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Abstract: We build a novel time-like coset sigma-model describing type-II superstring theory in a charged rotating black-brane background that interpolates between a local AdS_3 in the IR and a linear-dilaton geometry in the UV. This allows one to peform a systematic study of holography in non-AdS backgrounds which are smoothly connected to AdS_3. We construct massless closed string states vertex operators in the NS-NS sector, calculate the corresponding two-point correlation functions, and discuss holographic interpretation of our results from 4+1 dimensional boundary field theory point of view. Compactifying the theory on $\mathbb{T}^4$, we show that the spectrum of a single long string with unit winding agrees with the spectrum of a CFT_2 deformed by $T\bar{T}$. We also calculate correlation functions of operators of the dual 1+1 dimensional non-conformal boundary field theory using world-sheet techniques.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Long Strings, Renormalization Group

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1 Introduction

String theory on group manifolds and coset spaces supplies a rich class of models describing strings propagating on a non-trivial curved background, that are amenable to a formulation in terms of a two-dimensional world-sheet theory, exact in string length. This powerful feature follows from the potent combination of (super-)conformal and affine Kac-Moody symmetries enjoyed by these theories. The world-sheet action that respects these symmetries and describes string theory on group manifolds is given by the celebrated Wess-Zumino-Witten (WZW) model, while its generalization to group cosets is described by the gauged WZW (gWZW) model [1–3].

A particular class of the (g)WZW models, solvable in the limit of small string coupling g_s, yet exact in the string length ℓ_s, is given by the world-sheet models with the target space based on the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ group manifolds. An enormous technical simplification due to such a choice of the target space group manifold is afforded by the available properties of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ primary operators, in particular, the known structure of their correlation functions in the exact world-sheet theory. This knowledge assists one immensely in a calculation of various observables exact in the WZW level k that simultaneously translates into the stringy, beyond gravity, accessibility of the large curvature regime: a notoriously coveted goal of practitioners in the field.

One of the fruitful applications of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ based (g)WZW models is found in the field of holographic correspondence. Original examples are given by the holographic (bulk) description of a two-dimensional conformal field theory (CFT) [4–9], and holographic description of Little String Theory (LST) [10–23]. The former is given by the superconformal WZW model on $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ (where $AdS_3 \sim \mathrm{SL}(2,\mathbb{R})$, and $\mathbb{S}^3 \sim \mathrm{SU}(2)$), while the latter is described by the superconformal gWZW model on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{SU}(2) \times \mathbb{R}^5$. In the case when microscopic description of coset models is provided by a stack of k coincident NS5 branes wrapping $\mathbb{R}^5$ or $\mathbb{T}^4 \times \mathbb{S}^1$ and p fundamental strings (F1) along $\mathbb{R}^1$ or $\mathbb{S}^1$, the AdS_3 background found in the near-horizon limit of the F1 strings is described by the $\mathrm{SL}(2,\mathbb{R})$ WZW model at level k, while the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ describes the linear-dilaton background, sourced by a stack of k NS5-branes in its near-horizon limit. The corresponding dual field theory interpolates between a CFT_2, obtained by taking the decoupling limit of the F1-strings, and the world-volume theory of the NS5-branes, namely LST, obtained by taking the decoupling limit of the NS5 branes. The NS5-branes source the NS flux, equal to k, through the $\mathbb{S}^3 \sim \mathrm{SU}(2)$ sphere surrounding the NS5-branes in the ten-dimensional space-time. The radius of both AdS_3 and $\mathbb{S}^3$ behaves as $R = \sqrt{k}\,\ell_s$, which defines the small-curvature semi-classical limit (amenable to the supergravity approximation) at $k \to \infty$. However, an exact solvability of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ WZW model allows one to carry out calculations exactly in k, furnishing an exact world-sheet description.

One can construct a more complicated class of cosets describing dynamics of NS5-F1 system. Coset systems, described by the gWZW model, require one to impose anomaly-cancellation conditions. These can be formulated as time-like or null constraints (for cosets that include the time-like direction in target space) on the gauge parameters. Null cosets have been extensively studied in [24–27] and shown to describe micro-state geometries

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with world-sheet control. On the other hand, time-like cosets help us understand dynamics of black holes, scattering in a black hole background, and resolution of cosmological singularities [15, 28–32].

In [31, 33] collective excitations of the LST were studied holographically, by analyzing the pole structure of the stress-energy tensor two-point function, described in terms of the graviton scattering amplitudes in the bulk.[1] The latter were calculated exactly, in terms of the two-point correlation functions of vertex operators on the world-sheet, describing massless closed string states in the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of the supergravity multiplet. In particular, the diffusion mode was located, and its existence was independently verified by the supergravity analysis. Additionally, in [31] a finite-density version of this model was considered, by constructing the dual bulk configuration with a non-trivial gauge field profile. Such a background was explicitly demonstrated in [32] to appear in the configuration involving the near-horizon limit of a stack of NS5-branes with a large number of fundamental strings (F1).

In this paper, our goal is to further broaden our understanding of the asymptotically linear dilaton backgrounds, construct spectrum of string excitations on top of these backgrounds, calculate various correlation functions of vertex operators corresponding to massless string states, and comment on application of these results to holography. To this end, we consider type-II superstring theory on $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{U}(1)_x}{\mathrm{U}(1)}\times\mathbb{T}^4$ ten-dimensional coset space-time, and perform time-like gauging of the $\mathrm{U}(1)$ sub-group in denominator. We formulate exact world-sheet action of the superconformal gWZW model on this coset, describing type-II superstring theory in the corresponding background. We then proceed to deriving the exact in k gravity background, consisting of non-trivial configurations of the metric, dilaton, and B-field. The obtained background asymptotes to the two-dimensional linear dilaton background times $\mathbb{S}^3\times\mathbb{S}^1\times\mathbb{T}^4$. At the same time, the obtained geometry possesses an event horizon with non-vanishing angular velocities along two of the $\mathbb{S}^3$ and $\mathbb{S}^1$ directions.

Compactifying the $\mathrm{U}(1)_x\sim\mathbb{S}^1$ direction and performing the Kaluza-Klein (KK) reduction, we obtain the nine-dimensional space-time with two non-trivial $\mathrm{U}(1)$ gauge field profiles, non-vanishing angular velocities along two of the $\mathbb{S}^3$ directions, and an event horizon. Such a configuration describes a holographically dual theory at finite temperature, characterized by finite values of chemical potentials and charge densities corresponding to two $\mathrm{U}(1)$ charges. We provide a complete thermodynamic description of the ten- and nine-dimensional backgrounds, by calculating the corresponding temperature, entropy, angular momenta, energy, grand potential, and charges. We demonstrate that for the particular choice of the coset gauging parameters our results reproduce the known results in the literature, while largely extending those results to a more general case of a rotating charged asymptotically linear-dilaton background with a black brane event horizon.

We subsequently discuss collective excitations of the boundary theory by inspecting the structure of two-point correlation functions of the stress-energy tensor and the currents corresponding to two conserved $\mathrm{U}(1)$ charges. These can be determined holographically in terms of the two-point functions of the vertex operators of the world-sheet theory, describing

[1] See also [34] for related work.

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massless excitations of the NS-NS sector of the type-II supergravity background, involving the metric, dilaton, B-field, and gauge fields. We construct the corresponding vertex operators by performing the coset BRST quantization of the world-sheet theory, ensuring the U(1) symmetry at the quantum level. In particular, we determine the tachyon vertex operator, and study its asymptotic behavior. We then construct the most general massless NS-NS physical states, and calculate their two-point correlation functions.

The string theory model that we study in this paper, at zero temperature, interpolates between local $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ in the IR to a linear-dilaton times $\mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{T}^4$ in the UV. The dual boundary field theory interpolates between a certain vacua of LST in the UV to a CFT_2 in the IR. This allows us to systematically study holography in asymptotically non-AdS space-time. Furthermore, it has been argued that such models are intimately related to solvable models in string theory that go by the name of 'single-trace' $T\bar{T}$ deformation [32, 35–39]. We show that the spectrum of a single winding one long string in the interpolating coset background under investigation, exactly matches with the spectrum obtained in the case of $T\bar{T}$ deformed CFT_2 [40, 41] for the particular sign of the irrelevant coupling that gives rise to a real-valued spectrum. For a generic winding $\text{w} > 1$ sector, the spectrum agrees with the $\mathbb{Z}_\text{w}$ twisted sector of a symmetric orbifold theory where the block of the symmetric orbifold is obtained by $T\bar{T}$ deformation of a CFT_2. Using world-sheet string theory techniques we compute the two and three-point functions of operators of the boundary theory in momentum space. As one moves away from the IR conformal fixed point, the operators develop a momentum dependent ambiguity in the normalization of the operators, although it is not clear to us how to fix this normalization issue. The 'dimensions'[2] of the operators at a generic point on the renormalization group flow develop a momentum dependence which resonates with the idea of momentum dependent dimensions discussed in the context of non-local CFTs [42, 43].

The rest of this paper is organized as follows. In section 2 we review basic setup of superstring theory on $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ and introduce our conventions. In section 3 we construct the main ten-dimensional coset target space-time background that we are interested in this paper. We develop thermodynamics of this background in section 4. In section 5 we perform the Kaluza-Klein reduction of the $\text{U}(1)_x$ circle and construct the nine-dimensional background, characterized by two non-trivial U(1) gauge field profiles. We construct the spectrum of physical string excitation states on top of the background fields configuration in section 6, focusing specifically on the NS-NS sector. In section 7 we explore the holographic interpretation of our results, by considering two-point correlation functions of the dual boundary theory calculated in terms of the world-sheet vertex operators in the bulk. We discuss our results and comment on future directions in section 8. Appendix A is dedicated to review of relevant background concerning (superconformal) gauged WZW models. Additional ancillary material is provided in appendices B, C.

[2]Strictly speaking, as one moves away from the conformal fixed point, there is no notion of dimension of operators in the CFT sense. What we mean by 'dimension' is in the sense discussed in [42, 43].

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2 Review of superstrings on $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$

In this section, we will provide a brief summary of the type-II superstring theory on the $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ background, focusing on the low-lying states in the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector. In particular, we will set up conventions related to the $\mathrm{SL}(2,\mathbb{R})$ and SU(2) Wess-Zumino-Witten (WZW) models, that we will be following for the rest of the paper. We will also review the affine Kac-Moody symmetry algebra of the target space-time, discuss its representation space, normalizability of the string states, and the corresponding vertex operators. A complementary condensed glossary of our notations can be found in appendix A. For a detailed discussion of string theory on AdS_3 we refer the reader to [4–9].

The world-sheet superstring theory on $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ is given by the superconformal WZW model on the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$ group manifold. This world-sheet sigma-model is constrained by the left- and right-moving $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ supersymmetric affine Kac-Moody algebra, at supersymmetric level k. In the remainder of this section, we will discuss the affine symmetry algebra and the vertex operators of the $\mathrm{SL}(2,\mathbb{R}) \sim AdS_3$ and $\mathrm{SU}(2) \sim \mathbb{S}^3$ subspaces separately.

2.1 Symmetries and vertex operators of AdS_3

In this subsection, we will discuss the $\mathrm{SL}(2,\mathbb{R})$ superconformal WZW model, describing the world-sheet string theory on AdS_3. The classical supersymmetric WZW sigma-model consists of the bosonic sector, given by the $\mathrm{SL}(2,\mathbb{R})_{k+2}$ WZW model at the level $k+2$, and three pairs of free (anti)-holomorphic two-dimensional Majorana-Weyl fermions ψ^a, $\tilde{\psi}^a$ $a = 1,2,3$. Bosonic degrees of freedom comprise the (anti-)holomorphic Kac-Moody currents j_a, $\tilde{j}_a$, $a = 1,2,3$ at the level $k+2$. The fermionic sector can be bosonized, and described in terms of the $\mathrm{SL}(2,\mathbb{R})_{-2}$ WZW model at the level -2. Combining the bosonic and the fermionic degrees of freedom we obtain the total bosonic Kac-Moody currents forming the algebra at level k.

The exact (in the sense of world-sheet path integral) effective action for the bosonic field $g \in \mathrm{SL}(2,\mathbb{R})$ is given by the $\mathrm{SL}(2,\mathbb{R})_k$ WZW action,

$$S_{sl}[g] = k\, \mathtt{S}[g]\,, \tag{2.1}$$

where the $k = 1$ action $\mathtt{S}[g]$ is given by (A.1). The action (2.1) is invariant under the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ affine symmetry algebra. The global part of this symmetry algebra is generated by the Kac-Moody currents j^a and $\tilde{j}^a$, $a = 1,2,3$, given by

$$j^a = \frac{k+2}{2}\mathrm{Tr}\left(\partial g g^{-1}\,\tau^a\right), \qquad \tilde{j}^a = \frac{k+2}{2}\mathrm{Tr}\left(g^{-1}\bar{\partial} g\,\tau^a\right). \tag{2.2}$$

where

$$\tau^{1,2} = \sigma^{1,2}\,, \qquad \tau^3 = i\,\sigma^3\,, \tag{2.3}$$

and σ^a are Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

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At the same time, the total Kac-Moody currents, incorporating contributions both from the bosonic and the fermionic degrees of freedom, form the (anti-)holomorphic affine algebra $\mathtt{j}^a$, $\tilde{\mathtt{j}}^a$, $a = 1, 2, 3$ at level k, with the currents defined as[3]

$$\mathtt{j}^a = j^a - \frac{i}{k} f^a{}_{bc}\, \psi^b \psi^c\,, \qquad \tilde{\mathtt{j}}^a = \tilde{j}^a - \frac{i}{k} f^a{}_{bc}\, \tilde{\psi}^b \tilde{\psi}^c\,. \tag{2.5}$$

Using the Sugawara construction, we can write the (anti-) holomorphic components of the stress-energy tensor of the world-sheet sigma model. Focusing on the contributions from the bosonic degrees of freedom, we obtain[4]

$$T_{sl} \supset \frac{1}{k}\, j^a\, j^a\,, \qquad \tilde{T}_{sl} \supset \frac{1}{k}\, \tilde{j}^a\, \tilde{j}^a\,. \tag{2.6}$$

The quadratic Casimir of $\mathrm{SL}(2,\mathbb{R})_k$ is given by

$$c_2 = (j^3)^2 + \frac{1}{2}\{j^+, j^-\}, \tag{2.7}$$

where we defined

$$j^\pm = j^1 \pm i j^2. \tag{2.8}$$

Representations of $\mathrm{SL}(2,\mathbb{R})$ are labeled by the number j related to the quadratic Casimir as $c_2 = -j(j+1)$.

The spectrum of string states on AdS_3 can be constructed in terms of the unitary representations of $\mathrm{SL}(2,\mathbb{R})$. The representations relevant for studying string theory on AdS_3 can be largely classified into two groups: the principal discrete series representation[5] (lowest and highest weight) denoted by $\mathcal{D}^\pm_j$ and the principal continuous series representation denoted by $\mathcal{C}^\alpha_j$. The lowest weight principal discrete series states (incoming states) are built on states of the form $|j, j\rangle$ which are annihilated by j^+. Thus one can write

$$\mathcal{D}^+_j = \{|j, m\rangle;\ \ j \in \mathbb{R},\ \ m = j, j+1, j+2, \dots\}\,, \tag{2.9}$$

where m is the j^3 eigenvalue of $|j, m\rangle$. The higher weight discrete series states (outgoing states), $\mathcal{D}^-_j$, are obtained by taking the charge conjugate of the $\mathcal{D}^+_j$ states,

$$\mathcal{D}^-_j = \{|j, m\rangle;\ \ j \in \mathbb{R},\ \ m = -j, -j-1, -j-2, \dots\}\,, \tag{2.10}$$

Normalizability of the states requires

$$-\frac{1}{2} < j < \frac{k-1}{2}\,. \tag{2.11}$$

[3] See eq. (A.47) in appendix A.3 for details.

[4] The sign $\supset$ means that the terms to the right of this sign constitute some of the contributions to the expression to the left of this sign. In the case of (2.6), we focus on the contributions to the stress-energy tensor from the bosonic d.o.f., and omit writing explicitly contributions due to the fermions.

[5] The discrete series states are often referred to in the literature as the short strings. The short strings can be thought of as bound states of strings that are trapped deep inside the bulk of AdS_3.

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The principal continuous series[6] representation corresponds to states which are delta-function normalizable, given by

$$\mathcal{C}_j^\alpha = \left\{ |j,\alpha,m\rangle;\ \ j \in -\frac{1}{2} + i\mathbb{R},\ \ m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots \right\}, \tag{2.12}$$

where $|j,\alpha,m\rangle$ are eigenstates of j^3 with eigenvalue m. Without loss of generality, one can set $0 \leq \alpha < 1$. String theory on AdS_3 also contains operators that violate the bound (2.11). In terms of the AdS/CFT correspondence, such operators give rise to local operators of the boundary theory (see, e.g., [8] for a detailed discussion of the normalizability of the states in AdS_3).

Next, let us review construction of the $\mathrm{SL}(2,\mathbb{R})$ primaries. The vertex operators of $\mathrm{SL}(2,\mathbb{R})$ can always be locally decomposed into a product of vertex operators of $\mathrm{U}(1)_y$ and $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$. Such a decomposition of vertex operators is known as the parafermionic decomposition. Let us also assume that $\mathrm{U}(1)_y$ is generated by $\mathtt{j}^3$ and is parametrized by the scalar field y, satisfying the OPE $y(z)y(w) \simeq -\log(z-w)$. In terms of the field y, the current $\mathtt{j}^3$ takes the form

$$\mathtt{j}^3 = -i\sqrt{\frac{k}{2}}\partial y. \tag{2.13}$$

Thus, primary operator $V_{j;m,\bar{m}}$ of string theory on AdS_3, obtained by diagonalizing $\mathtt{j}^3, \tilde{\mathtt{j}}^3$ with eigenvalues $m, \bar{m}$ respectively, can be represented as

$$V_{j;m,\bar{m}} = e^{i\sqrt{\frac{2}{k}}(my+\bar{m}\bar{y})}\Psi_{j;m,\bar{m}}\,, \tag{2.14}$$

where $\Psi_{j;m,\bar{m}}$ is a primary vertex operator on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$. Using the OPE of $V_{j;m,\bar{m}}$ with the stress-energy tensor,

$$T_{sl}(z)V_{j;m,\bar{m}}(w) \simeq -\frac{j(j+1)}{k}\frac{1}{(z-w)^2}V_{j;m,\bar{m}}(w)\,, \tag{2.15}$$

one can read of the dimension of the vertex operator $V_{j;m,\bar{m}}$ as

$$\Delta_{sl} = -\frac{j(j+1)}{k}\,. \tag{2.16}$$

This implies that the dimensions of $\Psi_{j;m,\bar{m}}$ are given by

$$\Delta_\Psi = -\frac{j(j+1)}{k} + \frac{m^2}{k}\,, \qquad \bar{\Delta}_\Psi = -\frac{j(j+1)}{k} + \frac{\bar{m}^2}{k}\,. \tag{2.17}$$

The vertex operator $V_{j;m,\bar{m}}$ satisfies the following OPE relations with the generators of the $\mathrm{SL}(2,\mathbb{R})$ current algebra:

$$\begin{aligned} \mathtt{j}^3(z)V_{j;m,\bar{m}}(w) &\simeq \frac{m}{z-w}V_{j;m,\bar{m}}(w)\,, \\ \mathtt{j}^\pm(z)V_{j;m,\bar{m}}(w) &\simeq \frac{m\mp j}{z-w}V_{j;m\pm1,\bar{m}}(w)\,. \end{aligned} \tag{2.18}$$

[6]The continuous series states are often referred to in the literature as the long strings. Unlike the short strings, the long strings form the scattering states. They live very close to the boundary of AdS_3 and have a continuous momentum.

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Switching chirality of the currents, one can write a similar set of OPE algebra of $V_{j;m,\bar{m}}$ with the right-moving (anti-holomorphic) $\mathrm{SL}(2,\mathbb{R})$ currents.

It is often useful to transform the vertex operators from the momentum space $(m,\bar{m})$ to the position space $(x,\bar{x})$. Let us denote the position space vertex operators by $\Phi_j(z,\bar{z};x,\bar{x})$. Here, the position space coordinates $(x,\bar{x})$ can be thought of as coordinates of the boundary of AdS_3 on which the dual CFT_2 lives. The position space representations are related to the momentum space ones via the following transformation:

$$V_{j;m,\bar{m}} = \int_{\partial AdS_3} d^2x\; x^{j+m}\bar{x}^{j+\bar{m}}\Phi_j(z,\bar{z};x,\bar{x})\,. \tag{2.19}$$

One can invert the transformation (2.19) to calculate $\Phi_h(z,\bar{z};x,\bar{x})$:

$$\Phi_h(z,\bar{z};x,\bar{x}) = \sum_{m,\bar{m}} x^{-m-h}\bar{x}^{-\bar{m}-h}V_{h-1;m,\bar{m}}\,, \tag{2.20}$$

where $(h,h) = (j+1,j+1)$ is the spin of $\Phi_h(z,\bar{z};x,\bar{x})$ under space-time $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ symmetry.

The operator Φ_j satisfies the following reflection symmetry:

$$\Phi_{j+1}(z,\bar{z};x,\bar{x}) = \frac{2j+1}{\pi}\int d^2x'\;\frac{\Phi_{-j}(z,\bar{z};x',\bar{x}')}{|x-x'|^{4(j+1)}}\,. \tag{2.21}$$

Thus for real j one can restrict, without loss of generality, to $j > -1/2$, while for $j = -1/2 + is$ (2.21) relates wave function with $s > 0$ (out-going scattering waves) to those with $s < 0$ (in-coming scattering waves).

