated with quiver spaces are also introduced and studied.

2 Preliminaries

We begin by introducing several definitions that will be used throughout the remainder of the article.

Definition 2.1. [5] *An ultrametric space is a pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}_+$ is a mapping satisfying, for all $x, y, z \in X$:*

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq \sup\{d(x, y), d(y, z)\}$ (*ultrametric inequality*).

Definition 2.2. [3] *Let (X, d) be a metric space and let $x \in X$ and $r > 0$.*

- *The open ball centered at x with radius r is defined by*

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

- *The closed ball centered at x with radius r is defined by*

$$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}.$$

Definition 2.3. [4] *Let X be a set. A binary relation \leq on X is called a partial order if it satisfies the following three properties for all $x, y, z \in X$:*

1. *Reflexivity:* $x \leq x$,
2. *Antisymmetry:* if $x \leq y$ and $y \leq x$, then $x = y$,
3. *Transitivity:* if $x \leq y$ and $y \leq z$, then $x \leq z$.

The pair (X, \leq) is called a partially ordered set or poset.

Definition 2.4. Let (X, \leq) be a partially ordered set. The strict order associated with \leq is the binary relation $<$ on X defined by

$$x < y \iff x \leq y \text{ and } x \neq y.$$

The relation $<$ is called the strict order induced by \leq .

Definition 2.5. A magma is a set M equipped with a binary operation $*$: $M \times M \rightarrow M$. That is, for all $x, y \in M$, the element $x * y$ is also in M .

Definition 2.6. Let $(M, *)$ be a magma, then,

- The binary operation $*$ is called associative if for all $a, b, c \in M$, we have

$$(a * b) * c = a * (b * c).$$

- The binary operation $*$ is called idempotent if for every $a \in M$, we have

$$a * a = a.$$

Definition 2.7. [4] Let (X, \leq) be a partially ordered set. An element $e_0 \in X$ is called a least element, if for every $x \in X$, $e_0 \leq x$.

Definition 2.8. [4] Let (X, \leq) be a poset and let $*$: $X \times X \rightarrow X$ be a binary operation. The operation $*$ is said to be compatible with the order if for all $x, y, z, t \in X$,

$$x \leq y \text{ and } z \leq t \implies x * z \leq y * t.$$

Definition 2.9. [3] Let X be a set. A topological space is a pair (X, τ) , where $\tau \subseteq \mathcal{P}(X)$ is a collection of subsets of X , called open sets, such that:

- (i) $\emptyset \in \tau$ and $X \in \tau$;
- (ii) the union of any family of elements of τ belongs to τ ;
- (iii) the intersection of any finite family of elements of τ belongs to τ .

The collection τ is called a topology on X .

Definition 2.10. [3] Let (X, d) be a metric space. The topology induced by the metric d , denoted by τ_d , is the collection of subsets $\omega \subseteq X$ such that for every $x \in \omega$, there exists $r > 0$ satisfying, $B(x, r) \subseteq \omega$.

The concept of multiplicative metric space is rooted in multiplicative calculus, which was developed by A. E. Bashirov and his collaborators. Based on this theory, the notion of multiplicative metric space was formally introduced in 2012 by Mustafa Özavşar and Ali Cevikel [7]. Unlike classical metric spaces that rely on an additive triangle inequality, multiplicative metric spaces are defined using a multiplicative triangle inequality.

Definition 2.11. Let X be a non-empty set. A function

$$d : X \times X \rightarrow [1, +\infty)$$

is called a multiplicative metric if for all $x, y, z \in X$, the following hold:

- $d(x, y) = 1$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) \cdot d(y, z)$.

The pair (X, d) is called a multiplicative metric space.

