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Asymptotic separation-based tests for noise processes against first order diagonal bilinear dependence

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Abstract

This paper presents a consistent family of non-classical tests allowing us to distinguish between a simple order one diagonal bilinear model and a noise process. The consistence of the test is established by stating the asymptotic separation of the sequences of the probability laws defined by the hypotheses being tested. The family of separation sets, from which we deduce an acceptance region of the null, is based on the process trajectories and does not take into account the moments of the process. So, the methodology here presented is significantly different from those that use the moments of order greater than two to distinguish between such families of models.

AMS Classification: 62M10, 62F03

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Shortened title: Asymptotic separation-based tests for bilinear models

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1 Introduction

The asymptotic separation of two families of probability laws is a probabilistic notion that can be implemented in the statistical inference of stochastic processes to construct new kinds of convergent tests or estimators.

Let $X = (X_t, t \in \mathbb{Z})$ be a real stochastic process with a law belonging to a set of parametric laws $(P_\theta, \theta \in \Theta)$. Let $\{\Theta_1, \Theta_2\}$ be a partition of Θ . Following Geffroy (1976), we say that the two families of laws $(P_\theta, \theta \in \Theta_1)$ and $(P_\theta, \theta \in \Theta_2)$ are uniformly asymptotically separated if there exists a sequence of Borel sets of \mathbb{R}^T , $(A_T, T \in \mathbb{N})$, such that

$$\begin{cases} \inf_{\theta \in \Theta_1} P_\theta^T(A_T) \xrightarrow{T \rightarrow +\infty} 1 \\ \sup_{\theta \in \Theta_2} P_\theta^T(A_T) \xrightarrow{T \rightarrow +\infty} 0, \end{cases}$$

where P_θ^T denotes the probability law of (X_1, X_2, \dots, X_T) .

In Geffroy (1976, 1980), we find conditions on the two families of laws under which the uniform asymptotic separation is stated.

So, if the sets A_T only depend on the observable part of the process, we may use this procedure to construct sequences of convergent tests of the hypothesis $\theta \in \Theta_1$ against the alternative $\theta \in \Theta_2$, with acceptance regions A_T (Tassi and Legait, 1990).

This methodology was originally applied by Moché (1989) to test the signal function, $\theta \in \Theta$, of a model $X = (X_t, t \in \mathbb{Z})$, such that $X_t = \theta(t) + \varepsilon_t$, where $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a classical white noise.

If we deal with more general models like, for instance, ARMA models, Geffroy's results are not directly applied. Nevertheless, the procedure used to construct regions A_T leads to convergent tests of simple hypotheses for those kinds of models as was obtained by Massé and Viano (1995) for AR(1) models with independent error processes and Gonçalves and Mendes-Lopes (1998) for AR(p) models with a general non-independent error process.

More recently, Gonçalves, Jacob and Mendes-Lopes (2000) have shown the asymptotic separation between an error process and a fixed simple bilin-

ear order one model and a decision procedure to distinguish the two models was proposed, which is not a Neyman-Pearson classical one. In fact, as the size is not fixed *a priori*, a symmetrical role is attributed to both test hypotheses.

Taking into account this particular separation result, we will propose, in this paper, a test to distinguish between an error process $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ and the family of bilinear models $X = (X_t, t \in \mathbb{Z})$ defined by

$$X_t = \varphi X_{t-1} \varepsilon_{t-1} + \varepsilon_t,$$

with $\varphi > 0$ (or $\varphi < 0$) and where ε is a sequence of independent, identically distributed (i.i.d.) and absolutely continuous random variables.

The proposal developed here is a new methodology of testing for simple bilinear models. In fact, to our knowledge, the tests currently available for bilinear models are based on classical methods, like the Lagrange multiplier tests (e.g., Saikonnen and Luukonen, 1988). Non-classical approaches have been proposed, for instance, by Benghabrit and Hallin (1992, 1996), who have used the LAN property to construct optimal rank-based tests for testing autoregressive against bilinear dependence, or by Diebolt and Wandgi (1997) who have considered a nonparametric technic.

If compared with those proposals, our methodology presents two kinds of advantages: the weakness of the underlying model hypotheses and the easiness of its practical implementation. In fact, we point out that this test is essentially based on process trajectories and that the usual hypotheses on the process moments are not imposed here. This is a great advantage as we know that, for instance in moment approaches, moments of order greater than two are needed to distinguish the two families of models; so, methodologies using this kind of hypotheses will imply stronger restrictions in the families of error processes as well as in the domain of the parameter φ (Martins, 1999). Moreover, the simple and explicit form of the acceptance region leads us to a easily achievable test.