In the semiclassical approximation, i.e., in the limit $k \to \infty$, the two-point function of the operator $\Phi_h(z;x)$ can be determined via the world-sheet conformal invariance and the invariance under the global $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ symmetry of the space-time theory [8]

$$\langle\Phi_h(z_1;x_1)\Phi_{h'}(z_2;x_2)\rangle = \delta(h-h')\frac{B(h)}{|z_1-z_2|^{4\Delta_{sl}}|x_1-x_2|^{4h}}\,, \tag{2.22}$$

where

$$B(h) = \frac{k}{\pi}X^{2h-1}\gamma\left(1-\frac{2h-1}{k}\right), \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,, \tag{2.23}$$

and X is an arbitrary constant whose value can be adjusted by shifting the radial coordinate or rescaling the transverse field theory coordinates. In the momentum basis the two-point function takes the form

$$\langle V_{j;m,\bar{m}}(z)V_{j';-m,-\bar{m}}(w)\rangle = \frac{\pi\delta(j-j')}{|z-w|^{4\Delta_{sl}}}\frac{\Gamma\left(1-\frac{2j+1}{k}\right)\Gamma(-2j-1)\Gamma(j-m+1)\Gamma(j+\bar{m}+1)}{\Gamma\left(1+\frac{2j+1}{k}\right)\Gamma(2j+2)\Gamma(-j-m)\Gamma(-j+\bar{m})}. \tag{2.24}$$

The structure of three-point correlation function of the primary world-sheet operator Φ_h is fixed by the conformal symmetry of the world-sheet and the space-time theories. For

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the operator Φ_h normalized so that its two-point function is given by (2.22), one obtains [4]

$$\begin{aligned}\langle\Phi_{h_1}(z_1,x_1)\Phi_{h_2}(z_2,x_2)\Phi_{h_3}(z_3,x_3)\rangle \\ = \frac{C_{sl}(h_1,h_2,h_3)}{|z_{12}|^{2(\Delta_{h_1}+\Delta_{h_2}-\Delta_{h_3})}|z_{23}|^{2(\Delta_{h_2}+\Delta_{h_3}-\Delta_{h_1})}|z_{31}|^{2(\Delta_{h_3}+\Delta_{h_1}-\Delta_{h_2})}} \\ \times \frac{1}{|x_{12}|^{2(h_1+h_2-h_3)}|x_{23}|^{2(h_2+h_3-h_1)}|x_{31}|^{2(h_3+h_1-h_2)}}\end{aligned} \tag{2.25}$$

where $\Delta_h = \Delta_{sl}$ and $z_{ij} = z_i - z_j$ and $x_{ij} = x_i - x_j$. The OPE coefficient $C_{sl}(h_1,h_2,h_3)$ can be computed from the conformal Ward identity, giving

$$C_{sl}(h_1,h_2,h_3) = -\frac{G(1-h_1-h_2-h_3)G(h_3-h_1-h_2)G(h_3-h_3-h_1)G(h_1-h_2-h_3)}{2\pi^2 X(h_1+h_2+h_3-1)\gamma\left(\frac{k+1}{k}\right)G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)} \tag{2.26}$$

where

$$G(h) = k^{\frac{h(k+1-h)}{2k}}\Gamma_2(-h|1,k)\Gamma_2(k+1+h|,1,k) \tag{2.27}$$

with Γ_2 defined as

$$\log\Gamma_2(x|1,\omega) = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left[\sum_{m,n=0}^{\infty}(x+n+m\omega)^{-\epsilon} - \sum_{m,n=0,(m,n)\neq(0,0)}^{\infty}(n+m\omega)^{-\epsilon}\right]. \tag{2.28}$$

String theory on AdS_3 also contains operators obtained by a spectral flow automorphism of the symmetry algebra:

$$\begin{aligned}\mathtt{j}^3(z) &\to \mathtt{j}^3(z) - \frac{k\hat{\omega}}{2z}\,, \\ \mathtt{j}^{\pm}(z) &\to z^{\pm\hat{\omega}}\,\mathtt{j}^{\pm}(z)\,,\end{aligned} \tag{2.29}$$

where $\hat{\omega}\in\mathbb{Z}$ is the spectral flow parameter. Note that since we are working in the universal cover of $\mathrm{SL}(2,\mathbb{R})$, the left- and the right-moving spectral flow parameters are identical. Under spectral flow automorphism the Virasoro generators, obtained by mode expansion of the stress-energy tensor (2.6), transform as

$$L_n \to L_n + \hat{\omega}j^3_n - \frac{k\hat{\omega}^2}{4}\delta_{n,0} \qquad \forall n\in\mathbb{Z}\,, \tag{2.30}$$

where j^3_n are the Kac-Moody modes of the current j^3.

For each spectrally flowed sector $\hat{\omega}\in\mathbb{Z}$, string theory on AdS_3 contains twist operators given by

$$t^{\hat{\omega}}_{sl}(z,\bar{z}) = t^{\hat{\omega}}(z)\,\tilde{t}^{\hat{\omega}}(\bar{z}) = e^{-\hat{\omega}\left(\int \mathtt{j}^3(z) + \int\tilde{\mathtt{j}}^3(\bar{z})\right)} = e^{i\,\hat{\omega}\sqrt{\frac{k}{2}}\,(y(z)+\bar{y}(\bar{z}))}\,. \tag{2.31}$$

The twist field $t^{\hat{\omega}}_{sl}$ acts on the spectrally un-flowed (i.e., $\hat{\omega}=0$) vertex operators (2.14) to generate operators in the spectrally flowed sector:

$$\begin{aligned}t^{\hat{\omega}}_{sl}(z,\bar{z})V_{j;m,\bar{m}}(w,\bar{w}) &\simeq (z-w)^{-\hat{\omega}m}\,(\bar{z}-\bar{w})^{-\hat{\omega}\bar{m}}\,V^{\hat{\omega}}_{j;m,\bar{m}}(w,\bar{w}) \\ &\sim (z-w)^{-(m-\bar{m})\hat{\omega}}\,|z-w|^{2\bar{m}\hat{\omega}}\,V^{\hat{\omega}}_{j;m,\bar{m}}(w,\bar{w})\,,\end{aligned} \tag{2.32}$$

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where

$$V^{\hat{\omega}}_{j;m,\bar{m}} = e^{i\sqrt{\frac{2}{k}}\left[\left(m+\frac{k\hat{\omega}}{2}\right)y+\left(\bar{m}+\frac{k\hat{\omega}}{2}\right)\bar{y}\right]}\Psi_{j;m,\bar{m}}\,. \tag{2.33}$$

Note that the operator $\Psi_{j;m,\bar{m}}$ remains unchanged under spectral flow. In particular, its world-sheet dimension Δ_Ψ, given by (2.17), does not change under spectral flow. Demanding mutual locality of the spectrally flowed vertex operators with respect to the twist fields imposes the constraint

$$m-\bar{m}\in\mathbb{Z}\,. \tag{2.34}$$

It is easy to read off the world-sheet dimension of the vertex operator (2.33) as

$$\Delta_{sl} = \Delta_\Psi - \frac{\left(m+\frac{k\hat{\omega}}{2}\right)^2}{k} = -\frac{j(j+1)}{k} - m\hat{\omega} - \frac{k\hat{\omega}^2}{4}\,. \tag{2.35}$$

Switching the chiralities, one obtains the right-moving dimension as

$$\bar{\Delta}_{sl} = \bar{\Delta}_\Psi - \frac{\left(\bar{m}+\frac{k\hat{\omega}}{2}\right)^2}{k} = -\frac{j(j+1)}{k} - \bar{m}\hat{\omega} - \frac{k\hat{\omega}^2}{4}\,. \tag{2.36}$$

The OPEs of the spectrally flowed vertex operator (2.33) with the $\mathrm{SL}(2,\mathbb{R})$ currents take the form

$$\begin{aligned}\mathtt{j}^3(z)V^{\hat{\omega}}_{j;m,\bar{m}}(w) &\simeq \frac{m+\frac{k\hat{\omega}}{2}}{z-w}V_{j;m,\bar{m}}(w)\,,\\ \mathtt{j}^\pm(z)V^{\hat{\omega}}_{j;m,\bar{m}}(w) &\simeq \frac{m\mp j}{z-w}V_{j;m\pm1,\bar{m}}(w)\,.\end{aligned} \tag{2.37}$$

2.2 Symmetries and vertex operators of $\mathbb{S}^3$

Superstring theory on $\mathbb{S}^3$ is described by the superconformal SU(2) WZW model that is composed of the bosonic $\mathrm{SU}(2)_{k-2}$ WZW model at level $k-2$ and three pairs of free (anti-)holomorphic fermions that can be described by $\mathrm{SU}(2)_2$ WZW model at level 2. The bosonic and the fermionic sectors combine to give rise to the $\mathrm{SU}(2)_k$ affine symmetry algebra at level k.

For $g'\in\mathrm{SU}(2)$, the exact effective $\mathrm{SU}(2)_k$ WZW action is given by

$$S_{su} = k\,\mathtt{S}[g'] \tag{2.38}$$

where the level-one WZW action is defined in (A.1). The action (2.38) is invariant w.r.t. $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ affine symmetry algebra at level k. The global part of the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ symmetry algebra is generated by the Kac-Moody currents j'^a and $\tilde{j}'^a$, $a=1,2,3$, given by

$$j'^a = \frac{k}{2}\,\mathrm{Tr}\left(\partial g' g'^{-1}\sigma^a\right), \qquad \tilde{j}'^a = \frac{k}{2}\,\mathrm{Tr}\left(g'^{-1}\bar{\partial}g'\sigma^a\right). \tag{2.39}$$

These currents get combined with contributions from the fermions, rendering the $\mathrm{SU}(2)_k$ Kac-Moody algebra

$$\mathtt{j}'^a(z)\,\mathtt{j}'^b(w) = \frac{k}{2}\,\frac{\delta^{ab}}{(z-w)^2} + \frac{i\,\epsilon^{abc}}{z-w}\,\mathtt{j}'_c(w) + \dots\,. \tag{2.40}$$

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The world-sheet stress-energy tensor according to the Sugawara construction takes the form,

$$T_{su} \supset \frac{1}{k}\, j'^a\, j'^a\,, \qquad \tilde{T}_{su} \supset \frac{1}{k}\, \tilde{j}'^a\, \tilde{j}'^a\,, \tag{2.41}$$

where we omitted contributions from the fermions. The quadratic Casimir of $\mathrm{SU}(2)_k$ is given by

$$c_2' = (j'^3)^2 + \frac{1}{2}\{j'^+, j'^-\}\,, \tag{2.42}$$

where

$$j'^{\pm} = j'^1 \pm i j'^2\,. \tag{2.43}$$

All representations of SU(2) can be parametrized by j', that is related to the Casimir as $c_2' = j'(j'+1)$.

The group manifold $\mathrm{SU}(2)_k$, being compact, possesses only discrete series representation given by

$$\mathcal{D}'_{j'} = \left\{ |j'; m', \bar{m}'\rangle : \;\; j' = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2};\;\; m', \bar{m}' = -j', j'+1, \ldots, j' \right\}\,, \tag{2.44}$$

where $m', \bar{m}'$ are eigenvalues of $\mathtt{j}'^3$ and $\tilde{\mathtt{j}}'^3$.

Analogously to the case of $\mathrm{SL}(2,\mathbb{R})$, the primary vertex operators of SU(2) can be described in terms of the parafermionic decomposition into $\mathrm{U}(1)_{y_{su}}$ and $\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$. The subgroup $\mathrm{U}(1)_{y_{su}}$ is generated by $\mathtt{j}'^3$ and is parametrized by a scalar field y_{su}, satisfying the OPE $y_{su}(z) y_{su}(w) \simeq -\log(z-w)$. Thus one can write

$$\mathtt{j}'^3 = i\sqrt{\frac{k}{2}}\, \partial y_{su}\,. \tag{2.45}$$

The primary operators, $V'_{j';m',\bar{m}'}$, can be written as

$$V'_{j';m',\bar{m}'} = e^{i\sqrt{\frac{2}{k}}(m' y_{su} + \bar{m}' \bar{y}_{su})} \Psi'_{j';m',\bar{m}'}\,, \tag{2.46}$$

where $\Psi'_{j';m',\bar{m}'}$ is a primary vertex operator on $\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$. The vertex operator (2.46) transforms in the spin (j', j') representation of SU(2).

The SU(2) vertex operator $V'_{j';m',\bar{m}'}$ satisfies the following OPE with the stress-energy tensor:

$$T_{su}(z) V'_{j';m',\bar{m}'}(w) \simeq \frac{\Delta_{su}}{(z-w)^2} V_{j';m',\bar{m}'}(w)\,, \tag{2.47}$$

where we defined its scaling dimension as

$$\Delta_{su} = \frac{j(j+1)}{k}\,. \tag{2.48}$$

It follows that the dimension of $\Psi'_{j';m',\bar{m}'}$ is given by

$$\Delta_{\Psi'} = \frac{j(j+1)}{k} - \frac{m'^2}{k}, \qquad \bar{\Delta}_{\Psi'} = \frac{j(j+1)}{k} - \frac{\bar{m}'^2}{k}\,. \tag{2.49}$$

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The OPEs of the primary vertex operator $V'_{j';m',\bar{m}'}$ with the SU(2) currents are given by

$$\begin{aligned}
\mathtt{j}'^{3}(z)V'_{j';m',\bar{m}'}(w) &\simeq \frac{m'}{z-w}V'_{j';m',\bar{m}'}(w)\,,\\
\mathtt{j}'^{\pm}(z)V'_{j';m',\bar{m}'}(w) &\simeq \frac{\sqrt{j'(j'+1)-m'(m'\pm 1)}}{z-w}V'_{j';m'\pm 1,\bar{m}'}(w)\,.
\end{aligned} \tag{2.50}$$

Switching the chirality of the SU(2) currents, one can analogously write the OPE relations of $V'_{j';m',\bar{m}'}$ with the right-moving SU(2) currents.

Similarly to the vertex operators in AdS_3, the SU(2) vertex operators can be equivalently described in two different bases, related by

$$\Phi'_{j'}(z,\bar{z};u,\bar{u}) = \sum_{m',\bar{m}'=-j'}^{j'} \sqrt{C^{j'+m'}_{2j'}C^{j'+\bar{m}'}_{2j'}}\,u^{j'+m'}\bar{u}^{j'+\bar{m}'}V'_{j';m',\bar{m}'}(z,\bar{z})\,, \tag{2.51}$$

where

$$C^p_q = \frac{q!}{p!\,(q-p)!}\,. \tag{2.52}$$

The two-point functions of the SU(2) primaries are given by

$$\begin{aligned}
\langle \Phi'_{j'_1}(z,\bar{z};u,\bar{u})\Phi'_{j'_2}(w,\bar{w};v,\bar{v})\rangle &= \frac{\delta_{j'_1 j'_2}}{|z-w|^{4\Delta_{su}}|u-v|^{4j'_1}}\,,\\
\langle (V'_{j'_1;m'_1,\bar{m}'_1}(z,\bar{z}))^\dagger V'_{j'_2;m'_2,\bar{m}'_2}(w,\bar{w})\rangle &= \frac{\delta_{j'_1 j'_2}\delta_{m'_1 m'_2}\delta_{\bar{m}'_1\bar{m}'_2}}{|z-w|^{4\Delta_{su}}}\,,
\end{aligned} \tag{2.53}$$

where the conjugate operator is given by

$$(V'_{j';m',\bar{m}'})^\dagger = (-1)^{2j-m'-\bar{m}'}V'_{j';-m',-\bar{m}'}\,. \tag{2.54}$$

The three-point function of $V'_{j';m',\bar{m}'}$ is given by [44]

$$\begin{aligned}
&\langle V'_{j'_1;m'_1,\bar{m}'_1}(z_1), V'_{j'_2;m'_2,\bar{m}'_2}(z_2)V'_{j'_3;m'_3,\bar{m}'_3}(z_3)\rangle\\
&\qquad = \frac{C_{su}(j'_i,m'_i,\bar{m}'_i)}{|z_{12}|^{2(\Delta_{j'_1}+\Delta_{j'_2}-\Delta_{j'_3})}|z_{23}|^{2(\Delta_{j'_2}+\Delta_{j'_3}-\Delta_{j'_1})}|z_{31}|^{2(\Delta_{j'_3}+\Delta_{j'_1}-\Delta_{j'_2})}}
\end{aligned} \tag{2.55}$$

where $\Delta_{j'} = \Delta_{su}$ and

$$C_{su}(j'_i,m'_i,\bar{m}'_i) = \begin{bmatrix} j'_1 & j'_2 & j'_3\\ m'_1 & m'_2 & m'_3\end{bmatrix}\begin{bmatrix} j'_1 & j'_2 & j'_3\\ \bar{m}'_1 & \bar{m}'_2 & \bar{m}'_3\end{bmatrix}\rho(j'_1,j'_2,j'_3) \tag{2.56}$$

with the first two factors are the Wigner $3j$-symbols, and

$$\begin{aligned}
\frac{\rho^2(j'_1,j'_2,j'_3)}{(2j'_1+1)(2j'_2+1)(2j'_3+1)} &= \frac{\Gamma\left(\frac{k+1}{k}\right)\Gamma\left(1-\frac{2j'_1+1}{k}\right)\Gamma\left(1-\frac{2j'_2+1}{k}\right)\Gamma\left(1-\frac{2j'_3+1}{k}\right)}{\Gamma\left(\frac{k-1}{k}\right)\Gamma\left(1+\frac{2j'_1+1}{k}\right)\Gamma\left(1+\frac{2j'_2+1}{k}\right)\Gamma\left(1+\frac{2j'_3+1}{k}\right)}\\
&\times \frac{\Pi^2(j'_1+j'_2+j'_3+1)\Pi^2(j'_1+j'_2-j'_3)\Pi^2(j'_2+j'_3-j'_1)\Pi^2(j'_3+j'_1-j'_2)}{\Pi^2(2j'_1)\Pi^2(2j'_2)\Pi^2(2j'_3)}
\end{aligned} \tag{2.57}$$

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where Π denotes

$$\Pi(j') = \prod_{n=1}^{j'} \frac{\Gamma\left(1+\frac{n}{k}\right)}{\Gamma\left(1-\frac{n}{k}\right)}\,. \tag{2.58}$$

String theory on $\mathbb{S}^3$ contains states obtained by spectral flow automorphism of the SU(2) algebra:

$$\begin{aligned} \mathtt{j}'^3(z) &\to \mathtt{j}'^3(z) - \frac{k\hat{\omega}'}{2z}\,, \quad \tilde{\mathtt{j}}'^3(\bar{z}) \to \tilde{\mathtt{j}}'^3(z) - \frac{k\hat{\bar{\omega}}'}{2\bar{z}}\,, \\ \mathtt{j}'^{\pm}(z) &\to z^{\pm\hat{\omega}'}\mathtt{j}'^{\pm}(z)\,, \qquad \tilde{\mathtt{j}}'^{\pm}(\bar{z}) \to \bar{z}^{\pm\hat{\bar{\omega}}'}\tilde{\mathtt{j}}'^{\pm}(\bar{z})\,, \end{aligned} \tag{2.59}$$

where $\hat{\omega}', \hat{\bar{\omega}} \in \mathbb{Z}$ are the left- and right-moving SU(2) spectral flow parameters. Unlike spectral flow automorphism of $\mathrm{SL}(2,\mathbb{R})$, the left- and the right-moving SU(2) spectral flow parameters are different and independent of each other. Under spectral flow automorphism, the world-sheet Virasoro generators transform as

$$\begin{aligned} L'_n &\to L'_n - \hat{\omega}' j'^3_n + \frac{k\hat{\omega}'^2}{4}\delta_{n,0}\,, \\ \tilde{L}'_n &\to \tilde{L}'_n - \hat{\bar{\omega}}' \tilde{j}'^3_n + \frac{k\hat{\bar{\omega}}'^2}{4}\delta_{n,0}\,, \end{aligned} \tag{2.60}$$

where j'^3_n and $\tilde{j}'^3_n$ are the Kac-Moody modes of the currents j'^3 and $\tilde{j}'^3$.

In the parafermionic notation, the spectrally flowed vertex operators are given by

$$V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'} = e^{i\sqrt{\frac{2}{k}}\left[\left(m'+\frac{k\hat{\omega}'}{2}\right)y_{su}+\left(\bar{m}+\frac{k\hat{\bar{\omega}}'}{2}\right)\bar{y}_{su}\right]}\Psi'_{j';m',\bar{m}'}\,. \tag{2.61}$$

Its (anti-)holomorphic scaling dimensions are given by

$$\begin{aligned} \Delta_{su} &= \frac{j'(j'+1)}{k} + m'\hat{\omega}' + \frac{k\hat{\omega}'^2}{4}\,, \\ \bar{\Delta}_{su} &= \frac{j'(j'+1)}{k} + \bar{m}'\hat{\bar{\omega}}' + \frac{k\hat{\bar{\omega}}'^2}{4}\,. \end{aligned} \tag{2.62}$$

The OPEs of the spectrally flowed vertex operator (2.61) with the global SU(2) generators take the form

$$\begin{aligned} \mathtt{j}'^3(z)\, V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}(w) &= \frac{m' + \frac{k}{2}\hat{\omega}'}{z-w}\, V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}\,, \\ \tilde{\mathtt{j}}'^3(\bar{z})\, V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}(\bar{w}) &= \frac{\bar{m}' + \frac{k}{2}\hat{\bar{\omega}}'}{\bar{z}-\bar{w}}\, V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}\,. \end{aligned} \tag{2.63}$$

The OPE of $V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}$ with $\mathtt{j}'^{\pm}$ and $\tilde{\mathtt{j}}'^{\pm}$ remain unchanged under spectral flow.

3 Coset construction

The main model we are going to work with in this paper is given by the type-II superstring theory on the ten-dimensional space-time,

$$\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)} \times \mathrm{U}(1)^4\,, \tag{3.1}$$

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given by a product of the six-dimensional coset space-time and the four-dimensional flat compact space given by the torus $\mathbb{T}^4 = \mathrm{U}(1)^4$. The radii of $\mathbb{T}^4$ are free parameters of our model. In particular, we can take the limit of infinite radii, and consider the theory on the six-dimensional coset space times the $\mathbb{R}^4$ manifold.