3 Main Results

Having introduced the necessary definitions, we now present the main results. In what follows, $(G, \leq, *)$ will denote a partially ordered magma with a least element e_0 (abbreviated e_0 -Mpo). We begin with the following definition:

Definition 3.1. Let $(G, \leq, *)$ be an e_0 -Mpo, X a nonempty set, and f a mapping defined by:

$$\begin{aligned} f : X \times X &\rightarrow G \\ (x, y) &\mapsto f(x, y) = f_{x,y} \end{aligned}$$

Then (X, f) is called a quiver space over G if the following three axioms are satisfied:

1. $\forall x, y \in X, f_{x,y} = e_0 \Leftrightarrow x = y$.
2. $\forall x, y \in X, f_{x,y} = f_{y,x}$.
3. $\forall x, y, z \in X, f_{x,y} \leq f_{x,z} * f_{z,y}$.

Remark 3.1. Let \leq be the standard order relation on the real numbers, Therefore,

- (X, d) is a metric space, then, (X, d) is a quiver space over $(\mathbb{R}_+, \leq, +)$ a 0-Mpo.
- (X, d) is an ultrametric space, then, (X, d) is a quiver space over $(\mathbb{R}_+, \leq, \sup)$ a 0-Mpo.
- (X, d) be a multiplicative metric space, then (X, d) is a quiver space over $([1, \infty), \leq, \cdot)$, a 1-Mpo.

Proposition 3.1. *Let X be a nonempty set, $d : X \times X \rightarrow \mathbb{R}_+$, and let f_d be the mapping defined from $X \times X$ to $\mathcal{P}(\mathbb{R}_+)$ such that, for all $x, y \in X$,*

$$f_d(x, y) = f_{x,y} = (0, d(x, y)).$$

Thus,

- (X, d) is a metric space if and only if (X, f_d) is a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$, where $+$ is the operation defined by:

$$\forall A, B \in \mathcal{P}(\mathbb{R}_+) \quad A + B = \{a + b \mid a \in A, b \in B\}$$

- (X, d) is an ultrametric space if and only if (X, f_d) is a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, \cup)$.

Proof. $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$ and $(\mathcal{P}(\mathbb{R}_+), \subseteq, \cup)$ are \emptyset -Mpo. Let f_d be the mapping defined by

$$\begin{aligned} f_d : X \times X &\rightarrow \mathcal{P}(\mathbb{R}_+) \\ (x, y) &\mapsto f_d(x, y) = f_{x,y} = (0, d(x, y)) \end{aligned}$$

Then, for all $x, y \in X$,

- $x = y \Leftrightarrow d(x, y) = 0 \Leftrightarrow f_{x,y} = \emptyset$,
- $d(x, y) = d(y, x) \Leftrightarrow f_{x,y} = f_{y,x}$,
- For all $x, y, z \in X$,

$$\begin{aligned} d(x, y) \leq d(x, z) + d(z, y) &\Leftrightarrow (0, d(x, y)) \subseteq (0, d(x, z) + d(z, y)) \\ &\Leftrightarrow (0, d(x, y)) \subseteq (0, d(x, z)) + (0, d(z, y)) \\ &\Leftrightarrow f_{x,y} \subseteq f_{x,z} + f_{z,y}. \end{aligned}$$

- For all $x, y, z \in X$,

$$\begin{aligned} d(x, y) \leq \sup\{d(x, y), d(y, z)\} &\Leftrightarrow (0, d(x, y)) \subseteq (0, \sup\{d(x, y), d(y, z)\}) \\ &\Leftrightarrow (0, d(x, y)) \subseteq (0, d(x, z)) \cup (0, d(z, y)) \\ &\Leftrightarrow f_{x,y} \subseteq f_{x,z} \cup f_{z,y}. \end{aligned}$$

Hence the result. □

Example 3.1. *Let (X, d) be an ultrametric space and let $p \in X$. Let f_p be the mapping defined by:*

$$\begin{aligned} f_p : X \times X &\rightarrow \mathcal{P}(X) \\ (x, y) &\mapsto f_p(x, y) = \{z \in X \mid d(p, z) < d(x, y)\}. \end{aligned}$$

Then (X, f_p) is a quiver space over $(\mathcal{P}(X), \subseteq, \cup)$. Indeed,