In brief, we consider this methodology as a quick and symmetrical way of diagnostic between error processes and bilinear ones, for general type of data.

2 Stationarity and invertibility

Let us consider the simple bilinear real valued stochastic process $X = (X_t, t \in \mathbb{Z})$, defined by

$$X_t = \varphi X_{t-1} \varepsilon_{t-1} + \varepsilon_t, \quad (1)$$

where $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a sequence of i.i.d. absolutely continuous random variables and φ is a real parameter.

Under the conditions $E(|\log |\varepsilon_t||) < \infty$ and $E(\log |\varepsilon_t|) + \log |\varphi| < 0$, we obtain

$$X_t = \varepsilon_t + \sum_{n=1}^{\infty} \varphi^n \varepsilon_{t-n} \prod_{j=1}^n \varepsilon_{t-j} \quad (a.s.), \quad t \in \mathbb{Z}. \quad (2)$$

From Quinn (1982) and Azencott and Dacunha-Castelle (1984, pp. 30/32), (2) defines a strictly stationary and ergodic solution to (1).

Furthermore, model (1) is invertible if $E(|\log |X_t||) < \infty$ and $E(\log |X_t|) + \log |\varphi| < 0$; so, we obtain

$$\varepsilon_t = X_t + \sum_{n=1}^{\infty} (-\varphi)^n X_{t-n} \prod_{j=1}^n X_{t-j} \quad (a.s.), \quad t \in \mathbb{Z}. \quad (3)$$

Denoting, for a given stochastic process $U = (U_t, t \in \mathbb{Z})$, $\mathcal{B}_U^{(n)}$ the σ -field generated by (U_n, U_{n-1}, \dots) , we have from (2) and (3) $\mathcal{B}_X^{(t)} = \mathcal{B}_\varepsilon^{(t)}$.

3 Markovian representation and mixing

Following Diebolt and Wandji (1997), we state in theorem 3.1 a Markovian representation for model (1). This Markovian approach allows us to find sufficient conditions to ensure the geometric strong mixture (or α -mixture) property of this model.

Theorem 3.1 Let $(X_t, t \in \mathbb{Z})$ be the bilinear model defined by (1), where φ is a real parameter and $(\varepsilon_t, t \in \mathbb{Z})$ is a sequence of i.i.d. absolutely continuous random variables such that $E(|\log |\varepsilon_t||) < \infty$ and $E(\log |\varepsilon_t|) + \log |\varphi| < 0$. Then, there is a strictly stationary homogeneous Markovian process, $(Z_t, t \in \mathbb{Z})$, taking its values on \mathbb{R}^2 , such that

$$(i) \quad X_{t+1} = \mathbf{H}Z_t + \varepsilon_{t+1} \quad \text{and} \quad Z_t = \mathbf{A}Z_{t-1} + \mathbf{B}Z_{t-1}\varepsilon_t + \mathbf{C}\varepsilon_t + \mathbf{D}\varepsilon_t^2,$$

$$(ii) \quad X_t = \mathbf{K}Z_t,$$

where

$$\mathbf{H} = [0 \ 1], \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ \varphi \end{bmatrix}$$

and $\mathbf{K} = [1 \ 0]$.

Proof: Let us define

$$Z_t^{(1)} = X_t, \quad Z_t^{(2)} = \varphi X_t \varepsilon_t \quad \text{and} \quad Z_t = [Z_t^{(1)} \ Z_t^{(2)}]^T.$$

We have

$$Z_t^{(1)} = \varphi X_{t-1} \varepsilon_{t-1} + \varepsilon_t = Z_{t-1}^{(2)} + \varepsilon_t \quad (4)$$

and

$$\begin{aligned} \mathbf{H}Z_t + \varepsilon_{t+1} &= [0 \ 1] \begin{bmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{bmatrix} + \varepsilon_{t+1} \\ &= Z_t^{(2)} + \varepsilon_{t+1} \\ &= X_{t+1}, \text{ from } Z_t^{(1)} \text{ definition,} \end{aligned}$$

which proves the first part of (i). Concerning the second part, some simple algebra gives

$$\mathbf{A}Z_{t-1} + \mathbf{B}Z_{t-1}\varepsilon_t + \mathbf{C}\varepsilon_t + \mathbf{D}\varepsilon_t^2 = Z_t, \quad (5)$$

taking into account that

$$\begin{aligned} Z_t^{(2)} &= \varphi Z_t^{(1)} \varepsilon_t \\ &= \varphi Z_{t-1}^{(2)} \varepsilon_t + \varphi \varepsilon_t^2. \end{aligned}$$

Concerning (ii), it follows immediately that the result of the product KZ_t is $Z_t^{(1)}$, which is equal to X_t by definition.

