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Proof of uniform convergence for a cell-centered AP discretization of the hyperbolic heat equation on conical meshes

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Abstract

We prove here the uniform convergence of an AP scheme for the hyperbolic heat equation on conical meshes. The proof we propose is inspired from [1], where the same result is proved for polygonal meshes. We also give some geometrical results on conical meshes.

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1 Introduction

This work focuses on the properties of a numerical scheme that discretizes the following system, called the P_1 system:

$$\begin{cases} \partial_t \tilde{E}^\varepsilon + \frac{1}{\varepsilon} \operatorname{div} \tilde{\mathbf{F}}^\varepsilon = 0, \\ \partial_t \tilde{\mathbf{F}}^\varepsilon + \frac{1}{\varepsilon} \nabla \tilde{E}^\varepsilon = -\frac{\sigma}{\varepsilon^2} \tilde{\mathbf{F}}^\varepsilon. \end{cases} \quad (1)$$

where ε and $\sigma > 0$ are positive constants. The space variable is $\mathbf{x} \in \mathbb{R}^2$ and the time variable is $t \in \mathbb{R}_+$. The unknowns of system (1) are the radiative energy $\tilde{E}^\varepsilon(t, \mathbf{x})$ and the radiative flux $\tilde{\mathbf{F}}^\varepsilon(t, \mathbf{x}) \in \mathbb{R}^2$. This system is L^2 -stable:

$$\frac{d}{dt} \left(\int_{\Omega} (\tilde{E}^\varepsilon)^2 + \int_{\Omega} \|\tilde{\mathbf{F}}^\varepsilon\|^2 \right) \leq 0.$$

Moreover, as ε vanishes, the flux $\tilde{\mathbf{F}}^\varepsilon$ goes to 0 and \tilde{E}^ε converges toward the solution of the following diffusion equation:

$$\partial_t \tilde{E}^0 - \operatorname{div} \left(\frac{1}{\sigma} \nabla \tilde{E}^0 \right) = 0. \quad (2)$$

This property can be derived using a standard Hilbert expansion. A rigorous proof is given in [1]. Lemma 1.1 below gives the details of this result. A numerical scheme that discretises system (1) which is consistent with this diffusion limit is called *asymptotic preserving (AP)*. Figure 1 illustrates this property.

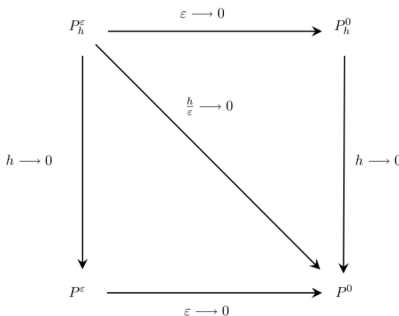


Figure 1: Definition of an AP scheme.

The space discretisation parameter is h , $\varepsilon > 0$ reflects the convergence of the model P^ε (system (1)) toward the limit model P^0 (equation (2)). A scheme P_h^ε consistent with P^ε is said to be *asymptotic preserving* if the scheme P_h^0 computed in the limit $\varepsilon \rightarrow 0$ is consistent with the limit model P^0 .

In this work, we study the AP scheme described in [2] for conical meshes and in [3] polygonal meshes. Our purpose is to prove the uniform convergence of the scheme with respect to ε . More precisely, we want to prove an estimate of the form:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq Ch^\eta, \quad (3)$$

where $\|P_h^\varepsilon - P^\varepsilon\|$ represents the error between the numerical solution and the exact solution for a given ε , and η is a positive constant that does not depend on ε . See Theorem 4.4 for the details of this result. In order to prove (3), we adapt the work [1] to conical meshes. In that work, the authors proved (3) for $\eta = 1/4$ on polygonal meshes. We propose here a proof adapted to conical meshes and which is simpler, yet leading to a rate of convergence of $\eta = 1/6$. Anyway, in practice, the rate of convergence is 1 (and 2 for the limit scheme P_h^0).

The structure of the proof we propose is similar to the one of [1]. We first prove a naive estimate of the form:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq C \sqrt{\frac{h}{\varepsilon}}. \quad (4)$$

The proof of (4) is given in Section 7. Then we derive another estimate using the triangle inequality:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq \|P_h^\varepsilon - P_h^0\| + \|P_h^0 - P^0\| + \|P^0 - P^\varepsilon\|, \quad (5)$$

where:

- $\|P_h^\varepsilon - P_h^0\|$ is the error between the numerical solution for a given ε and the solution computed with the limit scheme,
- $\|P_h^0 - P^0\|$ is the error between the numerical solution computed with the limit scheme and the exact solution to (2),
- $\|P^0 - P^\varepsilon\|$ is the difference between the exact solutions of (2) and (1).

We compute each term in the right hand side of (5) (see respectively Sections 6 and 5 and Lemma 1.1) and end up with:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq C \left(\sqrt{\varepsilon h} + \frac{\varepsilon}{\sqrt{h}} + h + \varepsilon \right),$$

thus leading to:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq C \min \left(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\varepsilon h} + \frac{\varepsilon}{\sqrt{h}} + h + \varepsilon \right). \quad (6)$$

Eventually, Lemma 1.2 gives the desired result (3) for $\eta = 1/6$.

Even though the structure of our proof is similar to the one of [1], we emphasize that it is simpler even in the case of a polygonal scheme. Indeed, in [1], the authors introduced another scheme (namely the Diffusion Approximation scheme DA_h^ε) which depends on h and ε and for which they prove the convergence toward P^0 . It comes down to writing:

$$\|P_h^\varepsilon - P^\varepsilon\| \leq \|P_h^\varepsilon - DA_h^\varepsilon\| + \|DA_h^\varepsilon - P^0\| + \|P^0 - P^\varepsilon\|.$$

Our proof does not need the introduction of an additional scheme and directly relies on the limit scheme and model.

In order to make a clear distinction between the numerical and exact solution, we denote by $(\tilde{E}^\varepsilon, \tilde{\mathbf{F}}^\varepsilon) = (\tilde{E}, \tilde{\mathbf{F}})$ the solution to (1), \tilde{E}^0 the solution to (2). The numerical solution is (E, \mathbf{F}) (which depends on ε but we do not write it so as to simplify the algebra) and E^0 is the solution to the limit scheme. Moreover, vectors are denoted in **bold** in the rest of the paper.

In the following, the constants $(C_m)_m$ that appear in the inequalities we prove are said to be *universal*: they do not depend on h , ε , the numerical solution or the final time (denoted as T), but they may depend on σ and \tilde{E}^0 and its derivatives.

Lemma 1.1. *There exists a universal constant $C_{1,1}$ such that, for any $T \geq 0$ and any $p \in \mathbb{N}$, one has:*

$$\|\tilde{E}^0 - \tilde{E}^\varepsilon\|_{L^2([0,T];H^p(\Omega))} + \|\tilde{\mathbf{F}}^\varepsilon\|_{L^2([0,T];H^p(\Omega))} \leq C_{1,1}(1+T)\varepsilon,$$

and:

$$\frac{d}{dt} \left(\|\tilde{E}^\varepsilon\|_{H^p(\Omega)}^2 + \|\tilde{\mathbf{F}}^\varepsilon\|_{H^p(\Omega)}^2 \right) \leq 0, \quad \frac{d}{dt} \left(\|\tilde{E}^0\|_{H^p(\Omega)}^2 \right) \leq 0.$$

The proof of Lemma 1.1 is given in [1].

Lemma 1.2. *Let $(h, \varepsilon) \in]0, 1]^2$, then:*

$$\min \left(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\varepsilon h} + \frac{\varepsilon}{\sqrt{h}} + h + \varepsilon \right) \leq 4h^{\frac{1}{6}}.$$

Proof. One has:

$$\min \left(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\varepsilon h} + \frac{\varepsilon}{\sqrt{h}} + h + \varepsilon \right) \leq \min \left(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\varepsilon h} \right) + \min \left(\sqrt{\frac{h}{\varepsilon}}, \frac{\varepsilon}{\sqrt{h}} \right) + \min \left(\sqrt{\frac{h}{\varepsilon}}, h \right) + \min \left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \right).$$

Since $\min(a, b) \leq \sqrt{ab}$ and $\varepsilon \leq 1$ and $h \leq 1$, the first and last terms can be bounded by:

$$\min \left(\sqrt{\frac{h}{\varepsilon}}, \sqrt{\varepsilon h} \right) \leq \left(\sqrt{\frac{h}{\varepsilon}} \sqrt{\varepsilon h} \right)^{\frac{1}{2}} = h^{\frac{1}{2}} \leq h^{\frac{1}{6}}, \quad \min \left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \right) \leq \left(\sqrt{\frac{h}{\varepsilon}} \varepsilon \right)^{\frac{1}{2}} \leq (h\varepsilon)^{\frac{1}{4}} \leq h^{\frac{1}{6}}.$$

Besides, the third term is bounded by:

$$\min \left(\sqrt{\frac{h}{\varepsilon}}, h \right) \leq h \leq h^{\frac{1}{6}}.$$

Eventually, using $\min(a, b) \leq a^{\frac{2}{3}} b^{\frac{1}{3}}$, the second term satisfies:

$$\min \left(\sqrt{\frac{h}{\varepsilon}}, \frac{\varepsilon}{\sqrt{h}} \right) \leq \left(\sqrt{\frac{h}{\varepsilon}} \right)^{\frac{2}{3}} \left(\frac{\varepsilon}{\sqrt{h}} \right)^{\frac{1}{3}} = h^{\frac{1}{6}}.$$

□

2 Conical meshes

In this section, we define conical meshes. We follow the presentation of [2, 4]. Important geometrical properties are presented and a quadrature formula is given in order to compute the integral of a smooth function on a conical cell.

2.1 Rational quadratic Bezier curve

A rational quadratic Bezier curve is a curve $\{\mathbf{M}^\omega(q), q \in [0, 1]\}$ such that:

$$\mathbf{M}^\omega(q) = \frac{(1-q)^2 \mathbf{M}_0 + 2\omega q(1-q) \mathbf{M}_1 + q^2 \mathbf{M}_2}{(1-q)^2 + 2\omega q(1-q) + q^2}, \quad (7)$$

where \mathbf{M}_0 and \mathbf{M}_2 are the extremities, \mathbf{M}_1 is the control point and $\omega \geq 0$ a scalar weight. The curve is said to be:

- degenerate if $\omega = 0$,
- elliptic if $\omega \in]0, 1[$,
- parabolic if $\omega = 1$,
- hyperbolic if $\omega > 1$.

In the first case ($\omega = 0$), the curve is equal to the segment $[\mathbf{M}_0, \mathbf{M}_2]$. Note that the control point \mathbf{M}_1 almost never belongs to the curve (except in some very particular cases). This is the main drawback of this parametrisation. This is why we prefer to parametrise the curve in a different way, using a point that lies on the curve. This point is named the *shoulder point* and is defined by:

$$\mathbf{S} = \mathbf{M}^\omega(0.5) = \frac{1}{2}(\mathbf{Q}_0 + \mathbf{Q}_2), \quad \mathbf{Q}_0 = \frac{1}{1+\omega}(\omega\mathbf{M}_1 + \mathbf{M}_0), \quad \mathbf{Q}_2 = \frac{1}{1+\omega}(\omega\mathbf{M}_1 + \mathbf{M}_2). \quad (8)$$

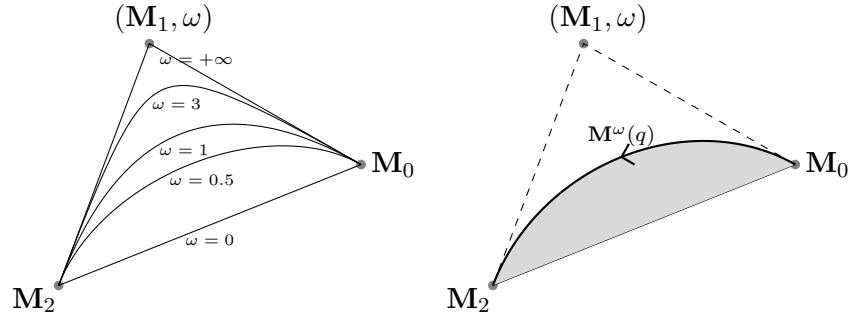


Figure 2: Rational quadratic Bezier curve.

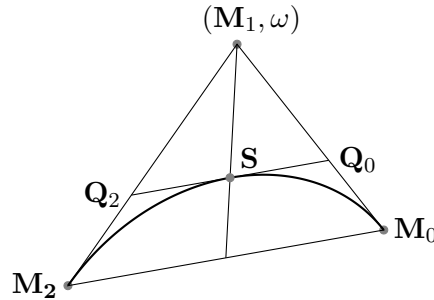


Figure 3: Shoulder point.

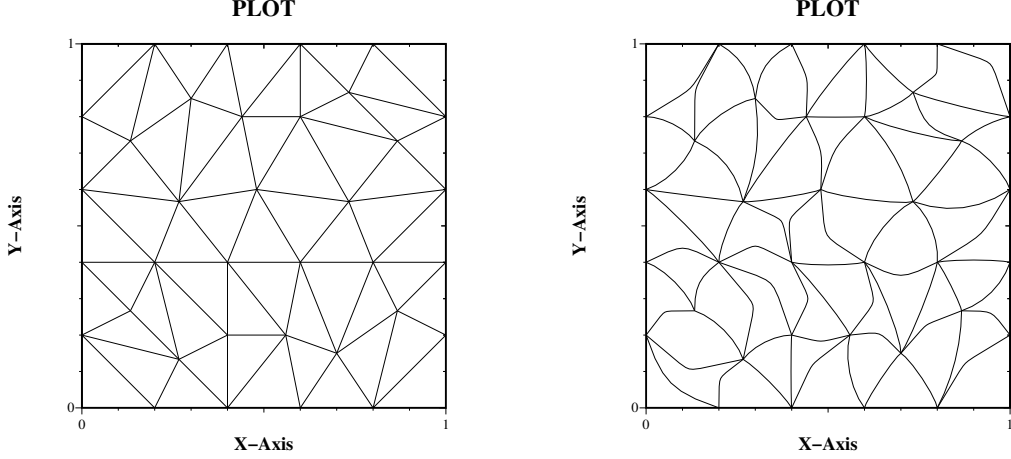


Figure 4: Polygonal (left) and conical (right) meshes.

2.2 Computing the area of a conical cell

Let Ω_j be a conical cell which center is denoted by \mathbf{x}_j . The area of Ω_j can be computed using the following formula:

$$|\Omega_j| = \int_{\Omega_j} d\mathbf{x} = \frac{1}{2} \int_{\partial\Omega_j} \langle \mathbf{x}(\mathbf{s}) - \mathbf{x}_j, \mathbf{N}(\mathbf{x}(\mathbf{s})) \rangle ds, \quad (9)$$

where $\mathbf{N}(\mathbf{x}(\mathbf{s}))$ is the unit normal vector to the edge at the curvilinear coordinate $\mathbf{s} \in \partial\Omega_j$ and ds is the surface measure on $\partial\Omega_j$. In the following, we express this area as a function of the *degrees of freedom (dof)* which are the vertices and the shoulder points. We define the following notation:

- $(\mathbf{x}_r)_r$ the coordinates of the vertices of the cell j ,
- $(\mathbf{M}_{1,r+1/2})_{r+1/2}$ the coordinates of the control points of the cell j ,
- $(\mathbf{x}_{r+1/2})_{r+1/2}$ the coordinates of the shoulder points of the cell j ,
- $\sum_r g_j^r$: sum over all the vertices of the cell j of the quantity g (g_j^r being the evaluation of the function g on the vertex r in cell j),
- $\sum_{r+1/2} g_j^{r+1/2}$: sum over all the shoulder points of the cell j of the quantity g ,
- $\sum_{\text{dof}} g_j^{\text{dof}} = \sum_r g_j^r + \sum_{r+1/2} g_j^{r+1/2}$: sum over all the degrees of freedom (*dof*) (nodes and shoulder points) of the cell j of the quantity g ,
- $N_{\text{dof}} = \sum_i 1$: number of cells that contains the given degree of freedom *dof*,

- $\sum_i g_i^{\text{dof}}$: sum, for a given degree of freedom, over all the cells that contains this degree of freedom,
- $\sum_{j^*} g_j$: sum over all the cells of the mesh,
- $\sum_{r^*} g^r$: sum over all the nodes of the mesh,
- $\sum_{r+1/2^*} g^{r+1/2}$: sum over all the shoulder points of the mesh,
- $\sum_{\text{dof}^*} g^{\text{dof}} = \sum_{r^*} g^r + \sum_{r+1/2^*} g^{r+1/2}$: sum over all the degrees of freedom (nodes and shoulder points) of the mesh,
- $\sum_{\text{dof}'} g^{\text{dof}} - g^{\text{dof}'}$: sum over the neighboring degrees of freedom of dof (those which share a common cell with dof).