- For all $x, y \in X$,

$$x = y \Leftrightarrow d(x, y) = 0 \Leftrightarrow f_p(x, y) = \{z \in X \mid d(p, z) < d(x, y)\} = \emptyset,$$

- For all $x, y \in X$, $d(x, y) = d(y, x) \Leftrightarrow f_p(x, y) = f_p(y, x)$,
- For all $x, y, z \in X$,

$$d(x, y) \leq \sup\{d(x, y), d(y, z)\} \Leftrightarrow f_p(x, y) \subseteq f_p(x, z) \cup f_p(z, y).$$

Thus, (X, f_p) is a quiver space over $(\mathcal{P}(X), \subseteq, \cup)$.

Conversely, under what conditions can a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$ (respectively over $(\mathcal{P}(\mathbb{R}_+), \subseteq, \cup)$) be a metric space (respectively an ultrametric space)?

Proposition 3.2. *Let (X, f) be a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$ (respectively over $(\mathcal{P}(\mathbb{R}_+), \subseteq, \cup)$). Assume that for all $x, y \in X$, the set $f(x, y)$ is bounded above and $f(x, y) \neq \{0\}$. Then (X, δ) is a metric space (respectively an ultrametric space), where δ is the mapping defined from $X \times X$ to \mathbb{R}_+ by:*

$$\delta(x, y) = \begin{cases} \sup(f(x, y)) & , \text{if } x \neq y \\ 0 & , \text{if } x = y \end{cases}$$

Proof. Let (X, f) be a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$. Let δ be the mapping defined from $X \times X$ to \mathbb{R}_+ by:

$$\delta(x, y) = \begin{cases} \sup(f(x, y)) & , \text{if } x \neq y \\ 0 & , \text{if } x = y \end{cases}$$

- Since for all $x, y \in X$, $f(x, y) \neq \{0\}$, we have

$$x \neq y \Leftrightarrow f(x, y) \neq \emptyset \Leftrightarrow \delta(x, y) \neq 0.$$

It follows that for all $x, y \in X$, $\delta(x, y) = 0 \Leftrightarrow x = y$,

- For all $x, y \in X$, $f(x, y) = f(y, x) \Leftrightarrow \delta(x, y) = \delta(y, x)$,
- Since for all $x, y, z \in X$, $f(x, y) \subseteq f(x, z) + f(z, y)$, we obtain

$$\sup(f(x, y)) \leq \sup\{f(x, z) + f(z, y)\} = \sup(f(x, z)) + \sup(f(z, y)).$$

Thus, for all $x, y, z \in X$, $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$. Consequently, (X, δ) is a metric space.

Respectively, if (X, f) is a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, \cup)$, then for all $x, y, z \in X$, $f(x, y) \subseteq f(x, z) \cup f(z, y)$, which implies

$$\sup(f(x, y)) \leq \sup\{f(x, z) \cup f(z, y)\} = \sup\{\sup(f(x, z)); \sup(f(z, y))\}.$$

Consequently, for all $x, y, z \in X$, $\delta(x, y) \leq \sup(\delta(x, z); \delta(z, y))$. Hence (X, δ) is an ultrametric space. \square

3.1 Notion of quiver

In this section, we define the analogue of the closed ball for a quiver space and establish some of its properties.

Definition 3.2. Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo. Let $x \in X$ and $g \in G$, then

- $\mathcal{H}(x, g) = \{z \in X \mid f(x, z) \leq g\}$ is called the closed quiver with center x and target g .
- $\mathcal{H}_s(x, g) = \{z \in X \mid f(x, z) < g\}$ is called the open quiver with center x and target g .