Under the conditions $E(|\log |\varepsilon_t||) < \infty$ and $E(\log |\varepsilon_t|) + \log |\varphi| < 0$, the components of Z_t are measurable functions of $(\varepsilon_t, \varepsilon_{t-1}, \dots)$. As $(\varepsilon_t, t \in \mathbb{Z})$ is strictly stationary, the components of Z_t are strictly stationary processes. So, equality (5) and the independence of ε_t and Z_{t-1} imply that the process $(Z_t, t \in \mathbb{Z})$ is Markovian and homogeneous.

So, the process $(Z_t, t \in \mathbb{Z})$ is strictly stationary. \diamond

The strong mixture property of the model is related to the geometric absolute regularity of $(Z_t, t \in \mathbb{Z})$, (Bradley, 1985). In order to establish this absolute regularity, we introduce now the following additional technical assumption (Mokkadem, 1990, th. 4.3):

(A) There exists a function, $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$, a compact $K \subset \mathbb{R}^2$ and real numbers $b, \varepsilon > 0, 0 < \delta < 1$, such that:

(i) $\lambda_2(K) > 0$, with λ_2 the Lebesgue measure on \mathbb{R}^2 .

(ii) $E(f(Z_t)/Z_{t-1} = z) \leq \delta f(z) - \varepsilon$, if $z \notin K$.

(iii) $E(f(Z_t)/Z_{t-1} = z) \leq b$, if $z \in K$.

According to theorem 4.3 of Mokkadem (1990), this assumption implies that the Markovian stationary process $(Z_t, t \in \mathbb{Z})$ defined in theorem 3.1 is geometrically absolutely regular and therefore geometrically α -mixing (Bradley, 1985). Then, the process $(X_t, t \in \mathbb{Z})$ is also geometrically α -mixing.

4 A consistent test for bilinear models

4.1 The test definition

Let us consider the model $X = (X_t, t \in \mathbb{Z})$ defined by

$$X_t = \varphi X_{t-1} \varepsilon_{t-1} + \varepsilon_t,$$

where φ is a real parameter and $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a sequence of i.i.d. random variables, absolutely continuous, such that $E(|\log |\varepsilon_t||) < \infty$ and $E(\log |\varepsilon_t|) + \log |\varphi| < 0$.

We suppose that, for each $t \in \mathbb{Z}$, the law of ε_t has a unique zero median.

In order to distinguish an error process from a bilinear one, we will construct a consistent test for the hypotheses

$$H_0 : \varphi = 0 \text{ against } H_1 : \varphi > 0.$$

We define the process $Y = (Y_t, t \in \mathbb{Z})$ by $Y_t = X_t \varepsilon_t$, i.e.,

$$Y_t = X_t \left(X_t + \sum_{n=1}^{\infty} (-\varphi)^n X_{t-n} \prod_{j=1}^n X_{t-j} \right) \quad (a.s.). \quad (6)$$

Obviously, the process $(Y_t, t \in \mathbb{Z})$ is also strictly stationary and ergodic. We note that $X_t = \varphi Y_{t-1} + \varepsilon_t$, according to (1) and (6).

To construct the acceptance regions of the test, we use T observations of the process X denoted by x_1, x_2, \dots, x_T , $T \geq 2$. Moreover, let g be a symmetrical, continuous and bounded weight-function defined on \mathbb{R} . Suppose also that g is strictly positive, non-increasing on \mathbb{R}^+ and P_φ -integrable, with P_φ the model distribution.

Defining the set

$$D(\varphi) = \left\{ (u, v) \in \mathbb{R}^2 : \left(u > 0 \wedge v < \frac{\varphi}{2} u \right) \vee \left(u < 0 \wedge v > \frac{\varphi}{2} u \right) \right\}, \quad (7)$$

we consider the regions

$$A_T(\varphi) = \left\{ x^{(T)} : \sum_{t=2}^T g(\varphi y_{t-1}) [2\mathbf{1}_{D(\varphi)}(y_{t-1}, x_t) - 1] \geq 0 \right\}, \quad T \geq 2, \quad (8)$$

where $x^{(t)} = (\dots, x_{t-1}, x_t) \in \prod_{-\infty}^t \mathbb{R}$ and y_t denotes the particular value of Y_t .