Our primary interest will be the case of asymmetric anomaly-free time-like gauging of the U(1) sub-group in the coset sub-space of (3.1). The goal of this section is to construct the world-sheet action, obtain the gauge currents, and derive the corresponding supergravity target space-time background geometry. We will demonstrate that the target space-time geometry has a black hole horizon, while its asymptotic region is given by the two-dimensional linear dilaton background times $\mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{T}^4$.[7]

We start by considering sigma-model on the group manifold

$$\mathcal{G} \times \mathrm{U}(1)^4 \qquad \text{where} \qquad \mathcal{G} = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)_x \,, \tag{3.2}$$

that is described by the action consisting of the WZW terms for the field $G \in \mathcal{G}$ and the Polyakov terms for the abelian components. Supersymmetric version of this model is obtained by supplementing the bosonic WZW/Polyakov sector with the free Majorana-Weyl fermions. The latter are split into (anti-)holomorphic sectors, and are labeled by indices in adjoint representation of the space-time symmetry group:

$$\psi^a,\ \tilde{\psi}^a \in sl(2,\mathbb{R})\,, \quad \psi'^b,\ \tilde{\psi}'^b \in su(2)\,, \quad \psi^x,\ \tilde{\psi}^x \in u(1)_x\,, \quad \psi^i,\ \tilde{\psi}^i \in u(1)^4\,. \tag{3.3}$$

The world-sheet fermions are then described by the action

$$S_f = \frac{k}{2\pi}\int d^2z \left(\psi^a\partial\psi^a + \tilde{\psi}^a\bar{\partial}\tilde{\psi}^a + \psi'^b\partial\psi'^b + \tilde{\psi}'^b\bar{\partial}\tilde{\psi}'^b + \psi^x\partial\psi^x + \tilde{\psi}^x\bar{\partial}\tilde{\psi}^x + \psi^i\partial\psi^i + \tilde{\psi}^i\bar{\partial}\tilde{\psi}^i\right) \tag{3.4}$$

where sums over repeated indices $a = 1,2,3$, $b = 1,2,3$, $i = 1,2,3,4$ are implied.

As mentioned above, bosonic sector of the model is described by the sum of WZW terms for the $\mathrm{SL}(2,\mathbb{R})$ and SU(2) sectors, plus the Polyakov terms for the $\mathrm{U}(1)_x$ and $\mathrm{U}(1)^4$ sub-spaces. Due to the nature of the coset (3.1), it is convenient to combine the group elements

$$g \in \mathrm{SL}(2,\mathbb{R})\,, \qquad g' \in \mathrm{SU}(2)\,, \qquad e^{i\sqrt{\frac{2}{k}}\,x} \in \mathrm{U}(1)_x \tag{3.5}$$

into one block-diagonal element of the sub-group $\mathcal{G}$ of (3.2), that we denote as

$$G = \mathrm{diag}\left[g,\ g',\ e^{i\sqrt{\frac{2}{k}}\,x}\right]\,. \tag{3.6}$$

The WZW action for the field G on the group manifold (3.2) is then given by

$$S_{\mathrm{WZW}}[g,g',x] = k\,\mathtt{S}[g] - k\,\mathtt{S}[g'] + \frac{1}{2\pi}\int d^2z\,\partial x\bar{\partial}x\,, \tag{3.7}$$

[7] We will denote indices of vectors in ten-dimensional space-time with capital-case Latin letters, and indices of the six-dimensional coset space-time with lower-case Greek letters. E.g., the space-time coordinates are given by $X^M = (X^\mu, z^1, z^2, z^3, z^4)$, where X^μ are coordinates of the six-dimensional coset space-time, and $z^i = (z^1, z^2, z^3, z^4)$ are coordinates of the torus $\mathbb{T}^4$. Throughout this paper, we will generally assume that all the coordinates are dimensionless, and therefore, e.g., the ten-dimensional metric is given by $ds^2_{10} = \ell_s^2\, G_{MN} dX^M dX^N$, where ℓ_s is the string length.

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where the level-one WZW action $\mathtt{S}$ is defined in appendix A. In terms of the field G the action (3.7) can be equivalently re-written as

$$\begin{aligned} S'_{\rm WZW}[G] \equiv S_{\rm WZW}[g,g',x] = \frac{k}{4\pi}\int d^2z\,{\rm Tr}\left(P\,G^{-1}\partial G\,G^{-1}\bar\partial G\right) \\ + \frac{ik}{24\pi}\int_{\mathcal{B}}{\rm Tr}\left(P\,G^{-1}dG\wedge G^{-1}dG\wedge G^{-1}dG\right)\,, \end{aligned} \tag{3.8}$$

where we introduced the projection operator

$$P = {\rm diag}\,[\,1,\ 1,\ -1,\ -1,\ -1\,]\,, \tag{3.9}$$

and prime in the WZW action notation in l.h.s. of (3.8) indicates the presence of this projection operator.

Together with the Polyakov terms for the $\mathrm{U}(1)^4$ sector,

$$S_{\mathbb{T}^4} = \frac{1}{2\pi}\int d^2z\,\partial z^i\,\bar\partial z^i\,, \tag{3.10}$$

total action of the supersymmetric sigma model on (3.2) is given by

$$S = S'_{\rm WZW} + S_{\mathbb{T}^4} + S_f\,. \tag{3.11}$$

We view the action (3.11) as exact (in k, that also means exact in ℓ_s) effective action of the world-sheet theory. In particular, the level k of the WZW terms that we wrote down for the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ sectors are exact in quantum theory.[8]

3.1 Gauging and gauge currents

Next, we are going to gauge the $\mathrm{U}(1)$ subgroup of the group $\mathcal{G}$. Let us start with the derivation in the bosonic sector, where an action of the gauged $\mathrm{U}(1)$ sub-group of the target space symmetry group $\mathcal{G}$ is defined by the following transformation laws

$$\begin{aligned} g &\sim e^{\frac{a_1}{\sqrt{k}}\xi\,\sigma_3}\,g\,e^{\frac{b_1}{\sqrt{k}}\xi\,\sigma_3}\,, \\ g' &\sim e^{i\frac{a_2}{\sqrt{k}}\xi\,\sigma_3}\,g'\,e^{i\frac{b_2}{\sqrt{k}}\xi\,\sigma_3}\,, \\ x_L &\sim x_L + a_3\,\xi\,, \qquad\qquad x_R \sim x_R + b_3\,\xi\,. \end{aligned} \tag{3.12}$$

Here ξ parametrizes the gauged $\mathrm{U}(1)$ group, and $x_{L,R}$ are (anti-)holomorphic components of the world-sheet field x. The model is defined by the parameters $a_{1,2,3}$, $b_{1,2,3}$, that in particular specify how the gauged $\mathrm{U}(1)$ sub-group is chosen within the group (3.2). In what follows, we will impose anomaly cancellation conditions on $a_{1,2,3}$, $b_{1,2,3}$, rendering four independent remaining parameters. These four parameters acquire a physical meaning in terms of conserved charges of the corresponding black hole background, given by the mass and three angular momenta, as we will demonstrate below in section 4.

[8] See appendix A.3 for a review of the corresponding formalism.

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We are going to be working with the following parametrization of the SL(2, ℝ) and SU(2) group elements

$$\begin{aligned} g &= e^{\alpha\sigma_3} e^{\theta\sigma_1} e^{\beta\sigma_3}\,, \\ g' &= e^{i\alpha'\sigma_3} e^{i\theta'\sigma_1} e^{i\beta'\sigma_3}\,, \end{aligned} \tag{3.13}$$

in terms of which the action (3.7) is given by (see sections A.1, A.2)

$$\begin{aligned} S_{\rm WZW}[g,g',x] = \frac{k}{2\pi}\int d^2z\, \Big(&\partial\alpha\bar\partial\alpha + \partial\beta\bar\partial\beta + \partial\theta\bar\partial\theta + 2\cosh(2\theta)\bar\partial\alpha\partial\beta \\ &+ \partial\alpha'\bar\partial\alpha' + \partial\beta'\bar\partial\beta' + \partial\theta'\bar\partial\theta' + 2\cos(2\theta')\bar\partial\alpha'\partial\beta' + \partial x\bar\partial x\Big)\,, \end{aligned} \tag{3.14}$$

while the transformations (3.12) are given by

$$\begin{aligned} &\alpha \sim \alpha + \frac{a_1}{\sqrt{k}}\,\xi\,, \qquad \beta \sim \beta + \frac{b_1}{\sqrt{k}}\,\xi\,, \quad \alpha' \sim \alpha' + \frac{a_2}{\sqrt{k}}\,\xi\,, \quad \beta' \sim \beta' + \frac{b_2}{\sqrt{k}}\,\xi\,, \\ &x \sim x + (a_3+b_3)\xi\,, \quad \theta \sim \theta\,, \qquad\qquad \theta' \sim \theta'\,. \end{aligned} \tag{3.15}$$

The U(1) transformation rules (3.12) can be compactly reformulated as

$$G \sim e^{\xi\, T_L}\, G\, e^{\xi\, T_R}\,, \tag{3.16}$$

where the left/right-moving generators of the corresponding $u(1)$ algebra are given by the following block-diagonal matrices

$$\begin{aligned} T_L &= \mathrm{diag}\left[\frac{a_1}{\sqrt{k}}\,\sigma_3,\ i\,\frac{a_2}{\sqrt{k}}\,\sigma_3,\ ia_3\sqrt{\frac{2}{k}}\right]\,, \\ T_R &= \mathrm{diag}\left[\frac{b_1}{\sqrt{k}}\,\sigma_3,\ i\,\frac{b_2}{\sqrt{k}}\,\sigma_3,\ ib_3\sqrt{\frac{2}{k}}\right]\,. \end{aligned} \tag{3.17}$$

Then introducing the compensator fields

$$U = e^{-u\,T_L}\,, \qquad V = e^{-v\,T_L}\,, \tag{3.18}$$

obeying the U(1) transformation rules

$$u \sim u + \xi\,, \qquad v \sim v + \xi\,, \tag{3.19}$$

we can construct U(1)-invariant field UGV. The corresponding locally U(1)-invariant WZW action can be written using the Polyakov-Wiegmann identity as

$$\begin{aligned} S'_{\rm WZW}[UGV] = S'_{\rm WZW}[G] + S'_{\rm WZW}[U] + S'_{\rm WZW}[V] \\ + \frac{k}{2\pi}\int d^2z\,\mathrm{Tr}\Big[P\Big(G^{-1}\bar\partial G\partial V V^{-1} + U^{-1}\bar\partial U\partial G G^{-1} + U^{-1}\bar\partial U G\partial V V^{-1}G^{-1}\Big)\Big], \end{aligned} \tag{3.20}$$

where $S'_{\rm WZW}[G]$ is given by (3.7), (3.8), while

$$\begin{aligned} S'_{\rm WZW}[U] &= \frac{k}{4\pi}\,\mathrm{Tr}(P\,T_L^2)\int d^2z\,\partial u\bar\partial u\,, \\ S'_{\rm WZW}[V] &= \frac{k}{4\pi}\,\mathrm{Tr}(P\,T_R^2)\int d^2z\,\partial v\bar\partial v\,. \end{aligned} \tag{3.21}$$

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Using (3.17) we obtain

$$\mathrm{Tr}(P\,T_L^2) = \frac{2}{k}\,(a_1^2+a_2^2+a_3^2)\,, \qquad \mathrm{Tr}(P\,T_R^2) = \frac{2}{k}\,(b_1^2+b_2^2+b_3^2)\,. \tag{3.22}$$

Notice that the action (3.20), while being invariant w.r.t. local U(1) gauge transformations (3.12), cannot be an action of the gauged WZW model. This is because it includes Polyakov kinetic terms (3.21) for the fields u, v, making these fields dynamical rather than auxiliary. One obvious way to eliminate these kinetic terms is given by the choice of null gauging[9]

$$a_1^2+a_2^2+a_3^2 = 0\,, \qquad b_1^2+b_2^2+b_3^2 = 0\,. \tag{3.23}$$

We are not going to pursue such a direction in this section. Instead we are going to consider the time-like gauging

$$a_1^2+a_2^2+a_3^2 = 1\,, \qquad b_1^2+b_2^2+b_3^2 = 1\,, \tag{3.24}$$

that ensures the anomaly-cancellation condition $\mathrm{Tr}(P\,T_L^2) = \mathrm{Tr}(P\,T_R^2)$, thereby allowing to eliminate kinetic terms for the fields u, v by considering the action

$$S_{\mathrm{gWZW}}[G] = S'_{\mathrm{WZW}}[UGV] - \frac{1}{2\pi}\int d^2z\,\partial w\bar\partial w\,. \tag{3.25}$$

Here we introduced the U(1)-invariant field $w = u - v$ described by the Polyakov action with the 'wrong sign'. In the time-like gauging (3.24) one easily derives

$$S'_{\mathrm{WZW}}[U] + S'_{\mathrm{WZW}}[V] - \frac{1}{2\pi}\int d^2z\,\partial w\bar\partial w = \frac{1}{\pi}\int d^2z\,A\tilde A\,, \tag{3.26}$$

where we denoted the auxiliary gauge field potentials as

$$A = -\partial v\,, \qquad \tilde A = -\bar\partial u\,. \tag{3.27}$$

Furthermore, introducing the $\mathrm{U}(1)_{L,R}$ gauge currents

$$J = k\,\mathrm{Tr}\left(P\,T_L\,\partial G\,G^{-1}\right)\,, \qquad \tilde J = k\,\mathrm{Tr}\left(P\,T_R\,G^{-1}\,\bar\partial G\right)\,, \tag{3.28}$$

we can re-write the action (3.25) as

$$S_{\mathrm{gWZW}}[G] = S'_{\mathrm{WZW}}[G] + \frac{1}{2\pi}\int d^2z\,\left(A\tilde J + \tilde A J + (M+2)\,A\tilde A\right)\,, \tag{3.29}$$

where we denoted

$$M = k\,\mathrm{Tr}\left(P\,T_L\,G\,T_R\,G^{-1}\right)\,. \tag{3.30}$$

Eliminating the non-dynamic fields A, $\tilde A$ via e.o.m., we obtain the action

$$S_{\mathrm{gWZW}}[G] = S'_{\mathrm{WZW}}[G] - \frac{1}{2\pi}\int d^2z\,\frac{J\tilde J}{M+2}\,, \tag{3.31}$$

[9] In case of null gauging, a_1 and b_1 are imaginary-valued, and so are α and β parameters of the $\mathrm{SL}(2,\mathbb{R})$ group element (3.13). Gauge-fixing, e.g., $\alpha = \beta = i\,t/2$ then gives Lorentzian time t. In case of time-like gauging, a_1, b_1, α, β are real-valued, but the target space-time metric of the considered gWZW model is Lorentzian.

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and a non-trivial dilaton field

$$\Phi = \Phi_0 - \frac{1}{2}\log\left(\frac{M}{2}+1\right)\,, \tag{3.32}$$

where Φ_0 is the dilaton background.

Let us denote

$$\Delta = \frac{M}{2}+1\,. \tag{3.33}$$

Using the parametrization (3.13) we obtain

$$\Delta = 1 + a_1 b_1 \cosh(2\theta) + a_2 b_2 \cos(2\theta') + a_3 b_3\,. \tag{3.34}$$

The dilaton background (3.32) can then be determined as

$$\Phi = \Phi_0 - \frac{1}{2}\log\left(1 + a_1 b_1 \cosh(2\theta) + a_2 b_2 \cos(2\theta') + a_3 b_3\right)\,. \tag{3.35}$$

For the currents (3.28) we obtain

$$J = 2\left(-i\frac{a_1}{\sqrt{k}}\,j^3 - i\frac{a_2}{\sqrt{k}}j'^3 + a_3\partial x\right)\,,\quad \tilde{J} = 2\left(-i\frac{b_1}{\sqrt{k}}\,\tilde{j}^3 - i\frac{b_2}{\sqrt{k}}\tilde{j}'^3 + b_3\bar{\partial}x\right)\,, \tag{3.36}$$

where we defined (recall that σ^a are Pauli matrices)

$$\begin{aligned} j^3 &= i\frac{k}{2}\,\mathrm{Tr}\left(\partial g g^{-1}\sigma^3\right)\,,\quad \tilde{j}^3 = i\frac{k}{2}\,\mathrm{Tr}\left(g^{-1}\bar{\partial}g\sigma^3\right)\,,\\ j'^3 &= \frac{k}{2}\,\mathrm{Tr}\left(\partial g' g'^{-1}\sigma^3\right)\,,\quad \tilde{j}'^3 = \frac{k}{2}\,\mathrm{Tr}\left(g'^{-1}\bar{\partial}g'\sigma^3\right)\,. \end{aligned} \tag{3.37}$$

In the parametrization (3.13) we obtain

$$\begin{aligned} j^3 &= ik\,\left(\partial\alpha + \cosh(2\theta)\,\partial\beta\right)\,,\quad \tilde{j}_3 = ik\,\left(\bar{\partial}\beta + \cosh(2\theta)\,\bar{\partial}\alpha\right)\,,\\ j'^3 &= ik\,\left(\partial\alpha' + \cos(2\theta')\,\partial\beta'\right)\,,\quad \tilde{j}'_3 = ik\,\left(\bar{\partial}\beta' + \cos(2\theta')\,\bar{\partial}\alpha'\right)\,. \end{aligned} \tag{3.38}$$

We will also use the notations

$$j^x = i\,\partial x\,,\ \tilde{j}^x = i\,\bar{\partial}x\,,\quad j^w = i\,\partial w\,,\ \tilde{j}^w = i\,\bar{\partial}w\,,\quad j^i = i\,\partial z^i\,,\ \tilde{j}^i = i\,\bar{\partial}z^i\,. \tag{3.39}$$

The current components j^3 and j'^3, and their anti-holomorphic counterparts, belong to the $sl(2,\mathbb{R})$ and $su(2)$ Kac-Moody algebras at levels $k+2$ and $k-2$, satisfying the OPEs[10]

$$\begin{aligned} j^a(z)\,j^b(w) &= \frac{k+2}{2}\,\frac{\delta^{ab}}{(z-w)^2} + \frac{\epsilon^{abc}}{z-w}\,j_c(w) + \dots\,,\\ j'^a(z)\,j'^b(w) &= \frac{k-2}{2}\,\frac{\delta^{ab}}{(z-w)^2} + \frac{i\,\epsilon^{abc}}{z-w}\,j'_c(w) + \dots\,, \end{aligned} \tag{3.40}$$

[10]In our conventions, both j^3 and j'^3 have real eigenvalues. An alternative convention for the $sl(2,\mathbb{R})$ currents, related to ours by redefinitions $j^3 \to ij^3$, $j_3 \to -ij_3$ would result in the $\eta_{ab} = \mathrm{diag}\{1,\,1,\,-1\}$ Cartan metric in (3.40) and its fermionic counterpart (3.45).

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where ϵ^{abc} is the Levi-Civita symbol, $\epsilon^{123} = 1$, and similarly for the anti-holomorphic components. The currents j^x, j^i, and j^w are abelian Kac-Moody currents at levels 1, 1, and -1 respectively,[11]

$$\langle j^x(z)\, j^x(w)\rangle = \frac{1}{2}\,\frac{1}{z-w}\,, \quad \langle j^i(z)\, j^j(w)\rangle = \frac{1}{2}\,\frac{\delta^{ij}}{z-w}\,, \quad \langle j^w(z)\, j^w(w)\rangle = -\frac{1}{2}\,\frac{1}{z-w}\,, \tag{3.42}$$

and similarly for the anti-holomorphic sector.

In the supersymmetric theory (3.11) one additionally has contributions to the gauge $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ currents (3.36) coming from the fermionic sector. The total gauge currents can be compactly reformulated in terms of the total bosonic currents of supersymmetric Kac-Moody algebra,[12]

$$\mathtt{j}^a = j^a + \frac{1}{k}\,\epsilon^a{}_{bc}\,\psi^b\,\psi^c\,, \qquad \mathtt{j}'^a = j'^a - \frac{i}{k}\,\epsilon^a{}_{bc}\,\psi'^b\,\psi'^c\,, \tag{3.43}$$

that in particular implies

$$\mathtt{j}^3 = j^3 + \frac{2}{k}\,\psi^1\,\psi^2\,, \qquad \mathtt{j}'^3 = j'^3 - \frac{2i}{k}\,\psi'^1\,\psi'^2\,. \tag{3.44}$$

Here the $sl(2,\mathbb{R})$ and $su(2)$ fermions satisfy the OPEs

$$\psi^a(z)\psi^b(w) = \frac{k}{2}\,\frac{\delta^{ab}}{z-w}\,, \qquad \psi'^a(z)\psi'^b(w) = \frac{k}{2}\,\frac{\delta^{ab}}{z-w}\,. \tag{3.45}$$

For the total bosonic currents, we have the following OPEs of the supersymmetric Kac-Moody algebras $sl(2,\mathbb{R})$ and $su(2)$

$$\begin{aligned} \mathtt{j}^a(z)\,\mathtt{j}^b(w) &= \frac{k}{2}\,\frac{\delta^{ab}}{(z-w)^2} + \frac{\epsilon^{abc}}{z-w}\,\mathtt{j}_c(w) + \dots\,, \\ \mathtt{j}'^a(z)\,\mathtt{j}'^b(w) &= \frac{k}{2}\,\frac{\delta^{ab}}{(z-w)^2} + \frac{i\,\epsilon^{abc}}{z-w}\,\mathtt{j}'_c(w) + \dots\,, \end{aligned} \tag{3.46}$$

both at level k. The supersymmetric theory counterpart of (3.36) is then given by

$$\mathtt{J} = -2i\left(\frac{a_1}{\sqrt{k}}\,\mathtt{j}^3 + \frac{a_2}{\sqrt{k}}\mathtt{j}'^3 + a_3\, j^x\right), \quad \tilde{\mathtt{J}} = -2i\left(\frac{b_1}{\sqrt{k}}\,\tilde{\mathtt{j}}^3 + \frac{b_2}{\sqrt{k}}\tilde{\mathtt{j}}'^3 + b_3\tilde{j}^x\right). \tag{3.47}$$

Taking into account anomaly-cancellation conditions (3.24), we then obtain

$$\langle \mathtt{J}(z)\mathtt{J}(w)\rangle = -2\,\frac{1}{(z-w)^2}\,, \qquad \langle \tilde{\mathtt{J}}(\bar z)\tilde{\mathtt{J}}(\bar w)\rangle = -2\,\frac{1}{(\bar z-\bar w)^2}\,. \tag{3.48}$$

Supersymmetric partners of the gauge currents (3.47) are given by the fermions

$$\psi = -2i\left(\frac{a_1}{\sqrt{k}}\,\psi^3 + \frac{a_2}{\sqrt{k}}\psi'^3 + a_3\psi^x\right), \quad \tilde\psi = -2i\left(\frac{b_1}{\sqrt{k}}\,\tilde\psi^3 + \frac{b_2}{\sqrt{k}}\tilde\psi'^3 + b_3\tilde\psi^x\right), \tag{3.49}$$

[11] These follow from the free field OPE

$$\langle \partial x(z)\partial x(w)\rangle = -\frac{1}{2}\,\frac{1}{(z-w)^2}\,, \qquad \langle \partial w(z)\partial w(w)\rangle = \frac{1}{2}\,\frac{1}{(z-w)^2}\,. \tag{3.41}$$

More comments on the sign of the $\langle \partial w \partial w\rangle$ two-point function will be given in section 6.1.

