



جامعة تونس  
Université de Tunis



# Transition from high school to university calculus: a study of connection

Ghedamsi, I., Lecorre, T. (2021) Transition from high school to university calculus: a study of connection. *ZDM Mathematics Education* **53**, 563–575.

<https://doi.org/10.1007/s11858-021-01262-1>

# Context: What?

□ Calculus 4 big ideas: limit, differentiation, integration & FTC, series

□ Calculus foundation based on the structure of the mathematics continuum:

-Archimedean continuum: standard analysis anticipated by the epsilon-delta limit definition of Weierstrass

-Infinitesimal-enriched continuum: non-standard analysis based not only on infinitesimals but also their inverses

□ Calculus instruction: real numbers, functions, sequences, rate of changes, derivatives, antiderivatives, integral, area, series...

# Context: What?

## ☐ Transition:

- Three levels of education: Primary, Secondary (Pre-Calculus, Calculus), Tertiary
- Beginning and ending of secondary-tertiary calculus transition: 2 or 3 last years of secondary education and 1 or 2 first years of tertiary education

## ☐ Transition: Numerous changes faced by students entering university calculus

- Learning expectations
- Teaching staff and practices
- Classroom environment
- Autonomy and self-government
- Social environment ...

# Context: What?

## ☐ Calculus Transition phenomenon: Meta analysis

- Described as either continuous or discontinuous / Not exactly like either
- Two “contradictory” pictures of reality / Separately or together
- Emerging trend of thoughts about transition: Search for potential continuous path in one or more of these changes (bridging both sides)
  - Disparities and possible connection between teachers’ pedagogical principles in either end of the transition
  - Bridging course or design (no evidence of positive impact)

# Project: Focus on change in cognitive requirements

❑ Key proposal / A more developmental perspective: Investigate teachers' teaching actions and the related mathematics knowledge for teaching (one, both sides?)

❑ New lenses/working hypothesis: methods and conceptual tools?

- **Step 1 (both levels):** to identify shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

- **Step 2 (both levels):** to get additional data from the teachers' perspective

- **Step 3 (which level?):** to engage in thinking on teachers' knowledge dynamics (*OECD working paper, 2017, Knowledge for teaching and the teaching nature of the teaching profession*)

# The study - Step1: How?

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

- ❑ Conceptual tools for characterizing /comparing cognitive demands
  - Memorization, procedures, procedures without connections, ...
  - Bloom taxonomy
  - Revised Bloom taxonomy - Two dimensions (Knowledge types & Cognitive processes):
    - Remember
    - Recall and apply procedures
    - Recognize and apply procedures
    - Understand
    - Apply understanding
    - Analyze
    - Evaluate
    - Create

Remember	Students are prompted to retrieve knowledge from long-term memory	Write the definition of the derivative of $f$ at $x = a$
Recall and Apply Procedure	Students must recall the algorithms for applying certain procedures and carry them out	Find the derivative of $f$ at $x = a$
Recognize and Apply Procedure <sup>a</sup>	Students must recognize what knowledge or procedures to recall without being directly prompted. Conceptual knowledge plays a plausible role in this venture, but some students may be able to answer the question with memorized procedures or formulas. Students may have to string together several procedures	At what value of $x$ does $f$ attain a minimum?
Understand	Students are prompted to make interpretations, provide explanations, make comparisons or make inferences that require an understanding of a mathematics concept	Is this graph a plausible graph for $f$ given the table for values of $f$ ? Explain why or why not
Apply Understanding	Students must recognize when to use (or apply) a concept when responding to a question or when working a problem. To recognize the need to apply, execute or implement a concept in the context of working a problem requires an understanding of the concept	Given a flask shaped like an inverted cone, write a function rule that expresses the relationship between the height of the water in the flask and number of ounces of water in the flask
Analyze	Students are prompted to break material into constituent parts and determine how parts relate to one another and to an overall structure or purpose. Differentiating, organizing, and attributing are characteristic cognitive processes at this level	Graph the given functions on a common screen. How are these graphs related? $y = 3^x$ , $y = 10^x$ , $y = (1/3)^x$ , $y = (1/10)^x$ . What general conclusions do you draw?
Evaluate	Students are prompted to make judgments based on criteria and standards. Checking and critiquing are characteristic cognitive processes at this level	Is it reasonable to use this model to predict the winning height [of the high jump] at the 2100 Olympics? (Stewart, 2012, p. 35)
Create	Students are prompted to put elements together to form a coherent or functional whole; reorganize elements into a new pattern or structure. Generating, planning, and producing are characteristic cognitive processes at this level	A student in our class wants to estimate the integral of a function using a Riemann sum with 5 partitions. Give a function and domain of integration such that the right-hand Riemann sum will be off of the true integral by more than 200 % but the left-hand Riemann sum will be exact