Considering a consistent estimator φ_T^* of φ , we state, in the following, that $A_T(\varphi_T^*)$ is the acceptance region of a consistent test of H_0 against H_1 .

4.2 The test consistence

For each $t \in \mathbb{Z}$, let us define

$$\psi(Y_{t-1}, X_t; \varphi) = \psi_t(\varphi) = g(\varphi Y_{t-1})[2\mathbf{1}_{D(\varphi)}(Y_{t-1}, X_t) - 1] \quad (9)$$

and

$$\bar{\Psi}_T(\varphi) = \frac{1}{T} \sum_{t=2}^T \psi(Y_{t-1}, X_t; \varphi). \quad (10)$$

In Gonçalves, Jacob and Mendes-Lopes (2000), a consistent decision procedure was presented to decide between the hypotheses $H_0 : \varphi = 0$ and $H_1 : \varphi = \beta$, ($\beta > 0$, arbitrarily fixed). We point out that an exponential decay of the size was obtained in this case. We also note that with a particular choice of the weight function we were led to the minimum upper bound of the size. In particular, this upper bound allows us to obtain, for a fixed $\alpha \in]0, 1[$, the number of observations, T , for which the test is of size α . So, a real comparison with a Neyman-Pearson methodology could be implemented. Moreover, some simulations results concerning the behaviour of the power have been obtained, allowing us to infer a similar exponential rate of convergence. Unfortunately, theoretical results assuring this behaviour haven't yet been obtained.

The consistence of that procedure was stated using the following convergence, which is a consequence of the ergodic theorem:

$$\lim_{T \rightarrow +\infty} \bar{\Psi}_T(\varphi) = E_\varphi(\psi_2(\varphi)), \quad \forall \varphi \geq 0, \quad P_\varphi - a.s.,$$

where E_φ denotes the expectation corresponding to the model distribution.

The proof of the consistence of the test here presented is analogous to that of theorem 1 of Gonçalves, Jacob and Mendes-Lopes (2000) after showing that

$$\lim_{T \rightarrow +\infty} \bar{\Psi}_T(\varphi_T^*) = E_\varphi(\psi_2(\varphi)), \quad P_\varphi - a.s.,$$

where φ_T^* is a convergent estimator of φ , as for instance the modified least squares estimator proposed by Pham and Tran (1981).

This convergence is the main theoretical result of this paper and is the key for using that procedure as a test; otherwise, it allows us to generalize the procedure presented in Gonçalves, Jacob and Mendes-Lopes (2000) to tests of composite hypotheses.

To establish this convergence, we will use a result of Gouriéroux and Monfort (1989), mentioned in the appendix as lemma A1, and we suppose that φ belongs to an interval $[-a, a]$ included in the stationarity region of the process X . The three following lemmas ensure the applicability of this result.

Let us define the process $W = (W_t, t \in \mathbb{Z})$ by

$$W_t = \sup_{\varphi} |\zeta_t(\varphi)| - E_{\varphi} \left(\sup_{\varphi} |\zeta_t(\varphi)| \right), \quad (11)$$

with

$$\zeta_t(\varphi) = \psi_t(\varphi) - E_{\varphi}(\psi_t(\varphi)), \quad \varphi \in [-a, a]. \quad (12)$$

Theorem 3.1 and the above-mentioned theorem 4.3 of Mokkadem, allow us to conclude that the process $(W_t, t \in \mathbb{Z})$ is strong mixing. This fact is stated in the next lemma.

Lemma 4.1 *Under assumption (A), the process $(W_t, t \in \mathbb{Z})$ defined by (11) is geometrically α -mixing.*

Proof: From part (i) of theorem 3.1, we may write

$$\varepsilon_t = X_t - \mathbf{H}Z_{t-1} \quad (13)$$

and from (ii) we have

$$X_t = \mathbf{K}Z_t.$$

Then

$$Y_t = X_t \varepsilon_t = \mathbf{K}Z_t (\mathbf{K}Z_t - \mathbf{H}Z_{t-1}). \quad (14)$$

So, Y_{t-1} is a measurable function of Z_{t-1} and Z_{t-2} .