[12] See appendix A.3 for a review.

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Using (3.45) in combination with the OPEs

$$\psi^x(z)\psi^x(w) = \frac{1}{2}\frac{1}{z-w}\,, \qquad \tilde{\psi}^x(\bar{z})\tilde{\psi}^x(\bar{w}) = \frac{1}{2}\frac{1}{\bar{z}-\bar{w}}\,, \tag{3.50}$$

we notice that the gauge fermions (3.49) that satisfy the OPEs

$$\langle\psi(z)\psi(w)\rangle = -2\frac{1}{z-w}\,, \qquad \langle\tilde{\psi}(\bar{z})\tilde{\psi}(\bar{w})\rangle = -2\frac{1}{\bar{z}-\bar{w}}\,. \tag{3.51}$$

3.2 Supergravity background

Combining (3.14), (3.34), (3.47), (3.38) in (3.31) we can write down the full world-sheet gauged WZW action for the considered parametrization (3.13). This action is invariant w.r.t. gauge transformation (3.15). We can fix the associated redundancy in the degrees of freedom by the gauge choice

$$\alpha = \frac{\nu}{2}\,y\,, \qquad \beta = \frac{1}{2}\,y\,, \tag{3.52}$$

where ν is an arbitrary parameter. For the sake of calculating supergravity background geometry, we set all the fermionic d.o.f. to zero. The resulting action is then given by

$$S_{\text{gWZW}}[G] = \frac{1}{2\pi}\int d^2z\,(\mathcal{G}_{MN}+\mathcal{B}_{MN})\,\partial X^M\bar{\partial}X^N\,, \tag{3.53}$$

where we defined the world-sheet field defining coordinates in ten-dimensional target space

$$X^M = (X^\mu,\ z^1,\ z^2,\ z^3,\ z^4)\,, \qquad X^\mu = (y,\ \theta,\ \alpha',\ \theta',\ \beta',\ x)\ , \tag{3.54}$$

with the non-vanishing components of the metric given by the six-dimensional coset subspace components

$$\begin{aligned}
\mathcal{G}_{yy} &= \frac{k}{4}\left(1+\nu^2+2\nu\cosh(2\theta)-\frac{2a_1b_1}{\Delta}\,(\nu+\cosh(2\theta))(1+\nu\cosh(2\theta))\right)\,,\\
\mathcal{G}_{y\alpha'} &= \mathcal{G}_{\alpha' y} = -\frac{k}{2\Delta}\,\left(a_1b_2\cos(2\theta')(\nu+\cosh(2\theta)) + a_2b_1(1+\nu\cosh(2\theta))\right)\,,\\
\mathcal{G}_{y\beta'} &= \mathcal{G}_{\beta' y} = -\frac{k}{2\Delta}\,(a_1b_2(\nu+\cosh(2\theta)) + a_2b_1\cos(2\theta')(1+\nu\cosh(2\theta)))\,,\\
\mathcal{G}_{yx} &= \mathcal{G}_{xy} = -\frac{\sqrt{k}}{2\Delta}\,(a_3b_1 + a_1b_3\nu + (a_1b_3+a_3b_1\nu)\cosh(2\theta))\,,\\
\mathcal{G}_{\theta\theta} &= \mathcal{G}_{\theta'\theta'} = k\,,\\
\mathcal{G}_{\alpha'\alpha'} &= \mathcal{G}_{\beta'\beta'} = k\left(1-\frac{2}{\Delta}\,a_2b_2\cos(2\theta')\right)\,,\\
\mathcal{G}_{\alpha'\beta'} &= \mathcal{G}_{\beta'\alpha'} = \frac{k}{\Delta}\,\left(\cos(2\theta')(1+a_3b_3+a_1b_1\cosh(2\theta)) - a_2b_2\right)\,,\\
\mathcal{G}_{\alpha' x} &= \mathcal{G}_{x\alpha'} = -\frac{\sqrt{k}}{\Delta}\,(a_2b_3 + a_3b_2\cos(2\theta'))\,,\\
\mathcal{G}_{\beta' x} &= \mathcal{G}_{x\beta'} = -\frac{\sqrt{k}}{\Delta}\,(a_3b_2 + a_2b_3\cos(2\theta'))\,,\\
\mathcal{G}_{xx} &= 1-\frac{2a_3b_3}{\Delta}\,,
\end{aligned} \tag{3.55}$$

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as well as the components of the metric on the torus $\mathbb{T}^4$, given by $\mathcal{G}_{z^1z^1} = \mathcal{G}_{z^2z^2} = \mathcal{G}_{z^3z^3} = \mathcal{G}_{z^4z^4} = 1$. Non-trivial components of the B-field are given by

$$\begin{aligned}
\mathcal{B}_{y\alpha'} &= -\mathcal{B}_{\alpha' y} = -\frac{k}{2\Delta}\left(a_1 b_2 \cos(2\theta')(\nu + \cosh(2\theta)) - a_2 b_1 (1 + \nu\cosh(2\theta))\right) , \\
\mathcal{B}_{y\beta'} &= -\mathcal{B}_{\beta' y} = -\frac{k}{2\Delta}\left(a_1 b_2 (\nu + \cosh(2\theta)) - a_2 b_1 \cos(2\theta')(1 + \nu\cosh(2\theta))\right) , \\
\mathcal{B}_{yx} &= -\mathcal{B}_{xy} = -\frac{\sqrt{k}}{2\Delta}\left(-a_3 b_1 + a_1 b_3 \nu + (a_1 b_3 - a_3 b_1 \nu)\cosh(2\theta)\right) , \\
\mathcal{B}_{\alpha'\beta'} &= -\mathcal{B}_{\beta'\alpha'} = -k\left(\cos(2\theta') + \frac{1}{\Delta} a_2 b_2 \sin(2\theta')^2\right) , \\
\mathcal{B}_{\alpha' x} &= -\mathcal{B}_{x\alpha'} = \frac{\sqrt{k}}{\Delta}\left(-a_2 b_3 + a_3 b_2 \cos(2\theta')\right) , \\
\mathcal{B}_{\beta' x} &= -\mathcal{B}_{x\beta'} = \frac{\sqrt{k}}{\Delta}\left(a_3 b_2 - a_2 b_3 \cos(2\theta')\right) .
\end{aligned} \tag{3.56}$$

Together with the dilaton (3.35), expressions (3.55), (3.56) represent the classical supergravity background, where the parameter ν is arbitrary and no physical quantities are expected to depend on it. One can explicitly verify that provided the anomaly cancellation conditions (3.24) are satisfied, this background obeys the classical equations of motions

$$R_{MN} + 2\nabla_M \partial_N \phi - \frac{1}{4}\mathcal{H}_{MLR}\mathcal{H}_N{}^{LR} = 0 , \tag{3.57}$$

$$4(\partial\Phi)^2 - 4\nabla^M \partial_M \Phi - R + \frac{1}{12}\mathcal{H}_{MNL}\mathcal{H}^{MNL} = 0 , \tag{3.58}$$

$$\nabla_L \mathcal{H}^L{}_{MN} - 2\partial_L \Phi \mathcal{H}^L{}_{MN} = 0 , \tag{3.59}$$

that can be derived from the corresponding bulk gravity action in the string frame

$$S = \frac{1}{2\kappa_0^2} V_{\mathbb{T}^4} \int d^6x \sqrt{-\det \mathcal{G}}\, e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}\right) , \tag{3.60}$$

where[13]

$$\mathcal{H}_{\mu\nu\lambda} = \partial_\mu \mathcal{B}_{\nu\lambda} + \partial_\lambda \mathcal{B}_{\mu\nu} + \partial_\nu \mathcal{B}_{\lambda\mu} , \tag{3.62}$$

and the dimensionful volume of the four-dimensional compactified torus (in string frame) is given by (here R_i are dimensionless radii of the circles of $\mathbb{T}^4$, where $z^i \sim z^i + 2\pi\ell_s R_i$, $i = 1, \ldots, 4$)

$$V_{\mathbb{T}^4} = (2\pi\ell_s)^4 \prod_{i=1}^{4} R_i . \tag{3.63}$$

The corresponding metric on target space-time is given by

$$ds_{10}^2 = ds_6^2 + \ell_s^2\left((dz^1)^2 + \cdots + (dz^4)^2\right) , \qquad ds_6^2 = \ell_s^2 \mathcal{G}_{\mu\nu}\, dX^\mu\, dX^\nu , \tag{3.64}$$

[13] Recall that the B-field is defined up to gauge transformations

$$\delta B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M , \tag{3.61}$$

that leaves the tensor (3.62) invariant.

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where we restored the string length ℓ_s, taking into account that in our conventions the space-time coordinates X^M are dimensionless. The metric in Einstein frame, $\tilde{\mathcal{G}}_{MN}$, is related to the metric in string frame, $\mathcal{G}_{MN}$, via

$$\mathcal{G}_{MN} = e^{\frac{\Phi-\Phi_0}{2}}\,\tilde{\mathcal{G}}_{MN}\,. \tag{3.65}$$

In Einstein frame normalization of the gravity action is given by $1/(2\kappa^2)$, where

$$\kappa = \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2}\,, \tag{3.66}$$

and G_N is the Newton's constant.[14] Notice that due to (3.65) the volume of the four-torus in the Einstein frame is given by $\tilde{V}_{\mathbb{T}^4} = e^{\Phi_0-\Phi}\,V_{\mathbb{T}^4}$.

To study the background (3.55), (3.56), (3.35) it is convenient to choose a different coordinate frame. Let us begin by rescaling the $\mathrm{U}(1)_x$ coordinate as follows,

$$x = \sqrt{k}\,x'\,. \tag{3.67}$$

Then every component of the metric and the B-field will be proportional to k, and therefore dependence of these tensors on the WZW level can be compactly expressed through an overall common factor. Next, we rescale the time coordinate y as follows,[15]

$$y = \frac{2a_1b_1}{a_1 - b_1\nu}\,t\,. \tag{3.68}$$

We also introduce a new radial coordinate $\rho(\theta)$ according to the rule (recall that $\Delta(\theta)$ was defined in (3.34))

$$\Delta = \rho^2 + 2a_3b_3 \quad \Rightarrow \quad \theta = \frac{1}{2}\,\cosh^{-1}\left(1 + \frac{\rho^2-\rho_+^2}{a_1b_1}\right)\,, \tag{3.69}$$

where we defined

$$\rho_\pm^2 = 1 \pm a_1b_1 - a_3b_3 + a_2b_2\cos(2\theta')\,. \tag{3.70}$$

For the choice of parameters

$$a_1b_1 > 0 \quad \Leftarrow \quad a_1 = \sqrt{1-a_2^2-a_3^2}\,, \quad b_1 = \sqrt{1-b_2^2-b_3^2} \tag{3.71}$$

the coordinate transformation (3.69) therefore implies the range of validity of the ρ coordinate given by

$$\rho \geq \rho_+\,. \tag{3.72}$$

In the new coordinates we then obtain the dilaton

$$\Phi = \Phi_0 - \frac{1}{2}\,\log(\rho^2 + 2a_3b_3)\,. \tag{3.73}$$

[14] See, e.g., [45].

[15] While this choice of the time coordinate can appear *ad hoc* at this point, the rationale behind it will be explained below, when we will show that in such a frame one obtains adjusted asymptotic value $\hat{\mathcal{G}}_{tt}/k = -1$.

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and the metric

$$ds_6^2 = \ell_s^2 \left(\hat{\mathcal{G}}_{tt}\, dt^2 + \mathcal{G}_{\rho\rho}\, d\rho^2 + 2\mathcal{G}_{\rho\theta'}\, d\rho\, d\theta' + \mathcal{G}_{ab}\, (dX^a + R^a\, dt)\, (dX^b + R^b\, dt)\right) , \tag{3.74}$$

Here we have defined

$$X^a = (\alpha',\ \beta',\ x') , \tag{3.75}$$

and used $\mathcal{G}_{ab}$, $a, b = \alpha', \beta', x'$ to denote projection of the $\mathcal{G}_{\mu\nu}$ tensor on the three-dimensional sub-space spanned by the vector (3.75). We have also re-arranged the terms in (3.74) in such a way that the off-diagonal metric elements $\mathcal{G}_{t\alpha'}$, $\mathcal{G}_{t\beta'}$, $\mathcal{G}_{tx'}$ are expressed in terms of the vector

$$R^a = \mathcal{G}^{ab}\, \mathcal{G}_{tb} , \tag{3.76}$$

where $\mathcal{G}^{ab}$ is the inverse metric on the three-dimensional sub-space spanned by (3.75). We then obtain the adjusted metric component

$$\begin{aligned} \hat{\mathcal{G}}_{tt} &= \mathcal{G}_{tt} - \mathcal{G}_{ab}\, R^a\, R^b \\ &= \mathcal{G}_{tt} - \mathcal{G}^{ab}\, \mathcal{G}_{ta}\, \mathcal{G}_{tb} . \end{aligned} \tag{3.77}$$

Introducing

$$\ell_\pm^2 = a_2 b_2 \cos(2\theta') - a_3 b_3 \pm \sqrt{(a_2^2 + a_3^2)(b_2^2 + b_3^2)} \tag{3.78}$$

we can express

$$\hat{\mathcal{G}}_{tt} = -k\, \frac{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}{(\rho^2 - \ell_+^2)(\rho^2 - \ell_-^2)} \tag{3.79}$$

We therefore observe that $\rho^2 = \rho_+^2$ is the outer horizon of the metric. Notice that $\rho_+^2 \geq \ell_+^2$, $\rho_-^2 \geq \ell_-^2$, and $\ell_+^2 \geq \ell_-^2$.[16] Therefore the apparent singularities of $\hat{\mathcal{G}}_{tt}$ at $\rho^2 = \ell_\pm^2$ are hidden behind the outer horizon $\rho^2 = \rho_+^2$.

Let us define

$$\begin{aligned} R^{\alpha'} &= \mathcal{R}^{\alpha'} - \frac{a_2 b_1 \nu}{a_1 - b_1 \nu} , \\ R^{\beta'} &= \mathcal{R}^{\beta'} - \frac{a_1 b_2}{a_1 - b_1 \nu} , \\ R^{x'} &= \mathcal{R}^{x'} - \frac{a_1 b_3 + a_3 b_1 \nu}{a_1 - b_1 \nu} , \end{aligned} \tag{3.80}$$

where we separated explicitly asymptotic values at $\rho \to \infty$, leaving us with $\mathcal{R}^a = \mathcal{O}(1/\rho)$, where

$$\begin{aligned} \mathcal{R}^{\alpha'} &= \frac{a_2\, (b_2^2 + b_3^2 - a_3 b_3 + a_2 b_2 \cos(2\theta') - \rho^2)}{(\rho^2 - \ell_+^2)(\rho^2 - \ell_-^2)} , \\ \mathcal{R}^{\beta'} &= \frac{b_2\, (-a_2^2 - a_3^2 + a_3 b_3 - a_2 b_2 \cos(2\theta') + \rho^2)}{(\rho^2 - \ell_+^2)(\rho^2 - \ell_-^2)} , \\ \mathcal{R}^{x'} &= \frac{a_3 b_2^2 - b_3 a_2^2 + (a_3 - b_3)(a_2 b_2 \cos(2\theta') - 2 a_3 b_3 - \rho^2)}{(\rho^2 - \ell_+^2)(\rho^2 - \ell_-^2)} , \end{aligned} \tag{3.81}$$

[16] This is easiest to see by taking $a_1 = \cos\psi$, $b_1 = \cos\psi'$.

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By performing the change of coordinates

$$\begin{aligned}\alpha' &= \hat{\alpha}' + \frac{a_2 b_1 \nu}{a_1 - b_1 \nu}\, t\,,\\ \beta' &= \hat{\beta}' + \frac{a_1 b_2}{a_1 - b_1 \nu}\, t\,,\\ x' &= \hat{x}' + \frac{a_1 b_3 + a_3 b_1 \nu}{a_1 - b_1 \nu}\, t\,,\end{aligned} \tag{3.82}$$

we can remove the off-diagonal $(t, \hat{\alpha}')$, $(t, \hat{\beta}')$, $(t, \hat{x}')$ metric elements in the asymptotic region. In the new coordinates we obtain

$$ds_6^2 = \ell_s^2 \left(\hat{\mathcal{G}}_{tt}\, dt^2 + \mathcal{G}_{\rho\rho}\, d\rho^2 + 2\mathcal{G}_{\rho\theta'}\, d\rho\, d\theta' + \mathcal{G}_{\theta'\theta'} d\theta'^2 + \mathcal{G}_{ab} \left(d\hat{X}^a + \mathcal{R}^a\, dt\right) \left(d\hat{X}^b + \mathcal{R}^b\, dt\right) \right), \tag{3.83}$$

where we defined

$$\hat{X}^a = \left(\hat{\alpha}',\ \hat{\beta}',\ \hat{x}'\right). \tag{3.84}$$

The rest of the components of the metric are given by

$$\begin{aligned}\frac{1}{k}\,\mathcal{G}_{\rho\rho} &= \frac{\rho^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}\,,\\ \frac{1}{k}\,\mathcal{G}_{\rho\theta} &= \frac{1}{k}\,\mathcal{G}_{\theta\rho} = \frac{\rho a_2 b_2 \sin(2\theta')}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}\,,\\ \frac{1}{k}\,\mathcal{G}_{\theta'\theta'} &= 1 - \frac{a_2^2 b_2^2 \sin(2\theta')^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}\,,\\ \frac{1}{k}\,\mathcal{G}_{\hat{\alpha}'\hat{\alpha}'} &= \frac{1}{k}\,\mathcal{G}_{\hat{\beta}'\hat{\beta}'} = 1 - \frac{2a_2 b_2 \cos(2\theta')}{\rho^2 + 2a_3 b_3}\,,\\ \frac{1}{k}\,\mathcal{G}_{\hat{\alpha}'\hat{\beta}'} &= \frac{1}{k}\,\mathcal{G}_{\hat{\beta}'\hat{\alpha}'} = \cos(2\theta') - \frac{a_2 b_2 (3 + \cos(4\theta'))}{2(\rho^2 + 2a_3 b_3)}\,,\\ \frac{1}{k}\,\mathcal{G}_{\hat{\alpha}'\hat{x}'} &= \frac{1}{k}\,\mathcal{G}_{\hat{x}'\hat{\alpha}'} = -\frac{a_2 b_3 + a_3 b_2 \cos(2\theta')}{\rho^2 + 2a_3 b_3}\,,\\ \frac{1}{k}\,\mathcal{G}_{\hat{\beta}'\hat{x}'} &= \frac{1}{k}\,\mathcal{G}_{\hat{x}'\hat{\beta}'} = -\frac{a_3 b_2 + a_2 b_3 \cos(2\theta')}{\rho^2 + 2a_3 b_3}\,,\\ \frac{1}{k}\,\mathcal{G}_{\hat{x}'\hat{x}'} &= \frac{\rho^2}{\rho^2 + 2a_3 b_3}\,.\end{aligned} \tag{3.85}$$

Notice that dependence on the arbitrary gauge parameter ν completely disappeared. In the asymptotic region $\rho \to \infty$ we obtain the linear dilaton background times $\mathbb{S}^3 \times \mathbb{S}^1$

$$ds_6^2 = k\,\ell_s^2 \left(-dt^2 + \frac{d\rho^2}{\rho^2} + ds_{\mathbb{S}^3}^2 + d\hat{x}'^2 \right), \tag{3.86}$$

where the $\mathrm{SU}(2) \sim \mathbb{S}^3$ metric is given by[17]

$$ds_{\mathbb{S}^3}^2 = d\hat{\alpha}'^2 + d\hat{\beta}'^2 + d\theta'^2 + 2\cos(2\theta') d\hat{\alpha}' d\hat{\beta}'\,. \tag{3.88}$$

[17] By performing the change of coordinates $\hat{\alpha}' = (\phi_2 - \phi_1)/2$, $\hat{\beta}' = (\phi_2 + \phi_1)/2$, we can bring this metric to the Hopf form

$$ds_{\mathbb{S}^3}^2 = d\theta'^2 + \sin^2(\theta')\, d\phi_1^2 + \cos^2(\theta')\, d\phi_2^2\,. \tag{3.87}$$