Proposition 3.3. Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo. Then,

- For all $x \in X$,

$$\mathcal{H}(x, e_0) = \{x\},$$

- If G contains a greatest element e_∞ , then

$$\forall x \in X, \mathcal{H}(x, e_\infty) = X,$$

- For all $x, y \in X$ and for all $g \in G$,

$$y \in \mathcal{H}(x, g) \Leftrightarrow x \in \mathcal{H}(y, g),$$

- For all $x \in X$, and for all $g, h \in G$,

$$g \leq h \Rightarrow \mathcal{H}(x, g) \subseteq \mathcal{H}(x, h).$$

Proof. Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo. Let $x \in X$ and $g, h \in G$. Then,

- $\mathcal{H}(x, e_0) = \{y \in X \mid f(x, y) \leq e_0\} = \{y \in X \mid f(x, y) = e_0\} = \{x\}$,
- $\mathcal{H}(x, e_\infty) = \{y \in X \mid f(x, y) \leq e_\infty\} = X$,
- By symmetry of f ,
- Assume that, $g \leq h$. By transitivity of the order relation \leq , it follows that $y \in \mathcal{H}(x, g) \Leftrightarrow f(x, y) \leq g \Rightarrow f(x, y) \leq h \Leftrightarrow y \in \mathcal{H}(x, h)$.

□

3.2 Quiver spaces over the power set of a given set

Proposition 3.4. *Let X and Y be two nonempty sets, and let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, *)$, a \emptyset -Mpo. Let $x \in X$ and $A, B \in \mathcal{P}(Y)$. Let $(A_i)_{i \in I}$ be a family of subsets of Y . Then,*

- $\forall x \in X, \quad \mathcal{H}(x, \emptyset) = \{x\},$
- $\forall x \in X, \quad \mathcal{H}(x, X) = X,$
- $\forall x, y \in X, \forall A \in \mathcal{P}(Y), \quad y \in \mathcal{H}(x, A) \Leftrightarrow x \in \mathcal{H}(y, A),$
- $\forall x \in X, \forall A, B \in \mathcal{P}(Y), \quad A \subseteq B \Rightarrow \mathcal{H}(x, A) \subseteq \mathcal{H}(x, B),$

•

$$\mathcal{H}(x, \bigcap_{i \in I} A_i) = \bigcap_{i \in I} \mathcal{H}(x, A_i),$$

•

$$\bigcup_{i \in I} \mathcal{H}(x, A_i) = \mathcal{H}(x, \bigcup_{i \in I} A_i).$$

Proof. Let X and Y be two nonempty sets, and let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, *)$, a \emptyset -Mpo. Let $x \in X$ and $A \in \mathcal{P}(Y)$, then, the closed quiver with center x and target A is defined by:

$$\mathcal{H}(x, A) = \{z \in X \mid f(x, z) \subseteq A\}.$$

The first four assertions are particular cases of Proposition 3.3, where $G = \mathcal{P}(Y)$, $\leq = \subseteq$, $e_0 = \emptyset$, and $e_\infty = X$. Let $x \in X$ and let $(A_i)_{i \in I}$ be a family of subsets of Y . Then,

•

$$\begin{aligned} y \in \mathcal{H}(x, \bigcap_{i \in I} A_i) &\Leftrightarrow f(x, y) \subseteq \bigcap_{i \in I} A_i \\ &\Leftrightarrow \forall i \in I, f(x, y) \subseteq A_i \\ &\Leftrightarrow \forall i \in I, y \in \mathcal{H}(x, A_i) \\ &\Leftrightarrow y \in \bigcap_{i \in I} \mathcal{H}(x, A_i). \end{aligned}$$

•

$$\begin{aligned} y \in \bigcup_{i \in I} \mathcal{H}(x, A_i) &\Leftrightarrow \exists i_0 \in I, y \in \mathcal{H}(x, A_{i_0}) \\ &\Leftrightarrow \exists i_0 \in I, f(x, y) \subseteq A_{i_0} \\ &\Leftrightarrow f(x, y) \subseteq \bigcup_{i \in I} A_i \\ &\Leftrightarrow y \in \mathcal{H}(x, \bigcup_{i \in I} A_i). \end{aligned}$$

□

Proposition 3.5. *Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo, such that the operation $*$ is associative, idempotent, and the order relation \leq is compatible with $*$. Then,*

(i) *For all $x, y \in X$ and for all $g, h \in G$,*

$$\mathcal{H}(x, g) \cap \mathcal{H}(y, h) \neq \emptyset \Rightarrow \mathcal{H}(y, h) \subseteq \mathcal{H}(x, g * h),$$