The integral (9) can be computed exactly using the coordinates of the vertices $(\mathbf{x}_r)_r$ and the control points $(\mathbf{M}_{1,r+1/2})_{r+1/2}$ (cf [4]):

$$|\Omega_j| = \frac{1}{2} \sum_r \langle \mathbf{C}_j^r, \mathbf{x}_r - \mathbf{x}_j \rangle + \frac{1}{2} \sum_{r+1/2, \text{ control points}} \langle \mathbf{C}_j^{r+1/2}, \mathbf{M}_{1,r+1/2} - \mathbf{x}_j \rangle, \quad (10)$$

it can also be expressed in terms of $(\mathbf{x}_r)_r$ and $(\mathbf{x}_{r+1/2})_{r+1/2}$ (cf [4]):

$$|\Omega_j| = \frac{1}{2} \sum_r \langle \tilde{\mathbf{C}}_j^r, \mathbf{x}_r - \mathbf{x}_j \rangle + \frac{1}{2} \sum_{r+1/2, \text{ shoulder points}} \langle \tilde{\mathbf{C}}_j^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_j \rangle = \frac{1}{2} \sum_{\text{dof}} \langle \tilde{\mathbf{C}}_j^{\text{dof}}, \mathbf{x}_{\text{dof}} - \mathbf{x}_j \rangle. \quad (11)$$

The coefficients \mathbf{C}_j^r , $\mathbf{C}_j^{r+1/2}$, $\tilde{\mathbf{C}}_j^r$ and $\tilde{\mathbf{C}}_j^{r+1/2}$ depend on the geometry of the cell and are given by:

$$\mathbf{C}_j^r = \frac{1}{2} ((1 - f(\omega_{r-1/2}))\mathbf{N}_{r-1,r} + (1 - f(\omega_{r+1/2}))\mathbf{N}_{r,r+1} + f(\omega_{r-1/2})\mathbf{N}_{r-1/2,r} + f(\omega_{r+1/2})\mathbf{N}_{r,r+1/2}),$$

$$\mathbf{C}_j^{r+1/2} = \frac{f(\omega_{r+1/2})}{2} \mathbf{N}_{r,r+1},$$

$$\tilde{\mathbf{C}}_j^r = \frac{1}{2} ((1 - h(\omega_{r-1/2}))\mathbf{N}_{r-1,r} + (1 - h(\omega_{r+1/2}))\mathbf{N}_{r,r+1} + h(\omega_{r-1/2})\tilde{\mathbf{N}}_{r-1/2,r} + h(\omega_{r+1/2})\tilde{\mathbf{N}}_{r,r+1/2}), \quad (12)$$

and:

$$\tilde{\mathbf{C}}_j^{r+1/2} = \frac{h(\omega_{r+1/2})}{2} \mathbf{N}_{r,r+1}, \quad (13)$$

and:

$$\mathbf{N}_{r,r+1} = (\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp, \quad \mathbf{N}_{r,r+1/2} = (\mathbf{M}_{1,r+1/2} - \mathbf{x}_r)^\perp, \quad \mathbf{N}_{r+1/2,r+1} = (\mathbf{x}_{r+1} - \mathbf{M}_{1,r+1/2})^\perp,$$

$$\tilde{\mathbf{N}}_{r,r+1/2} = (\mathbf{x}_{r+1/2} - \mathbf{x}_r)^\perp, \quad \tilde{\mathbf{N}}_{r+1/2,r+1} = (\mathbf{x}_{r+1} - \mathbf{x}_{r+1/2})^\perp,$$

with:

$$h(\omega) = f(\omega) \frac{1+\omega}{\omega}, \quad f(\omega) = \begin{cases} \frac{2\omega}{1-\omega^2} \left(\frac{1}{\sqrt{1-\omega^2}} \arctan \left(\sqrt{\frac{1-\omega}{1+\omega}} \right) - \frac{\omega}{2} \right) & \text{if } \omega \in [0, 1[, \\ \frac{2}{3} & \text{if } \omega = 1, \\ \frac{\omega}{\omega^2-1} \left(\omega + \frac{1}{\sqrt{\omega^2-1}} \log(\omega - \sqrt{\omega^2-1}) \right) & \text{if } \omega > 1, \end{cases} \quad (14)$$

and for any vector $\xi \in \mathbb{R}^2$:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi^\perp = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}.$$

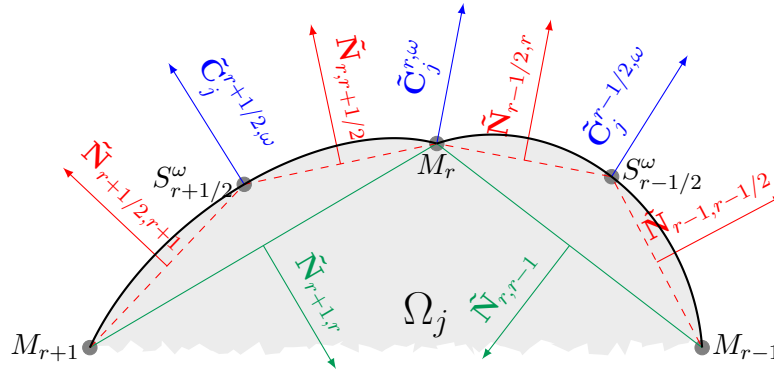


Figure 5: Normal vector from a degree of freedom defined on boundary cell Ω_j . Two types: endpoints \mathbf{M}_r denoted by $\tilde{\mathbf{C}}_j^r$ or shoulder point denoted by $\tilde{\mathbf{C}}_j^{r+1/2}$

Figure 6 displays the curves of functions f and h in (14).

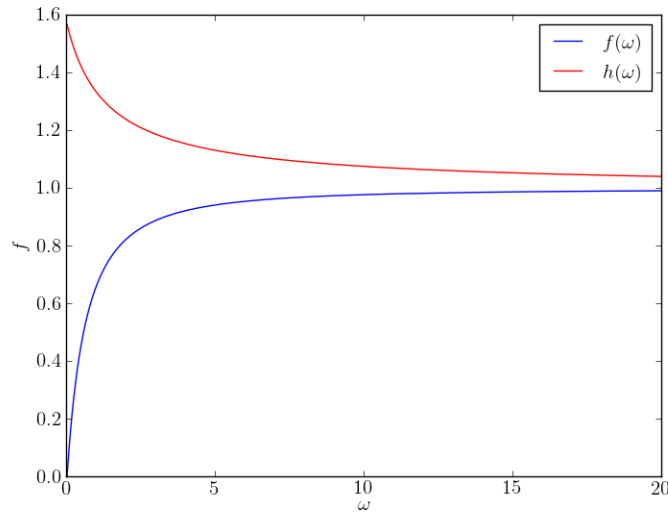


Figure 6: functions f and h .

Equations (10) and (11) are a consequence of Theorem 2.1. Moreover, the following properties are satisfied:

- for any cell j :

$$\sum_{\text{dof}} \tilde{\mathbf{C}}_j^{\text{dof}} = \mathbf{0},$$

- for any inner degree of freedom dof :

$$\sum_i \tilde{\mathbf{C}}_i^{\text{dof}} = \mathbf{0}. \quad (15)$$

For a given quantity defined on the degrees of freedom $(u_{\text{dof}})_{\text{dof}}$, we define its L^2 norm by:

$$\|u_{\text{dof}}\|_{L^2(\Omega)} = \left(\sum_{\text{dof}^*} h^2 |u_{\text{dof}}|^2 \right)^{\frac{1}{2}},$$

For a quantity that is defined on the cells $(u_j)_{j < J}$ and for a given function \tilde{u} , we define:

$$\|u\|_{L^2(\Omega)} = \left(\sum_{j^*} |\Omega_j| |u_j|^2 \right)^{\frac{1}{2}}, \quad \|u - \tilde{u}\|_{L^2(\Omega)} = \left(\sum_{j^*} |\Omega_j| |u_j - \tilde{u}(\mathbf{x}_j)|^2 \right)^{\frac{1}{2}}.$$

Similarly, for a given quantity given on the shoulder points $(u_{r+1/2})_{r+1/2}$:

$$\|u_{r+1/2}\|_{L^2(\Omega)} = \left(\sum_{r+1/2^*} h^2 |u_{r+1/2}|^2 \right)^{\frac{1}{2}}.$$

We also define:

$$\|u_{\text{dof}} - u_j\|_{L^2(\Omega)} = \left(\sum_{\text{dof}^*} h^2 \sum_i |u_{\text{dof}} - u_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{j^*} h^2 \sum_{\text{dof}} |u_{\text{dof}} - u_j|^2 \right)^{\frac{1}{2}}, \quad (16)$$

To conclude, the following theorem is used in the proof of the consistency of the limit scheme. It states that the quadrature formula (11) is of order 2:

Theorem 2.1. *Let $\mathbf{g} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ be a vector-valued function. Then:*

$$\frac{1}{|\Omega_j|} \int_{\partial\Omega_j} \langle \mathbf{g}, \mathbf{N} \rangle = \frac{1}{|\Omega_j|} \sum_{\text{dof}} \langle \mathbf{g}(\mathbf{x}_{\text{dof}}), \tilde{\mathbf{C}}_j^{\text{dof}} \rangle + O(h).$$

More precisely:

$$\left| \int_{\partial\Omega_j} \langle \mathbf{g}, \mathbf{N} \rangle - \sum_{\text{dof}} \langle \mathbf{g}(\mathbf{x}_{\text{dof}}), \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \right| \leq Ch |\Omega_j| \|\nabla^2 \mathbf{g}\|_{L^\infty(\Omega_j)},$$

where C is independant of the h , Ω_j and \mathbf{g} .

The result is also true for any scalar function $g \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$:

$$\frac{1}{|\Omega_j|} \int_{\partial\Omega_j} g \mathbf{N} = \frac{1}{|\Omega_j|} \sum_{\text{dof}} g(\mathbf{x}_{\text{dof}}) \tilde{\mathbf{C}}_j^{\text{dof}} + O(h), \quad \left| \int_{\partial\Omega_j} g \mathbf{N} - \sum_{\text{dof}} g(\mathbf{x}_{\text{dof}}) \tilde{\mathbf{C}}_j^{\text{dof}} \right| \leq Ch |\Omega_j| \|\nabla^2 g\|_{L^\infty(\Omega_j)}.$$

The following result is a consequence of the previous theorem.

Corollary 2.2. *The following equality holds true for any cell j :*

$$\sum_{\text{dof}} \beta_j^{\text{dof}} = |\Omega_j| I_2,$$

where $\beta_j^{\text{dof}} = \tilde{\mathbf{C}}_j^{\text{dof}} \otimes (\mathbf{x}_{\text{dof}} - \mathbf{x}_j)$.

3 Numerical method

We refer to [2] for the details of the construction of this scheme (JLb scheme). It writes:

$$\begin{cases} |\Omega_j| \frac{d}{dt} E_j + \frac{1}{\varepsilon} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = 0, \\ |\Omega_j| \frac{d}{dt} \mathbf{F}_j + \frac{1}{\varepsilon} \sum_{\text{dof}} \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}) = \mathbf{0}, \end{cases} \quad (17)$$

where \mathbf{F}_{dof} satisfies:

$$\left(\alpha_{\text{dof}} + \frac{\sigma}{\varepsilon} \beta_{\text{dof}} \right) \mathbf{F}_{\text{dof}} = \sum_i E_i \tilde{\mathbf{C}}_i^{\text{dof}} + \alpha_i^{\text{dof}} \mathbf{F}_i. \quad (18)$$

The matrices α_j^{dof} and β_j^{dof} are defined as:

$$\alpha_j^{\text{dof}} = \tilde{\mathbf{C}}_j^{\text{dof}} \otimes \tilde{\mathbf{n}}_j^{\text{dof}}, \quad \beta_j^{\text{dof}} = \tilde{\mathbf{C}}_j^{\text{dof}} \otimes (\mathbf{x}_{\text{dof}} - \mathbf{x}_j), \quad \tilde{\mathbf{n}}_j^{\text{dof}} = \frac{\tilde{\mathbf{C}}_j^{\text{dof}}}{\ell_j^{\text{dof}}}, \quad \ell_j^{\text{dof}} = \|\tilde{\mathbf{C}}_j^{\text{dof}}\|, \quad (19)$$

and:

$$\alpha_{\text{dof}} = \sum_i \alpha_i^{\text{dof}}, \quad \beta_{\text{dof}} = \sum_i \beta_i^{\text{dof}}. \quad (20)$$

For any node r , this reads as:

$$\left(\alpha_r + \frac{\sigma}{\varepsilon} \beta_r \right) \mathbf{F}_r = \sum_i E_i \tilde{\mathbf{C}}_i^r + \alpha_i^r \mathbf{F}_i. \quad (21)$$

We present in the next sections some sufficient conditions on the mesh so that the matrix of (21) is positive definite, and thus invertible. However, for any shoulder point $r + 1/2$, the matrix of system (18) has rank 1 and thus it is not invertible. The vector $\mathbf{F}_{r+1/2}$ is computed as the solution to:

$$\begin{cases} \langle \mathbf{F}_{r+1/2}, \mathbf{D}_{r+1/2} \rangle = s_{r+1/2}, \\ \langle \mathbf{F}_{r+1/2}, \mathbf{D}_{r+1/2}^\perp \rangle = \frac{1}{2} \langle \mathbf{F}_r + \mathbf{F}_{r+1}, \mathbf{D}_{r+1/2}^\perp \rangle, \end{cases}$$

with:

$$\mathbf{D}_{r+1/2} = 2\tilde{\mathbf{n}}_j^{r+1/2} + \frac{\sigma}{\varepsilon} (\mathbf{x}_i - \mathbf{x}_j), \quad s_{r+1/2} = E_j - E_i + \left\langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{F}_j + \mathbf{F}_i \right\rangle. \quad (22)$$

where i and j are the two cells that contain the shoulder point $r + 1/2$.

The limit scheme writes:

$$|\Omega_j| \frac{d}{dt} E_j^0 + \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}}^0, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = 0, \quad (23)$$

with, for any node r and any shoulder point $r + 1/2$:

$$\sigma\beta_r\mathbf{F}_r^0 = \sum_i E_i^0 \tilde{\mathbf{C}}_i^r, \quad \begin{cases} \langle \mathbf{F}_{r+1/2}^0, \mathbf{x}_i - \mathbf{x}_j \rangle = \frac{E_j^0 - E_i^0}{\sigma}, \\ \langle \mathbf{F}_{r+1/2}^0, (\mathbf{x}_i - \mathbf{x}_j)^\perp \rangle = \frac{1}{2} \langle \mathbf{F}_r^0 + \mathbf{F}_{r+1}^0, (\mathbf{x}_i - \mathbf{x}_j)^\perp \rangle. \end{cases} \quad (24)$$

Remark 1. Notice that, for any $(\xi_1, \xi_2) \in (\mathbb{R}^2)^2$:

$$\langle \alpha_j^{dof} \xi_1, \xi_2 \rangle = \ell_j^{dof} \langle \xi_1, \tilde{\mathbf{n}}_j^{dof} \rangle \langle \xi_2, \tilde{\mathbf{n}}_j^{dof} \rangle, \quad \langle \beta_j^{dof} \xi_1, \xi_2 \rangle = \ell_j^{dof} \langle \xi_1, \mathbf{x}_{dof} - \mathbf{x}_j \rangle \langle \xi_2, \tilde{\mathbf{n}}_j^{dof} \rangle,$$

4 Geometric properties

4.1 Assumptions on the mesh

In this section, we present the assumptions on the mesh that are used in the rest of the paper.