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By picking a new radial coordinate χ we can eliminate the $\mathcal{G}_{\chi\theta}$ off-diagonal term in the metric tensor. For instance, let us choose

$$\chi = \rho^2 - \rho_+^2 + a_1 b_1 \,. \tag{3.89}$$

The new radial coordinate in the domain of parameters (3.71) is then defined on the interval $\chi \in [\chi_+, +\infty)$, where

$$\chi_\pm = \pm a_1 b_1 \,, \tag{3.90}$$

and χ_+ represents the outer black hole horizon. The interval in the new coordinates is given by

$$\begin{aligned} \frac{1}{k\ell_s^2}\, ds_6^2 = &-\frac{(\chi-\chi_+)(\chi-\chi_-)}{\ell^2}\, dt^2 + \frac{d\chi^2}{4(\chi-\chi_+)(\chi-\chi_-)} + d\theta'^2 \\ &+ \frac{\mathcal{G}_{ab}}{k}\, (d\hat{X}^a + \mathcal{R}^a\, dt)\, (d\hat{X}^b + \mathcal{R}^b\, dt) \,, \end{aligned} \tag{3.91}$$

where we defined

$$\ell^2 = (\chi+1)^2 - (1-a_1^2)(1-b_1^2) \,. \tag{3.92}$$

In the new coordinates we can also express

$$\begin{aligned} \mathcal{R}^{\hat{\alpha}'} &= -\frac{a_2\,(\chi + b_1^2)}{\ell^2} \,, \\ \mathcal{R}^{\hat{\beta}'} &= \frac{b_2\,(\chi + a_1^2)}{\ell^2} \,, \\ \mathcal{R}^{\hat{x}'} &= \frac{a_1^2 b_3 - b_1^2 a_3 + (b_3 - a_3)\,\chi}{\ell^2} \,, \end{aligned} \tag{3.93}$$

and[18]

$$\frac{\mathcal{G}_{ab}}{k} = \begin{pmatrix} 1 - \frac{2a_2b_2\cos(2\theta')}{\ell_0^2} & \frac{\cos(2\theta')(a_3b_3+\chi+1)-a_2b_2}{\ell_0^2} & -\frac{a_2b_3+a_3b_2\cos(2\theta')}{\ell_0^2} \\ \frac{\cos(2\theta')(a_3b_3+\chi+1)-a_2b_2}{\ell_0^2} & 1 - \frac{2a_2b_2\cos(2\theta')}{\ell_0^2} & -\frac{a_2b_3\cos(2\theta')+a_3b_2}{\ell_0^2} \\ -\frac{a_2b_3+a_3b_2\cos(2\theta')}{\ell_0^2} & -\frac{a_2b_3\cos(2\theta')+a_3b_2}{\ell_0^2} & 1 - \frac{2a_3b_3}{\ell_0^2} \end{pmatrix} , \tag{3.94}$$

where we denoted

$$\ell_0^2 = \chi + 1 + a_3 b_3 + a_2 b_2 \cos(2\theta') \,. \tag{3.95}$$

The metric (3.91) defines a rotating black hole with the event horizon at $\chi \equiv \chi_+$, $\theta' \equiv$ const, $d\hat{X}^a = -\mathcal{R}^a dt$, $a = \hat{\alpha}'$, $\hat{\beta}'$, $\hat{x}'$. The latter indicates that the angular velocity at the horizon is given by[19]

$$\Omega^a = -\sqrt{k}\, \mathcal{R}^a|_{\chi=\chi_+} \,, \qquad a = \hat{\alpha}',\ \hat{\beta}',\ \hat{x}' \,. \tag{3.96}$$

Using (3.93) we then obtain

$$\Omega^{\hat{\alpha}'} = \frac{\sqrt{k}\, a_2 b_1}{a_1 + b_1} \,, \qquad \Omega^{\hat{\beta}'} = -\frac{\sqrt{k}\, a_1 b_2}{a_1 + b_1} \,, \qquad \Omega^{\hat{x}'} = \sqrt{k}\, \frac{a_3 b_1 - a_1 b_3}{a_1 + b_1} \,. \tag{3.97}$$

[18] In Hopf coordinates (3.87) we have $\mathcal{G}_{\phi_1\phi_2} = 0$.

[19] Here the factor of $\sqrt{k}$ originates from the fact that our time coordinate t is normalized so that the asymptotic behavior of the metric is $ds^2 = -k\,\ell_s^2\, dt^2 + \ldots$

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These can be further split into the left- and right-moving components

$$\begin{aligned}
\Omega_L^{\hat{\alpha}'} &= \frac{\sqrt{k}\, a_2 b_1}{a_1+b_1}\,, \quad \Omega_R^{\hat{\alpha}'} = 0\,, \quad \Omega_L^{\hat{\beta}'} = 0\,, \quad \Omega_R^{\hat{\beta}'} = -\frac{\sqrt{k}\, a_1 b_2}{a_1+b_1}\,, \\
\Omega_L^{\hat{x}'} &= \frac{\sqrt{k}\, a_3 b_1}{a_1+b_1}\,, \quad \Omega_R^{\hat{x}'} = -\frac{\sqrt{k}\, a_1 b_3}{a_1+b_1}\,,
\end{aligned} \tag{3.98}$$

so that

$$\Omega^{\hat{\alpha}'} = \Omega_L^{\hat{\alpha}'} + \Omega_R^{\hat{\alpha}'}\,, \qquad \Omega^{\hat{\beta}'} = \Omega_L^{\hat{\beta}'} + \Omega_R^{\hat{\beta}'}\,, \qquad \Omega^{\hat{x}'} = \Omega_L^{\hat{x}'} + \Omega_R^{\hat{x}'}\,. \tag{3.99}$$

The full metric can also be written as (recall that ℓ_0 is given by (3.95))[20]

$$\frac{\mathcal{G}_{\mu\nu}}{k} = \begin{pmatrix}
-1+\frac{2}{\ell_0^2} & 0 & \frac{b_2\cos(2\theta')-a_2}{\ell_0^2} & 0 & \frac{b_2-a_2\cos(2\theta')}{\ell_0^2} & \frac{b_3-a_3}{\ell_0^2} \\
0 & \frac{1}{4(\chi^2-a_1^2b_1^2)} & 0 & 0 & 0 & 0 \\
\frac{b_2\cos(2\theta')-a_2}{\ell_0^2} & 0 & 1-\frac{2a_2b_2\cos(2\theta')}{\ell_0^2} & 0 & \frac{\cos(2\theta')(a_3b_3+\chi+1)-a_2b_2}{\ell_0^2} & -\frac{a_2b_3+a_3b_2\cos(2\theta')}{\ell_0^2} \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{b_2-a_2\cos(2\theta')}{\ell_0^2} & 0 & \frac{\cos(2\theta')(a_3b_3+\chi+1)-a_2b_2}{\ell_0^2} & 0 & 1-\frac{2a_2b_2\cos(2\theta')}{\ell_0^2} & -\frac{a_2b_3\cos(2\theta')+a_3b_2}{\ell_0^2} \\
\frac{b_3-a_3}{l^2} & 0 & -\frac{a_2b_3+a_3b_2\cos(2\theta')}{\ell_0^2} & 0 & -\frac{a_2b_3\cos(2\theta')+a_3b_2}{\ell_0^2} & 1-\frac{2a_3b_3}{\ell_0^2}
\end{pmatrix} \tag{3.100}$$

The dilaton is given by

$$\Phi = \Phi_0 - \log \ell_0\,. \tag{3.101}$$

The Ricci scalar is given by

$$kR = 2\frac{3\chi^2+10(1+a_3b_3)(\chi-a_2b_2\cos(2\theta'))+7(a_1^2(1-b_3^2)-b_2^2(1-a_3^2))-3a_2^2b_2^2\cos(2\theta')^2}{\ell_0^4}\,. \tag{3.102}$$

Finally, for the independent components of the anti-symmetric B-field we obtain

$$\begin{aligned}
\frac{1}{k}\,\mathcal{B}_{t\hat{\alpha}'} &= \frac{a_2b_1\nu}{a_1-b_1\nu} + \frac{b_2\cos(2\theta')+a_2}{\ell_0^2}\,, \\
\frac{1}{k}\,\mathcal{B}_{t\hat{\beta}'} &= -\frac{a_1b_2}{a_1-b_1\nu} + \frac{a_2\cos(2\theta')+b_2}{\ell_0^2}\,, \\
\frac{1}{k}\,\mathcal{B}_{t\hat{x}'} &= \frac{a_3b_1\nu-a_1b_3}{a_1-b_1\nu} + \frac{a_3+b_3}{\ell_0^2}\,, \\
\frac{1}{k}\,\mathcal{B}_{\hat{\alpha}'\hat{\beta}'} &= -\frac{a_2b_2+\cos(2\theta')(a_3b_3+\chi+1)}{\ell_0^2}\,, \\
\frac{1}{k}\,\mathcal{B}_{\hat{\alpha}'\hat{x}'} &= \frac{a_3b_2\cos(2\theta')-a_2b_3}{\ell_0^2}\,, \\
\frac{1}{k}\,\mathcal{B}_{\hat{\beta}'\hat{x}'} &= \frac{a_3b_2-a_2b_3\cos(2\theta')}{\ell_0^2}\,.
\end{aligned} \tag{3.103}$$

Notice that the unphysical ν-dependent terms can be readily eliminated using the gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$, for the following choice of transformation parameters

$$\Lambda_{\hat{\alpha}'} = -k\,\frac{a_2b_1\nu}{a_1-b_1\nu}\,t\,, \quad \Lambda_{\hat{\beta}'} = k\frac{a_1b_2}{a_1-b_1\nu}\,t\,, \quad \Lambda_{\hat{x}'} = -k\,\frac{a_3b_1\nu-a_1b_3}{a_1-b_1\nu}\,t\,, \tag{3.104}$$

[20] Setting $a_1 = \cos\psi$, $a_3 = \sin\psi$, $b_1 = 1$, $a_2 = b_2 = b_3 = 0$, changing the radial coordinate to $\chi = 2\,e^{2r}\cos\psi - 1$, and performing Kaluza-Klein reduction of the x coordinate we recover the metric of two-dimensional charged black hole in string frame, $ds^2 = k\ell_s^2\left(-f(r)dt^2 + dr^2/f(r) + ds^2_{\mathbb{S}^3}\right)$, where $f(r) = 1-e^{-2r}\sec\psi+\frac{1}{4}e^{-4r}\tan^2\psi$. The dilaton is then given by $\Phi = \Phi_0 - \frac{1}{2}\log(2\cos\psi) - r$, in agreement with [30].

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with the other components being trivial, $\Lambda_i = 0$, $i \neq \hat{\alpha}'$, $\hat{\beta}'$, $\hat{x}'$.[21] Finally, anomaly-free conditions (3.71) need to be imposed on the derived expressions (3.100), (3.101), (3.103) for the metric, dilaton, and the B-field.

3.3 Conserved charges

The black brane background (3.100), (3.101), (3.103) possess symmetries associated with translations in time t, angular coordinates $\hat{\alpha}'$, $\hat{\beta}'$ of the $\mathbb{S}^3$ and compact coordinate $\hat{x}'$ of the $\mathbb{S}^1$. In this subsection, we are going to derive the corresponding expressions for the conserved energy and angular momenta.

Angular momenta components are defined via the 1-forms

$$k^{G(a)} = \mathcal{G}_{a\mu}\, dX^\mu\,, \qquad k^{B(a)} = \mathcal{B}_{a\mu}\, dX^\mu\,, \qquad a = \hat{\alpha}',\ \hat{\beta}',\ \hat{x}'\,, \tag{3.105}$$

by the Komar formula (see, e.g., [46, 47]) in string frame

$$\mathcal{J}_a^{(G,B)} = \frac{1}{\kappa_0^2}\, V_{\mathbb{T}^4}\, \ell_s^3 \int \star(dk^{G,B(a)})\, e^{-2\phi}\,, \qquad a = \hat{\alpha}',\ \hat{\beta}',\ \hat{x}'\,, \tag{3.106}$$

as the linear combinations

$$\mathcal{J}_a^{(L,R)} = \frac{\mathcal{J}_a^{(G)} \pm \mathcal{J}_a^{(B)}}{2}\,. \tag{3.107}$$

In (3.106) the star denotes the usual Hodge dual of a p-form,

$$\begin{aligned} dk^{G,B(a)} &= k^{G,B(a)}_{bc}\, d\hat{X}^b \wedge d\hat{X}^c\,, \quad k^{G,B(a)}_{bc} = \partial_{[b} k^{G,B(a)}_{c]} = \frac{1}{2}\left(\partial_b(\mathcal{G},\mathcal{B})_{ca} - \partial_c(\mathcal{G},\mathcal{B})_{ba}\right), \\ \star(dk^{G,B(a)}) &= \frac{1}{4!}\sqrt{\det\mathcal{G}}\,\epsilon_{a_1a_2a_3a_4a_5a_6}\,\mathcal{G}^{a_5b_5}\mathcal{G}^{a_6b_6}\,k^{(a)G,B}_{b_5b_6}\, dX^{a_1}\wedge dX^{a_2}\wedge dX^{a_3}\wedge dX^{a_4}\,, \end{aligned} \tag{3.108}$$

and $\epsilon_{\tau\chi\hat{\alpha}'\theta'\hat{\beta}'\hat{x}'} = 1$. In (3.106) the integral is performed over the asymptotic boundary $\mathbb{S}^3 \times \mathbb{S}^1$, which selects the corresponding component $(\hat{\alpha}'\theta'\hat{\beta}'\hat{x}')$ of the 4-form $\star(dk_a)$. We then obtain

$$\begin{aligned} \mathcal{J}^G_{\hat{\alpha}'} &= \mathcal{J}^B_{\hat{\alpha}'} = \frac{4\pi^2}{\kappa^2}\, k\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, a_2\,, \\ \mathcal{J}^G_{\hat{\beta}'} &= -\mathcal{J}^B_{\hat{\beta}'} = -\frac{4\pi^2}{\kappa^2}\, k\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, b_2\,, \\ \mathcal{J}^G_{\hat{x}'} &= \frac{4\pi^2}{\kappa^2}\, k\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, (a_3 - b_3)\,, \qquad \mathcal{J}^B_{\hat{x}'} = \frac{4\pi^2}{\kappa^2}\, k\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, (a_3 + b_3)\,. \end{aligned} \tag{3.109}$$

For the energy/mass we can analogously use the Komar formula

$$\mathcal{E} = -\frac{1}{2\kappa_0^2}\, V_{\mathbb{T}^4}\, \ell_s^3 \int \star(dk^{G(t)})\, e^{-2\phi}\,, \tag{3.110}$$

where $k^{G(t)} = \mathcal{G}_{t\mu}\, dX^\mu$, which gives[22,23]

$$\mathcal{E} = \frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}}\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\,. \tag{3.111}$$

[21] In general, adding constants to the components of the B-field cannot affect supergravity equations of motion, that depend on the B-field via the tensor $H = dB$.

[22] Here the factor of $\sqrt{k}$ is due to our choice of normalization of the time coordinate, which is dual to the energy $\mathcal{E}$, according to which $ds^2 = -k\,\ell_s^2\, dt^2 + \dots$ in the asymptotic region.

[23] Recall that relation between κ and κ_0 is given by (3.66).

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In the next section, we will demonstrate that the derived expressions for energy and angular momenta are consistent with the first law of thermodynamics.

4 Thermodynamics

In this section, we are going to study thermodynamics of the background (3.100), (3.101), (3.103). This background describes a black brane, rotating with non-vanishing angular velocities along two of the $\mathbb{S}^3$ as well as the $\mathbb{S}^1$ directions. We are going to determine the temperature and derive the associated Bekenstein-Hawking entropy. Using expressions for the energy, angular velocities, and angular momenta calculated in the previous section, we will derive the free energy, and demonstrate explicitly the first law of thermodynamics. Additionally, by explicit calculation of the regularized on-shell action, we will demonstrate that the free energy vanishes, in agreement with the general expectation for an asymptotically linear dilaton space-time.

The near-horizon limit of the metric (3.91) can be taken by expanding

$$\chi = \chi_+ \left(1 + r^2\right). \tag{4.1}$$

around $r = 0$. This gives

$$ds_6^2 = \ell_s^2 \left(k \left(\frac{1}{2} \left(dr^2 - \frac{4 a_1^2 b_1^2}{(a_1+b_1)^2} r^2 dt^2 \right) + d\theta'^2 \right) + \mathcal{G}_{ab} \left(d\hat{X}^a + \mathcal{R}^a\, dt \right) \left(d\hat{X}^b + \mathcal{R}^b\, dt \right) \right). \tag{4.2}$$

The null interval at the horizon is then defined by

$$\chi = \chi_+ \,, \qquad d\theta' = 0 \,, \qquad d\hat{X}^a = -\mathcal{R}^a dt \,, \tag{4.3}$$

while regularity at the horizon requires that Euclidean time has the period expressed in terms of the black hole temperature as $\tau \sim \tau + 1/T$. Recall that our Lorentzian time coordinate t is normalized so that the asymptotic metric at the boundary is determined by $ds^2 = -k\, \ell_s^2\, dt^2 + \ldots$. Then the temperature is given by

$$T = \frac{1}{\pi \ell_s \sqrt{k}} \frac{a_1 b_1}{a_1 + b_1} \,. \tag{4.4}$$

The temperature (4.4) takes values in the range $0 \leq T \leq T_H$, where $T_H = \frac{1}{2\pi\sqrt{k}\,\ell_s}$ is the Hagedorn temperature of two-dimensional black hole. The zero-temperature limit, $T = 0$, is achieved for the choice of parameters $a_1 b_1 = 0$, in which case we obtain $\chi_+ = \chi_-$. In the limit $a_1 \to 1$, $b_1 \to 1$ we get $T \to T_H$, which corresponds to sending the outer horizon to asymptotic linear dilaton region of space-time.

At the outer horizon we obtain for the Einstein frame metric

$$\tilde{V}_{\mathbb{T}^4} \sqrt{\det ||\tilde{\mathcal{G}}_{ij}||} = e^{2(\Phi_0 - \Phi)}\, V_{\mathbb{T}^4} \sqrt{\det ||\mathcal{G}_{ij}||} \Big|_{\chi=\chi_+} = k^2 \ell_s^4 \left(a_1 + b_1\right) V_{\mathbb{T}^4}\, \sin(2\theta') \,, \tag{4.5}$$

where $i, j = \hat{\alpha}', \theta', \hat{\beta}', \hat{x}'$, and $\mathcal{G}_{ij}$ is projection of the metric on the associated sub-space, and we substituted expression (3.101) for the dilaton. The horizon area, obtained by integrating over $\theta' \in [0, \frac{\pi}{2}]$, $\alpha' \in [0, \pi]$, $\beta' \in [0, 2\pi]$ is therefore given by

$$\mathcal{A} = 2\pi^2\, k^2 \ell_s^3 \left(a_1 + b_1\right) V_{\mathbb{S}^1}\, V_{\mathbb{T}^4} \,, \tag{4.6}$$

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where we also denoted dimensionful length of the $\mathrm{U}(1)_x$ circle (in string frame) as

$$V_{\mathbb{S}^1} = 2\pi \ell_s R_x \,, \tag{4.7}$$

following from the periodicity condition for the dimensionless coordinate $x \sim x + 2\pi R_x$. The Bekenstein-Hawking entropy is therefore given by[24]

$$\mathcal{S} = \frac{\mathcal{A}}{4G_N} = \frac{2\pi \mathcal{A}}{\kappa^2} = \frac{4\pi^2}{\kappa^2}\,\frac{1}{T}\,k^{\frac{3}{2}}\ell_s^2\,V_{\mathbb{S}^1}\,V_{\mathbb{T}^4}\,a_1 b_1 \,. \tag{4.10}$$

Now let us proceed to calculation of the Euclidean on-shell action

$$S = S_0 + S_{\mathrm{GHY}} + S_{\mathrm{c.t.}} \,, \tag{4.11}$$

where S_0 is obtained by plugging in the background solution into the bulk action (3.60), S_{GHY} is the Gibbons-Hawking-York (GHY) boundary term, and $S_{\mathrm{c.t.}}$ is the counter-term action. Let us start with the former. Using (3.58) we can re-write (3.60) on shell as[25]

$$\begin{aligned} S_0 &= -\frac{2}{\kappa_0^2}\,\ell_s^4\,V_{\mathbb{T}^4}\int d^6x\,\sqrt{\det\mathcal{G}}\,e^{-2\Phi}\,\left(\nabla^\mu\partial_\mu\Phi - 2\,\partial_\mu\Phi\partial^\mu\Phi\right) \\ &= -\frac{2}{\kappa_0^2}\,\ell_s^4\,V_{\mathbb{T}^4}\int d^6x\,\partial_\mu\left(\sqrt{\det\mathcal{G}}\,e^{-2\Phi}\,\partial^\mu\Phi\right) \,. \end{aligned} \tag{4.12}$$

Using the Stokes's theorem, we obtain

$$S_0 = -\frac{2}{\kappa_0^2}\,\ell_s^4\,V_{\mathbb{T}^4}\int d^5x\,\sqrt{\det\gamma}\,e^{-2\Phi}\,n_\mu\,\partial^\mu\Phi \,, \tag{4.13}$$

where

$$\gamma_{ij} = \mathcal{G}_{ij} \,, \qquad i,j = t,\hat{\alpha}',\theta',\hat{\beta}',\hat{x}' \,, \tag{4.14}$$

is the metric on the boundary $\chi = \Lambda$,[26] and n_μ, given by (B.7), is the outward-pointing unit vector normal to the boundary. We can then re-write (4.13) as

$$S_0 = -\frac{2}{\kappa_0^2}\,\ell_s^4\,V_{\mathbb{T}^4}\int_0^{1/(\sqrt{k}\,\ell_s\,T)} d\tau \int d^4x\,\sqrt{\det\mathcal{G}}\,e^{-2\Phi}\,\mathcal{G}^{\chi\chi}\,\partial_\chi\Phi\Big|_{\chi=\Lambda} \,, \tag{4.15}$$

where, as before, $\det\mathcal{G}$ is determinant of the six-dimensional metric. Using (3.100), (3.101), we obtain for any χ,

$$\sqrt{\det\mathcal{G}}\,e^{-2\Phi} = \frac{k^3}{2}\,\sin(2\theta')\,e^{-2\Phi_0} \,. \tag{4.16}$$

[24] One can parametrize the asymmetric gauging of the U(1) sub-group as (see [32] and references therein)

$$a_1 = \cos(\chi - \psi) \,, \quad b_1 = \cos\chi \,, \quad a_2 = b_2 = 0 \,, \tag{4.8}$$

i.e., decoupling the SU(2) sub-group. Taking the same limit in (4.10), we can compare it with the entropy of rotating black string following from the expressions for the metric $\mathcal{G}_{xx}$ and dilaton $\Phi - \Phi_0$ given by eq. (3.36) in [32]. It suffices to point out that the horizon area is given by

$$\mathcal{A} = e^{2(\Phi_0-\Phi)}\,2\pi^2\,k^2\ell_s^3\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\,\sqrt{\mathcal{G}_{xx}} = \sqrt{(1+\cos\psi)(1+\cos(2\chi-\psi))} \,, \tag{4.9}$$

that after a simple manipulation one can see to coincide with (4.6) for the parameters (4.8).