# The study - Step1: How?

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

## □ ATD and TDS constructs

TDS	ATD
Socio-constructivist vision of learning and teaching mathematics	Socio-cultural vision of learning and teaching mathematics
<p>Fundamental situations of mathematical objects</p> <p>Mathematical game (or problem).</p> <p>The problem should satisfy the same criteria as those developed in mathematical games theory-basically; mathematical object targeted by the problem should not be explicitly referenced and must provide an optimal method for solving it.</p>	<p>Epistemological Reference Model (ERM) related to mathematical objects</p> <p>Organization of local and regional mathematical praxeologies through sequences of connected mathematical praxeologies.</p> <p>A praxeology models human activity including mathematical one.</p>
Didactical variables namely parameters that influence the way students may solve tasks	Questions with generating power OR NOT for praxeologies (type of tasks,...)



# The study - Step1: How?

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

## ❑ Previous studies and state of the art

- practical demands vs. analytical and theoretical demands
  - computation and visualisation vs. formalization
  - instantiation vs. generalization
  - conception and image vs. definition
  - calculus notion viewed as a process vs. calculus notion encapsulated in an object ...
- No global exhaustive framework is assumed because there are too many **didactical variables** to consider

## ❑ TDS choice: A search for second-order variable

- Our previous experience with didactical variables to describe differences in the cognitive demands of tasks: the use of technical methods, the use of visualisation techniques, the use of formalism, the use of reasoning and proof setting ...
- A focus not on the variables themselves but on relationships among them (enable description of possible continuous paths)

# The study - Step1: How?

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

❑ TDS choice: A search for second-order variable

➤ Shared cornerstone teaching goals

- students' conceptual understanding of calculus and 3 skills concerned with this goal: calculus reasoning, association of essential calculus ideas, and knowing of each idea through given definitions and theorems
  - limit notion connects almost all foundational calculus: (1) real numbers as a study of limits of sequences/series; (2) differential calculus as a study of limits of rates of change; (3) integral calculus as a study of limits of accumulated changes; and (4) the fundamental theorem of calculus (FTC) as a study of the relationships between two limits
  - unifying idea of limit: (1) a 'precise' definition of limit; (2) in each application, a proof of existence of this limit, and (3) a method for 'calculating' the limit.
- successive approximation methods: use of infinity and infinitesimals, stress the difference between algebra and calculus, and build bridges with fundamental structural constructs

# The study - Step1: How?

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

- ❑ TDS choice: A search for second-order variable (next)
  
- Didactic variables concerned with the learning expectations related to this teaching goal?
  - Focus on two aspects: (1) reasoning with the limit notion, including inductive reasoning, counterexamples, reductio ad absurdum, necessary condition, or sufficient condition; (2) linking the limit idea to real numbers, derivatives, integrals, or the FTC
  - 2<sup>nd</sup> order variable: (1) semiotic representations of limits (e.g., numeric, algebraic, graphic, and formal representation), (2) limit status (process, procept, object), and (3) reasoning and proof setting (e.g., algebraic formulas, numerical approach through inductive reasoning, graphical representations and counterexamples, formal proof and deductive reasoning)
  