Therefore, $\zeta_t(\varphi)$ defined by (12) is also a measurable function of Z_t, Z_{t-1} and Z_{t-2} as well as W_t . Then, as the process Z is geometrically α -mixing under assumption (\mathcal{A}) , the same happens to the process W . \diamond

Lemma 4.2 *Under the conditions of lemma 4.1, we have*

$$\lim_{T \rightarrow +\infty} \bar{\Psi}_T(\varphi) = E_\varphi(\psi_2(\varphi)), \quad P_\varphi - a.s., \quad (15)$$

uniformly in $\varphi, \varphi \in [-a, a]$.

Proof: To prove this lemma, let's use a result of Carbon (1988), which is presented in the appendix as theorem A2.

Firstly, note that the ergodic theorem allows us to obtain the convergence (15).

Defining

$$H_T(\varphi) = \frac{T}{T-1} \bar{\Psi}_T(\varphi), \quad \varphi \in [-a, a],$$

the uniform convergence (15) is established, proving that

$$\lim_{T \rightarrow +\infty} \sup_{\varphi} |H_T(\varphi) - E_\varphi(H_T(\varphi))| = 0, \quad P_\varphi - a.s.,$$

as $E_\varphi(H_T(\varphi)) = E_\varphi(\psi_2(\varphi))$.

Note that

$$\frac{1}{T-1} \sum_{t=2}^T \zeta_t(\varphi) = H_T(\varphi) - E(H_T(\varphi)), \quad (16)$$

with $\zeta_t(\varphi)$ defined by (12).

Let us verify that the process $(W_t, t \geq 2)$ satisfies the conditions of theorem A2.

Firstly, we remember that, according to lemma 4.1, $(W_t, t \in \mathbb{Z})$ is geometrically α -mixing and so the process $(W_t, t \geq 2)$ is also α -mixing (Bradley, 1985, p. 170). It is clear that the α -mixing coefficients of $(W_t, t \geq 2)$ do not exceed the corresponding coefficients of $(W_t, t \in \mathbb{Z})$. So, we may conclude that the process $(W_t, t \geq 2)$ is geometrically α -mixing.

Moreover, $E_\varphi(W_t) = 0$.

Let us denote $S_t = \sup_\varphi |\zeta_t(\varphi)|$. As g is bounded, there exists $L > 0$ such that $g(x) \leq L$, $x \in \mathbb{R}$. Then

$$\begin{aligned} |\zeta_t(\varphi)| &\leq g(\varphi Y_{t-1}) + |E_\varphi \{g(\varphi Y_{t-1})[2\mathbf{1}_{D_\varphi}(Y_{t-1}, X_t) - 1]\}| \\ &\leq g(\varphi Y_{t-1}) + E_\varphi(g(\varphi Y_{t-1})) \\ &\leq 2L, \end{aligned}$$

which implies

$$|W_t| = |S_t - E(S_t)| \leq 4L.$$

Finally,

$$E_\varphi(W_t^2) \leq E_\varphi(S_t^2) \leq 4L^2.$$

Let ϵ be any positive real number.

$$\begin{aligned} P_\varphi \left(\sup_\varphi |H_T(\varphi) - E_\varphi(H_T(\varphi))| > \epsilon \right) &\leq P_\varphi \left(\sup_\varphi \frac{1}{T-1} \sum_{t=2}^T |\zeta_t(\varphi)| > \epsilon \right) \\ &\leq P_\varphi \left(\frac{1}{T-1} \sum_{t=2}^T S_t > \epsilon \right) \\ &= P_\varphi \left\{ \frac{1}{T-1} \sum_{t=2}^T [S_t - E_\varphi(S_t)] + \frac{1}{T-1} \sum_{t=2}^T E_\varphi(S_t) > \epsilon \right\} \\ &\leq P_\varphi \left(\frac{1}{T-1} \sum_{t=2}^T W_t + 2L > \epsilon \right). \end{aligned}$$

Using Tchebychev inequality, we obtain

$$P_\varphi \left(\frac{1}{T-1} \sum_{t=2}^T W_t + 2L > \epsilon \right) \leq e^{2L} e^{-\epsilon} E_\varphi \left[\exp \left(\frac{1}{T-1} \sum_{t=2}^T W_t \right) \right]. \quad (17)$$