We denote by h the characteristic length of the mesh ($h = \Delta x = \Delta y$ for a cartesian mesh). We assume that there exists a numerical constant $C_1 > 0$ independent of h such that, for any cell j and any dof :

$$\frac{1}{C_1} h^2 \leq |\Omega_j| \leq C_1 h^2, \quad \frac{1}{C_1} h \leq \|\tilde{\mathbf{C}}_j^{dof}\| \leq C_1 h, \quad N_{dof} \leq C_1.$$

We define: $\alpha_j = \sum_{dof} \alpha_j^{dof}$. The matrices α_r and α_j are symmetric positive, we also assume that they are invertible, meaning that:

$$\forall \xi \in \mathbb{R}^2, \langle \alpha_r \xi, \xi \rangle \geq \frac{1}{C_1} h \|\xi\|^2, \quad \langle \alpha_j \xi, \xi \rangle \geq \frac{1}{C_1} h \|\xi\|^2, \quad (25)$$

In addition, we assume that, for any cell j and any dof of j :

$$\frac{1}{C_1} h \leq \|\mathbf{x}_{dof} - \mathbf{x}_j\| \leq C_1 h, \quad (26)$$

and for any neighboring cells i and j :

$$\frac{1}{C_1} h \leq \|\mathbf{x}_i - \mathbf{x}_j\| \leq C_1 h. \quad (27)$$

Eventually, we define:

$$\gamma_1 = \frac{1}{h^2} \min_{j, dof} \langle \tilde{\mathbf{C}}_j^{dof}, \mathbf{x}_{dof} - \mathbf{x}_j \rangle, \quad \gamma_2 = \frac{1}{h^2} \max_{j, dof} |\langle \tilde{\mathbf{C}}_j^{dof}, (\mathbf{x}_{dof} - \mathbf{x}_j)^\perp \rangle|, \quad (28)$$

and we assume that:

$$\gamma_1 > \tilde{C}_1 \gamma_2, \quad \frac{1}{C_1} - \tilde{C}_1 (\gamma_2 + \gamma_2^2) > 0, \quad \tilde{C}_1 = 8C_1^3. \quad (29)$$

4.2 Important results

Under the assumptions of the previous section, we can prove some very important results that are essential for the rest of the paper. The proofs are given in the Appendix. The first one is the fact that the matrices $(\beta_r)_r$ are positive definite:

Theorem 4.1. *There exists a constant C such that, for any node r :*

$$\forall \xi \in \mathbb{R}^2, \langle \beta_r \xi, \xi \rangle \geq C_{4.1} h^2 \|\xi\|^2. \quad (30)$$

The following theorem is very useful since it allows to bound from below the sum $\sum_{dof} \langle \beta_{dof} \mathbf{u}_{dof}, \mathbf{u}_{dof} \rangle$ by $\|\mathbf{u}_{dof}\|_{L^2(\Omega)}^2$. This inequality is easy to prove for the polygonal scheme since the matrices $(\beta_r)_r$ are positive definite. However, in the conical case, the matrices $(\beta_{r+1/2})_{r+1/2}$ are never positive definite.

Theorem 4.2. Let $(\mathbf{u}_{dof})_{dof}, (u_{r+1/2})_{r+1/2}$ be such that, for any shoulder point $r + 1/2$:

$$\langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle = \frac{1}{2} \langle \mathbf{u}_r + \mathbf{u}_{r+1}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle + u_{r+1/2},$$

with:

$$\tilde{\mathbf{D}}_{r+1/2} = \frac{\mathbf{D}_{r+1/2}}{\|\mathbf{D}_{r+1/2}\|} = \frac{2\varepsilon \tilde{\mathbf{n}}_j^{r+1/2} + \sigma(\mathbf{x}_i - \mathbf{x}_j)}{\|2\varepsilon \tilde{\mathbf{n}}_j^{r+1/2} + \sigma(\mathbf{x}_i - \mathbf{x}_j)\|}.$$

Then there exists a universal constant $C_{4.2} > 0$ such that:

$$\sum_{dof^*} \langle \beta_{dof} \mathbf{u}_{dof}, \mathbf{u}_{dof} \rangle \geq \frac{1}{C_{4.2}} \|\mathbf{u}_{dof}\|_{L^2(\Omega)}^2 - C_{4.2} \|u_{r+1/2}\|_{L^2(\Omega)}^2.$$

The following lemma is used in the proof of Theorem 4.2.

Lemma 4.3. For any $(\sigma, \varepsilon) \in (\mathbb{R}_+^*)^2$, one has:

$$C_{\mathbf{D}} \left(1 + \frac{\sigma h}{\varepsilon}\right) \geq \|\mathbf{D}_{r+1/2}\| \geq \frac{1}{C_{\mathbf{D}}} \left(1 + \frac{\sigma h}{\varepsilon}\right),$$

with $C_{\mathbf{D}} = \max(2, C_1)$.

Eventually, the aim of the rest of the paper is to prove the following:

Theorem 4.4. We assume that the assumptions of Section 4.1 are fulfilled. We also assume that the initial energies $\tilde{E}(t=0)$ and $E(t=0)$ do not depend on ε and that they satisfy:

$$\|E(t=0) - \tilde{E}(t=0)\|_{L^2(\Omega)} \leq C_{init} h, \quad (31)$$

and that the initial fluxes satisfy:

$$\|\mathbf{F}(t=0) - \tilde{\mathbf{F}}(t=0)\|_{L^2(\Omega)} \leq C_{init} h, \quad \|\mathbf{F}(t=0)\|_{L^2(\Omega)} + \|\tilde{\mathbf{F}}(t=0)\|_{L^2(\Omega)} \leq C_{init} \varepsilon. \quad (32)$$

Then there exists a constant C such that, for any final time $T \geq 0$:

$$\|E - \tilde{E}\|_{L^2([0, T] \times \Omega)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{L^2([0, T] \times \Omega)} \leq C \left(1 + T^{\frac{3}{2}}\right) h^{\frac{1}{6}}.$$

5 Convergence of the limit diffusion scheme (23)

In this section, we prove the convergence of the limit diffusion scheme (23) toward the solution of (2) for sufficiently smooth data. To this aim, we adapt the proof of [5]. We prove an estimate of the form:

$$\|P_h^0 - P^0\| \leq Ch,$$

where the meaning of this notation is explained in the Introduction. More precisely, we define the following errors:

$$e_j^0(t) = E_j^0(t) - \tilde{E}^0(t, \mathbf{x}_j), \quad \mathbf{f}_{\text{dof}}^0(t) = \mathbf{F}_{\text{dof}}^0(t) + \frac{1}{\sigma} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}).$$

Therefore, using the notation of Section 2.2:

$$\|e^0(t)\|_{L^2(\Omega)}^2 = \sum_{j^*} |\Omega_j| |E_j^0(t) - \tilde{E}^0(t, \mathbf{x}_j)|^2, \quad \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 = \sum_{\text{dof}^*} h^2 \|\mathbf{f}_{\text{dof}}^0\|^2,$$

and we prove the following theorem:

Theorem 5.1. *Under the assumptions of Theorem 4.4, the diffusion limit scheme (23) is first order convergent:*

$$\|e^0\|_{L^2([0,T] \times \Omega)} + \|\mathbf{f}_{\text{dof}}^0\|_{L^2([0,T] \times \Omega)} \leq C_{5.1} h \left(1 + T^{\frac{3}{2}}\right).$$

To this aim, we first prove that the limit scheme (23) is stable and consistent.

Lemma 5.2. *The limit scheme (23) is L^2 stable:*

$$\frac{d}{dt} \|E^0\|_{L^2(\Omega)}^2 + C_{5.2} \|\mathbf{F}_{\text{dof}}^0\|_{L^2(\Omega)}^2 \leq 0, \quad (33)$$

and:

$$\|E^0(t)\|_{L^2(\Omega)}^2 + C_{5.2} \sigma \int_0^t \|\mathbf{F}_{\text{dof}}^0\|_{L^2(\Omega)}^2 \leq \|E^0(t=0)\|_{L^2(\Omega)}^2. \quad (34)$$

Moreover, defining:

$$a_j(t) = \partial_t \tilde{E}^0(t, \mathbf{x}_j) - \frac{1}{|\Omega_j|} \frac{1}{\sigma} \sum_{\text{dof}} \langle \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}), \tilde{\mathbf{C}}_j^{\text{dof}} \rangle,$$

$$\mathbf{b}_{\text{dof}}(t) = \frac{1}{\sigma} \left(\beta_{\text{dof}} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}) + \sum_i \tilde{E}^0(t, \mathbf{x}_i) \tilde{\mathbf{C}}_i^{\text{dof}} \right),$$

and:

$$b_{r+1/2}(t) = \left\langle \nabla \tilde{E}^0(t, \mathbf{x}_{r+1/2}) - \frac{1}{2} [\nabla \tilde{E}^0(t, \mathbf{x}_r) + \nabla \tilde{E}^0(t, \mathbf{x}_{r+1})], \frac{(\mathbf{x}_i - \mathbf{x}_j)^\perp}{\|\mathbf{x}_i - \mathbf{x}_j\|} \right\rangle,$$

we have the following result:

Lemma 5.3. *The limit scheme is consistent of order 1, that is to say, there exists a universal constant $C_{5.3} > 0$ such that:*

$$\|a\|_{L^2(\Omega)} \leq C_{5.3} h, \quad \|\mathbf{b}_{\text{dof}}\|_{L^2(\Omega)} \leq C_{5.3} h^3, \quad \|b_{r+1/2}\|_{L^2(\Omega)} \leq C_{5.3} h^2.$$

The idea is to prove that $\beta_r^{-1} \sum_i \tilde{E}^0(t, \mathbf{x}_i) \tilde{\mathbf{C}}_i^{\text{dof}}$ is an approximation to $-(\nabla \tilde{E}^0)_r$.

5.1 Proof of Lemma 5.2

Using (23), one has:

$$\frac{1}{2} \frac{d}{dt} \left(\|E^0\|_{L^2(\Omega)}^2 \right) = \sum_{j^*} |\Omega_j| E_j^0 \frac{d}{dt} E_j^0 = - \sum_{j^*} \sum_{\text{dof}} E_j^0 \langle \mathbf{F}_{\text{dof}}^0, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = - \sum_{\text{dof}^*} \sum_i E_i^0 \langle \mathbf{F}_{\text{dof}}^0, \tilde{\mathbf{C}}_i^{\text{dof}} \rangle.$$

According to (24), one can write, for any *dof*:

$$\sum_i E_i^0 \langle \mathbf{F}_{\text{dof}}^0, \tilde{\mathbf{C}}_i^{\text{dof}} \rangle = \sigma \langle \beta_{\text{dof}} \mathbf{F}_{\text{dof}}^0, \mathbf{F}_{\text{dof}}^0 \rangle.$$

Therefore, exchanging the sum over the cells and the sum over the degrees of freedom, one can write:

$$\frac{1}{2} \frac{d}{dt} \left(\|E^0\|_{L^2(\Omega)}^2 \right) = -\sigma \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{F}_{\text{dof}}^0, \mathbf{F}_{\text{dof}}^0 \rangle,$$

which is negative by virtue of Theorem 4.2. This gives equation (33). Equation (34) is obtained by integrating (33) in time.

5.2 Proof of Lemma 5.3

Since the solution to the diffusion equation (2) is smooth (see [6] for instance), one can write:

$$\partial_t \tilde{E}^0(t, \mathbf{x}_j) = \frac{1}{|\Omega_j|} \int_{\Omega_j} \partial_t \tilde{E}^0(t, \mathbf{x}) d\mathbf{x} + O(h^2),$$

since \mathbf{x}_j is the barycenter of cell Ω_j . Therefore:

$$\partial_t \tilde{E}^0(t, \mathbf{x}_j) = \frac{1}{|\Omega_j|} \int_{\Omega_j} \text{div} \left(\frac{1}{\sigma} \nabla \tilde{E}^0 \right) (t, \mathbf{x}) d\mathbf{x} + O(h^2) = \frac{1}{|\Omega_j|} \int_{\partial\Omega_j} \frac{1}{\sigma} \langle \nabla \tilde{E}^0(t, \mathbf{x}), \mathbf{N} \rangle d\mathbf{x} + O(h^2),$$

and according to Theorem 2.1:

$$\partial_t \tilde{E}^0(t, \mathbf{x}_j) = \frac{1}{|\Omega_j|} \frac{1}{\sigma} \sum_{\text{dof}} \langle \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}), \tilde{\mathbf{C}}_j^{\text{dof}} \rangle + O(h^2).$$

Therefore:

$$|a_j(t)| \leq C_{35} h^2, \quad \|a\|_{L^2(\Omega)} \leq C_{35} h^2, \quad (35)$$

here the constant C_{30} depends on the gradient of \tilde{E}^0 , which is bounded uniformly in time according to Lemma 1.1. In addition, we have:

$$\mathbf{b}_{\text{dof}} = \sum_i \underbrace{\left(\tilde{E}^0(t, \mathbf{x}_i) - \tilde{E}^0(t, \mathbf{x}_{\text{dof}}) - \langle \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}), \mathbf{x}_i - \mathbf{x}_{\text{dof}} \rangle \right)}_{=O(h^2)} \tilde{\mathbf{C}}_j^{\text{dof}} = O(h^3),$$

which leads to:

$$\|\mathbf{b}_{\text{dof}}\|_{L^2(\Omega)} \leq C_{36} h^3. \quad (36)$$

Here again the constant C_{31} depends on the gradient of \tilde{E}^0 . Eventually, one has:

$$\nabla \tilde{E}^0(t, \mathbf{x}_{r+1/2}) - \frac{1}{2} (\nabla \tilde{E}^0(t, \mathbf{x}_r) + \nabla \tilde{E}^0(t, \mathbf{x}_{r+1})) = O(h^2), \quad \|\mathbf{b}_{r+1/2}\|_{L^2(\Omega)} \leq C_{37} h^2. \quad (37)$$

5.3 Proof of Theorem 5.1

First and foremost, one can write, using the linearity of the scheme:

$$\begin{cases} \frac{d}{dt}e_j^0 + \frac{1}{|\Omega_j|} \sum_{\text{dof}} \langle \tilde{\mathbf{C}}_j^{\text{dof}}, \mathbf{f}_{\text{dof}}^0 \rangle = -a_j, \\ \beta_{\text{dof}} \mathbf{f}_{\text{dof}}^0 = \frac{1}{\sigma} \sum_i e_i^0 \tilde{\mathbf{C}}_i^{\text{dof}} - \mathbf{b}_{\text{dof}}, \\ \left\langle \mathbf{f}_{r+1/2}^0, \frac{(\mathbf{x}_i - \mathbf{x}_j)^\perp}{\|\mathbf{x}_i - \mathbf{x}_j\|} \right\rangle = \left\langle \frac{\mathbf{f}_r^0 + \mathbf{f}_{r+1}^0}{2}, \frac{(\mathbf{x}_i - \mathbf{x}_j)^\perp}{\|\mathbf{x}_i - \mathbf{x}_j\|} \right\rangle - b_{r+1/2}. \end{cases} \quad (38)$$

Using the first equation of (38), one has:

$$\frac{1}{2} \frac{d}{dt} \left(\|e^0\|_{L^2(\Omega)}^2 \right) = \sum_{j^*} |\Omega_j| e_j^0 \frac{d}{dt} e_j^0 = - \sum_{j^*} |\Omega_j| e_j^0 a_j - \sum_{j^*} e_j^0 \sum_{\text{dof}} \langle \tilde{\mathbf{C}}_j^{\text{dof}}, \mathbf{f}_{\text{dof}}^0 \rangle.$$

Exchanging the sum over the cells and th sum over the degrees of freedom leads to:

$$\frac{1}{2} \frac{d}{dt} \left(\|e^0\|_{L^2(\Omega)}^2 \right) = - \sum_{j^*} |\Omega_j| e_j^0 a_j - \sigma \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}^0, \mathbf{f}_{\text{dof}}^0 \rangle - \sum_{\text{dof}^*} \langle \mathbf{b}_{\text{dof}}, \mathbf{f}_{\text{dof}}^0 \rangle.$$