[25] In the second line we have applied $\sqrt{\det\mathcal{G}}\,\nabla_\mu A^\mu = \partial_\mu\left(\sqrt{\det\mathcal{G}}\,A^\mu\right)$ to $A^\mu = \partial^\mu(e^{-2\Phi})$.

[26] The corresponding contribution from the lower limit of integration $\chi = \chi_+$ vanishes.

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Combining this with

$$\mathcal{G}^{\chi\chi}\,\partial_\chi\Phi|_{\chi=\Lambda} = -2\Lambda + 2(1 + a_2 b_2\cos(2\theta') + a_3 b_3) + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{4.17}$$

and integrating over $\theta' \in [0, \pi/2]$, etc., we obtain

$$S_0 = \frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}} \ell_s^2\, \frac{1}{T}\, V_{\mathbb{S}^1}\, V_{\mathbb{T}^4}\, (\Lambda - 1 - a_3 b_3)\,. \tag{4.18}$$

We can remove the divergence with the following counter-term,

$$S_{\text{c.t.}} = -\frac{2}{\kappa_0^2}\,\ell_s^4\, V_{\mathbb{T}^4} \int_{\chi=\Lambda} d^5x\,\sqrt{\det\gamma}\, e^{-2\Phi} = -\frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}} \ell_s^2\, \frac{1}{T}\, V_{\mathbb{S}^1}\, V_{\mathbb{T}^4}\, \Lambda \tag{4.19}$$

without affecting the finite part.

The GHY term at the boundary $\chi = \Lambda$ in string frame is given by (B.10),

$$S_{\text{GHY}} = \frac{1}{\kappa_0^2}\,\ell_s^4\, V_{\mathbb{T}^4} \int_{\chi=\Lambda} d^5x\,\sqrt{\det\gamma}\, e^{-2\Phi}\, K\,. \tag{4.20}$$

In (4.20) we used the induced metric (4.14) on the boundary $\chi = \Lambda$. At the same time, $K = \nabla_\mu n^\mu$ is trace of the extrinsic curvature tensor, defined by the normalized outward-pointing unit vector (B.7) normal to the surface $\chi = \Lambda$. We then evaluate

$$\sqrt{\det\gamma}\, K = 2k^2 \left(\chi - \frac{(\chi-\chi_+)(\chi-\chi_-)}{\ell_0^2}\right) \sin(2\theta')\,, \tag{4.21}$$

that finally gives

$$S_{\text{GHY}} = \frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}} \ell_s^2\, \frac{1}{T}\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, (1 + a_3 b_3)\,. \tag{4.22}$$

The free energy

$$\mathcal{F} = T\,(S_0 + S_{\text{GHY}} + S_{\text{c.t.}}) \tag{4.23}$$

can be calculated by combing (4.18), (4.19), (4.22), giving

$$\mathcal{F} = 0\,, \tag{4.24}$$

consistently with the Hagedorn growth of density of states, and in agreement with analogous result for the two-dimensional charged black hole [48–50].

On the other hand, the free energy is given by

$$\mathcal{F} = \mathcal{E} - T\,\mathcal{S} - \Omega\cdot\mathcal{J}\,, \tag{4.25}$$

where we introduced short-hand notation

$$\Omega\cdot\mathcal{J} = \Omega_L^{\hat\alpha'}\,\mathcal{J}^L_{\hat\alpha'} + \Omega_R^{\hat\alpha'}\,\mathcal{J}^R_{\hat\alpha'} + \Omega_L^{\hat\beta'}\,\mathcal{J}^L_{\hat\beta'} + \Omega_R^{\hat\beta'}\,\mathcal{J}^R_{\hat\beta'} + \Omega_L^{\hat x'}\,\mathcal{J}^L_{\hat x'} + \Omega_R^{\hat x'}\,\mathcal{J}^R_{\hat x'}\,, \tag{4.26}$$

where components of the angular velocity have been derived in (3.98), and angular momenta are given by (3.109). Using these expressions and taking into account the anomaly-cancellation constraints, we derive

$$\Omega\cdot\mathcal{J} = \frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}}\, \ell_s^2\, V_{\mathbb{T}^4}\, V_{\mathbb{S}^1}\, (1 - a_1 b_1)\,. \tag{4.27}$$

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Combining expressions for the entropy (4.10), angular velocities and momenta (4.27), and mass (3.111) in the expression for the free energy (4.25) we can explicitly reproduce that the free energy vanishes, in agreement with (4.24).

We can also verify that the first law of thermodynamics is satisfied,

$$d\mathcal{E} - T\, d\mathcal{S} - \Omega_L^{\hat{\alpha}'}\, d\mathcal{J}^L_{\hat{\alpha}'} - \Omega_R^{\hat{\beta}'}\, d\mathcal{J}^R_{\hat{\beta}'} - \Omega_L^{\hat{x}'}\, d\mathcal{J}^L_{\hat{x}'} - \Omega_R^{\hat{x}'}\, d\mathcal{J}^R_{\hat{x}'} = 0\,. \tag{4.28}$$

This identity can be demonstrated for arbitrary $da_{1,2}$, $db_{1,2}$ by substituting $\Omega_R^{\hat{\alpha}'} = \Omega_L^{\hat{\beta}'} = 0$, using $d\mathcal{E} = 0$, and taking into account that anomaly cancellation conditions imply

$$da_3 = -\frac{a_1 da_1 + a_2 da_2}{a_3}\,, \qquad db_3 = -\frac{b_1 db_1 + b_2 db_2}{b_3}\,. \tag{4.29}$$

Lastly, we notice that the four independent parameters defining the gauging of the U(1) sub-group that remain after the anomaly-cancellation conditions have been imposed on $a_{1,2,3}$, $b_{1,2,3}$, can be expressed in terms of the temperature and angular momenta.

5 Two-charge background

In section 3.2, we derived the metric, dilaton, and B-field ten-dimensional background configuration (3.100), (3.101), (3.103) for string theory on the coset space (3.1). This background possesses an event horizon, that rotates with the angular velocities given by (3.97) in the directions of $\hat{\alpha}'$, $\hat{\beta}'$, $\hat{x}'$. It asymptotes to a static linear dilaton two-dimensional space-time times $\mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{T}^4$.

In this section, we are going to construct the nine-dimensional background obtained from (3.100), (3.101), (3.103) by compactifying the $\mathrm{U}(1)_x$ coordinate $\hat{x}'$. To be precise, due to the non-vanishing angular velocity $\Omega^{\hat{x}'}$ at the horizon, we first need to perform a change of coordinates $\hat{x}' \to \tilde{x}'$, defined by[27]

$$\hat{x}' = \tilde{x}' + \Omega^{\hat{x}'}\, t/\sqrt{k}\,, \tag{5.1}$$

such that the new coordinate $\tilde{x}'$ is not rotating at the horizon. By performing the KK reduction of $\tilde{x}'$, from the ten-dimensional background $\mathcal{G}_{MN}$, $\mathcal{B}_{MN}$, Φ, we can derive the nine-dimensional background

$$\begin{aligned} \mathrm{G}_{mn} &= \mathcal{G}_{mn} - \frac{\mathcal{G}_{\tilde{x}'m}\,\mathcal{G}_{\tilde{x}'n}}{\mathcal{G}_{\tilde{x}'\tilde{x}'}}\,, \quad \mathrm{B}_{mn} = \mathcal{B}_{mn}\,, \qquad \Phi_9 = \Phi - \frac{1}{4}\log(\mathcal{G}_{\tilde{x}'\tilde{x}'})\,, \\ \mathrm{A}_m &= \frac{\mathcal{G}_{\tilde{x}'m}}{\mathcal{G}_{\tilde{x}'\tilde{x}'}}\,, \qquad\qquad \mathrm{B}_m = \frac{1}{k}\,\mathcal{B}_{\tilde{x}'m}\,, \end{aligned} \tag{5.2}$$

where we chose indices m, n to label coordinates of the nine-dimensional non-compact background. This background features two U(1) gauge fields A_m, B_m, that are sourced by the corresponding charges Q_A, Q_B at the chemical potentials μ_A, μ_B that we will calculate below.

[27] An extra factor of $1/\sqrt{k}$ is due to our conventions of separating an explicit common factor of k in expression for the interval ds^2.

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We can write the five-dimensional coset sub-space metric of the nine-dimensional background in the form analogous to (3.91),

$$\begin{aligned}\frac{1}{k\ell_s^2}\,ds_5^2 = &-\frac{(\chi-\chi_+)(\chi-\chi_-)}{\ell^2}\,dt^2 + \frac{d\chi^2}{4(\chi-\chi_+)(\chi-\chi_-)} + d\theta'^2\\ &+\frac{\mathrm{G}_{rs}}{k}\,(d\tilde{X}^r + \tilde{\mathcal{R}}^r\,dt)\,(d\tilde{X}^s + \tilde{\mathcal{R}}^s\,dt)\,,\end{aligned} \tag{5.3}$$

where indices r, s take values $\hat{\alpha}', \hat{\beta}'$, the two-dimensional metric G_{rs} is given by

$$\begin{aligned}\mathrm{G}_{\hat{\alpha}'\hat{\alpha}'} &= \frac{\ell^2 + b_2^2(1-a_1^2)\sin(2\theta')^2}{\ell_0^2(\ell_0^2 - 2a_3b_3)}\,,\\ \mathrm{G}_{\hat{\alpha}'\hat{\beta}'} &= \frac{\ell^2\cos(2\theta') - a_2b_2(\chi+1)\sin(2\theta')^2}{\ell_0^2(\ell_0^2 - 2a_3b_3)}\,,\\ \mathrm{G}_{\hat{\beta}'\hat{\beta}'} &= \frac{\ell^2 + a_2^2(1-b_1^2)\sin(2\theta')^2}{\ell_0^2(\ell_0^2 - 2a_3b_3)}\,,\end{aligned} \tag{5.4}$$

and we have defined

$$\tilde{\mathcal{R}}^{\hat{\alpha}'} = -\frac{a_2(\chi+b_1^2)}{\ell^2}\,, \qquad \tilde{\mathcal{R}}^{\hat{\beta}'} = \frac{b_2(\chi+a_1^2)}{\ell^2}\,. \tag{5.5}$$

In the asymptotic region $\chi \to \infty$ we have a static background, $\tilde{\mathcal{R}}^r = \mathcal{O}(1/\chi)$, while the metric G_{rs} approaches

$$\mathrm{G}_{rs} = \begin{pmatrix} 1 & \cos(2\theta') \\ \cos(2\theta') & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\chi}\right)\,. \tag{5.6}$$

The nine-dimensional background then asymptotes to a static linear dilaton two-dimensional space-time times $\mathbb{S}^3 \times \mathbb{T}^4$.

We can calculate angular velocities at the horizon according to $\Omega^r = -\sqrt{k}\,\tilde{\mathcal{R}}^r|_{\chi=\chi_+}$, $r = \hat{\alpha}', \hat{\beta}'$, analogously to (3.96), yielding

$$\Omega^{\hat{\alpha}'} = \frac{\sqrt{k}\,a_2b_1}{a_1+b_1}\,, \qquad \Omega^{\hat{\beta}'} = -\frac{\sqrt{k}\,a_1b_2}{a_1+b_1}\,. \tag{5.7}$$

in agreement with (3.97). We can determine the corresponding angular momenta $\mathcal{J}_r^{(G,B)}$, $r = \hat{\alpha}', \hat{\beta}'$ using (3.106) applied to the nine-dimensional background, rendering

$$\begin{aligned}\mathcal{J}^G_{\hat{\alpha}'} = \mathcal{J}^B_{\hat{\alpha}'} &= \frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,a_2\,,\\ \mathcal{J}^G_{\hat{\beta}'} = -\mathcal{J}^B_{\hat{\beta}'} &= -\frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,b_2\,,\end{aligned} \tag{5.8}$$

in agreement with (3.109).

One can see that at the horizon the time-like components of the gauge fields in co-rotating reference frame vanish, $\tilde{\mathrm{A}}_t|_{\chi=\chi_+} = 0$, $\tilde{\mathrm{B}}_t|_{\chi=\chi_+} = 0$, up to a gauge transformation. Recall that since the horizon is rotating in the directions of $\hat{\alpha}'$, $\hat{\beta}'$, we actually need to calculate gauge fields in the co-rotating coordinates

$$\hat{\alpha}' = \tilde{\alpha}' + \Omega^{\hat{\alpha}'}\,t/\sqrt{k}\,, \qquad \hat{\beta}' = \tilde{\beta}' + \Omega^{\hat{\beta}'}\,t/\sqrt{k}\,, \tag{5.9}$$

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resulting in the transformation laws

$$\tilde{\mathrm{A}}_t = \mathrm{A}_t + \Omega^{\hat{\alpha}'} A_{\hat{\alpha}'} + \Omega^{\hat{\beta}'} A_{\hat{\beta}'}\,, \qquad \tilde{\mathrm{B}}_t = \mathrm{B}_t + \Omega^{\hat{\alpha}'} B_{\hat{\alpha}'} + \Omega^{\hat{\beta}'} B_{\hat{\beta}'}\,. \tag{5.10}$$

This gives

$$\begin{aligned} \tilde{\mathrm{A}}_t &= \frac{(a_3 b_1 - a_1 b_3)(\chi - \chi_+)}{(a_1 + b_1)(\ell_0^2 - 2a_3 b_3)}\,, \\ \tilde{\mathrm{B}}_t &= \frac{(a_3 b_1 + a_1 b_3)(\chi - \chi_+)}{(a_1 + b_1)\ell_0^2}\,, \end{aligned} \tag{5.11}$$

where we have also performed a constant shift

$$\tilde{\mathrm{B}}_t \to \tilde{\mathrm{B}}_t + \frac{a_3 b_1 + a_1 b_3}{a_1 + b_1}\,. \tag{5.12}$$

The remaining non-trivial components of the gauge fields are given by

$$\begin{aligned} \tilde{\mathrm{A}}_{\hat{\alpha}'} &= -\frac{a_2 b_3 + a_3 b_2 \cos(2\theta')}{\ell_0^2 - 2a_3 b_3}\,, \quad \tilde{\mathrm{A}}_{\hat{\beta}'} = -\frac{a_3 b_2 + a_2 b_3 \cos(2\theta')}{\ell_0^2 - 2a_3 b_3}\,, \\ \tilde{\mathrm{B}}_{\hat{\alpha}'} &= \frac{a_2 b_3 - a_3 b_2 \cos(2\theta')}{\ell_0^2}\,, \quad \tilde{\mathrm{B}}_{\hat{\beta}'} = \frac{-a_3 b_2 + a_2 b_3 \cos(2\theta')}{\ell_0^2}\,. \end{aligned} \tag{5.13}$$

From asymptotic behavior of the gauge potentials we can read off the chemical potential

$$\mu_A = \sqrt{k}\,\frac{a_3 b_1 - a_1 b_3}{a_1 + b_1}\,, \qquad \mu_B = \sqrt{k}\,\frac{a_3 b_1 + a_1 b_3}{a_1 + b_1}\,, \tag{5.14}$$

that we can subsequently split into the left-moving and right-moving components as

$$\mu_L = \frac{\mu_A + \mu_B}{2} = \sqrt{k}\,\frac{a_3 b_1}{a_1 + b_1}\,, \qquad \mu_R = \frac{\mu_A - \mu_B}{2} = -\sqrt{k}\,\frac{a_1 b_3}{a_1 + b_1}\,. \tag{5.15}$$

Expressions for the chemical potential (5.15) agree with their angular velocities counterparts $\Omega^{\hat{x}'}_{L,R}$ before compactification, given by (3.98),

$$\mu_{L,R} = \Omega^{\hat{x}'}_{L,R}\,. \tag{5.16}$$

The corresponding charges $Q_{L,R}$ can be found analogously to [51, 52]. We first calculate

$$Q_{G,B} = -\frac{1}{\kappa_0^2}\,\frac{2\pi^2 \ell_s^2}{\sqrt{k}}\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1} \lim_{\chi\to\infty} \sqrt{-\det \mathrm{G}}\, e^{-2\Phi_9}\, \mathcal{G}_{\hat{x}'\hat{x}'}\, F^{\chi t}_{G,B}\,, \tag{5.17}$$

where the field strength tensors are defined as $F^{rs}_G = \partial^r \mathrm{A}^s - \partial^s \mathrm{A}^r$, $F^{rs}_B = \partial^r \mathrm{B}^s - \partial^s \mathrm{B}^r$, obtaining

$$Q_G = \frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\,(a_3 - b_3)\,, \qquad Q_B = \frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\,(a_3 + b_3)\,. \tag{5.18}$$

Notice that these expressions match the angular momenta $\mathcal{J}^{G,B}_{\hat{x}'}$ of the ten-dimensional background (3.109). The corresponding left- and right-moving charges can then be found according to

$$\begin{aligned} Q_L &= \frac{Q_G + Q_B}{2} = \frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\,a_3\,, \\ Q_R &= \frac{Q_G - Q_B}{2} = -\frac{4\pi^2}{\kappa^2}\,k\,\ell_s^2\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\,b_3\,. \end{aligned} \tag{5.19}$$

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The free energy of the nine-dimensional background is given by expression analogous to its ten-dimensional counterpart (4.25), except that now we have rotations only in $\hat{\alpha}'$, $\hat{\beta}'$ directions, as well as two U(1) charges,

$$\mathcal{F} = \mathcal{E} - T\,\mathcal{S} - \Omega_L^{\hat{\alpha}'}\,\mathcal{J}^L_{\hat{\alpha}'} - \Omega_R^{\hat{\beta}'}\,\mathcal{J}^R_{\hat{\beta}'} - \mu_L\,Q_L - \mu_R\,Q_R\,. \tag{5.20}$$

Here we can calculate

$$\Omega_L^{\hat{\alpha}'}\,\mathcal{J}^L_{\hat{\alpha}'} + \Omega_R^{\hat{\beta}'}\,\mathcal{J}^R_{\hat{\beta}'} + \mu_L\,Q_L + \mu_R\,Q_R = \frac{4\pi^2}{\kappa^2}\,k^{\frac{3}{2}}\,\ell_s^2\,V_{\mathbb{T}^4}\,V_{\mathbb{S}^1}\;\;(1 - a_1 b_1)\;, \tag{5.21}$$

in agreement with (4.27). The values for temperature T, entropy $\mathcal{S}$, energy $\mathcal{E}$, and grand potential $\mathcal{F}$ of the nine-dimensional background are the same as for the ten-dimensional background, and are given by (4.4), (4.10), (4.24), (3.111). Finally, one can explicitly verify that the first law of thermodynamics is satisfied,

$$d\mathcal{E} - T\,d\mathcal{S} - \Omega_L^{\hat{\alpha}'}\,d\mathcal{J}^L_{\hat{\alpha}'} - \Omega_R^{\hat{\beta}'}\,d\mathcal{J}^R_{\hat{\beta}'} - \mu_L\,dQ_L - \mu_R\,dQ_R = 0\,. \tag{5.22}$$

6 Spectrum

In this section, we will discuss construction of spectrum of type-II superstring theory on the coset space (3.1). As we will review below, physical states of superstring theory satisfy the super-Virasoro constraints, ensuring superconformal invariance of quantum theory on the world-sheet. Additionally, physical states of the coset sigma-model satisfy gauge-invariant conditions, that can be formulated via the BRST formalism [53].[28] We will derive the corresponding constraints in section 6.1.

It is important to emphasize that our conventions in this section are to use the coset BRST formalism to impose the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ gauge symmetry in the spectrum of quantum theory. At the same time, we will use the covariant quantization approach (without employing the superconformal BRST formalism) to ensure superconformal invariance of the spectrum in terms of the (super-)Virasoro constraints.

We are mostly interested in the bosonic excitations of the NS-NS sector, in particular, the lowest tachyon state, and the massless states composing the gravity multiplet of the type-II supergravity. Therefore the distinction between type-IIA and type-IIB superstrings, originating from the choice of space-time chirality of the fermionic states in the R-NS and NS-R sectors, will not play a role in most of the subsequent discussion. We will also not consider bosonic excitations coming from the R-R sector, leaving these topics for future work.

6.1 Coset BRST constraints

In this subsection, we are going to derive the coset BRST constraints, that have to be satisfied by the physical states on the coset (3.1). These constraints impose the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry on the states of the physical spectrum of the coset theory. (We stress once again, that in this paper the superconformal BRST formalism is not used to impose superconformal symmetry on the physical states.)

[28] See [31, 32] for a recent discussion.