Applying inequality (21) in the appendix to the process $(W_t, t \geq 2)$, we have

$$\forall \epsilon > 0, \forall \gamma, 0 < \gamma < 1, \exists T_0 = T_0(\gamma) \geq 0: \forall T \geq T_0,$$

$$e^{-\epsilon} E_\varphi \left[\exp \left(\frac{1}{T-1} \sum_{t=2}^T W_t \right) \right] \leq 2 \exp \left(\frac{-\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right). \quad (18)$$

From (17) and (18), we conclude that

$$\forall \epsilon > 0, \forall \gamma, 0 < \gamma < 1, \exists T_0 = T_0(\gamma) \geq 0: \forall T \geq T_0,$$

$$P_\varphi \left(\sup_\varphi |H_T(\varphi) - E_\varphi(H_T(\varphi))| > \epsilon \right) \leq 2e^{2L} \exp \left(\frac{-\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right). \quad (19)$$

So, there exists $T^* > 0$ such that, for $T > T^*$,

$$\exp \left(\frac{\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right) \geq \left(\frac{\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right)^{\frac{4}{1-\gamma}}.$$

This inequality is equivalent to

$$\exp \left(\frac{-\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right) \leq \left(\frac{\epsilon}{20Le} \right)^{-\frac{4}{1-\gamma}} (T-1)^{-2}$$

The series with general term $2e^{2L} \exp \left(-\frac{\epsilon}{20Le} (T-1)^{\frac{1-\gamma}{2}} \right)$, $T \geq 2$, is convergent. Consequently, using (19), the series

$$\sum_{T=2}^{+\infty} P_\varphi \left(\sup_\varphi |H_T(\varphi) - E_\varphi(H_T(\varphi))| > \epsilon \right)$$

is also convergent, for any positive ϵ . This fact implies

$$\lim_{T \rightarrow +\infty} \sup_\varphi |H_T(\varphi) - E_\varphi(H_T(\varphi))| = 0, \quad P_\varphi \quad a.s.,$$

which ends the proof. \diamond

The following lemma ensures the uniform continuity of the function $H(\varphi) = E_\varphi(\psi_2(\varphi))$.

Lemma 4.3 *The function $H(\varphi) = E_\varphi(\psi_2(\varphi))$, $\varphi \in [-a, a]$, is continuous.*

Proof: Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\lim_{n \rightarrow +\infty} \varphi_n = \varphi$.

From (2) we have the P_0 -measurability of $(X_t, t \in \mathbb{Z})$.

Let us denote

$$x_t = f_t(\varphi) = \sum_{k=1}^{+\infty} \varphi^k e_{t-k}^2 \prod_{i=1}^k e_{t-i} + e_t,$$

where e_t represents a particular value of ε_t , $t \in \mathbb{Z}$. Denote

$$h(\varphi) = g(\varphi f_1(\varphi) e_1) [2\mathbb{1}_{D(\varphi)}(f_1(\varphi) e_1, f_2(\varphi)) - 1].$$

We start by proving that $\lim_{n \rightarrow +\infty} h(\varphi_n) = h(\varphi)$, P_0 -a.s., i.e.,

$$\begin{aligned} \lim_{n \rightarrow +\infty} g(\varphi_n f_1(\varphi_n) e_1) [2\mathbb{1}_{D(\varphi_n)}(f_1(\varphi_n) e_1, f_2(\varphi_n)) - 1] \\ = g(\varphi f_1(\varphi) e_1) [2\mathbb{1}_{D(\varphi)}(f_1(\varphi) e_1, f_2(\varphi)) - 1], \quad P_0\text{-a.s.} \end{aligned}$$

The function $f_t(\varphi)$ is continuous for each $t \in \mathbb{Z}$. In fact, considering

$$f_{t,M}(\varphi) = \sum_{k=1}^M \varphi^k e_{t-k}^2 \prod_{i=1}^k e_{t-i} + e_t, \quad M \in \mathbb{N},$$

we know that $\lim_{M \rightarrow +\infty} f_{t,M}(\varphi) = f_t(\varphi)$, P_0 -a.s. .

As $f_{t,M}(\varphi)$ are continuous functions and

$$\sup_{\varphi} |f_{t,M}(\varphi) - f_t(\varphi)| \leq \sum_{k=M+1}^{+\infty} a^k e_{t-k}^2 \prod_{i=1}^{k-1} |e_{t-i}|,$$

which converges a.s. to zero when $M \rightarrow +\infty$ (Gonçalves, Jacob and Mendes-Lopes, 2000), we conclude that the convergence is uniform and, consequently, that $f_t(\varphi)$ is continuous.