(ii) *For all $x, y \in X$ and for all $g \in G$,*

$$\mathcal{H}(x, g) \cap \mathcal{H}(y, g) \neq \emptyset \Leftrightarrow \mathcal{H}(x, g) = \mathcal{H}(y, g)$$

Proof. (i) Let $x, y \in X$ and $g, h \in G$. Let $s \in \mathcal{H}(y, h)$. Since the intersection of $\mathcal{H}(x, g)$ and $\mathcal{H}(y, h)$ is nonempty, there exists $z \in X$ such that $f(x, z) \leq g$ and $f(y, z) \leq h$. Since (X, f) is a quiver space over $(G, \leq, *)$, we have $f(x, y) \leq f(x, z) * f(z, y)$, and it follows that $f(x, y) \leq g * h$. Moreover, since $s \in \mathcal{H}(y, h)$, we have

$$f(x, s) \leq f(x, y) * f(y, s) \leq (g * h) * h = g * (h * h) = g * h,$$

because $*$ is associative, idempotent, and the order relation \leq is compatible with $*$. Thus, $s \in \mathcal{H}(x, g * h)$, which proves the result.

(ii) Since $*$ is idempotent and for all $x \in X$ and $g \in G$, $\mathcal{H}(x, g) \neq \emptyset$, the result follows immediately from (i) by taking $g = h$.

□

Corollary 3.1. *Let X and Y be two nonempty sets, and let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, \cup)$. Then:*

• *For all $x, y \in X$ and for all $A, B \in \mathcal{P}(Y)$,*

$$\mathcal{H}(x, A) \cap \mathcal{H}(y, B) \neq \emptyset \Rightarrow \mathcal{H}(y, B) \subseteq \mathcal{H}(x, A \cup B)$$

• *For all $x, y \in X$ and for all $A \in \mathcal{P}(Y)$,*

$$\mathcal{H}(x, A) \cap \mathcal{H}(y, A) \neq \emptyset \Leftrightarrow \mathcal{H}(x, A) = \mathcal{H}(y, A)$$

Proof. Let X and Y be two nonempty sets, and let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, \cup)$. The operation \cup is associative, idempotent, and \subseteq is compatible with \cup . Hence, this is a particular case of Proposition 3.4, with $G = \mathcal{P}(Y)$, $\leq = \subseteq$, and $*$ = \cup . □

Proposition 3.6. *Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo. Assume that the operation $*$ is associative, idempotent, and that \leq is compatible with $*$. Then, the collection of closed quivers with target $g \in G$ forms a partition of X .*

Proof. Let (X, f) be a quiver space over $(G, \leq, *)$, an e_0 -Mpo. Let $g \in G$. For all $x \in X$, we have $x \in \mathcal{H}(x, g)$, hence $\mathcal{H}(x, g) \neq \emptyset$ for all $x \in X$. Let $x \in X$. Since $x \in \mathcal{H}(x, g)$, we have

$$X \subseteq \bigcup_{x \in X} \mathcal{H}(x, g),$$

thus

$$X = \bigcup_{x \in X} \mathcal{H}(x, g).$$

Let $x \neq y \in X$. Since the operation $*$ is associative, idempotent, and \leq is compatible with $*$, it follows from (ii) of Proposition 3.5 that

$$\mathcal{H}(x, g) \cap \mathcal{H}(y, g) = \emptyset \quad \text{or} \quad \mathcal{H}(x, g) = \mathcal{H}(y, g).$$

Consequently, the collection of closed quivers with target g forms a partition of X . \square

Remark 3.2. Let $g_0 \in G$. The binary relation defined on X by

$$\forall x, y \in X, x \sim_{g_0} y \Leftrightarrow \mathcal{H}(x, g_0) = \mathcal{H}(y, g_0),$$

is an equivalence relation.