Moreover, applying Theorem 4.2 to $\mathbf{u}_{\text{dof}} = \mathbf{f}_{\text{dof}}^0$ and $u_{r+1/2} = b_{r+1/2}$, one can write:

$$\sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}^0, \mathbf{f}_{\text{dof}}^0 \rangle \geq \frac{1}{C_{4.2}} \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 - C_{4.2} \|b_{r+1/2}\|_{L^2(\Omega)}^2.$$

Using Lemma 5.3, one has:

$$\sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}^0, \mathbf{f}_{\text{dof}}^0 \rangle \geq \frac{1}{C_{39}} \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 - C_{39} h^4, \quad (39)$$

and:

$$\left| \sum_{j^*} |\Omega_j| e_j^0 a_j \right| \leq C_{40} h \|e^0\|_{L^2(\Omega)}, \quad \left| \sum_{\text{dof}^*} \langle \mathbf{b}_{\text{dof}}, \mathbf{f}_{\text{dof}}^0 \rangle \right| \leq C_{40} h \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}. \quad (40)$$

This leads to:

$$\frac{d}{dt} \left(\|e^0\|_{L^2(\Omega)}^2 \right) + \frac{1}{C_{41}} \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 \leq C_{41} [h \|e^0\|_{L^2(\Omega)} + h^4 + h \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}], \quad (41)$$

for some universal constant $C_{41} > 0$. Moreover, one can write, for any $\gamma > 0$:

$$h \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)} \leq \gamma \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 + \frac{1}{\gamma} h^2.$$

Thus choosing γ small enough (and independently of h), equation (41) implies:

$$\frac{d}{dt} \left(\|e^0\|_{L^2(\Omega)}^2 \right) + \frac{1}{C_{42}} \|\mathbf{f}_{\text{dof}}^0\|_{L^2(\Omega)}^2 \leq C_{42} [h \|e^0\|_{L^2(\Omega)} + h^2 + h^4], \quad (42)$$

for some universal constant $C_{42} > 0$. Using $h \leq 1$, we thus have:

$$\frac{d}{dt} \left(\|e^0\|_{L^2(\Omega)}^2 \right) \leq C_{42} h \left(\|e^0\|_{L^2(\Omega)} + h \right).$$

Therefore, applying Lemma 9.1 to $f = \|e^0\|_{L^2(\Omega)}$, $k_1 = Ch$ and $k_2 = h$ and using the assumption on the initial data (31), there exists a constant $C_{43} > 0$ such that:

$$\|e^0(t)\|_{L^2(\Omega)}^2 \leq C_{43} h^2 (1 + t^2), \quad \|e^0\|_{L^2([0,T] \times \Omega)}^2 \leq C_{43} h^2 (1 + T^3). \quad (43)$$

Integrating equation (42) in time gives:

$$\|\mathbf{f}_{\text{dof}}^0\|_{L^2([0,T]\times\Omega)}^2 \leq C_{44}h^2(T^2 + 1), \quad (44)$$

for some universal constant $C_{44} > 0$.

6 Study of $\|P_h^0 - P_h^\varepsilon\|$

In this section, we prove an estimate of the form:

$$\|P_h^0 - P_h^\varepsilon\| \leq C \left(\sqrt{h\varepsilon} + \frac{\varepsilon}{\sqrt{h}} \right).$$

More precisely, we denote by (E, \mathbf{F}) the solution to (17) (which depends on ε but we do not make it explicit). We denote by E^0 the solution to the limit diffusion scheme (23). We define: $e = E - E^0$ and:

$$\mathbf{F}_j^0 = \alpha_j^{-1} \left(\sum_{\text{dof}} \alpha_j^{\text{dof}} \mathbf{F}_{\text{dof}}^0 \right), \quad \text{that is to say: } \sum_{\text{dof}} \alpha_j^{\text{dof}} (\mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0) = \mathbf{0}, \quad (45)$$

as well as:

$$\mathbf{f}_{\text{dof}} = \mathbf{F}_{\text{dof}} - \varepsilon \mathbf{F}_{\text{dof}}^0, \quad \mathbf{f}_j = \mathbf{F}_j - \varepsilon \mathbf{F}_j^0.$$

We prove the following:

Theorem 6.1. *Under the assumptions of Theorem 4.4, there exists a constant $C_{6.1} > 0$ such that:*

$$\|e\|_{L^2([0,T] \times \Omega)}^2 + \|\mathbf{f}\|_{L^2([0,T] \times \Omega)}^2 \leq C_{6.1} (1 + T^3) \left(\varepsilon h + \frac{\varepsilon^2}{h} \right).$$

In order to prove Theorem 6.1, we prove the following lemmas:

Lemma 6.2. *The scheme (17) is L^2 stable, that is, the solution (E, \mathbf{F}) to (17) satisfies:*

$$\frac{d}{dt} \left(\|E\|_{L^2(\Omega)}^2 + \|\mathbf{F}\|_{L^2(\Omega)}^2 \right) + C_{6.2} \frac{1}{\varepsilon^2} \|\mathbf{F}_{\text{dof}}\|_{L^2(\Omega)}^2 \leq 0, \quad (46)$$

and:

$$\|E(t)\|_{L^2(\Omega)}^2 + \|\mathbf{F}(t)\|_{L^2(\Omega)}^2 + C_{6.2} \frac{1}{\varepsilon^2} \int_0^t \|\mathbf{F}_{\text{dof}}\|_{L^2(\Omega)}^2 \leq \|E(t=0)\|_{L^2(\Omega)}^2 + \|\mathbf{F}(t=0)\|_{L^2(\Omega)}^2. \quad (47)$$

Lemma 6.3. *Under the assumptions of Theorem 4.4, there exists a universal constant $C_{6.3} > 0$ such that, for any $T \geq 0$, the solution (E, \mathbf{F}) to (17) satisfies:*

$$\|\mathbf{F}\|_{L^2([0,T] \times \Omega)} \leq C_{6.3} \varepsilon.$$

Lemma 6.4. *Under the assumptions of Theorem 5.1, there exists a constant $C_{6.4} > 0$ such that:*

$$\|\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2([0,T] \times \Omega)} \leq C_{6.4} (1 + T) h, \quad (48)$$

where $\|\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2([0,T] \times \Omega)}$ is defined in (16), and:

$$\|\mathbf{F}_j^0\|_{L^2([0,T] \times \Omega)} \leq C_{6.4}, \quad (49)$$

$$\left\| \frac{d}{dt} \mathbf{F}_j^0 \right\|_{L^2([0,T] \times \Omega)} \leq \frac{C_{6.4}}{h}. \quad (50)$$

6.1 Proof of Lemma 6.2

This proof is done in the polygonal case in [5]. Using (17), one has:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|E\|_{L^2(\Omega)}^2 + \|\mathbf{F}\|_{L^2(\Omega)}^2 \right) &= \sum_j |\Omega_j| \left(E_j \frac{d}{dt} E_j + \left\langle \mathbf{F}_j, \frac{d}{dt} \mathbf{F}_j \right\rangle \right) \\ &= -\frac{1}{\varepsilon} \left[\sum_j \sum_{\text{dof}} E_j \langle \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle + \sum_{\text{dof}} \langle \mathbf{F}_j, \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}) \rangle \right]. \end{aligned}$$

According to (18), one can write, for any *dof*:

$$\sum_i E_i \langle \tilde{\mathbf{C}}_i^{\text{dof}}, \mathbf{F}_{\text{dof}} \rangle = \frac{\sigma}{\varepsilon} \langle \beta_{\text{dof}} \mathbf{F}_{\text{dof}}, \mathbf{F}_{\text{dof}} \rangle - \left\langle \sum_i \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}), \mathbf{F}_{\text{dof}} \right\rangle,$$

and therefore:

$$\frac{1}{2} \frac{d}{dt} \left(\|E\|_{L^2(\Omega)}^2 + \|\mathbf{F}\|_{L^2(\Omega)}^2 \right) = -\frac{\sigma}{\varepsilon^2} \sum_{\text{dof}} \langle \beta_{\text{dof}} \mathbf{F}_{\text{dof}}, \mathbf{F}_{\text{dof}} \rangle - \frac{1}{\varepsilon} \sum_{\text{dof}} \underbrace{\left\langle \sum_i \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}), \mathbf{F}_i - \mathbf{F}_{\text{dof}} \right\rangle}_{\geq 0},$$

which is nonpositive in virtue of Theorem 4.2. Using Theorem 4.2 leads to (46). Integrating (46) gives (47).

6.2 Proof of Lemma 6.3

Multiplying the second equation of system (17) by \mathbf{F}_j leads to:

$$\frac{1}{2} \frac{d}{dt} (|\Omega_j| \|\mathbf{F}_j\|^2) + \frac{1}{\varepsilon} \langle \mathbf{F}_j, \alpha_j \mathbf{F}_j \rangle - \frac{1}{\varepsilon} \sum_{\text{dof}} \langle \mathbf{F}_j, \alpha_j^{\text{dof}} \mathbf{F}_{\text{dof}} \rangle = 0.$$

According to assumption (25), one can write:

$$\frac{1}{2} \frac{d}{dt} (|\Omega_j| \|\mathbf{F}_j\|^2) + \frac{1}{C_1} \frac{h}{\varepsilon} \|\mathbf{F}_j\|^2 \leq \frac{C_1 h}{\varepsilon} \|\mathbf{F}_j\| \left(\sum_{\text{dof}} \|\mathbf{F}_{\text{dof}}\| \right). \quad (51)$$

Moreover, using:

$$\|\mathbf{F}_j\| \|\mathbf{F}_{\text{dof}}\| \leq \gamma \|\mathbf{F}_j\|^2 + \frac{1}{\gamma} \|\mathbf{F}_{\text{dof}}\|^2,$$

and choosing γ small enough ($\gamma = 1/(2C_1^2 N_j)$), equation (51) gives:

$$\frac{d}{dt} (|\Omega_j| \|\mathbf{F}_j\|^2) + \frac{1}{C_{52}} \frac{h}{\varepsilon} \|\mathbf{F}_j\|^2 \leq C_{52} \frac{h}{\varepsilon} \left(\sum_{\text{dof}} \|\mathbf{F}_{\text{dof}}\|^2 \right), \quad (52)$$

for some universal constant $C_{31} > 0$. Summing (52) over the cells gives:

$$\frac{d}{dt} \left(\|\mathbf{F}\|_{L^2(\Omega)}^2 \right) + \frac{1}{C_{53}} \frac{1}{\varepsilon h} \|\mathbf{F}\|_{L^2(\Omega)}^2 \leq C_{53} \frac{1}{\varepsilon h} \|\mathbf{F}_{\text{dof}}\|_{L^2(\Omega)}^2. \quad (53)$$

Equation (53) can be written as:

$$\frac{d}{dt} \left[\|\mathbf{F}(t)\|_{L^2(\Omega)}^2 \exp \left(\frac{1}{C_{53}} \frac{t}{\varepsilon h} \right) \right] \leq C_{53} \frac{1}{\varepsilon h} \|\mathbf{F}_{\text{dof}}(t)\|_{L^2(\Omega)}^2 \exp \left(\frac{1}{C_{53}} \frac{t}{\varepsilon h} \right). \quad (54)$$

Integrating (54) twice leads to:

$$\|\mathbf{F}\|_{L^2([0,T]\times\Omega)}^2 - \varepsilon h C_{53} \|\mathbf{F}(0)\|_{L^2(\Omega)}^2 \left[1 - \exp\left(-\frac{1}{C_{53}} \frac{t}{\varepsilon h}\right) \right] \leq C_{53} \frac{1}{\varepsilon h} \int_0^T \int_0^t \|\mathbf{F}_{\text{dof}}(s)\|_{L^2(\Omega)}^2 \exp\left(-\frac{1}{C_{53}} \frac{t-s}{\varepsilon h}\right) ds. \quad (55)$$

According to the Young inequality for convolution products (see [7]):

$$\begin{aligned} \int_0^T \int_0^t \|\mathbf{F}_{\text{dof}}(s)\|_{L^2(\Omega)}^2 \exp\left(-\frac{1}{C_{53}} \frac{t-s}{\varepsilon h}\right) ds dt &\leq \left(\int_0^T \|\mathbf{F}_{\text{dof}}(s)\|_{L^2(\Omega)}^2 ds \right) \left(\int_0^T \exp\left(\frac{1}{C_{53}} \frac{-s}{\varepsilon h}\right) ds \right) \\ &\leq C_{53} \varepsilon h \int_0^T \|\mathbf{F}_{\text{dof}}(s)\|_{L^2(\Omega)}^2 ds. \end{aligned}$$

Therefore, owing to Lemma 6.2, equation (55) comes down to:

$$\|\mathbf{F}\|_{L^2([0,T]\times\Omega)}^2 - \varepsilon h C_{53} \|\mathbf{F}(t=0)\|_{L^2(\Omega)}^2 \leq C_{53}^2 \int_0^T \|\mathbf{F}_{\text{dof}}(s)\|_{L^2(\Omega)}^2 ds \leq C_{53}^2 \varepsilon^2 \left(\|E(t=0)\|_{L^2(\Omega)}^2 + \|\mathbf{F}(t=0)\|_{L^2(\Omega)}^2 \right).$$

As a conclusion, since $\varepsilon h \leq 1$ and according to assumption (31) and (32), there exists a universal constant $C_{6.3} > 0$ such that:

$$\|\mathbf{F}\|_{L^2([0,T]\times\Omega)}^2 \leq C_{6.3} \varepsilon^2.$$

6.3 Proof of lemma 6.4

We first show (48). In fact \mathbf{F}_j^0 converges toward $\sigma^{-1}(\nabla \tilde{E}^0)(t, \mathbf{x}_j)$. According to (45), we have:

$$\mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0 = \alpha_j^{-1} \sum_{\text{dof}'} \alpha_j^{\text{dof}'} (\mathbf{F}_{\text{dof}'}^0 - \mathbf{F}_{\text{dof}}^0),$$

thus:

$$\begin{aligned} \mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0 &= \alpha_j^{-1} \left(\sum_{\text{dof}'} \alpha_j^{\text{dof}'} [\mathbf{F}_{\text{dof}'}^0 + \sigma^{-1} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}'})] \right) - (\sigma^{-1} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}) + \mathbf{F}_{\text{dof}}^0) \\ &\quad + \alpha_j^{-1} \left(\sum_{\text{dof}'} \alpha_j^{\text{dof}'} [\sigma^{-1} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}}) - \sigma^{-1} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}'})] \right) \end{aligned}$$

The last term is $O(h)$ since the exact solution \tilde{E}^0 is smooth. The two firsts terms are also bounded by $O(h)$ according to Theorem 5.1. More precisely, there exists a positive constant C_{37} such that:

$$\begin{aligned} \|\mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2([0,T]\times\Omega)} &\leq C_{56} \|\mathbf{F}_{\text{dof}}^0 + \sigma^{-1} \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}})\|_{L^2([0,T]\times\Omega)} \\ &\quad + C_{56} \left(\int_0^T h^2 \sum_{\text{dof}^*} \sum_{\text{dof}'} \|\nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}'}) - \nabla \tilde{E}^0(t, \mathbf{x}_{\text{dof}})\|^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (56)$$

Hence:

$$\|\mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2([0,T]\times\Omega)} \leq C_{57} \left(1 + T^{\frac{3}{2}}\right) h + C_{57} h \sqrt{T} \leq 2C_{57} \left(1 + T^{\frac{3}{2}}\right) h, \quad (57)$$

for some positive constant C_{57} . For the inequality (49), we use the L^2 stability of the limit scheme (Lemma 5.2):

$$\|\mathbf{F}_{\text{dof}}^0\|_{L^2([0,T]\times\Omega)} \leq C_{58}. \quad (58)$$

Using (25) and (45), there exists a constant C_{59} such that, for any cell j :

$$\|\mathbf{F}_j^0\| \leq C_{59} \sum_{\text{dof}} \|\mathbf{F}_{\text{dof}}^0\|, \quad (59)$$

thus one can write:

$$\|\mathbf{F}_j^0\|_{L^2([0,T] \times \Omega)} \leq C_{60}. \quad (60)$$

Eventually, for the inequality (50), we compute the time derivative of (45) and we use (25):

$$\left\| \frac{d}{dt} \mathbf{F}_j^0 \right\|_{L^2([0,T] \times \Omega)} \leq C_{61} \left\| \frac{d}{dt} \mathbf{F}_{\text{dof}}^0 \right\|_{L^2([0,T] \times \Omega)}. \quad (61)$$