- Sample: (1) introductory tasks for defining sequence convergence, derivative number, and integral, (2) introductory tasks for setting theorems of existence of mainstream notions related to sequence convergence, derivative number and integral, and (3) all application tasks for investigating the numerical existence of limits through approximation procedures

12 <sup>th</sup> grade	13 <sup>th</sup> grade	1 <sup>st</sup> year of university
Algebraic properties of functions	Continuity and limit	Generalities on fundamental techniques
Continuity	Sequences' convergence (8T)	Real numbers and sequences (3T)
Continuity and limit	Derivatives (7T)	Limit, continuity, differentiability (4T)
Asymptotic behaviours	Inverse functions	Convex functions
Derivative numbers (3T)	Antiderivatives	Asymptotic behaviours
Derivative of functions	Integral (3T)	Integration (includes antiderivatives) (1T)
Calculus of algebraic functions	Logarithmic function	Series
Trigonometric functions	Exponential function	
Sequences	Differential equations	
Sequences' convergence (6T)		

- approximation of a rational number given by means of its improper representation (TA1.1)
- approximation of an irrational number (TA.1.2)
- approximation of polygonal area, perimeter and length (TA1.3)
- approximation of volume of circular solids (TA1.4).
- approximation of zero of algebraic functions (TA2.1)
- approximation of zero of transcendental functions (TA2.2)
- approximation of a real number represented by successive terms of a series (TA3.1)
- approximation of non polygonal area (TA3.2)

# The study - Step1: Meta-result

Shared cornerstone teaching goals (Curricula, textbooks, tasks, etc.)

❑ This study case does not fit the narrative of transition from high school to university mathematics as a shift from informal to formal mathematics:

- School trend is not a deductive top-down approach such as new maths,
- School trend is not a bottom-up approach where the use of “good” or “generic” examples lead to calculus notions
- School trend is a mixture of both with an emphasis on exactness in defining concepts and on moderate formal rigour... (many things to say about the output of such choices!)

❑ A linkage through opportunities given to the students: (1) to master different types of reasoning and proofs (e.g. numerical methods and inductive reasoning, graphical representations and counterexamples, formal proof and deductive reasoning); and (2) to think about the link between formal and informal calculus statements

# The study – Step2: How?

## Additional data from the teachers' perspective

- Teachers mathematical knowledge for teaching: Conceptual tools
- Knowledge-oriented teachers education curricula: (1) General pedagogical knowledge, (2) Pedagogical content knowledge, (3) Content knowledge
- Knowledge situated in the context of teaching mathematics : (1) Common mathematics Knowledge, (2) Specialized mathematics Knowledge, (3) Knowledge of mathematics and teaching , (4) Knowledge of students' learning of mathematics
- Teachers mathematical knowledge for teaching : TDS Construct of the milieu
  - To understand teachers' teaching actions
  - Distinction between knowing and knowledge (Knowing connotes activities of a knower, while knowledge connotes facts—justified true “values”)
  - Do not separate what teachers know from their thinking, deciding, and acting

# The study – Step2: How?

Additional data from the teachers' perspective

## □ Network choice: 'Calculus' Knowledge for Teaching CaKT

- Common Calculus Knowledge 'CCaK': e.g. scholarly-related knowledge-knowledge on topology (Schulman)
- Specialized Calculus Knowledge 'SCaK': e.g. knowledge to define calculus notions, paradigmatic examples, fundamental calculus reasoning, calculus knowledge of the prior/next grade level
- Knowledge of Calculus and Teaching 'KCaT': e.g. Knowledge for deciding on the use of proofs of general statements, for explaining calculus ideas (different representations, examples, or counterexamples)
- Knowledge of Calculus and Students 'KCaS': e.g. knowledge on students' difficulties with calculus notions, students' conceptions of calculus notions, school knowledge students should enhance to meet university calculus