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To lighten up the discussion, we will begin by performing derivation of the BRST constraints for the six-dimensional coset subspace of the target space-time (3.1), working in the purely bosonic sector. We will subsequently formulate the supersymmetric version of the obtained constraints. Partition function of the gauged WZW model (3.29) is given by

$$Z = \int [dG][dA][d\tilde{A}]\, e^{-S_{\rm gWZW}[G,A,\tilde{A}]}\,. \tag{6.1}$$

Using (3.25), (3.27), we can re-write it as

$$Z = \int [dG][du][dv]\ \det\,\partial\ \det\,\bar{\partial}\, e^{-S'_{\rm WZW}[UGV]+S_{\rm P}[w]}\,, \tag{6.2}$$

where

$$S_{\rm P}[w] = \frac{1}{2\pi}\int d^2z\,\partial w\bar{\partial}w \tag{6.3}$$

is the Polyakov action for the field $w = u - v$. Introducing anti-commuting ghost fields b, c, $\tilde{b}$, $\tilde{c}$, one can express the functional determinants in (6.2) as

$$\det\,\partial\ \det\,\bar{\partial} = \int [Db][Dc][D\tilde{b}][D\tilde{c}]\, e^{-\frac{1}{2\pi}\int d^2z\,(b\bar{\partial}c+\tilde{b}\partial\tilde{c})}\,. \tag{6.4}$$

Ghost fields satisfy the following non-trivial OPEs

$$c(z)b(w) \simeq \frac{1}{z-w}\,, \qquad \tilde{c}(\bar{z})\tilde{b}(\bar{w}) \simeq \frac{1}{\bar{z}-\bar{w}}\,. \tag{6.5}$$

The ghosts b and c are conformal primaries of dimensions 1 and 0 respectively [16, 54],[29] with the following expansions in integer modes,

$$b(z) = \sum_n \frac{b_n}{z^{n+1}}\,, \quad c(z) = \sum_n \frac{c_n}{z^n}\,, \quad \tilde{b}(\bar{z}) = \sum_n \frac{\tilde{b}_n}{\bar{z}^{n+1}}\,, \quad \tilde{c}(\bar{z}) = \sum_n \frac{\tilde{c}_n}{\bar{z}^n}\,. \tag{6.6}$$

Due to the OPEs (6.5), these modes satisfy the anti-commutation relations

$$\{c_n,\, b_m\} = \delta_{mn}\,, \qquad \{\tilde{c}_n,\, \tilde{b}_m\} = \delta_{mn}\,. \tag{6.7}$$

Conformal dimensions of the ghosts can also be reproduced from their the stress-energy tensor,

$$T_{bc}(z) = b(z)\,\partial c(z)\,, \tag{6.8}$$

where the product in the r.h.s. is normally-ordered.

Finally, let us change integration variables $u \to u + w$, $G \to U^{-1}GV^{-1}$ and divide the partition function (6.2) by the constant factor of $\int [Dv]$, rendering

$$Z = \int [dG][dw][Db][Dc][D\tilde{b}][D\tilde{c}]\, e^{-S_{\rm q}}\,, \tag{6.9}$$

[29] As a side note, in the BRST quantization approach to superstring theory one employs the superconformal ghosts comprised of the fermions $\mathtt{b}$, $\mathtt{c}$, and bosons β, γ. Dimensions of these pairs of ghosts are $(2,1)$ and $(3/2,-1/2)$, respectively. In particular, this is reflected in contributions of $\mathtt{b}$, $\mathtt{c}$ to the stress-energy tensor, $T_{\mathtt{bc}} = -2\,\mathtt{b}\,\partial\,\mathtt{c} + \mathtt{c}\,\partial\,\mathtt{b}$, which is manifestly different from its coset BRST ghosts counterpart (6.8). We do not perform superconformal BRST quantization of superstring theory, and therefore the superconformal ghosts $\mathtt{b}$, $\mathtt{c}$, β, γ do not appear in our derivation of the spectrum.

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where the full action of the model is given by

$$S_{\rm q} = S'_{\rm WZW}[G] - S_{\rm P}[w] + \frac{1}{2\pi}\int d^2z\,(b\bar\partial c + \tilde b\partial\tilde c)\,. \tag{6.10}$$

Notice that from the action (6.10) it follows that the two-point correlation function of the field w has the 'wrong' sign

$$\langle\partial w(z_1)\partial w(z_2)\rangle = \frac{1}{2}\,\frac{1}{(z_1-z_2)^2}\,. \tag{6.11}$$

This does not pose a problem, since the physical state (coset BRST) conditions will eliminate the corresponding negative norm states from the spectrum. These conditions are derived as follows. Consider the holomorphic BRST transformation defined by the fermionic parameter η, with the non-trivial variations given by [55]

$$\delta G = i\eta\, c\, T_L G\,, \qquad \delta w = i\eta\, c\,, \qquad \delta b = i\eta\,(J + 2\partial w)\,, \tag{6.12}$$

where the gauge current J is given by (3.28). Using (A.12) we can derive the corresponding variation of the action (6.10) as

$$\begin{aligned}\delta S_{\rm q} &= -\frac{1}{2\pi}\int d^2z\,\left(k\,{\rm Tr}\,\left(P\,\partial G G^{-1}\bar\partial(\delta G G^{-1})\right) + 2\partial w\,\bar\partial\delta w - \delta b\bar\partial c\right)\\ &= -\frac{i}{2\pi}\int d^2z\,\bar\partial\eta\, c\,(J+2\partial w)\,.\end{aligned} \tag{6.13}$$

Analogous calculation can be done for the anti-holomorphic BRST transformation,

$$\delta G = -i\eta\,\tilde c\, G T_R\,, \qquad \delta w = i\eta\,\tilde c\,, \qquad \delta b = i\eta\,(-\tilde J + 2\bar\partial w)\,, \tag{6.14}$$

giving variation of the action

$$\delta S_{\rm q} = -\frac{i}{2\pi}\int d^2z\,\partial\eta\,\tilde c\,\left(-\tilde J + 2\bar\partial w\right)\,. \tag{6.15}$$

For a global parameter η the transformations (6.12), (6.14) are therefore a symmetry of the action (6.10). Choosing a local parameter η we obtain the coset BRST currents $J^{\rm BRST} = c\,J^{(0)}$, $\tilde J^{\rm BRST} = \tilde c\,\tilde J^{(0)}$, where we denoted the corresponding null currents as[30]

$$J^{(0)} = i\,J + 2j^w\,, \qquad \tilde J^{(0)} = -i\,\tilde J + 2\tilde j^w\,. \tag{6.16}$$

Let us now switch to discussion of the full supersymmetric model. To begin with, recall that the $\mathrm{U}(1)_L\times\mathrm{U}(1)_R$ currents J, $\tilde J$ of the purely bosonic sector, given by (3.28), are to be replaced with their supersymmetric theory counterparts $\mathtt{J}$, $\tilde{\mathtt{J}}$, given by (3.47). These total bosonic gauge currents incorporate contributions from the fermions in adjoint representation of the $sl(2,\mathbb{R})$ and $su(2)$ algebras. The corresponding null currents in the supersymmetric coset theory are then given by

$$\mathtt{J}^{(0)} = i\,\mathtt{J} + 2j^w\,, \qquad \tilde{\mathtt{J}}^{(0)} = -i\,\tilde{\mathtt{J}} + 2\tilde j^w\,. \tag{6.17}$$

[30]Here we have taken the defiinitions (3.39) into account.

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Indeed, combining (3.48), (6.11) we recover the BRST null conditions

$$\langle \mathtt{J}^{(0)}(z)\mathtt{J}^{(0)}(w)\rangle = 0\,, \qquad \langle \tilde{\mathtt{J}}^{(0)}(\bar{z})\tilde{\mathtt{J}}^{(0)}(\bar{w})\rangle = 0\,. \tag{6.18}$$

The associated BRST charges

$$Q_{\rm BRST} = \frac{1}{2\pi i}\oint dz\, \mathtt{J}^{\rm BRST}\,, \qquad \tilde{Q}_{\rm BRST} = \frac{1}{2\pi i}\oint dz\, \tilde{\mathtt{J}}^{\rm BRST}\,, \tag{6.19}$$

of the currents $\mathtt{J}^{\rm BRST} = c\,\mathtt{J}^{(0)}$, $\tilde{\mathtt{J}}^{\rm BRST} = \tilde{c}\,\tilde{\mathtt{J}}^{(0)}$ are therefore nilpotent, $Q^2_{\rm BRST} = 0$, $\tilde{Q}^2_{\rm BRST} = 0$. Physical states in the spectrum of the coset model (6.10) are to be closed w.r.t. these charges,

$$Q_{\rm BRST}|{\rm phys}\rangle = 0\,, \qquad \tilde{Q}_{\rm BRST}|{\rm phys}\rangle = 0\,, \tag{6.20}$$

and are defined up to the BRST-exact states. Expanding the null currents (6.17),

$$\mathtt{J}^{(0)}(z) = \sum_n \frac{\mathtt{J}^{(0)}_n}{z^{n+1}}\,, \qquad \tilde{\mathtt{J}}^{(0)}(\bar{z}) = \sum_n \frac{\tilde{\mathtt{J}}^{(0)}_n}{\bar{z}^{n+1}}\,, \tag{6.21}$$

the physical state conditions are expressed as

$$\mathtt{J}^{(0)}_n|{\rm phys}\rangle = 0\,, \qquad \tilde{\mathtt{J}}^{(0)}_n|{\rm phys}\rangle = 0\,, \qquad n \geq 0\,, \tag{6.22}$$

while the BRST exact states are obtained as $\mathtt{J}^{(0)}_{-1}|0\rangle$, $\tilde{\mathtt{J}}^{(0)}_{-1}|0\rangle$, where $|0\rangle$ is the physical vacuum. Using (6.18) we also obtain the commutation relations

$$[\mathtt{J}^{(0)}_m,\, \mathtt{J}^{(0)}_n] = 0\,, \qquad [\tilde{\mathtt{J}}^{(0)}_m,\, \tilde{\mathtt{J}}^{(0)}_n] = 0\,. \tag{6.23}$$

The (anti-)holomorphic superpartners of the gauge currents $\mathtt{J}$, $\tilde{\mathtt{J}}$ are given by the fermions (3.49). At the same time, the superpartners of the bosonic currents j^w, $\tilde{j}^w$ are given by the fermions ψ^w, $\tilde{\psi}^w$, such that[31]

$$\psi^w(z)\psi^w(w) = -\frac{1}{2}\,\frac{1}{z-w}\,, \qquad \tilde{\psi}^w(z)\tilde{\psi}^w(w) = -\frac{1}{2}\,\frac{1}{\bar{z}-\bar{w}}\,. \tag{6.24}$$

We can use these to write down the fermionic null currents superpartners [56][32]

$$\psi^{(0)} = i\,\psi + 2\,\psi^w\,, \qquad \tilde{\psi}^{(0)} = -i\,\tilde{\psi} + 2\,\tilde{\psi}^w\,, \tag{6.25}$$

that satisfy the null conditions

$$\langle \psi^{(0)}(z)\psi^{(0)}(w)\rangle = 0\,, \qquad \langle \tilde{\psi}^{(0)}(\bar{z})\tilde{\psi}^{(0)}(\bar{w})\rangle = 0\,. \tag{6.26}$$

Physical states in the spectrum of NS states of the supersymmetric coset theory should satisfy the constraints

$$\psi^{(0)}_r|{\rm phys}\rangle = 0\,, \qquad \tilde{\psi}^{(0)}_r|{\rm phys}\rangle = 0\,, \qquad r = \frac{1}{2}, \frac{3}{2}, \dots\,, \tag{6.27}$$

[31] Notice that the sign of these correlation functions is opposite to the sign of the two-point functions of physical fermions (3.50). This is analogous to having opposite signs of physical and auxiliary bosonic currents (3.42). Unphysical w polarization is eliminated from the spectrum by the BRST constraints.

[32] The fermionic sector possesses its own ghosts, which are bosonic counterparts of b and c. We skip detailed discussion of this formalism.

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where we defined the NS sector mode expansion amplitudes as

$$\psi^{(0)} = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\psi_r^{(0)}}{z^{r+\frac{1}{2}}}\,, \qquad \tilde{\psi}^{(0)} = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\tilde{\psi}_r^{(0)}}{\bar{z}^{r+\frac{1}{2}}}\,, \tag{6.28}$$

where r is half-integer in the NS sector. Using (6.26) we obtain

$$\{\psi_r^{(0)},\, \psi_s^{(0)}\} = 0\,, \qquad \{\tilde{\psi}_r^{(0)},\, \tilde{\psi}_s^{(0)}\} = 0\,. \tag{6.29}$$

6.2 Virasoro constraints

In section 6.1, we used the coset BRST formalism to derive the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ null physical state constraints $\mathtt{J}_n^{(0)}|\mathrm{phys}\rangle = 0$, $n \geq 0$, $\psi_r^{(0)}|\mathrm{phys}\rangle = 0$, $r = 1/2,\, 3/2, \ldots$, that are satisfied by the states in the spectrum of the coset theory (3.1). At the same time, the physical states are defined up to the exact states, $\mathtt{J}_{-n}^{(0)}\,|0\rangle$ and $\psi_{-r}^{(0)}\,|0\rangle$ where $|0\rangle$ is the physical vacuum.

Besides these constraints, the physical states of superstring theory have to satisfy the (super-)Virasoro constraints. These constraints reflect superconformal invariance of the quantum world-sheet theory. The superconformal transformations are generated by the stress energy tensor $T(z)$, and the super-current $G(z)$, and their anti-holomorphic counterparts $\tilde{T}(\bar{z})$, $\tilde{G}(\bar{z})$. These currents form (anti-)holomorphic affine superconformal algebra, that we briefly review in appendix A.3.

Components of the stress-energy tensor of the coset world-sheet theory (3.1) are given by

$$\begin{aligned} T &= \frac{1}{k} j^a j^a + \frac{1}{k} j'^a j'^a + j^x j^x - j^w j^w + T_f + T_{\mathbb{T}^4} + T_{\mathrm{ghosts}}, \\ \tilde{T} &= \frac{1}{k} \tilde{j}^a \tilde{j}^a + \frac{1}{k} \tilde{j}'^a \tilde{j}'^a + \tilde{j}^x \tilde{j}^x - \tilde{j}^w \tilde{j}^w + \tilde{T}_f + \tilde{T}_{\mathbb{T}^4} + \tilde{T}_{\mathrm{ghosts}}\,, \end{aligned} \tag{6.30}$$

where $T_{\mathbb{T}^4}$, $\tilde{T}_{\mathbb{T}^4}$ are the components of the world-sheet stress-energy tensor with target space $\mathbb{T}^4$ parametrized by coordinates z^i, $i = 1, 2, 3, 4$,

$$T_{\mathbb{T}^4} = \sum_{i=1}^{4} j^i\, j^i\,, \qquad \tilde{T}_{\mathbb{T}^4} = \sum_{i=1}^{4} \tilde{j}^i\, \tilde{j}^i\,. \tag{6.31}$$

We have also denoted contributions from the fermionic sector as

$$T_f = -\frac{1}{k}\,\psi^a \partial\psi^a - \frac{1}{k}\,\psi'^a \partial\psi'^a - \psi^i \partial\psi^i + \psi^w \partial\psi^w\,, \tag{6.32}$$

and similarly for the anti-holomorphic sector, while contributions from coset BRST ghosts, T_{ghosts} comes from the ghosts sector, that includes T_{bc} given by (6.8), and its supersymmetric counterpart.

The (anti-)holomorphic supercurrents $G(z)$, $\tilde{G}(\bar{z})$ are fermionic operators. We are mostly interested in the NS sector, and therefore the fermions are expanded in half-integer modes. Specifically, expanding the stress-energy tensor and the supercurrent, we obtain

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}\,, & \tilde{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \frac{\tilde{L}_n}{\bar{z}^{n+2}}\,, \\ G(z) &= \sum_{r\in\mathbb{Z}+1/2} \frac{G_r}{z^{r+\frac{3}{2}}}\,, & \tilde{G}(\bar{z}) &= \sum_{r\in\mathbb{Z}+1/2} \frac{\tilde{G}_r}{\bar{z}^{r+\frac{3}{2}}}\,. \end{aligned} \tag{6.33}$$

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The physical state conditions in terms of the corresponding mode expansion amplitudes are given by[33]

$$\begin{gathered}\left(L_n-\frac{1}{2}\,\delta_{n,0}\right)|\text{phys}\rangle=0\,,\quad \left(\tilde{L}_n-\frac{1}{2}\,\delta_{n,0}\right)|\text{phys}\rangle=0\,,\quad n=0,1,2,\dots\\ G_r|\text{phys}\rangle=0\,,\qquad\qquad \tilde{G}_r|\text{phys}\rangle=0\,,\quad r=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots\,.\end{gathered}\tag{6.34}$$

Additionally, the physical states are defined up to exact states $L_{-n}\,|0\rangle=0$, $G_{-r}\,|0\rangle=0$, and similarly for the anti-holomorphic sector.

6.3 Ground state vertex operator

We now proceed to construction of the primary vertex operator of the (3.1) coset model. This operator describes the ground state of the model, that we write as

$$V^{(0)}\tilde{V}^{(0)}=W^{(0)}(z,\bar{z})\,V^{(0)}_{\mathbb{T}^4}(z,\bar{z})\,,\tag{6.35}$$

where $W^{(0)}(z,\bar{z})$ is the ground-state vertex operator of the six-dimensional coset sub-space of (3.1), while[34]

$$V^{(0)}_{\mathbb{T}^4}=e^{i\ell_s\sum_{i=1}^{4}(p^i_L z^i_L+p^i_R z^i_R)}\,.\tag{6.36}$$

is due to the four-dimensional flat sub-space $\mathbb{T}^4=\mathrm{U}(1)^4$.

The corresponding ground state $V^{(0)}\tilde{V}^{(0)}|0\rangle$ should satisfy physical conditions, that include the coset BRST constraints and the superconformal super-Virasoro constraints. In this subsection we will focus on the former.

We start by expressing the coset ground state vertex operator $W^{(0)}(z,\bar{z})$ as a product of the $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SU}(2)$, $\mathrm{U}(1)_x$, and $\mathrm{U}(1)_w$[35] primary operators,

$$W^{(0)}(z,\bar{z})=V^{\hat{\omega}}_{j;m,\bar{m}}(z,\bar{z})\,V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j;m',\bar{m}'}(z,\bar{z})\,e^{i\ell_s(p_L x_L(z,\bar{z})+p_R x_R(z,\bar{z}))}\,e^{i\ell_s\kappa w}\,,\tag{6.37}$$

where $\hat{\omega}\in\mathbb{Z}$ is the $\mathrm{SL}(2,\mathbb{R})$ spectral flow parameter and $\hat{\omega}'$, $\hat{\bar{\omega}}'\in\mathbb{Z}$ are the $\mathrm{SU}(2)$ left- and right-moving spectral flow parameters. The left- and right-moving momenta on the compact $\mathrm{U}(1)_x$ circle with radius R_x are given by

$$p_{L,R}=\frac{n}{R_x}\pm\frac{\mathrm{w}\,R_x}{\ell_s^2}\,,\tag{6.38}$$

where n and w are the momentum and winding numbers respectively.

Recall that the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ primary operators satisfy (2.37), (2.63)

$$\begin{aligned}\mathtt{j}_3(z)V^{\hat{\omega}}_{j;m,\bar{m}}(0)&=\frac{m+\frac{k}{2}\hat{\omega}}{z}\,V^{\hat{\omega}}_{j;m,\bar{m}}\,,&\qquad \tilde{\mathtt{j}}_3(\bar{z})V^{\hat{\omega}}_{j;m,\bar{m}}(0)&=\frac{\bar{m}+\frac{k}{2}\hat{\omega}}{\bar{z}}\,V^{\hat{\omega}}_{j;m,\bar{m}}\,,\\ \mathtt{j}'_3(z)V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}(0)&=\frac{m'+\frac{k}{2}\hat{\omega}'}{z}\,V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}\,,&\qquad \tilde{\mathtt{j}}'_3(\bar{z})V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}(0)&=\frac{\bar{m}'+\frac{k}{2}\hat{\bar{\omega}}'}{\bar{z}}\,V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'}\,,\end{aligned}\tag{6.39}$$

[33] In terms of the vertex operator $\mathcal{V}$ corresponding to the state $|\text{phys}\rangle$, one calculates, e.g., $L_n\cdot\mathcal{V}(0)\simeq\frac{1}{2\pi i}\oint z^n\,T(z)\,\mathcal{V}(0)$ and sets it to zero.

[34] Recall that in our conventions all the coordinates are dimensionless. At the same time, we will define all the momenta to be dimensionful.

[35] The $\mathrm{U}(1)_w$ is non-compact, and therefore $\kappa_L=\kappa_R=\kappa$.