Since the scheme is linear and its coefficients are constant, the time derivative of the unknowns, namely $(\frac{d}{dt} E_j^0)_j$ and $(\frac{d}{dt} \mathbf{F}_{\text{dof}}^0)_{\text{dof}}$ satisfy the same equations as $(E_j^0)_j$ and $(\mathbf{F}_{\text{dof}}^0)_{\text{dof}}$ (only the initial conditions change). Therefore we can apply Lemma 5.2 to the quantity $(\frac{d}{dt} \mathbf{F}_{\text{dof}}^0)_{\text{dof}}$:

$$\left\| \frac{d}{dt} \mathbf{F}_{\text{dof}}^0 \right\|_{L^2([0,T] \times \Omega)} \leq C_{62} \left\| \frac{d}{dt} E^0(t=0) \right\|_{L^2(\Omega)}. \quad (62)$$

The right hand side of (62) can be computed as follows:

$$\frac{d}{dt} E_j^0(t=0) = -\frac{1}{|\Omega_j|} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}}^0(t=0), \tilde{\mathbf{C}}_j^{\text{dof}} \rangle,$$

thus leading to:

$$\left\| \frac{d}{dt} E^0(t=0) \right\|_{L^2(\Omega)} \leq \frac{C_{63}}{h} \|\mathbf{F}_{\text{dof}}^0(t=0)\|_{L^2(\Omega)}. \quad (63)$$

Moreover, by assumption (31), at initial time one can write:

$$E_j^0(t=0) = \tilde{E}^0(t=0, \mathbf{x}_j) + O(h).$$

This leads to:

$$\sum_{\text{dof}^*} \left\| \sum_i E_i^0(t=0) \tilde{\mathbf{C}}_i^{\text{dof}} \right\|^2 \leq \sum_{\text{dof}^*} \left\| \sum_i \tilde{E}^0(t=0, \mathbf{x}_i) \tilde{\mathbf{C}}_i^{\text{dof}} \right\|^2 + C_{64} h^2. \quad (64)$$

Using $\sum_i \tilde{\mathbf{C}}_i^{\text{dof}} = \mathbf{0}$ and the fact that \tilde{E} is smooth:

$$\sum_{\text{dof}^*} \left\| \sum_i \tilde{E}^0(t=0, \mathbf{x}_i) \tilde{\mathbf{C}}_i^{\text{dof}} \right\|^2 = \sum_{\text{dof}^*} \left\| \sum_i \underbrace{[\tilde{E}^0(t=0, \mathbf{x}_i) - \tilde{E}^0(t=0, \mathbf{x}_{\text{dof}})]}_{=O(h)} \tilde{\mathbf{C}}_i^{\text{dof}} \right\|^2 \leq C_{65} h^2. \quad (65)$$

Therefore, according to the definition (24) of $\mathbf{F}_{\text{dof}}^0(t=0)$, one can write:

$$\|\mathbf{F}_{\text{dof}}^0(t=0)\|_{L^2(\Omega)}^2 \leq C_{66} \sum_{\text{dof}^*} \frac{1}{h^2} \left\| \sum_i \tilde{E}_i^0(t=0) \tilde{\mathbf{C}}_i^{\text{dof}} \right\|^2 \leq C'_{66}. \quad (66)$$

Hence, in view of (63):

$$\left\| \frac{d}{dt} E^0(t=0) \right\|_{L^2(\Omega)} \leq \frac{C_{67}}{h}. \quad (67)$$

Collecting (61), (62) and (67), we find (50).

6.4 Proof of theorem 6.1

Using equations (18) and (24), one can write:

$$\frac{\sigma}{\varepsilon} \beta_{\text{dof}} \mathbf{f}_{\text{dof}} = \sum_i e_i \tilde{\mathbf{C}}_i^{\text{dof}} + \sum_i \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}). \quad (68)$$

In addition, (17) and (23) imply:

$$\frac{1}{2} \frac{d}{dt} (|\Omega_j| e_j^2) + \frac{1}{\varepsilon} \sum_{\text{dof}} \langle \mathbf{f}_{\text{dof}}, e_j \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = 0. \quad (69)$$

Therefore, summing (69) over the cells gives:

$$\frac{1}{2} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 \right) = -\frac{1}{\varepsilon} \sum_{\text{dof}^*} \left\langle \mathbf{f}_{\text{dof}}, \sum_i e_i \tilde{\mathbf{C}}_i^{\text{dof}} \right\rangle,$$

which, using (68), can be written as:

$$\frac{1}{2} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 \right) = -\frac{\sigma}{\varepsilon^2} \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}, \mathbf{f}_{\text{dof}} \rangle + \frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}) \rangle.$$

Hence:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 \right) &= -\frac{\sigma}{\varepsilon^2} \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}, \mathbf{f}_{\text{dof}} \rangle + \frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle \\ &\quad + \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle. \end{aligned}$$

Using the definition (45) of \mathbf{F}_i^0 , this reads as:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 \right) &= -\frac{\sigma}{\varepsilon^2} \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}, \mathbf{f}_{\text{dof}} \rangle + \frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle \\ &\quad + \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}} - \mathbf{f}_i, \alpha_i^{\text{dof}} (\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle. \end{aligned}$$

Moreover, the second line of (17) gives:

$$\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{f}\|_{L^2(\Omega)}^2 \right) = -\frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_i, \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}) \rangle - \varepsilon \sum_{j^*} |\Omega_j| \left\langle \mathbf{f}_j, \frac{d}{dt} \mathbf{F}_j^0 \right\rangle. \quad (70)$$

Using (45), one has:

$$\sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_i, \alpha_i^{\text{dof}} (\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle = \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{f}_j, \alpha_j^{\text{dof}} (\mathbf{F}_j^0 - \mathbf{F}_{\text{dof}}^0) \rangle = 0.$$

Therefore (70) reads as:

$$\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{f}\|_{L^2(\Omega)}^2 \right) = -\frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_i, \alpha_i^{\text{dof}} (\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle - \varepsilon \sum_{j^*} |\Omega_j| \left\langle \mathbf{f}_j, \frac{d}{dt} \mathbf{F}_j^0 \right\rangle,$$

thus leading to:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2 \right) &= -\frac{\sigma}{\varepsilon^2} \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{f}_{\text{dof}}, \mathbf{f}_{\text{dof}} \rangle - \frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_i - \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle \\ &\quad + \sum_{\text{dof}^*} \sum_i \langle \mathbf{f}_{\text{dof}} - \mathbf{f}_i, \alpha_i^{\text{dof}} (\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle - \varepsilon \sum_{j^*} |\Omega_j| \left\langle \mathbf{f}_j, \frac{d}{dt} \mathbf{F}_j^0 \right\rangle. \end{aligned}$$

Now, for any $\gamma > 0$, we have:

$$\langle \mathbf{f}_{\text{dof}} - \mathbf{f}_i, \alpha_i^{\text{dof}}(\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle \leq \frac{\gamma}{\varepsilon} \langle \mathbf{f}_i - \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}}(\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle + \frac{\varepsilon}{\gamma} \langle \mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0, \alpha_i^{\text{dof}}(\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle.$$

Hence, choosing γ small enough (independently of h and ε) and using Theorem 4.2, one can finally write:

$$\begin{aligned} \frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2 \right) + \frac{1}{C_{71}} \frac{1}{\varepsilon^2} \|\mathbf{f}_{\text{dof}}\|_{L^2(\Omega)}^2 + \frac{1}{C_{71}} \frac{1}{\varepsilon} \sum_{\text{dof}^*} \sum_i \underbrace{\langle \mathbf{f}_i - \mathbf{f}_{\text{dof}}, \alpha_i^{\text{dof}}(\mathbf{f}_i - \mathbf{f}_{\text{dof}}) \rangle}_{\geq 0} \\ \leq C_{71} \varepsilon \sum_{\text{dof}^*} \sum_i \langle \mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0, \alpha_i^{\text{dof}}(\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0) \rangle + \varepsilon \|\mathbf{f}_j\|_{L^2(\Omega)} \left\| \frac{d}{dt} \mathbf{F}_j^0 \right\|_{L^2(\Omega)}. \end{aligned} \quad (71)$$

Therefore, since $\|\alpha_j^{\text{dof}}\| \leq Ch$:

$$\frac{d}{dt} \left(\|e\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2 \right) \leq C_{72} \frac{\varepsilon}{h} \|\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2(\Omega)}^2 + \varepsilon \|\mathbf{f}\|_{L^2(\Omega)} \left\| \frac{d}{dt} \mathbf{F}^0 \right\|_{L^2(\Omega)}. \quad (72)$$

Integrating (72) over $[0, T]$ and using the assumptions on the initial data (31) and (32) gives:

$$\|e\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2 \leq C_{73} \frac{\varepsilon}{h} \|\mathbf{F}_i^0 - \mathbf{F}_{\text{dof}}^0\|_{L^2([0, T] \times \Omega)}^2 + \varepsilon \|\mathbf{f}\|_{L^2([0, T] \times \Omega)} \left\| \frac{d}{dt} \mathbf{F}^0 \right\|_{L^2([0, T] \times \Omega)} + C_{73} \varepsilon^2. \quad (73)$$

As a conclusion, using Lemma 6.4:

$$\|e\|_{L^2([0, T] \times \Omega)}^2 + \|\mathbf{f}\|_{L^2([0, T] \times \Omega)}^2 \leq C_{74} (1 + T^3) \left[\varepsilon h + \frac{\varepsilon^2}{h} \right]. \quad (74)$$

7 Proof of the naive estimation (4)

We recall that we remove the exponent ε for clarity and we note $(\tilde{E}, \tilde{\mathbf{F}}) = (\tilde{E}^\varepsilon, \tilde{\mathbf{F}}^\varepsilon)$ the solution to (1). We define:

$$\bar{e}^2 = \frac{1}{2} \int_{\Omega} (E - \tilde{E})^2 + \|\mathbf{F} - \tilde{\mathbf{F}}\|^2 = \sum_{j^*} \int_{\Omega_j} (E_j - \tilde{E})^2 + \|\mathbf{F}_j - \tilde{\mathbf{F}}\|^2,$$

and:

$$\begin{aligned} \tilde{E}_j &= \frac{1}{|\Omega_j|} \int_{\Omega_j} \tilde{E}, & \tilde{\mathbf{F}}_j &= \frac{1}{|\Omega_j|} \int_{\Omega_j} \tilde{\mathbf{F}}, \\ \tilde{E}_{\text{dof}} &= \tilde{E}(\mathbf{x}_{\text{dof}}), & \tilde{\mathbf{F}}_{\text{dof}} &= \tilde{\mathbf{F}}(\mathbf{x}_{\text{dof}}), & \delta_{j, \text{dof}}(\tilde{E}) &= \tilde{E}_j - \tilde{E}_{\text{dof}}, & \delta_{j, \text{dof}}(\tilde{\mathbf{F}}) &= \tilde{\mathbf{F}}_j - \tilde{\mathbf{F}}_{\text{dof}}, \\ \tilde{\delta}_j(\tilde{E}) &= \int_{\partial\Omega_j} \tilde{E} \mathbf{n} - \sum_{\text{dof}} \tilde{E}_{\text{dof}} \tilde{\mathbf{C}}_j^{\text{dof}}, & \tilde{\delta}_j(\tilde{\mathbf{F}}) &= \int_{\partial\Omega_j} \langle \tilde{\mathbf{F}}, \mathbf{n} \rangle - \sum_{\text{dof}} \langle \tilde{\mathbf{F}}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle. \end{aligned}$$

Lemma 7.1. *The following results hold true:*

$$\int_{\Omega} \|\tilde{\mathbf{F}}\|^2 \geq \sum_{j^*} |\Omega_j| \|\tilde{\mathbf{F}}_j\|^2, \quad \|\tilde{\mathbf{F}}\|_{L^2(\Omega)} \leq C_{7.1} \varepsilon. \quad (75)$$

$$\|\delta(\tilde{E})\|_{L^2(\Omega)} + \|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)} \leq C_{7.1} h, \quad \|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)} \leq C_{7.1} \varepsilon h \quad (76)$$

$$\|\tilde{\delta}(\tilde{E})\|_{L^2(\Omega)} + \|\tilde{\delta}(\tilde{\mathbf{F}})\|_{L^2(\Omega)} \leq C_{7.1} h^3. \quad (77)$$

Theorem 7.2. *Under the assumptions of Theorem 4.4, there exists a constant $C_{7.2}$ such that:*

$$\|\bar{e}\|_{L^2([0, T])} \leq C_{7.2} (1 + T) \frac{h}{\varepsilon}.$$

Proof of Lemma 7.1

For (75), we use the Cauchy-Schwarz inequality:

$$\int_{\Omega} \|\tilde{\mathbf{F}}\|^2 = \sum_{j^*} \int_{\Omega_j} \|\tilde{\mathbf{F}}\|^2 \geq \sum_{j^*} \left\| \int_{\Omega_j} \tilde{\mathbf{F}} \right\|^2 \frac{1}{|\Omega_j|} = \sum_{j^*} |\Omega_j| \|\tilde{\mathbf{F}}_j\|^2 = \|\tilde{\mathbf{F}}\|_{L^2(\Omega)}^2.$$

According to Lemma 1.1, one has:

$$\int_{\Omega} \|\tilde{\mathbf{F}}\|^2 \leq C_{1.1} \varepsilon^2,$$

which gives (75).

For (76), we use the fact that \tilde{E} and $\tilde{\mathbf{F}}$ are smooth:

$$\delta_{j, \text{dof}}(\tilde{E}) = O(h), \quad \delta_{j, \text{dof}}(\tilde{\mathbf{F}}) = O(h).$$

Moreover:

$$\|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)} \leq C_{78} h \left(\int_{\Omega} \|\nabla \tilde{\mathbf{F}}\|^2 \right)^{\frac{1}{2}}, \quad (78)$$

and according to Lemma 1.1, one has:

$$\left(\int_{\Omega} \|\nabla \tilde{\mathbf{F}}\|^2 \right)^{\frac{1}{2}} \leq C_{1.1} \varepsilon.$$

The estimate (77) is proved using Theorem 2.1.