As g is continuous, the function $g(\varphi f_1(\varphi) e_1)$ is also continuous in φ .

Now we study

$$\mathbf{I}_{D(\varphi_n)}(f_1(\varphi_n)e_1, f_2(\varphi_n)) = \mathbf{I}_{D_0}\left(f_1(\varphi_n)e_1, f_2(\varphi_n) - \frac{\varphi_n}{2}f_1(\varphi_n)e_1\right),$$

with $D_0 =]0, +\infty[\times]-\infty, 0[\cup]-\infty, 0[\times]0, +\infty[$.

Let $p: \mathbb{R} \rightarrow \mathbb{R}^2$ be the function defined by

$$p(\varphi) = (f_1(\varphi)e_1, f_2(\varphi) - \frac{\varphi}{2}f_1(\varphi)e_1).$$

This function is continuous as f_t is continuous for all $t \in \mathbb{Z}$.

Let us show now that the function

$$\mathbf{I}_{D_0}(x, y), \quad (x, y) \in \mathbb{R}^2,$$

is continuous on $\mathbb{R}^2 \setminus \{(x, y) : xy = 0\}$.

Firstly, we may write

$$\mathbf{I}_{D_0}(x, y) = \mathbf{I}_{D_{1,0}}(x, y) + \mathbf{I}_{D_{2,0}}(x, y)$$

where $D_{1,0} =]-\infty, 0[\times]0, +\infty[$ and $D_{2,0} =]0, +\infty[\times]-\infty, 0[$.

We consider $(x, y) \in D_{1,0}$ and a sequence $(x_n, y_n) \rightarrow (x, y)$, when $n \rightarrow +\infty$.

So,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0, \max(|x_n - x|, |y_n - y|) < \varepsilon,$$

and then

$$\forall n \geq n_0, x_n < x + \varepsilon \implies x_n \leq x + \inf_{\varepsilon > 0} \varepsilon = x < 0$$

and

$$\forall n \geq n_0, y - \varepsilon < y_n < y + \varepsilon \implies y_n \geq \sup_{\varepsilon > 0} (y - \varepsilon) = y > 0.$$

Then

$$\forall n \geq n_0, \mathbb{I}_{D_{1,0}}(x_n, y_n) = 1.$$

If $(x, y) \in D_{1,0}^C \setminus \{(x, y) : xy = 0\}$, there exists three possibilities for the values of x and y : $x > 0, y > 0$, $x > 0, y < 0$ and $x < 0, y < 0$.

Let us analyse the first case.

Obviously, $\mathbb{I}_{D_{1,0}}(x, y) = 0$. Considering $(x_n, y_n) \rightarrow (x, y)$, $n \rightarrow +\infty$, we conclude as before that $x > 0$ and $y_n \geq y > 0$.

So, $\forall n \geq n_0, \mathbb{I}_{D_{1,0}}(x_n, y_n) = 0$.

In the other two cases, we conclude in the same way.

The analysis for $(x, y) \in D_{2,0}$ follows the same steps as this one.

Using the composition of continuous functions and the absolute continuity of P_0 , we conclude that

$$\lim_{n \rightarrow +\infty} h(\varphi_n) = h(\varphi), P_0 - a.s. \quad (20)$$

As g is bounded

$$\sup_n \int g^2(\varphi_n f_1(\varphi_n) e_1) dP_0 < +\infty,$$

which implies

$$\sup_n \int h^2(\varphi_n) dP_0 < +\infty.$$

Then, the sequence $(h(\varphi_n))_{n \in \mathbb{N}}$ is bounded in L^2 and so it is P_0 -equi-integrable (Marle, 1974, chap 4). In consequence, as in (20) $(h(\varphi_n))_{n \in \mathbb{N}}$ converges *a.s.* to $h(\varphi)$, we have

$$\lim_{n \rightarrow +\infty} \int h(\varphi_n) dP_0 = \int h(\varphi) dP_0.$$

◇

We note that, as the function $H(\varphi)$ is continuous on $[-a, a]$, it is also uniformly continuous.