4 Topologies defined on quiver spaces over the power set of a given set

In this section, we exhibit some examples of topologies of quiver spaces over the power set of a given set. Let X, Y be two nonempty sets. Let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, *)$, a \emptyset -Mpo, and let $\mathcal{A} = (A_i)_{i \in I}$ be a family of subsets of Y stable under intersection, i.e.,

$$\forall i, j \in I, \exists k \in I, A_i \cap A_j = A_k.$$

Then we define

$$\tau_{\mathcal{A}}(f) = \{\omega \in \mathcal{P}(X) \mid \forall x \in \omega, \exists i \in I, \mathcal{H}(x, A_i) \subseteq \omega\}.$$

Proposition 4.1. Let X, Y be two nonempty sets. Let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, *)$, a \emptyset -Mpo. Let $\mathcal{A} = (A_i)_{i \in I}$ be a family of subsets of Y stable under intersection. Then $\tau_{\mathcal{A}}(f)$ is a topology on X .

Proof. We have $\emptyset \in \tau_{\mathcal{A}}(f)$ and $X \in \tau_{\mathcal{A}}(f)$. Let $(\omega_i)_{1 \leq i \leq p}$ be a finite family of elements of $\tau_{\mathcal{A}}(f)$. Let $x \in \bigcap_{i=1}^p \omega_i$. Then for all $1 \leq i \leq p$, $x \in \omega_i$, hence for each $1 \leq i \leq p$ there exists $n_i \in I$ such that

$$\mathcal{H}(x, A_{n_i}) \subseteq \omega_i.$$

By Proposition 3.4, it follows that

$$\bigcap_{i=1}^p \mathcal{H}(x, A_{n_i}) = \mathcal{H}(x, \bigcap_{i=1}^p A_{n_i}) \subseteq \bigcap_{i=1}^p \omega_i.$$

Since $\mathcal{A} = (A_i)_{i \in I}$ is stable under intersection, there exists $k \in I$ such that

$$\bigcap_{i=1}^p A_{n_i} = A_k.$$

Thus,

$$\forall x \in \bigcap_{i=1}^p \omega_i, \exists k \in I, \mathcal{H}(x, A_k) \subseteq \bigcap_{i=1}^p \omega_i.$$

Hence,

$$\bigcap_{i=1}^p \omega_i \in \tau_{\mathcal{A}}(f).$$

Let $x \in \bigcup_{i \in I} \omega_i$. Then there exists $i_0 \in I$ such that $x \in \omega_{i_0}$, hence

$$\exists n_0 \in I, \mathcal{H}(x, A_{n_0}) \subseteq \omega_{i_0} \subseteq \bigcup_{i \in I} \omega_i.$$

Consequently,

$$\bigcup_{i \in I} \omega_i \in \tau_{\mathcal{A}}(f).$$

Thus, $\tau_{\mathcal{A}}(f)$ is a topology on X . □

Remark 4.1. Proposition 4.1 still holds if the family $\mathcal{A} = (A_i)_{i \in I}$ is increasing with respect to inclusion, i.e., for all $i, j \in I$,

$$i \leq j \implies A_i \subseteq A_j.$$

Corollary 4.1. Let (X, d) be a metric space, and let f_d be the map from $X \times X$ to $\mathcal{P}(\mathbb{R}_+)$ defined by

$$f_d(x, y) = (0, d(x, y)) \quad \text{for all } x, y \in X.$$

Let $\mathcal{A} = (A_r)_{r \in \mathbb{R}_+^*}$ be the family defined by $A_r = (0, r)$ for all $r > 0$. Then,

$$\tau_{\mathcal{A}}(f_d) = \{\omega \in \mathcal{P}(X) \mid \forall x \in \omega, \exists r > 0, \mathcal{H}(x, (0, r)) \subseteq \omega\}$$

is a topology on X .

Proof. Let (X, d) be a metric space. By Proposition 3.1, (X, f_d) is a quiver space over $(\mathcal{P}(\mathbb{R}_+), \subseteq, +)$. Let $\mathcal{A} = (A_r)_{r \in \mathbb{R}_+^*}$ with $A_r = (0, r)$ for all $r > 0$.