Proof of Theorem 7.2

One has:

$$\frac{d}{dt} \bar{e}^2 = D_1 + D_2 + D_3 + D_4,$$

with:

$$D_1 = \int_{\Omega} E \frac{d}{dt} E + \left\langle \frac{d}{dt} \mathbf{F}, \mathbf{F} \right\rangle = \sum_{j^*} |\Omega_j| \left(E_j \frac{d}{dt} E_j + \left\langle \mathbf{F}_j, \frac{d}{dt} \mathbf{F}_j \right\rangle \right), \quad D_2 = \int_{\Omega} \tilde{E} \partial_t \tilde{E} + \langle \tilde{\mathbf{F}}, \partial_t \tilde{\mathbf{F}} \rangle,$$

and:

$$D_3 = - \int_{\Omega} \tilde{E} \frac{d}{dt} E + \left\langle \tilde{\mathbf{F}}, \frac{d}{dt} \mathbf{F} \right\rangle, \quad D_4 = - \int_{\Omega} E \partial_t \tilde{E} + \langle \mathbf{F}, \partial_t \tilde{\mathbf{F}} \rangle.$$

According to Lemma 6.2:

$$D_1 = - \frac{\sigma}{\varepsilon^2} \sum_{\text{dof}^*} \langle \beta_{\text{dof}} \mathbf{F}_{\text{dof}}, \mathbf{F}_{\text{dof}} \rangle - \frac{1}{\varepsilon} \sum_{\text{dof}^*} \left\langle \sum_i \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}), \mathbf{F}_i - \mathbf{F}_{\text{dof}} \right\rangle, \quad (79)$$

and it can be easily shown that:

$$D_2 = - \frac{\sigma}{\varepsilon^2} \int_{\Omega} \|\tilde{\mathbf{F}}\|^2.$$

Study of D_3

Using (17), we have:

$$D_3 = \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \tilde{E}_j + \langle \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}), \tilde{\mathbf{F}}_j \rangle.$$

Since, for any cell j , $\sum_{\text{dof}} \langle \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = 0$, this implies:

$$D_3 = \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \tilde{E}_j + \langle \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}), \tilde{\mathbf{F}}_j \rangle.$$

Hence:

$$D_3 = \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \delta_{j,\text{dof}}(\tilde{E}) + \langle \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}), \delta_{j,\text{dof}}(\tilde{\mathbf{F}}) \rangle \quad (80)$$

$$+ \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \tilde{E}_{\text{dof}} + \langle \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}), \tilde{\mathbf{F}}_{\text{dof}} \rangle. \quad (81)$$

Study of D_4

Using (1), one has:

$$D_4 = \frac{1}{\varepsilon} \sum_{j^*} \left[E_j \int_{\partial\Omega_j} \langle \tilde{\mathbf{F}}, \mathbf{n} \rangle + \left\langle \mathbf{F}_j, \int_{\partial\Omega_j} \tilde{E} \mathbf{n} \right\rangle \right] + \frac{\sigma}{\varepsilon^2} \sum_{j^*} |\Omega_j| \langle \mathbf{F}_j, \tilde{\mathbf{F}}_j \rangle.$$

Hence:

$$D_4 = \frac{1}{\varepsilon} \sum_{j^*} [E_j \tilde{\delta}_j(\tilde{\mathbf{F}}) + \langle \mathbf{F}_j, \tilde{\delta}_j(\tilde{E}) \rangle] + \frac{\sigma}{\varepsilon^2} \sum_{j^*} |\Omega_j| \langle \mathbf{F}_j, \tilde{\mathbf{F}}_j \rangle + \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} E_j \langle \tilde{\mathbf{F}}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle + \tilde{E}_{\text{dof}} \langle \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle. \quad (82)$$

Since $\sum_i \tilde{E}_{\text{dof}} \langle \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_i^{\text{dof}} \rangle = 0$ for any dof , the last term of (82) is equal to:

$$\sum_{j^*} \sum_{\text{dof}} \tilde{E}_{\text{dof}} \langle \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = \sum_{j^*} \sum_{\text{dof}} \tilde{E}_{\text{dof}} \langle \mathbf{F}_j - \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle.$$

This leads to:

$$D_4 = \frac{1}{\varepsilon} \sum_{j^*} [E_j \tilde{\delta}_j(\tilde{\mathbf{F}}) + \langle \mathbf{F}_j, \tilde{\delta}_j(\tilde{E}) \rangle] + \frac{\sigma}{\varepsilon^2} \sum_{j^*} |\Omega_j| \langle \mathbf{F}_j, \tilde{\mathbf{F}}_j \rangle \quad (83)$$

$$+ \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} E_j \langle \tilde{\mathbf{F}}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle + \tilde{E}_{\text{dof}} \langle \mathbf{F}_j - \mathbf{F}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle. \quad (84)$$

According to (18):

$$\sum_{j^*} \sum_{\text{dof}} E_j \langle \tilde{\mathbf{F}}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = \sum_{\text{dof}^*} \left\langle \tilde{\mathbf{F}}_{\text{dof}}, \sum_i E_i \tilde{\mathbf{C}}_i^{\text{dof}} \right\rangle = \sum_{\text{dof}^*} \sum_i \langle \tilde{\mathbf{F}}_{\text{dof}}, \alpha_i^{\text{dof}} (\mathbf{F}_{\text{dof}} - \mathbf{F}_i) \rangle + \frac{\sigma}{\varepsilon} \sum_{\text{dof}^*} \langle \tilde{\mathbf{F}}_{\text{dof}}, \beta_{\text{dof}} \mathbf{F}_{\text{dof}} \rangle. \quad (85)$$

Eventually, summing (80)-(81) and (83)-(84) and using (85):

$$D_3 + D_4 = \underbrace{\frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_j, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \delta_{j,\text{dof}}(\tilde{E}) + \langle \alpha_j^{\text{dof}} (\mathbf{F}_j - \mathbf{F}_{\text{dof}}), \delta_{j,\text{dof}}(\tilde{\mathbf{F}}) \rangle}_{=D_5} + \underbrace{\frac{1}{\varepsilon} \sum_{j^*} [E_j \tilde{\delta}_j(\tilde{\mathbf{F}}) + \langle \mathbf{F}_j, \tilde{\delta}_j(\tilde{E}) \rangle]}_{=D_6}$$

$$+ \frac{\sigma}{\varepsilon^2} \left[\sum_{\text{dof}^*} \langle \tilde{\mathbf{F}}_{\text{dof}}, \beta_{\text{dof}} \mathbf{F}_{\text{dof}} \rangle + \sum_{j^*} |\Omega_j| \langle \mathbf{F}_j, \tilde{\mathbf{F}}_j \rangle \right].$$

and:

$$\frac{d}{dt} \bar{e}^2 = D_5 + D_6 + \frac{\sigma}{\varepsilon^2} D_7 + D_8, \quad (86)$$

with:

$$D_7 = \sum_{\text{dof}^*} \langle \tilde{\mathbf{F}}_{\text{dof}}, \beta_{\text{dof}} \mathbf{F}_{\text{dof}} \rangle + \sum_{j^*} |\Omega_j| \langle \mathbf{F}_j, \tilde{\mathbf{F}}_j \rangle - \sum_{\text{dof}^*} \langle \mathbf{F}_{\text{dof}}, \beta_{\text{dof}} \mathbf{F}_{\text{dof}} \rangle - \int_{\Omega} \|\tilde{\mathbf{F}}\|^2. \quad (87)$$

$$D_8 = -\frac{1}{\varepsilon} \sum_{\text{dof}^*} \left\langle \sum_i \alpha_i^{\text{dof}} (\mathbf{F}_i - \mathbf{F}_{\text{dof}}), \mathbf{F}_i - \mathbf{F}_{\text{dof}} \right\rangle. \quad (88)$$

Study of D_5 and D_6

Using Remark 1, one can write, for any $\gamma > 0$:

$$|D_5| \leq \gamma \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \ell_j^{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_j, \tilde{\mathbf{n}}_j^{\text{dof}} \rangle^2 + \frac{C_{89} h}{\gamma} \frac{1}{\varepsilon} \sum_{j^*} \sum_{\text{dof}} \delta_{j,\text{dof}}(\tilde{E})^2 + \|\delta_{j,\text{dof}}(\tilde{\mathbf{F}})\|^2. \quad (89)$$

According to Lemma 7.1, the right hand side of (89) can be bounded by:

$$\frac{h}{\varepsilon} \sum_j \sum_{\text{dof}} \delta_{j,\text{dof}}(\tilde{E})^2 + \|\delta_{j,\text{dof}}(\tilde{\mathbf{F}})\|^2 \leq C_{90} \frac{1}{\varepsilon h} \left(\|\delta(\tilde{E})\|_{L^2(\Omega)}^2 + \|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)}^2 \right) \leq C'_{90} \frac{h}{\varepsilon}. \quad (90)$$

This leads to:

$$|D_5| \leq -\gamma D_8 + C'_{90} \frac{h}{\varepsilon} \frac{1}{\gamma}. \quad (91)$$

Moreover, using the Cauchy-Schwarz inequality and Lemma 7.1, one can easily show:

$$|D_6| \leq C_{92} \frac{h}{\varepsilon} (\|E\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^2(\Omega)}) \leq C'_{92} \frac{h}{\varepsilon}. \quad (92)$$

Study of D_7

Using Corollary 2.2 one can write:

$$D_7 = \sum_{\text{dof}^*} \left[\langle \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}, \beta_{\text{dof}} \mathbf{F}_{\text{dof}} \rangle + \sum_i \langle \beta_i^{\text{dof}} \mathbf{F}_i, \tilde{\mathbf{F}}_i \rangle \right] - \int_{\Omega} \|\tilde{\mathbf{F}}\|^2.$$

Using Lemma 7.1 and Corollary 2.2, we can notice that:

$$- \int_{\Omega} \|\tilde{\mathbf{F}}\|^2 \leq - \sum_{j^*} |\Omega_j| \|\tilde{\mathbf{F}}_j\|^2 = - \sum_{j^*} \sum_{\text{dof}} \langle \beta_j^{\text{dof}} \tilde{\mathbf{F}}_j, \tilde{\mathbf{F}}_j \rangle.$$

Therefore we can write:

$$D_7 \leq - \sum_{\text{dof}^*} \langle \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}, \beta_{\text{dof}} (\tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}) \rangle + \tilde{D}_7, \quad (93)$$

with:

$$\tilde{D}_7 = \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_{\text{dof}}, \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}} \rangle - \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_i, \tilde{\mathbf{F}}_i - \mathbf{F}_i \rangle$$

Applying Theorem 4.2 to $\mathbf{u}_{\text{dof}} = \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}$ and $u_{r+1/2} = O(h^2)$, the sum in the right hand side of (93) can be bounded from below by:

$$\sum_{\text{dof}^*} \langle \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}, \beta_{\text{dof}}(\tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}) \rangle \geq \frac{1}{C_{94}} \|\tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}}\|_{L^2(\Omega)}^2 - C_{94} h^4. \quad (94)$$

Moreover, one has:

$$\tilde{D}_7 = \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \delta_{i,\text{dof}}(\tilde{\mathbf{F}}), \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}} \rangle + \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_i, \delta_{i,\text{dof}}(\tilde{\mathbf{F}}) \rangle + \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_i, \mathbf{F}_{\text{dof}} - \mathbf{F}_i \rangle. \quad (95)$$

The first sum of (95) can be bounded by:

$$\left| \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \delta_{i,\text{dof}}(\tilde{\mathbf{F}}), \tilde{\mathbf{F}}_{\text{dof}} - \mathbf{F}_{\text{dof}} \rangle \right| \leq \left| \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \delta_{i,\text{dof}}(\tilde{\mathbf{F}}), \delta_{i,\text{dof}}(\tilde{\mathbf{F}}) \rangle \right| + \left| \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \delta_{i,\text{dof}}(\tilde{\mathbf{F}}), \tilde{\mathbf{F}}_i - \mathbf{F}_{\text{dof}} \rangle \right| \quad (96)$$

$$\leq C_{97} \underbrace{\|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)}^2}_{\leq h^2 \varepsilon^2} + C_{97} \underbrace{\|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)}}_{\leq h\varepsilon} \left(\underbrace{\|\tilde{\mathbf{F}}\|_{L^2(\Omega)}}_{\leq \varepsilon} + \underbrace{\|\mathbf{F}_{\text{dof}}\|_{L^2(\Omega)}}_{\leq \varepsilon} \right) \leq C'_{97} h \varepsilon^2. \quad (97)$$

The second sum of (95) can be bounded by:

$$\left| \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_i, \delta_{i,\text{dof}}(\tilde{\mathbf{F}}) \rangle \right| \leq C_{98} \underbrace{\|\tilde{\mathbf{F}}\|_{L^2(\Omega)}}_{\leq \varepsilon} \underbrace{\|\delta(\tilde{\mathbf{F}})\|_{L^2(\Omega)}}_{\leq h\varepsilon} \leq C'_{98} h \varepsilon^2. \quad (98)$$

Using Remark 1, the third sum of (95) can be bounded by:

$$\left| \sum_{\text{dof}^*} \sum_i \langle \beta_i^{\text{dof}} \tilde{\mathbf{F}}_i, \mathbf{F}_{\text{dof}} - \mathbf{F}_i \rangle \right| \leq \gamma \varepsilon \sum_{\text{dof}^*} \sum_i \ell_i^{\text{dof}} \langle \mathbf{F}_{\text{dof}} - \mathbf{F}_i, \tilde{\mathbf{n}}_i^{\text{dof}} \rangle^2 + \frac{C_{99} h}{\varepsilon \gamma} \underbrace{\|\tilde{\mathbf{F}}\|_{L^2(\Omega)}^2}_{\leq C \varepsilon^2}. \quad (99)$$

Summing equations (96), (97), (98) and (99) and using (88) leads to:

$$D_7 \leq C_{100} \left(1 + \frac{1}{\gamma} \right) h \varepsilon - \gamma \varepsilon^2 D_8. \quad (100)$$

Conclusion

Using (91), (92), (100) and (86), one can write, for any $\gamma > 0$:

$$\frac{d}{dt} \bar{e}^2 \leq C_{101} \frac{h}{\varepsilon} \left(1 + \frac{1}{\gamma} \right) + D_8 (1 - C_{101} \gamma). \quad (101)$$

Choosing γ small enough and using $D_8 \leq 0$ leads to:

$$\frac{d}{dt} \bar{e}^2(t) \leq C_{102} \frac{h}{\varepsilon}. \quad (102)$$

Integrating this inequality on $[0, T]$, we conclude the proof of Theorem 7.2.

8 Conclusion and proof of Theorem 4.4

Using Theorem 7.2, there exists a constant C_{103} such that:

$$\|E - \tilde{E}\|_{L^2([0,T] \times \Omega)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{L^2([0,T] \times \Omega)} \leq C_{103} (1+T) \frac{h}{\varepsilon}. \quad (103)$$

Moreover, using the triangle inequality, we have:

$$\begin{aligned} \|E - \tilde{E}\|_{L^2([0,T] \times \Omega)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{L^2([0,T] \times \Omega)} &\leq \|E - E^0\|_{L^2([0,T] \times \Omega)} + \|E^0 - \tilde{E}^0\|_{L^2([0,T] \times \Omega)} + \|\tilde{E}^0 - \tilde{E}\|_{L^2([0,T] \times \Omega)} \\ &\quad + \|\mathbf{F} - \varepsilon \mathbf{F}^0\|_{L^2([0,T] \times \Omega)} + \varepsilon \|\mathbf{F}^0\|_{L^2([0,T] \times \Omega)} + \|\tilde{\mathbf{F}}\|_{L^2([0,T] \times \Omega)}. \end{aligned}$$

Therefore, according to Theorem 5.1, Theorem 6.1 and Lemma 1.1 and using $1+T \leq 2(1+T^{\frac{3}{2}})$, we can write:

$$\|E - \tilde{E}\|_{L^2([0,T] \times \Omega)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{L^2([0,T] \times \Omega)} \leq C_{104} \left(1+T^{\frac{3}{2}}\right) \left(\sqrt{\varepsilon h} + \frac{\varepsilon}{\sqrt{h}} + h + \varepsilon\right) \quad (104)$$

Using Lemma 1.2 concludes the proof.