# The study – Step2: Method

## Additional data from the teachers' perspective

□ Questionnaire towards university teachers: Multiple-choice and multiple-response questions for collecting information on KCaT and KCaS

- (1) explanation of relationships between calculus ideas (KCaT<sub>1</sub>): use of graphics to support the introduction of definitions (QT1), use of examples when enunciating theorems (QT2), and use of formalism to define notions (QT3)

- (2) what particular tasks should be emphasized to overcome students' difficulties (KCaS<sub>1</sub>): procedural tasks (QS1), tasks involving mathematics logic rules (QS2), graphical tasks (QS3), tasks promoting the use of numerical calculator (Q5), tasks promoting the use of graphing calculator (QS4), experimental tasks (QS5), and tasks where students are expected to make conjectures (QS6)

- (3) what calculus knowledge encountered in school should be improved during secondary education (KCaS<sub>2</sub>): formal statements (QSS1), mathematics logic rules (QSS2), reasoning (QSS3), and understanding notions (QSS5)

- A paired t-test has been used to test the signification, via mean comparisons, of the association between teachers' responses to questions



# The study – Step2: Some results

## Additional data from the teachers' perspective

### ❑ Questionnaire towards university teachers: Limitation

- Should contain concrete calculus tasks?
- A priori assumption on teachers' knowledge
- Declarative statements and little evidence on teachers' knowledge could be given from the data

### ❑ From the project perspective

- In line with secondary textbooks and curricula, teachers calculus knowledge for students do not involve experimental tasks
- Calculus knowledge for students includes students' shortcomings; university teachers declare that they enhance students' manipulation of formal statements and devolve to secondary education the responsibility to overcome calculus reasoning problems

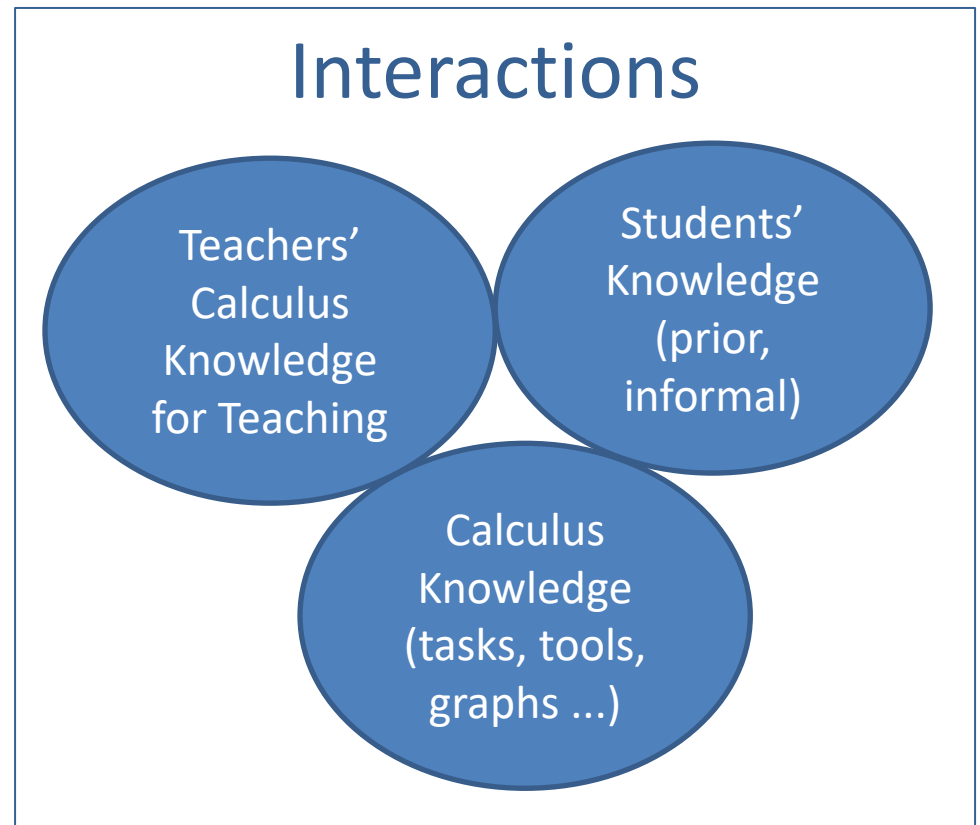
# The study – Step3: How?