For $r_1, r_2 \in \mathbb{R}_+^*$, we have

$$A_{r_1} \cap A_{r_2} = (0, r_1) \cap (0, r_2) = (0, \min(r_1, r_2)) = A_{\min(r_1, r_2)} \in \mathcal{A}.$$

Thus, the family \mathcal{A} is stable under intersection. Hence, by Proposition 4.1, $\tau_{\mathcal{A}}(f_d)$ is a topology on X . \square

Definition 4.1. Let X and Y be two non-empty sets, and let (X, f) be a quiver space over $(\mathcal{P}(Y), \subseteq, *)$, a \emptyset -Mpo. Let $\mathcal{A} = (A_i)_{i \in I}$ be a family of subsets of Y stable under intersection. Define

$$\tau_{\mathcal{A}_s}(f) = \{\omega \in \mathcal{P}(X) \mid \forall x \in \omega, \exists i \in I, \mathcal{H}_s(x, A_i) \subseteq \omega\}.$$

Proposition 4.2. Let (X, d) be a metric space and f_d the map defined by $f_d(x, y) = (0, d(x, y))$ for all $x, y \in X$. Let $\mathcal{A} = (A_r)_{r \in \mathbb{R}_+^*}$ with $A_r = (0, r)$ for all $r > 0$. Then, the topology $\tau_{\mathcal{A}_s}(f_d)$ coincides with the topology induced by the metric, τ_d .

Proof. • Let $\omega \in \tau_d$ and $x \in \omega$. Then, there exists $r_x > 0$ such that $B(x, r_x) \subseteq \omega$, i.e., for all $z \in X$, $d(x, z) < r_x$ implies $z \in \omega$. Equivalently, for all $z \in X$, $(0, d(x, z)) \subsetneq (0, r_x)$ implies $z \in \omega$. Hence, $\mathcal{H}_s(x, (0, r_x)) \subseteq \omega$, so $\omega \in \tau_{\mathcal{A}_s}(f_d)$.

• Conversely, let $\omega \in \tau_{\mathcal{A}_s}(f_d)$ and $x \in \omega$. Then, there exists $r_x > 0$ such that $\mathcal{H}_s(x, (0, r_x)) \subseteq \omega$, i.e., for all $z \in X$, $f_d(x, z) \subsetneq (0, r_x)$ implies $z \in \omega$. Equivalently, for all $z \in X$, $d(x, z) < r_x$ implies $z \in \omega$. Hence, $B(x, r_x) \subseteq \omega$, so $\omega \in \tau_d$. \square

4.1 Quiver topology

Let (X, f) be a quiver space over $(\mathcal{P}(X), \subseteq, *)$, a \emptyset -Mpo. Define

$$\mathcal{Q}^f = \{\omega \in \mathcal{P}(X) \mid \forall x \in X, \omega \subseteq \mathcal{H}(x, \omega)\}.$$

Proposition 4.3. Let (X, f) be a quiver space over $(\mathcal{P}(X), \subseteq, *)$, a \emptyset -Mpo. Then, \mathcal{Q}^f is a topology on X .

Proof. Let (X, f) be a quiver space over $(\mathcal{P}(X), \subseteq, *)$, a \emptyset -Mpo.

• By Proposition 5.4, for all $x \in X$, $\mathcal{H}(x, \emptyset) = \{x\}$ and $\mathcal{H}(x, X) = X$. Hence, $\emptyset \in \mathcal{Q}^f$ and $X \in \mathcal{Q}^f$.

- Let $\omega = (\omega_i)_{1 \leq i \leq p}$ be a finite family of elements in \mathcal{Q}^f . For all $x \in X$ and all $1 \leq i \leq p$, $\omega_i \subseteq \mathcal{H}(x, \omega_i)$. By Proposition 3.4,

$$\bigcap_{i=1}^p \omega_i \subseteq \bigcap_{i=1}^p \mathcal{H}(x, \omega_i) = \mathcal{H}\left(x, \bigcap_{i=1}^p \omega_i\right).$$

Hence, $\bigcap_{i=1}^p \omega_i \in \mathcal{Q}^f$.