9 Appendix

9.1 Proof of lemma 4.3

One has, according to (22):

$$\|\mathbf{D}_{r+1/2}\| \leq 2\|\tilde{\mathbf{n}}_j^{r+1/2}\| + \frac{\sigma}{\varepsilon}\|\mathbf{x}_i - \mathbf{x}_j\| \leq 2 + C_1 \frac{\sigma}{\varepsilon} h \leq C_{\mathbf{D}} \left(1 + \frac{\sigma}{\varepsilon} h\right),$$

and:

$$\|\mathbf{D}_{r+1/2}\|^2 = 4 + \left(\frac{\sigma}{\varepsilon}\right)^2 \|\mathbf{x}_i - \mathbf{x}_j\|^2 + 4\frac{\sigma}{\varepsilon} \langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle.$$

Moreover:

$$\langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle \geq \frac{2}{hC_1} \gamma_1 \geq 0.$$

Indeed:

$$\langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle = \langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_j \rangle - \langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_i \rangle,$$

since $-\tilde{\mathbf{n}}_j^{r+1/2} = \tilde{\mathbf{n}}_i^{r+1/2}$ and using the definition of γ_1 (28):

$$\langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle = \langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_j \rangle + \langle \tilde{\mathbf{n}}_+^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_i \rangle \geq \frac{1}{hC_1} 2\gamma_1.$$

Therefore:

$$\|\mathbf{D}_{r+1/2}\|^2 \geq 4 + \left(\frac{\sigma h}{C_1 \varepsilon}\right)^2, \quad \text{and: } \|\mathbf{D}_{r+1/2}\| \geq \frac{1}{C_{\mathbf{D}}} \left(1 + \frac{\sigma}{\varepsilon} h\right).$$

9.2 Proof of Theorem 4.2

We recall that, for any shoulder point $r + 1/2$:

$$\tilde{\mathbf{D}}_{r+1/2} = \frac{1}{\|\mathbf{D}_{r+1/2}\|} \mathbf{D}_{r+1/2}.$$

One has:

$$\langle \mathbf{u}_{r+1/2}, \beta_{r+1/2} \mathbf{u}_{r+1/2} \rangle = \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{C}}_j^{r+1/2} \rangle \langle \mathbf{u}_{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle.$$

We decompose $\tilde{\mathbf{C}}_j^{r+1/2}$ and $\mathbf{x}_i - \mathbf{x}_j$ on the basis $(\tilde{\mathbf{D}}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp)$, finding:

$$\langle \mathbf{u}_{r+1/2}, \beta_{r+1/2} \mathbf{u}_{r+1/2} \rangle = \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 \lambda_0 + \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle \lambda_1 + \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle^2 \lambda_2,$$

with:

$$\lambda_0 = \langle \tilde{\mathbf{C}}_j^{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle \langle \mathbf{x}_i - \mathbf{x}_j, \tilde{\mathbf{D}}_{r+1/2} \rangle, \quad \lambda_2 = \langle \tilde{\mathbf{C}}_j^{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle \langle \mathbf{x}_i - \mathbf{x}_j, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle,$$

and:

$$\lambda_1 = \langle \tilde{\mathbf{C}}_j^{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle \langle \mathbf{x}_i - \mathbf{x}_j, \tilde{\mathbf{D}}_{r+1/2} \rangle + \langle \tilde{\mathbf{C}}_j^{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle \langle \mathbf{x}_i - \mathbf{x}_j, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle.$$

Using (27), (28) and Lemma 4.3:

$$\langle \tilde{\mathbf{C}}_j^{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle = \frac{1}{\|\mathbf{D}_{r+1/2}\|} \left[2 \|\tilde{\mathbf{C}}_j^{r+1/2}\| + \frac{\sigma}{\varepsilon} \langle \tilde{\mathbf{C}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle \right] \geq \frac{1}{C_{\mathbf{D}}} \frac{C_1^{-1} h + \frac{\sigma}{\varepsilon} h^2 \gamma_1}{1 + \frac{\sigma h}{\varepsilon}},$$

and:

$$\langle \mathbf{x}_i - \mathbf{x}_j, \tilde{\mathbf{D}}_{r+1/2} \rangle = \frac{1}{\|\mathbf{D}_{r+1/2}\|} \left[2 \langle \tilde{\mathbf{n}}_j^{r+1/2}, \mathbf{x}_i - \mathbf{x}_j \rangle + \frac{\sigma}{\varepsilon} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \right] \geq \frac{1}{C_{\mathbf{D}}} \frac{C_1^{-1} h \gamma_1 + \frac{\sigma}{\varepsilon} C_1^{-2} h^2}{1 + \frac{\sigma h}{\varepsilon}}.$$

This leads to:

$$\lambda_0 \geq \frac{1}{C_1^2 C_{\mathbf{D}}} h^2 \frac{(1 + \frac{\sigma}{\varepsilon} h \gamma_1) (\gamma_1 + \frac{\sigma}{\varepsilon} h)}{1 + (\frac{\sigma h}{\varepsilon})^2} \geq \frac{1}{C_1^2 C_{\mathbf{D}}} h^2 \gamma_1 \geq 0.$$

Hence:

$$\langle \mathbf{u}_{r+1/2}, \beta_{r+1/2} \mathbf{u}_{r+1/2} \rangle \geq \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 (\lambda_0 - |\lambda_1|) - \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle^2 (|\lambda_2| + |\lambda_1|).$$

In addition, using Lemma 4.3 again:

$$|\lambda_1| \leq C_{\mathbf{D}} \frac{1}{1 + \frac{\sigma h}{\varepsilon}} \left(\frac{\sigma}{\varepsilon} \gamma_2 h^2 \right) C_1 h \leq C_{\mathbf{D}} C_1 \gamma_2 h^2, \quad |\lambda_2| \leq C_{\mathbf{D}}^2 \frac{1}{1 + (\frac{\sigma h}{\varepsilon})^2} \left(\frac{\sigma}{\varepsilon} \gamma_2 h^2 \right) C_1 \gamma_2 h \leq C_{\mathbf{D}}^2 C_1 \gamma_2^2 h^2.$$

Using:

$$\langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle^2 \leq 3 \left(\|\mathbf{u}_r\|^2 + \|\mathbf{u}_{r+1}\|^2 + u_{r+1/2}^2 \right), \quad (105)$$

and noticing $\max(C_{\mathbf{D}}^2 C_1, \frac{3}{2} C_{\mathbf{D}} C_1^2) \leq 4 C_1^3$, one can write:

$$\langle \mathbf{u}_{r+1/2}, \beta_{r+1/2} \mathbf{u}_{r+1/2} \rangle \geq h^2 \left[\langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 \left(\frac{1}{\tilde{C}} \gamma_1 - \tilde{C} \gamma_2 \right) - \tilde{C} (\gamma_2 + \gamma_2^2) (\|\mathbf{u}_r\|^2 + \|\mathbf{u}_{r+1}\|^2) - \tilde{C} (\gamma_2 + \gamma_2^2) u_{r+1/2}^2 \right],$$

with $\tilde{C} = 8C_1^3$. Summing over the degrees of freedom and using (25), one finally has:

$$\sum_{\text{dof}^*} \langle \mathbf{u}_{\text{dof}}, \beta_{\text{dof}} \mathbf{u}_{\text{dof}} \rangle \geq h^2 \left(\frac{1}{\tilde{C}} - \tilde{C}(\gamma_2 + \gamma_2^2) \right) \sum_{r^*} \|\mathbf{u}_r\|^2 \quad (106)$$

$$+ h^2 \left(\frac{1}{\tilde{C}} \gamma_1 - \tilde{C} \gamma_2 \right) \sum_{r+1/2^*} \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 - h^2 \tilde{C}(\gamma_2 + \gamma_2^2) \sum_{r+1/2^*} u_{r+1/2}^2. \quad (107)$$

Using the assumptions:

$$\frac{1}{\tilde{C}} - \tilde{C}(\gamma_2 + \gamma_2^2) > 0, \quad \frac{1}{\tilde{C}} \gamma_1 - \tilde{C} \gamma_2 > 0,$$

and denoting

$$C' = \max \left[\left(\frac{1}{\tilde{C}} - \tilde{C}(\gamma_2 + \gamma_2^2) \right)^{-1}, \left(\frac{1}{\tilde{C}} \gamma_1 - \tilde{C} \gamma_2 \right)^{-1}, \tilde{C}(\gamma_2 + \gamma_2^2) \right],$$

one can write (106)-(107) as:

$$\sum_{\text{dof}^*} \langle \mathbf{u}_{\text{dof}}, \beta_{\text{dof}} \mathbf{u}_{\text{dof}} \rangle \geq h^2 \frac{1}{C'} \left(\sum_{r^*} \|\mathbf{u}_r\|^2 + \sum_{r+1/2^*} \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 \right) - C' h^2 \sum_{r+1/2^*} u_{r+1/2}^2. \quad (108)$$

Using:

$$\|\mathbf{u}_{r+1/2}\|^2 = \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 + \langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2}^\perp \rangle^2 \leq 3 \left(\langle \mathbf{u}_{r+1/2}, \tilde{\mathbf{D}}_{r+1/2} \rangle^2 + \|\mathbf{u}_r\|^2 + \|\mathbf{u}_{r+1}\|^2 + u_{r+1/2}^2 \right), \quad (109)$$

one can eventually write:

$$\sum_{\text{dof}^*} \langle \mathbf{u}_{\text{dof}}, \beta_{\text{dof}} \mathbf{u}_{\text{dof}} \rangle \geq h^2 \frac{1}{C''} \left(\sum_{r^*} \|\mathbf{u}_r\|^2 + \sum_{r+1/2^*} \|\mathbf{u}_{r+1/2}\|^2 \right) - h^2 C'' \sum_{r+1/2^*} u_{r+1/2}^2,$$

for some $C'' \geq 1$, which is the desired result.

9.3 Proof of theorem 4.1

Let $\xi \in \mathbb{R}^2$.

$$\langle \xi, \beta_j^r \xi \rangle = \langle \xi, \tilde{\mathbf{C}}_j^r \rangle \langle \xi, \mathbf{x}_r - \mathbf{x}_j \rangle.$$

We decompose $\mathbf{x}_r - \mathbf{x}_j$ on the basis $\tilde{\mathbf{n}}_j^r, (\tilde{\mathbf{n}}_j^r)^\perp$ and get:

$$\langle \xi, \beta_j^r \xi \rangle = \langle \xi, \tilde{\mathbf{C}}_j^r \rangle \left[\langle \xi, \tilde{\mathbf{n}}_j^r \rangle \langle \mathbf{x}_r - \mathbf{x}_j, \tilde{\mathbf{n}}_j^r \rangle + \langle \xi, (\tilde{\mathbf{n}}_j^r)^\perp \rangle \langle \mathbf{x}_r - \mathbf{x}_j, (\tilde{\mathbf{n}}_j^r)^\perp \rangle \right]. \quad (110)$$

Owing to (27) and (28), one has:

$$\langle \mathbf{x}_r - \mathbf{x}_j, \tilde{\mathbf{n}}_j^r \rangle \geq \frac{1}{C_1 h} \langle \mathbf{x}_r - \mathbf{x}_j, \tilde{\mathbf{C}}_j^r \rangle \geq \frac{h}{C_1} \gamma_1,$$

Thus the first term of (110) can be bounded from below as follows:

$$\langle \xi, \tilde{\mathbf{C}}_j^r \rangle \langle \xi, \tilde{\mathbf{n}}_j^r \rangle \langle \mathbf{x}_r - \mathbf{x}_j, \tilde{\mathbf{n}}_j^r \rangle = \underbrace{\langle \xi, \alpha_j^r \xi \rangle}_{\geq 0} \langle \mathbf{x}_r - \mathbf{x}_j, \tilde{\mathbf{n}}_j^r \rangle \geq \langle \xi, \alpha_j^r \xi \rangle \gamma_1 \frac{h}{C_1}.$$

Using (28) again, the second term of (110) is bounded by:

$$|\langle \xi, \tilde{\mathbf{C}}_j^r \rangle \langle \xi, (\tilde{\mathbf{n}}_j^r)^\perp \rangle \langle \mathbf{x}_r - \mathbf{x}_j, (\tilde{\mathbf{n}}_j^r)^\perp \rangle| = |\langle \xi, \tilde{\mathbf{n}}_j^r \rangle \langle \xi, (\tilde{\mathbf{n}}_j^r)^\perp \rangle \langle \mathbf{x}_r - \mathbf{x}_j, (\tilde{\mathbf{C}}_j^r)^\perp \rangle| \leq \gamma_2 h^2 \|\xi\|^2.$$

We thus have:

$$\langle \xi, \beta_r \xi \rangle \geq \langle \xi, \alpha_r \xi \rangle \gamma_1 \frac{h}{C_1} - \sum_i \gamma_2 h^2 \|\xi\|^2 \geq h^2 \underbrace{\left(\gamma_1 \frac{1}{C_1^2} - C_1 \gamma_2 \right)}_{>0} \|\xi\|^2.$$

9.4 Sufficient condition on the mesh (independent of ω) ensuring the assumptions of Theorem 4.1

The assumptions of Theorem 4.1 depend on the parameters $(\omega_{r+1/2})_{r+1/2}$ of the edges and on the coordinates of the vertices and the shoulder points (which also depend on the $(\omega_{r+1/2})_{r+1/2}$). In this section, we present some sufficient conditions on the vertices and the control points that ensure the assumptions of Theorem 4.1. We define:

$$\begin{aligned} \tilde{\gamma}_1 &= \frac{1}{h^2} \min_{j, r} \min(\langle (\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp, \mathbf{x}_r - \mathbf{x}_j \rangle, \langle (\mathbf{x}_r - \mathbf{x}_{r-1})^\perp, \mathbf{x}_r - \mathbf{x}_j \rangle), \\ \tilde{\gamma}_2 &= \frac{1}{h^2} \max_{j, r} \max(|\langle \mathbf{x}_{r+1} - \mathbf{x}_r, \mathbf{x}_r - \mathbf{x}_j \rangle|, |\langle \mathbf{x}_r - \mathbf{x}_{r-1}, \mathbf{x}_r - \mathbf{x}_j \rangle|), \\ \delta &= \frac{1}{h} \max_{r+1/2} \left\| \mathbf{M}_{r+1/2} - \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} \right\|, \end{aligned} \quad (111)$$

and we assume $\tilde{\gamma}_1 > 0$.

According to the definition of the shoulder points (8), it reads as a convex combination of the middle of the edge and of the control point:

$$\mathbf{x}_{r+1/2} = \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} + \lambda_{r+1/2} \left(\mathbf{M}_{r+1/2} - \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} \right), \quad \lambda_{r+1/2} = \frac{\omega_{r+1/2}}{1 + \omega_{r+1/2}}.$$

Using (12), one can write:

$$\begin{aligned} 2\tilde{\mathbf{C}}_j^r &= \left(1 - \frac{h(\omega_{r+1/2})}{2} \right) (\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp + \left(1 - \frac{h(\omega_{r-1/2})}{2} \right) (\mathbf{x}_r - \mathbf{x}_{r-1})^\perp \\ &+ h(\omega_{r+1/2}) \lambda_{r+1/2} \left(\mathbf{M}_{r+1/2} - \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} \right)^\perp - h(\omega_{r-1/2}) \lambda_{r-1/2} \left(\mathbf{M}_{r-1/2} - \frac{\mathbf{x}_r + \mathbf{x}_{r-1}}{2} \right)^\perp. \end{aligned}$$

Since $\tilde{\gamma}_1 > 0$ and $1 - h(\omega)/2 \geq 1 - \pi/4 > 0$, we have:

$$\langle \tilde{\mathbf{C}}_j^r, \mathbf{x}_r - \mathbf{x}_j \rangle \geq \frac{1}{C_{112}} \tilde{\gamma}_1 h^2 - C_{112} \delta h^2. \quad (112)$$

Besides, one has:

$$|\langle (\tilde{\mathbf{C}}_j^r)^\perp, \mathbf{x}_r - \mathbf{x}_j \rangle| \leq C_{113} \tilde{\gamma}_2 h^2 + C_{113} \delta h^2. \quad (113)$$

According to (13), one has:

$$\tilde{\mathbf{C}}_j^{r+1/2} = \frac{h(\omega_{r+1/2})}{2} (\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp,$$

thus leading to:

$$\langle \tilde{\mathbf{C}}_j^{r+1/2}, \mathbf{x}_{r+1/2} - \mathbf{x}_j \rangle \geq \frac{1}{C_{114}} \left\langle (\mathbf{x}_{r+1} - \mathbf{x}_r)^\perp, \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} - \mathbf{x}_j \right\rangle - C_{114} \delta h^2 \geq \frac{1}{C_{114}} \tilde{\gamma}_1 h^2 - C_{114} \delta h^2 \quad (114)$$

$$\left| \langle \tilde{\mathbf{C}}_j^{r+1/2}, (\mathbf{x}_{r+1/2} - \mathbf{x}_j)^\perp \rangle \right| \leq C_{115} \left(\left| \left\langle \mathbf{x}_{r+1} - \mathbf{x}_r, \frac{\mathbf{x}_{r+1} + \mathbf{x}_r}{2} - \mathbf{x}_j \right\rangle \right| + \delta h^2 \right) \leq C_{115} h^2 (\tilde{\gamma}_2 + \delta) \quad (115)$$

Eventually, the condition $\tilde{\gamma}_1 \geq \tilde{C}(\tilde{\gamma}_2 + \delta)$ for some $\tilde{C} \geq 1$, implies the assumptions (29).