Engage in thinking on teachers' knowledge dynamics

- ❑ A Method that permits to engage both researcher and teachers in a collaborative investigation of a didactical phenomenon in such a way that this investigation leads to both research findings and developmental findings (it is a kind of R&D methodology)
  
- ❑ From the project perspective
  - The investigation is related to the possibility to reduce differences in learning expectations before entering university
  - Preliminary studies on high school teachers teaching actions in calculus classrooms are needed before engaging in this method

TDS alignment: Teaching actions through three learning phases

- Phase of students' action on calculus notions
- Phase of students' oral and written formulations about calculus notions
- Phase of students' validation of calculus statements



# The study – Step3: How?

Engage in thinking on teachers' knowledge dynamics

## □ Reflection on teaching actions (Dewey ; Schön)

- Conscious considerations of their consequences on the progress of students' learning
- Deep examination of the causes embodied in the related CaKT
- Three types of reflection contents:
  - (1) technical, e.g. school and programs
  - (2) practical, e.g. experience as both teacher and graduate in mathematics
  - (3) theoretical including general assumptions and rules

## □ Guided reflection

- Collaborative effort between teachers and researchers
- Record and transcript of calculus classes
- Questions for stimulating interview are prepared by the researchers:
  - (1) Confrontation to the dilemma and verbalization of thoughts
  - (2) Revisit CaKT and explore possibilities to adjust teaching actions

# The study – Step3: Method

Engage in thinking on teachers' knowledge dynamics

□ Analysis of 6 successive calculus classes on sequence and sequence convergence (each class lasted two hours and involves 25 high school students):

- Actions related to the management of students' use of informal statements and quantified rules
- Actions related to the management of students' reasoning and proving
- Actions related to the management of the operationalisation of notions

□ Planning of a series of interviews

- A priori evaluation of CaKT (Lack of knowledge...)
- Crucial classroom situations
- Basis of discussion: (1) Is there anything that strikes you in this sequence? (2) What kinds of understanding does this sequence involve? (3) Why did you do [this]? (4) What might have happened, if you have done same before? (5) How can you improve what is not working?

# Results of two successive interviews (2x2h)

## □ Give evidence of teacher's reflective thinking

- Focus on answers to Q3 and Q4 (reflection contents)
- Reasons for actions: meaning of related KCaT and KCaS
- Change knowledge and adjust actions

## □ Repartition of teacher and researcher utterances

	Operation	Formalism	Validation
Researcher	23 (9%)	44 (16%)	77 (27%)
Teacher	20 (9%)	38 (14%)	70 (25%)

## □ Global results

- Symmetrical interventions between the researcher and the teacher
- Disparities between the times allowed for each set of actions
- Teacher's CCK and SCaK helps him adapting and enriching KCaT and KCaS
- The meaning of teacher's CaKT is embodied in its experience as both a teacher and a graduate in mathematics

## Informal statements: Typical classroom situation

*P10: what do you think about this sequence?*

*22.Hedi: its terms are approaching  $\sqrt{3}$*

*P11: and ...*

*23.Nada: they are closed to  $\sqrt{3}$  and never reach it ...*

***P12: it isn't clear with the graphic... let's move to another tool...use your calculator***

*24.Wiem: terms are approaching more and more...*

***25.Hedi: and never reach it!***

## Typical teacher's responses

Q(1) *"Is there any problem?"; "why students suggest such properties (of limit)?"*