We may now establish the consistence of the test of the hypotheses

$$H_0 : \varphi = 0 \text{ against } H_1 : \varphi > 0,$$

with acceptance region given by

$$A_T(\varphi_T^*) = \left\{ x^{(T)} : \sum_{t=2}^T g(\varphi_T^* y_{t-1}) [2\mathbf{1}_{D(\varphi_T^*)}(y_{t-1}, x_t) - 1] \geq 0 \right\}, \quad T \geq 2.$$

Theorem 4.1 *Let $(X_t, t \in \mathbb{Z})$ be a real stochastic process defined by (1) under all the conditions previously imposed. Let $\varphi_T^* = \varphi_T^*(x_1, \dots, x_T)$ be a consistent estimate of φ . Then $A_T(\varphi_T^*)$ is the acceptance region of a consistent test of $H_0 : \varphi = 0$ against $H_1 : \varphi > 0$.*

Proof: Lemmas 4.2 and 4.3 and lemma A1 allow us to conclude that, if $\varphi_T^*(X_1, \dots, X_T)$ converges a.s. to φ , when $T \rightarrow +\infty$, then

$$\lim_{T \rightarrow +\infty} \bar{\Psi}_T(\varphi_T^*(X_1, \dots, X_T)) = E_\varphi(\psi_2(\varphi)), \quad P_\varphi - a.s..$$

The consistence is established by showing that under H_1 we have $E_\varphi(\psi_2(\varphi)) < 0$ (Gonçalves, Jacob and Mendes-Lopes, 2000), i.e.,

$$\lim_{T \rightarrow +\infty} \bar{\Psi}_T(\varphi_T^*) < 0, \quad (P_\varphi - a.s.),$$

which implies

$$\lim_{T \rightarrow +\infty} P_\varphi(A_T(\varphi_T^*)) = 0$$

by the dominated convergence theorem. ◇

Remark. Concerning the hypotheses

$$H_0 : \varphi = 0 \text{ against } H_1' : \varphi < 0,$$

we obtain an analogous result defining the sequence of regions $(A'_T(\varphi_T^*), T \geq 2)$ by

$$A'_T(\varphi_T^*) = \left\{ x^{(T)} : \sum_{t=2}^T g(\varphi_T^* y_{t-1}) [2\mathbf{1}_{D'(\varphi_T^*)}(y_{t-1}, x_t) - 1] \geq 0 \right\},$$

where

$$D'(\varphi) = \left\{ (u, v) \in \mathbb{R}^2 : \left(u < 0 \wedge v < \frac{\varphi}{2}u \right) \vee \left(u > 0 \wedge v > \frac{\varphi}{2}u \right) \right\}.$$

Appendix

I. Lemma A1.

Lemma (Gouriéroux and Monfort, 1989, p. 436) *Let $\tilde{Q}_n(\alpha, \beta)$ be a sequence of functions that converges uniformly in (α, β) to $\tilde{Q}_\infty(\alpha, \beta)$. If the limit function \tilde{Q}_∞ is uniformly continuous in (α, β) and if (α_n) is a sequence that converges to α_0 , then the sequence of functions $(\tilde{Q}_n(\alpha_n, \beta))$ converges uniformly in β to $\tilde{Q}_\infty(\alpha_0, \beta)$.*

II. Theorem A2.

The following inequality is obtained from the proof of proposition 23 of Carbon (1988, p. 138-143).

Theorem (Carbon, 1988) *Let $(U_t, t \in \mathbb{N})$ be a sequence of real random variables that is geometrically α -mixing and verifies $E(U_t) = 0$, $|U_t| \leq d$ and $E(U_t^2) \leq D$, $t \in \mathbb{N}$. Then, $\forall \epsilon > 0$, $\forall \gamma$, $0 < \gamma < 1$, $\exists N = N(\gamma) \geq 0$ such that $\forall n \geq N$,*

$$e^{-\epsilon} E \left[\exp \left(\frac{1}{n} \sum_{t=1}^n U_t \right) \right] \leq 2 \exp \left(\frac{-\epsilon}{5de} n^{\frac{1-\gamma}{2}} \right). \quad (21)$$

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Abstract

Given a realizations of a random sum S_N , we investigate the estimations of its probability density function f_N and generating function G by kernel and empirical methods respectively. In each case, we discuss two estimators and make comparisons based on their central limit theorems.

1. Introduction.

Statistical inferences for stochastic process data grow rapidly recently. Developments in this field can be found in Taniguchi and Kakizawa [14] for time series, Prakasa Rao [10] for diffusion type processes, Prakasa Rao [11] for