- Let $\omega = (\omega_i)_{i \in I}$ be a (possibly infinite) family of elements in \mathcal{Q}^f . By Proposition 3.4,

$$\bigcup_{i \in I} \omega_i \subseteq \bigcup_{i \in I} \mathcal{H}(x, \omega_i) = \mathcal{H}\left(x, \bigcup_{i \in I} \omega_i\right).$$

Hence, $\bigcup_{i \in I} \omega_i \in \mathcal{Q}^f$.

Therefore, \mathcal{Q}^f is a topology on X . □

Definition 4.2. Let (X, f) be a quiver space over $(\mathcal{P}(X), \subseteq, *)$, a \emptyset -Mpo. Then, \mathcal{Q}^f is called the quiver topology on X relative to the operation $*$.

5 Lipschitzian mapping of a quiver space

Definition 5.1. Let (X, f) be a quiver space over $(G, \leq, *)$ an e_0 -Mpo. Let g be a mapping from X into itself. Then g is said to be h -Lipschitzian if there exists h in G , with $e_0 \neq h$, such that

$$\forall x, y \in X, f(g(x), g(y)) \leq h * f(x, y).$$

Proposition 5.1. Let (X, f) be a quiver space over $(G, \leq, *)$ an e_0 -Mpo. Suppose that the magma $(G, *)$ has a neutral element $i_0 \neq e_0$. Let g be an i_0 -Lipschitzian mapping from X into itself. Then, for all $x \in X$ and all $h \in G$,

$$g(\mathcal{H}(x, h)) \subseteq \mathcal{H}(g(x), h).$$

Proof. Let (X, f) be a quiver space over $(G, \leq, *)$ an e_0 -Mpo. Let $x \in X$ and $h \in G$. Since g is i_0 -Lipschitzian from X into itself, then for all y in X , $f(g(x), g(y)) \leq i_0 * f(x, y) = f(x, y)$. Let $z \in \mathcal{H}(x, h)$, so $f(x, z) \leq h$. It follows that $f(g(x), g(z)) \leq h$, hence $g(z) \in \mathcal{H}(g(x), h)$. Consequently, for all $z \in \mathcal{H}(x, h)$, $g(z) \in \mathcal{H}(g(x), h)$. □

Remark 5.1. In the proposition 5.1, If, x is a fixed point of g , then for all $h \in G$, $\mathcal{H}(x, h)$ is invariant under g .

Let X be a set and $g : X \rightarrow X$ a mapping. We define g^n as the n -th iterate of g , where $n \in \mathbb{N}$, by $g^0 = id_X$ and $g^{n+1} = g \circ g^n$.

Proposition 5.2. *Let (X, f) be a quiver space over $(G, \leq, *)$ an e_0 -Mpo. Suppose that the magma $(G, *)$ has a neutral element $i_0 \neq e_0$. Let g be an i_0 -Lipschitzian mapping from X into itself. Then, for all $x \in X$ and all $h \in G$,*

$$\forall n \in \mathbb{N}^*, g^n(\mathcal{H}(x, h)) \subseteq \mathcal{H}(g^n(x), h).$$

Proof. Let $x \in X$ and $h \in G$. By induction, for $n = 1$, according to the proposition 5.1, it follows that, $g(\mathcal{H}(x, h))$ is a subset of $\mathcal{H}(g(x), h)$. Suppose that, the property hold for a given $n \in \mathbb{N}^*$. Let $z \in g^{n+1}(\mathcal{H}(x, h))$ which means that, $z \in g(g^n(\mathcal{H}(x, h))) \subseteq g(\mathcal{H}(g^n(x), h)) \subseteq \mathcal{H}(g^{n+1}(x), h)$. We conclude that, pour tout $x \in X$ et pour tout $h \in G$, $\forall n \in \mathbb{N}^*$, $g^n(\mathcal{H}(x, h)) \subseteq \mathcal{H}(g^n(x), h)$. \square

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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