Q(3) *"n increases and become larger..."; "should I avoid the using of terms similar to "approach"...this could help students to comprehend inequalities involved in limit definition..."; "I have tried to make a link to the graphic (with algebraic inequalities)"; "Examples from the program of sequences reaching its limit"*

KCaT Knowledge to connect visualisation to the results of numerical computations or statements involving algebraic inequalities

KCaS Knowledge related to informal understanding and the development of conceptions

## Quantifiers' rules: Typical classroom situation

105. Nada (blackboard):  $f$  is bounded from above then for every  $x \in D_f$  there exists  $M$  such that  $f(x) \leq M$ .

**P84: look at your statement... should you find  $M$  for each  $x$ ?**

106. Jed: No!

P85: then  $M$  exists...

107. Nada: euhhh...for every  $x$ ...

108. Nada (blackboard): [correction of statement]

**109. Students all together: what's the difference?**

## Typical teacher's responses

Q(1) "I didn't spent enough time to speak about quantifiers rules ..."

Q(3) "Students are not expected to know these rules... they are not useful for the following..."; "It's very difficult to find examples that illustrate differences between quantifiers statements"; "How students could understand the need to reverse quantifiers..."; "I have not enough materials (examples of sequences expected by the program)"

KCaT Knowledge of paradigmatic examples to illustrate quantifiers rules and differences

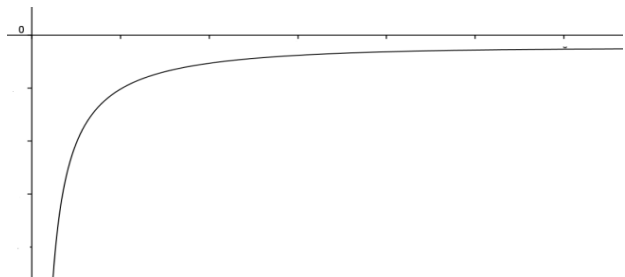
KCaS Knowledge of the cognitive demands of tasks using conversion of quantifiers



## Reasoning (NSC): Typical classroom situation

103. Wiem: *the function  $f$  has an upper bounded so it has a maximum*

$P_{88}$ : *This function has an upper bounded; **do you think** that  $f$  admits a maximum*



104. Wiem: **No...**

## Typical teacher's responses

Q(1) *"Hmm...this is a big problem (differences between SC and NC and giving evidence of these differences)"; "This needs a good counterexample..."*

Q(3) *"I'm not sure that graphics are sufficient...I used to find algebraic example..."; "I have not enough materials (examples of sequences expected by the program)"*

# Final comments

- ❑ An attempt to go beyond the *dualistic view* of the research in calculus transition towards university by engaging high school teachers in reforming their actions and making a linkage between the two levels
  
- ❑ Our collaborative method permits to: (1) model problematic teaching actions regarding predominant students' learning expectations in the transition; (2) allow the teacher to experience confusion in these actions, and to think on the meanings of calculus knowledge for teaching which have been implicit in these actions:
  - Teachers' CaKT influences teaching actions and reciprocally teaching actions depend on a combination of the different domains of this knowledge
  - Teacher's thinking on the meaning of her/his knowledge is essential to initiate change
  - Potential changes engage the teacher to go beyond basic high school expectations

# Final comments

## ❑ However ...

- This collaborative method is a time intensive process for both teacher and researcher
- Many interviews are needed to improve our understanding of the meaning given by the teacher to Calculus knowledge for Teaching
- Transferability of the teacher's experience to the level of his school: techniques to reflect critically on his actions, and contribute to initiate changes in the teaching actions of his colleagues

❑ The application of our collaborative method shows that it is possible to analyze the extent to which high school teachers' can make changes in their teaching actions for meeting university expectations. Ideas for future investigations include examination of crucial teachers' calculus knowledge for teaching in both sides of the transition, the meanings that found these knowledge and their potential connections

**THANK YOU!**