9.5 Proof of Corollary 2.2

Let $\xi \in \mathbb{R}^2$, one has:

$$|\Omega_j| \xi = \int_{\Omega_j} \xi = \int_{\partial\Omega_j} \langle \xi, \mathbf{x} - \mathbf{x}_j \rangle \mathbf{n}$$

Moreover, according to Theorem 2.1:

$$\int_{\partial\Omega_j} \langle \xi, \mathbf{x} - \mathbf{x}_j \rangle \mathbf{n} = \sum_{\text{dof}} \langle \xi, \mathbf{x}_{\text{dof}} - \mathbf{x}_j \rangle \tilde{\mathbf{C}}_j^{\text{dof}},$$

and the relation $\langle \xi, \mathbf{x}_{\text{dof}} - \mathbf{x}_j \rangle \tilde{\mathbf{C}}_j^{\text{dof}} = \beta_j^{\text{dof}} \xi$ gives the result.

9.6 Technical lemma

Lemma 9.1. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a C^1 function and $k_1, k_2 > 0$ such that:*

$$\frac{d}{dt}(f^2) \leq k_1(f(t) + k_2),$$

then:

$$\forall t \geq 0, f^2(t) \leq 2(k_1^2 t^2 + f^2(t=0) + k_2^2)$$

Proof. One has:

$$\frac{d}{dt}(f^2 + k_2^2) \leq 2k_1 \sqrt{f^2 + k_2^2}, \quad \text{hence:} \quad \frac{d}{dt} \sqrt{f^2 + k_2^2} \leq k_1,$$

therefore:

$$f^2(t) \leq 2(k_1^2 t^2 + f^2(t=0) + k_2^2).$$

□

9.7 Proof of Theorem 2.1

In order to prove Theorem 2.1, we prove that the quadrature formula is exact for affine vector-valued functions. More precisely:

$$\forall A \in \mathbb{R}^{2 \times 2}, \int_{\partial\Omega_j} \langle A\mathbf{x}, \mathbf{N} \rangle = \sum_{\text{dof}^*} \langle A\mathbf{x}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \quad (116)$$

To do so, we first write:

$$\int_{\partial\Omega_j} \langle A\mathbf{x}, \mathbf{N} \rangle = \sum_{\text{edges}} \int_e \langle A\mathbf{x}, \mathbf{N} \rangle.$$

Each edge e can be parametrised by the following: $e = \{\mathbf{M}(q), q \in [0, 1]\}$ with:

$$\mathbf{M}(q) = \sum_{i=0}^2 \lambda_i(q) \mathbf{M}_i, \quad \lambda_i(q) \geq 0, \quad \sum_{i=0}^2 \lambda_i(q) = 1,$$

where $\mathbf{M}_0 = \mathbf{x}_r$ and $\mathbf{M}_2 = \mathbf{x}_{r+1}$ are the vertices and $\mathbf{M}_1 = \mathbf{M}_{1,r+1/2}$ is the control point, and:

$$\lambda_0(q) = \frac{(1-q)^2}{\ell(q)}, \quad \lambda_1(q) = 2\omega \frac{(1-q)q}{\ell(q)}, \quad \lambda_2(q) = \frac{q^2}{\ell(q)},$$

and:

$$\ell(q) = (1-q)^2 + 2\omega(1-q)q + q^2.$$

Therefore we can write:

$$\int_e \langle A\mathbf{x}, \mathbf{N} \rangle = \int_0^1 \left\langle A\mathbf{M}(q), \frac{d}{dq}\mathbf{M}^\perp(q) \right\rangle dq = \sum_{i=0}^2 \sum_{j=0}^2 \left(\int_0^1 \lambda_j \frac{d\lambda_i}{dq} \right) \langle A\mathbf{M}_j, \mathbf{M}_i^\perp \rangle.$$

We explain in the following how the computation of the integrals $\left(\int_0^1 \lambda_j \frac{d\lambda_i}{dq} \right)_{i,j}$ can be reduced to the computation of a single integral. Using:

$$\lambda_0(0) = 1 \quad \lambda_1(0) = \lambda_2(0) = 0 \quad \lambda_2(1) = 1, \quad \lambda_1(1) = \lambda_0(1) = 0,$$

we deduce that, for $i \neq j$:

$$\int_0^1 \lambda_j \frac{d\lambda_i}{dq} = - \int_0^1 \lambda_i \frac{d\lambda_j}{dq},$$

and for $i = j$:

$$\int_0^1 \lambda_i \frac{d\lambda_i}{dq} = \begin{cases} -\frac{1}{2} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \frac{1}{2} & \text{if } i = 2 \end{cases}$$

Moreover, using $\lambda_0(q) = \lambda_2(1-q)$ and $\lambda_1(q) = \lambda_1(1-q)$, one has :

$$\int_0^1 \frac{d\lambda_0}{dq} \lambda_1 = - \int_0^1 \frac{d\lambda_2}{dq} \lambda_1.$$

In addition, since $\lambda_0 = 1 - \lambda_1 - \lambda_2$:

$$\int_0^1 \frac{d\lambda_2}{dq} \lambda_0 = \underbrace{\int_0^1 \frac{d\lambda_2}{dq}}_{=1} - \underbrace{\int_0^1 \frac{d\lambda_2}{dq} \lambda_2}_{=\frac{1}{2}} - \int_0^1 \frac{d\lambda_2}{dq} \lambda_1 = \frac{1}{2} - \int_0^1 \frac{d\lambda_2}{dq} \lambda_1.$$

Therefore we only need to compute the integral:

$$\mathcal{I}_1 = \int_0^1 \frac{d\lambda_2}{dq} \lambda_1.$$

The expression of $\frac{d\lambda_2}{dq} \lambda_1$ is quite complicated. This is why we show in the following that computing \mathcal{I}_1 is equivalent to computing a simpler integral.

Writing:

$$1 = q^2 + (1-q)^2 + 2q(1-q), \quad \text{we deduce: } \frac{1}{\ell(q)} = \lambda_2(q) + \lambda_0(q) + \frac{1}{\omega} \lambda_1(q). \quad (117)$$

We define $\mathcal{I}_2 = \int_0^1 \frac{d\lambda_2}{dq} \frac{1}{\ell}$. Multiplying (117) by $\frac{d\lambda_2}{dq}$ and integrating, we find, using the above equalities:

$$\mathcal{I}_2 = \int_0^1 \frac{d\lambda_2}{dq} \frac{1}{\ell} = \frac{1}{2} + \frac{1}{2} - \int_0^1 \frac{d\lambda_2}{dq} \lambda_1 + \frac{1}{\omega} \int_0^1 \frac{d\lambda_2}{dq} \lambda_1 = 1 + \frac{1-\omega}{\omega} \mathcal{I}_1.$$

This leads to:

$$\mathcal{I}_1 = \frac{\omega}{1-\omega} (\mathcal{I}_2 - 1).$$

Hence we only need to compute \mathcal{I}_2 . An integration by part leads to:

$$\mathcal{I}_2 = \int_0^1 \frac{d}{dq} \left(\frac{q^2}{\ell(q)} \right) \frac{1}{\ell(q)} dq = \left[\frac{q^2}{\ell^2(q)} \right]_0^1 - \underbrace{\int_0^1 \frac{q^2}{\ell(q)} \frac{d}{dq} \left(\frac{1}{\ell(q)} \right) dq}_{=\mathcal{I}_3} = 1 - \mathcal{I}_3,$$

and:

$$\mathcal{I}_3 = \frac{1}{2} \int_0^1 q^2 \frac{d}{dq} \left(\frac{1}{\ell(q)^2} \right) dq = \frac{1}{2} \left[\left[\frac{q^2}{\ell^2(q)} \right]_0^1 - \int_0^1 \frac{2q}{\ell(q)^2} dq \right] = \frac{1}{2} \left[1 - \underbrace{\int_0^1 \frac{2q-1}{\ell(q)^2} dq}_{=\mathcal{I}_4} - \underbrace{\int_0^1 \frac{1}{\ell(q)^2} dq}_{=\mathcal{I}_5} \right].$$

Besides:

$$\mathcal{I}_4 = \frac{1}{2(1-\omega)} \left[\frac{1}{\ell} \right]_0^1 = 0.$$

Eventually, we find:

$$\mathcal{I}_1 = \frac{\omega}{2(\omega-1)} (1 - \mathcal{I}_5) = \frac{\omega}{2(1-\omega)} (\mathcal{I}_5 - 1).$$

We thus need to compute the integral $\mathcal{I}_5 = \int_0^1 1/\ell^2$. This can be done using a partial fraction decomposition, we do not give the details of this computation. In the end we find $\mathcal{I}_1 = f(\omega)/2$ where f is defined in (14).

Now we explain why this gives the desired result. Using:

$$\mathbf{C}_j^r = \frac{1}{2} (\mathbf{N}_{r-1/2}^+ + \mathbf{N}_{r+1/2}^-),$$

with:

$$\mathbf{N}_{r+1/2}^- = (1 - f_{r+1/2}) \mathbf{N}_{r,r+1} + f_{r+1/2} \mathbf{N}_{r,r+1/2}, \quad \mathbf{N}_{r+1/2}^+ = (1 - f_{r+1/2}) \mathbf{N}_{r,r+1} + f_{r+1/2} \mathbf{N}_{r+1/2,r+1},$$

we obtain:

$$\sum_r \langle \mathbf{A}\mathbf{x}_r, \mathbf{C}_j^r \rangle + \sum_r \langle \mathbf{A}\mathbf{M}_{r+1/2}, \mathbf{C}_j^{r+1/2} \rangle = \sum_{r+1/2} B_{r+1/2}, \quad (118)$$

with:

$$B_{r+1/2} = \left\langle \frac{1}{2} \mathbf{A}\mathbf{x}_r, \mathbf{N}_{r+1/2}^- \right\rangle + \left\langle \frac{1}{2} \mathbf{A}\mathbf{x}_{r+1}, \mathbf{N}_{r+1/2}^+ \right\rangle + \left\langle \frac{1}{2} \mathbf{A}\mathbf{M}_{r+1/2}, f_{r+1/2} \mathbf{N}_{r,r+1} \right\rangle.$$

and $f_{r+1/2} = f(\omega_{r+1/2})$. Notice that $B_{r+1/2}$ can also be written as a function of the vertices and the shoulder point:

$$B_{r+1/2} = \left\langle \frac{1}{2} \mathbf{A}\mathbf{x}_r, \tilde{\mathbf{N}}_{r+1/2}^- \right\rangle + \left\langle \frac{1}{2} \mathbf{A}\mathbf{x}_{r+1}, \tilde{\mathbf{N}}_{r+1/2}^+ \right\rangle + \left\langle \frac{1}{2} \mathbf{A}\mathbf{x}_{r+1/2}, h_{r+1/2} \mathbf{N}_{r,r+1} \right\rangle,$$

with:

$$\tilde{\mathbf{N}}_{r+1/2}^- = (1 - h_{r+1/2})\mathbf{N}_{r,r+1} + h_{r+1/2}\tilde{\mathbf{N}}_{r,r+1/2}, \quad \tilde{\mathbf{N}}_{r+1/2}^+ = (1 - h_{r+1/2})\mathbf{N}_{r,r+1} + h_{r+1/2}\tilde{\mathbf{N}}_{r+1/2,r+1}.$$

and $h_{r+1/2} = h(\omega_{r+1/2})$. This leads to:

$$\sum_{\text{dof}^*} \langle A\mathbf{x}_{\text{dof}}, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle = \sum_{r+1/2} B_{r+1/2} = \sum_r \langle A\mathbf{x}_r, \mathbf{C}_j^r \rangle + \sum_r \langle A\mathbf{M}_{r+1/2}, \mathbf{C}_j^{r+1/2} \rangle. \quad (119)$$

Moreover, one has:

$$\begin{aligned} B_{r+1/2} &= \left\langle \frac{1}{2}A\mathbf{x}_r, (1 - f_{r+1/2})\mathbf{x}_{r+1}^\perp + f_{r+1/2}\mathbf{M}_{r+1/2}^\perp - \mathbf{x}_r^\perp \right\rangle \\ &\quad + \left\langle \frac{1}{2}A\mathbf{M}_{r+1/2}, f_{r+1/2}\mathbf{x}_{r+1}^\perp - f_{r+1/2}\mathbf{x}_r^\perp \right\rangle \\ &\quad + \left\langle \frac{1}{2}A\mathbf{x}_{r+1}, \mathbf{x}_{r+1}^\perp - f_{r+1/2}\mathbf{M}_{r+1/2}^\perp - (1 - f_{r+1/2})\mathbf{x}_r^\perp \right\rangle. \end{aligned}$$

To conclude, we notice that the coefficient in front of $\langle A\mathbf{M}_{r+1/2}, \mathbf{x}_{r+1}^\perp \rangle$ writes exactly $f_{r+1/2}/2$. More generally, we can notice that the coefficient in front of $\langle A\mathbf{M}_i, \mathbf{M}_j^\perp \rangle$ writes exactly as $\int_0^1 \lambda_i \frac{d\lambda_j}{dq}$. This leads to:

$$B_{r+1/2} = \int_0^1 \left\langle A\mathbf{M}(q), \frac{d}{dq}\mathbf{M}^\perp(q) \right\rangle dq,$$

with:

$$\mathbf{M}_0 = \mathbf{M}(q=0) = \mathbf{x}_r, \quad \mathbf{M}_2 = \mathbf{M}(q=1) = \mathbf{x}_{r+1},$$

and $\mathbf{M}_1 = \mathbf{M}_{r+1/2}$ is the control point of the edge $[\mathbf{x}_r, \mathbf{x}_{r+1}]$. This concludes the proof for the vector-valued functions and proves (116).

For the scalar case, it is sufficient to show:

$$\forall \xi \in \mathbb{R}^2, \int_{\partial\Omega_j} \langle \xi, \mathbf{N} \rangle \mathbf{x} = \sum_{\text{dof}^*} \langle \xi, \tilde{\mathbf{C}}_j^{\text{dof}} \rangle \mathbf{x}_{\text{dof}}.$$

This can be done by choosing $A = \xi \otimes \mathbf{e}$ for any $\mathbf{e} \in \mathbb{R}^2$ in (116).

Remark 2. (*Quadrature formula using direct edge parametrization*)

We recall the parametrization of conical curve using $\mathbf{M}_0, \mathbf{M}_1$ (and a non negative scalar weight ω), \mathbf{M}_2 in terms of CAD points is given by (7). Changing the view point and taking now the shoulder points \mathbf{S} instead of control point M_1 as reference node, the parametrization of conical curve (using $\mathbf{M}_0, \mathbf{S}, \mathbf{M}_2$) (see Figure 3):

$$\mathbf{M}^\omega(q) = \tilde{\lambda}_0(q)\mathbf{M}_0 + \tilde{\lambda}_1(q)\mathbf{S} + \tilde{\lambda}_2(q)\mathbf{M}_2, \quad q \in [0, 1]. \quad (120)$$

with:

$$\tilde{\lambda}_0(q) = \frac{(1-q)(1-2q)}{\ell(q)}, \quad \tilde{\lambda}_1(q) = 2(\omega+1)\frac{q(1-q)}{\ell(q)}, \quad \tilde{\lambda}_2(q) = \frac{q(2q-1)}{\ell(q)}, \quad \sum_{l=0}^2 \alpha_l(q) = 1.$$

Using this parametrization, we could have directly proved:

$$B_{r+1/2} = \int_0^1 \left\langle A\mathbf{M}(q), \frac{d}{dq}\mathbf{M}^\perp(q) \right\rangle dq,$$

with:

$$B_{r+1/2} = \left\langle \frac{1}{2} \mathbf{A} \mathbf{x}_r, \tilde{\mathbf{N}}_{r+1/2}^- \right\rangle + \left\langle \frac{1}{2} \mathbf{A} \mathbf{x}_{r+1}, \tilde{\mathbf{N}}_{r+1/2}^+ \right\rangle + \left\langle \frac{1}{2} \mathbf{A} \mathbf{x}_{r+1/2}, h_{r+1/2} \mathbf{N}_{r,r+1} \right\rangle,$$

and concluded the proof using:

$$\sum_{dof^*} \left\langle \mathbf{A} \mathbf{x}_{dof}, \tilde{\mathbf{C}}_j^{dof} \right\rangle = \sum_{r+1/2} B_{r+1/2}.$$

However this approach would have led to much more complicated computations, the computation of the integrals $\left(\int_0^1 \tilde{\lambda}_j \frac{d\tilde{\lambda}_i}{dq} \right)_{i,j}$ being more difficult.

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