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while for the $\mathrm{U}(1)_x$, $\mathrm{U}(1)_w$ primary operators one finds

$$\begin{aligned} j^x(z)\, e^{i\ell_s(p_L x_L(0)+p_R x_R(0))} &= \frac{\ell_s p_L}{2z}\, e^{i\ell_s(p_L x_L+p_R x_R)}\,, \quad j^w(z)\, e^{i\ell_s\kappa w(0)} = -\frac{\ell_s\kappa}{2z}\, e^{i\ell_s\kappa w} \\ \tilde{j}^x(\bar{z})\, e^{i\ell_s(p_R x_L(0)+p_R x_R(0))} &= \frac{\ell_s p_R}{2\bar{z}}\, e^{i\ell_s(p_L x_L+p_R x_R)}\,, \quad \tilde{j}^w(\bar{z})\, e^{i\ell_s\kappa w(0)} = -\frac{\ell_s\kappa}{2\bar{z}}\, e^{i\ell_s\kappa w}\,. \end{aligned} \tag{6.40}$$

Using (3.47), (6.17) we then obtain

$$\begin{aligned} \mathtt{J}^{(0)}(z)W^{(0)}(0) &= 2\left(\frac{1}{\sqrt{k}}\left[a_1\left(m+\frac{k}{2}\hat{\omega}\right)+a_2\left(m'+\frac{k}{2}\hat{\omega}'\right)\right]+\frac{\ell_s p_L a_3}{2}-\frac{\ell_s\kappa}{2}\right)\frac{W^{(0)}}{z}\,, \\ \tilde{\mathtt{J}}^{(0)}(\bar{z})W^{(0)}(0) &= 2\left(-\frac{1}{\sqrt{k}}\left[b_1\left(\bar{m}+\frac{k}{2}\hat{\omega}\right)+b_2\left(\bar{m}'+\frac{k}{2}\hat{\bar{\omega}}'\right)\right]-\frac{\ell_s p_R b_3}{2}-\frac{\ell_s\kappa}{2}\right)\frac{W^{(0)}}{\bar{z}}\,, \end{aligned} \tag{6.41}$$

which is consistent with $m, \bar{m} \in \mathbb{R}$, $m', \bar{m}' \in \mathbb{R}$. The BRST constraints (6.22) therefore take the form

$$\begin{aligned} \frac{1}{\sqrt{k}}\left[a_1\left(m+\frac{k}{2}\hat{\omega}\right)+a_2\left(m'+\frac{k}{2}\hat{\omega}'\right)\right]+\frac{\ell_s p_L a_3}{2}-\frac{\ell_s\kappa}{2} &= 0\,, \\ \frac{1}{\sqrt{k}}\left[b_1\left(\bar{m}+\frac{k}{2}\hat{\omega}\right)+b_2\left(\bar{m}'+\frac{k}{2}\hat{\bar{\omega}}'\right)\right]+\frac{\ell_s p_R b_3}{2}+\frac{\ell_s\kappa}{2} &= 0\,. \end{aligned} \tag{6.42}$$

Combining the constraints (6.42) we obtain

$$\begin{aligned} \frac{1}{\sqrt{k}}\left[a_1\left(m+\frac{k}{2}\hat{\omega}\right)+a_2\left(m'+\frac{k}{2}\hat{\omega}'\right)+b_1\left(\bar{m}+\frac{k}{2}\hat{\omega}\right)+b_2\left(\bar{m}'+\frac{k}{2}\hat{\bar{\omega}}'\right)\right] \\ +\frac{\ell_s p_L a_3}{2}+\frac{\ell_s p_R b_3}{2}=0\,. \end{aligned} \tag{6.43}$$

6.4 Asymptotic behavior of vertex operators

In this subsection, we are going to explore asymptotic behavior of vertex operators describing string excitation modes. We begin with the ground state vertex operator (6.37). Specifically, we are interested in the plane-wave behavior in the asymptotic region, that allows one to read off energy of the string excitation mode. In fact, the same energy characterizes any string excitation obtained by acting with the creation operators on the given ground state vertex operator.

In the $\mathrm{SL}(2,\mathbb{R})$ parametrization given by (3.13), the asymptotic region is found by taking the limit $\theta \to \infty$. The $\mathrm{SL}(2,\mathbb{R})$ ground state vertex operator in this limit behaves as [28]

$$V^{\hat{\omega}}_{j;m,\bar{m}}(\theta\to\infty) \simeq e^{iy\left[(m+\frac{k\hat{\omega}}{2})\nu+(\bar{m}+\frac{k\hat{\omega}}{2})\right]}\left[e^{2\theta j}+(-1)^{-(2j+1)}R(j;m,\bar{m})e^{-2\theta(j+1)}\right]\,, \tag{6.44}$$

where (this coincides with the coefficient in (2.24))

$$R(j;m,\bar{m}) = \frac{\Gamma\left(1-\frac{2j+1}{k}\right)\Gamma(j+1-m)\Gamma(j+1+\bar{m})\Gamma(-2j-1)}{\Gamma\left(1+\frac{2j+1}{k}\right)\Gamma(-j-m)\Gamma(-j+\bar{m})\Gamma(2j+1)}\,. \tag{6.45}$$

In the parafermionic decomposition one represents locally $\mathrm{SU}(2) \sim \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \mathrm{U}(1)_{\mathrm{y}}$, that allows one to re-write the SU(2) ground state vertex operator as

$$V'^{\hat{\omega}',\hat{\bar{\omega}}'}_{j';m',\bar{m}'} = e^{i\left[\left(m'+\frac{k\hat{\omega}'}{2}\right)y_{\mathrm{su}}+\left(\bar{m}+\frac{k\hat{\bar{\omega}}'}{2}\right)\bar{y}_{\mathrm{su}}\right]}\Psi_{j';m,\bar{m}'}\,, \tag{6.46}$$

where $\Psi_{j';m,\bar{m}'}$ is the $\mathrm{SU}(2)/\mathrm{U}(1)$ vertex operator.

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Combining expressions (6.44), (6.46) for the SL(2, $\mathbb{R}$) and SU(2) ground state vertex operators in the expression (6.37) for the ground vertex operator of the coset sub-space of (3.1) and focusing on the plane-wave phase in the asymptotic region, we obtain

$$W^{(0)} \simeq e^{i\left[\left(\left(m+\frac{k\hat{\omega}}{2}\right)\nu+\left(\bar{m}+\frac{k\hat{\omega}}{2}\right)\right)y+\left(m'+\frac{k\hat{\omega}'}{2}\right)y_{\rm su}+\left(\bar{m}+\frac{k\hat{\bar{\omega}}'}{2}\right)\bar{y}_{\rm su}+\ell_s(p_L x_L+p_R x_R+\kappa w)\right]}. \tag{6.47}$$

Here we can substitute expressions for m, $\bar{m}$, obtained by solving the BRST constraints (6.42), which after some re-arrangement gives

$$W^{(0)} \simeq e^{i\left[\sqrt{k}\,\kappa t+\left(m'+\frac{k\hat{\omega}'}{2}\right)\hat{y}_{\rm su}+\left(\bar{m}'+\frac{k\hat{\bar{\omega}}'}{2}\right)\hat{\bar{y}}_{\rm su}+\ell_s(p_L\hat{x}_L+p_R\hat{x}_R+\kappa w)\right]}, \tag{6.48}$$

where we defined the time coordinate t

$$y = \frac{2a_1b_1t}{a_1-b_1\nu}, \tag{6.49}$$

as well as the moving frame coordinates

$$\begin{aligned} \hat{x}_L &= x_L - \frac{a_3b_1\nu\sqrt{k}\,t}{a_1-b_1\nu}, \quad & \hat{x}_R &= x_R - \frac{a_1b_3\sqrt{k}\,t}{a_1-b_1\nu}, \\ \hat{y}_{\rm su} &= y_{\rm su} - \frac{2a_2b_1\nu\,t}{a_1-b_1\nu}, & \hat{\bar{y}}_{\rm su} &= \bar{y}_{\rm su} - \frac{2a_1b_2\,t}{a_1-b_1\nu}. \end{aligned} \tag{6.50}$$

Notice that the time coordinate change (6.49) and the moving frame coordinates change (6.50) agree with the corresponding expressions (3.68), (3.82) obtained by analyzing the asymptotic region in derivation of the coset background geometry.[36]

Defining time as $t \to \frac{t}{\sqrt{k}} - \ell_s w$, we conclude from the asymptotic plane-wave expression (6.48) for the vertex operator $W^{(0)}$ that the frequency of the excited string state defined by this vertex operator is given by κ. Notice that this is true regardless of whether we consider a spectrally flowed sector.

6.5 Mass-shell condition and spectrum

In subsection 6.3, we constructed ground state vertex operator (6.35) of the coset model (3.1), and derived coset BRST physical state constraints (6.42) that such an operator needs to satisfy. In subsection 6.2 we wrote down the (super-) Virasoro constraints that need to be satisfied by the (NS sector) states of superstring theory on the coset (3.1). Only a finite number of non-trivial constraints, depending on the string excitation number, need to be imposed for each state, with the rest of infinitely many constraints satisfied automatically. However, every state needs to satisfy the lowest Virasoro L_0, $\tilde{L}_0$ constraints, known as the mass-shell condition. In this subsection we will derive the mass-shell condition for string states on the coset (3.1).

Let us denote the (anti-)holomorphic superstring vertex operator at the level ($\bar{N}$) N as ($\tilde{V}^{(\bar{N})}$) $V^{(N)}$. It is obtained by acting on the ground state vertex operator $V^{(0)}\tilde{V}^{(0)}$ (that

[36] The relative factor of $\sqrt{k}$ in the transformation laws for $x = x_L + x_R$ is due to the relation between x and x' given by (3.67).

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corresponds to $N = \bar{N} = 0$) with the product $(\tilde{\alpha}_{\bar{N}})\ \alpha_N$ of creation operators such as $(\tilde{\psi}^M_{-r},$ $\tilde{\mathtt{j}}^L_{-n})\ \psi^M_{-r}$, $\mathtt{j}^L_{-n}$, $r > 0$, $n > 0$,

$$V^{(N)} = \alpha_N\, V^{(0)}\,, \qquad \tilde{V}^{(N)} = \tilde{\alpha}_{\bar{N}}\, \tilde{V}^{(0)}\,, \tag{6.51}$$

such that[37]

$$[L_0,\, \alpha_N] = N\, \alpha_N\,, \qquad [\tilde{L}_0,\, \tilde{\alpha}_{\bar{N}}] = \bar{N}\, \tilde{\alpha}_{\bar{N}}\,. \tag{6.52}$$

Using (6.52) together with the OPE of the ground state vertex operator (6.35) with the stress-energy tensor, we derive

$$\begin{aligned} L_0 \cdot V^{(N)} &= \left(\Delta_{sl} + \Delta_{su} + \frac{\ell_s^2 p_L^2}{4} - \frac{\ell_s^2 \kappa^2}{4} + N + \Delta_{\mathbb{T}^4}\right) V^{(N)}, \\ \tilde{L}_0 \cdot \tilde{V}^{(\bar{N})} &= \left(\bar{\Delta}_{sl} + \bar{\Delta}_{su} + \frac{\ell_s^2 p_R^2}{4} - \frac{\ell_s^2 \kappa^2}{4} + \bar{N} + \bar{\Delta}_{\mathbb{T}^4}\right) \tilde{V}^{(\bar{N})}\,, \end{aligned} \tag{6.53}$$

where $N, \bar{N} = 1, 2, \ldots$ are the string excitation numbers,

$$\begin{aligned} \Delta_{sl} &= -\frac{j(j+1)}{k} - m\hat{\omega} - \frac{k\hat{\omega}^2}{4}, \qquad \bar{\Delta}_{sl} = -\frac{j(j+1)}{k} - \bar{m}\hat{\omega} - \frac{k\hat{\omega}^2}{4}, \\ \Delta_{su} &= \frac{j'(j'+1)}{k} + m'\hat{\omega}' + \frac{k\hat{\omega}'^2}{4}, \qquad \bar{\Delta}_{su} = \frac{j'(j'+1)}{k} + \bar{m}'\hat{\bar{\omega}}' + \frac{k\hat{\bar{\omega}}'^2}{4} \end{aligned} \tag{6.54}$$

are scaling dimensions of spectrally-flowed ground state vertex operators of the $\mathrm{SL}(2,\mathbb{R})$ and SU(2) WZW models (see (2.35), (2.36), (2.62)), and

$$\Delta_{\mathbb{T}^4} = \frac{\ell_s^2}{2} \sum_{i=1}^{4} (p_L^i)^2\,, \qquad \bar{\Delta}_{\mathbb{T}^4} = \frac{\ell_s^2}{2} \sum_{i=1}^{4} (p_R^i)^2 \tag{6.55}$$

are scaling dimensions of the ground state vertex operator (6.36) on $\mathbb{T}^4$. The mass-shell condition, given by the L_0, $\tilde{L}_0$ Virasoro constraints of (6.34), then reads

$$\begin{aligned} \frac{\ell_s^2 \kappa^2}{4} &= \Delta_{sl} + \Delta_{su} + \frac{\ell_s^2 p_L^2}{4} + \Delta_{\mathbb{T}^4} + N - \frac{1}{2}\,, \\ \frac{\ell_s^2 \kappa^2}{4} &= \bar{\Delta}_{sl} + \bar{\Delta}_{su} + \frac{\ell_s^2 p_R^2}{4} + \bar{\Delta}_{\mathbb{T}^4} + \bar{N} - \frac{1}{2}\,. \end{aligned} \tag{6.56}$$

In section 6.4 we established that the energy of string state, defined by the plane-wave behavior of the vertex operator in the asymptotic region,[38] is given by κ. The L_0, $\tilde{L}_0$ Virasoro constraints (6.56) then have the physical interpretation of the mass-shell conditions, with the terms in the r.h.s. of these expressions given by the sum of the momenta (on $\mathrm{SL}(2,\mathbb{R})$ and SU(2) the role of momenta are played by the representation numbers j,

[37] See (A.65) for commutation relations of the Virasoro amplitudes with the super-Kac-Moody currents.

[38] The plane-wave calculation is true for any excitation numbers $N = \bar{N}$, and is determined only by asymptotic behavior of the ground state vertex operator $W^{(0)}$.

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m, j', m') and oscillator numbers $N-1/2$, $\bar{N}-1/2$. Substituting (6.54) in (6.56) and imposing the BRST constraints (6.42) to eliminate $m, \bar{m}$, one obtains[39]

$$\begin{aligned}\left(\kappa-\frac{(a_1-b_1)\sqrt{k}}{2a_1b_1\ell_s}\hat{\omega}\right)^2 &= \left(\frac{n}{R_x}\right)^2+\left(\frac{\mathrm{w}R_x}{\ell_s^2}\right)^2+\frac{2}{\ell_s^2}\Bigg(-2\frac{j(j+1)-j'(j'+1)}{k}\\ &+\frac{k\hat{\omega}^2}{8}\left[\left(\frac{1}{a_1}-\frac{1}{b_1}\right)^2+4\right]+(m'\hat{\omega}'+\bar{m}'\hat{\bar{\omega}}')+\frac{k}{4}(\hat{\omega}'^2+\hat{\bar{\omega}}'^2)\\ &+\frac{a_2\hat{\omega}}{a_1}\left(m'+\frac{k\hat{\omega}'}{2}\right)+\frac{b_2\hat{\omega}}{b_1}\left(\bar{m}'+\frac{k\hat{\bar{\omega}}'}{2}\right)\\ &+\frac{\sqrt{k}\ell_s\hat{\omega}}{2a_1b_1}\left((a_3b_1+a_1b_3)\frac{n}{R_x}+(a_3b_1-a_1b_3)\frac{\mathrm{w}R_x}{\ell_s^2}\right)+\\ &+\Delta_{\mathbb{T}^4}+\bar{\Delta}_{\mathbb{T}^4}+\left(N-\frac{1}{2}\right)+\left(\bar{N}-\frac{1}{2}\right)\Bigg).\end{aligned} \tag{6.57}$$

For the ground state vertex operator (6.35) in the NS-NS sector we have $N=\bar{N}=0$.[40] The state is then tachyonic, $\kappa^2<0$, at zero momenta. Such a state is removed from the physical spectrum by the GSO projection. The first excited states of the NS sector, with ($\bar{N}=\frac{1}{2}$) $N=\frac{1}{2}$ defined by the (anti-)holomorphic vertex operators ($\tilde{V}^{(\frac{1}{2})}$) $V^{(\frac{1}{2})}$ are massless, and represent the lowest-lying state in the physical spectrum of string excitation. Irreducible representations of tensor product of these states form the NS-NS sector of the supergravity multiplet, and will be studied below in section 6.6.

6.6 Massless states

Massless states of the type-II superstring theory form the type-II supergravity multiplet. Half of its bosonic degrees of freedom are contained in the NS-NS sector, and form a gravity multiplet, consisting of the graviton, dilaton, and B-field. In section 3, we have derived background values of these fields, (3.100), (3.101), (3.103), for superstring theory on the coset (3.1), exactly in k.

In this subsection, we are interested in excitations of the massless NS-NS fields on top of the background (3.100), (3.101), (3.103). Performing the KK reduction on the $\mathrm{U}(1)_x$ circle, one can also describe excitations on top of the nine-dimensional background (5.2). These excitations can be obtained by first separately constructing (anti-)holomorphic massless vertex operators in the NS sectors of the (right-) left-moving string states, and then taking the direct product of these vertex operarors. Finally, the resulting closed string states can be grouped into irreducible representations of the target space-time symmetry group, rendering the graviton, dilaton, and B-field excitations.

The massless holomorphic vertex operator $V^{(\frac{1}{2})}$ in the NS sector on the coset (3.1) is obtained by acting with the creation operators $\psi^M_{-1/2}$ on the tachyon ground state vertex operator (6.35). For the most general polarizations ξ_M in the target space-time we obtain

[39] Recall that the parameters $a_{1,2,3}$, $b_{1,2,3}$ need to satisfy the anomaly-free conditions (3.24).

[40] Also setting spectral flow parameters and the $\mathrm{U}(1)_x$ winding number to zero, $\hat{\omega}=\hat{\omega}'=\hat{\bar{\omega}}'=\mathrm{w}=0$.

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the state

$$V^{(\frac{1}{2})} = \Big[\xi_+\,\psi^+_{-\frac{1}{2}}\,\mathtt{j}^-_0 + \xi_-\,\psi^-_{-\frac{1}{2}}\,\mathtt{j}^+_0 + \xi\,\psi^{(0)}_{-\frac{1}{2}} \tag{6.58}$$
$$+\,\xi'_+\,\psi'^+_{-\frac{1}{2}}\,\mathtt{j}'^-_0 + \xi'_-\,\psi'^-_{-\frac{1}{2}}\,\mathtt{j}'^+_0 + \xi'_3\,\psi'^3_{-\frac{1}{2}} + \xi_w\,\psi^w_{-\frac{1}{2}} + \xi_x\,\psi^x_{-\frac{1}{2}} + \sum_{i=1}^{4}\xi_i\,\psi^i_{-\frac{1}{2}}\Big]\,V^{(0)}\,,$$

where we introduced linear combinations of the $sl(2,\mathbb{R})$ and $su(2)$ fermions

$$\psi^\pm = \psi^1 \pm i\psi^2\,, \qquad \psi'^\pm = \psi'^1 \pm i\psi'^2\,. \tag{6.59}$$

In (6.58) we have chosen to replace the basis target space direction ψ^3 with the null spinor $\psi^{(0)}$ given by (6.25). Physical state conditions, that include the coset BRST constraints and super-Virasoro constraints, and remove the pure-gauge exact states, restrict the allowed polarizations ξ_M, rendering eight independent polarizations.

Due the coset BRST constraints, the state (6.58) needs to be closed w.r.t. the null spinor $\psi^{(0)}$, given by (6.25). In terms of the amplitudes (6.28) this condition imposes

$$\psi^{(0)}_{\frac{1}{2}}\,V^{(0)} = 0\,. \tag{6.60}$$

In terms of the basis fermionic operators, this can be written explicitly as

$$\left(\frac{a_1}{\sqrt{k}}\,\psi^3_{\frac{1}{2}} + \frac{a_2}{\sqrt{k}}\,\psi'^3_{\frac{1}{2}} + a_3\,\psi^x_{\frac{1}{2}} + \psi^w_{\frac{1}{2}}\right)\,V^{(0)} = 0\,. \tag{6.61}$$

The BRST constraint (6.61) can be solved as

$$\xi_w = a_3\,\xi_x + a_2\sqrt{k}\,\xi'_3\,. \tag{6.62}$$

The most general massless holomorphic BRST closed state is then given by

$$V^{(\frac{1}{2})} = \Big[\xi_+\,\psi^+_{-\frac{1}{2}}\,\mathtt{j}^-_0 + \xi_-\,\psi^-_{-\frac{1}{2}}\,\mathtt{j}^+_0 + \xi\,\psi^{(0)}_{-\frac{1}{2}} + \sum_{i=1}^{4}\xi_i\,\psi^i_{-\frac{1}{2}} \tag{6.63}$$
$$+\,\xi'_+\,\psi'^+_{-\frac{1}{2}}\,\mathtt{j}'^-_0 + \xi'_-\,\psi'^-_{-\frac{1}{2}}\,\mathtt{j}'^+_0 + \xi'_3\,(\psi'^3_{-\frac{1}{2}} + a_2\sqrt{k}\,\psi^w_{-\frac{1}{2}}) + \xi_x\,(\psi^x_{-\frac{1}{2}} + a_3\,\psi^w_{-\frac{1}{2}})\Big]\,V^{(0)}\,.$$

Here the polarization ξ is un-physical, and represents a pure-gauge exact state on the coset. After imposing the coset BRST physical state conditions, the vertex operator (6.63) then possesses ten independent polarizations.

Next, let us discuss the super-Virasoro constraints (6.34). For the massless state, $V^{(\frac{1}{2})}$, the only non-trivial constraints are given by

$$\left(L_0 - \frac{1}{2}\right)\,V^{(\frac{1}{2})} = 0\,, \qquad G_{\frac{1}{2}}\,V^{(\frac{1}{2})} = 0\,, \tag{6.64}$$

and its counterparts in the anti-holomorphic sector. The L_0 constraint is simply (6.56) with $N = \bar{N} = 1/2$, giving the mass-shell condition for the massless state. The

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super-Virasoro $G_{1/2}$ constraint restricts allowed physical polarizations. First, using (6.39), (6.40), (A.66), (6.42) we obtain

$$\{G_{\frac{1}{2}},\, \psi^{(0)}_{-\frac{1}{2}}\}\, V^{(0)} = 0\,, \tag{6.65}$$

and therefore the BRST-exact state $\psi^{(0)}_{-\frac{1}{2}}\, V^{(0)}$ in (6.63) automatically satisfies the super-Virasoro $G_{1/2}$ constraint. Using (2.37), (2.50) we obtain

$$\begin{aligned} \mathtt{j}_0^{\mp}\, \mathtt{j}_0^{\pm}\, V_{j;m,\bar m} &= (m \mp j)(m \pm 1 \pm j)\, V_{j;m,\bar m}\,, \\ \mathtt{j}_0^{\prime\mp}\, \mathtt{j}_0^{\prime\pm}\, V'_{j';m',\bar m'} &= (j'(j'+1) - m'(m' \pm 1))\, V'_{j';m',\bar m'}\,. \end{aligned} \tag{6.66}$$

Then using (A.66) we arrive at

$$G_{\frac{1}{2}}\, V^{(\frac{1}{2})} = \left[\hat\xi_+ + \hat\xi_- + \hat\xi'_+ + \hat\xi'_- + \xi'_3 \left(m' - \frac{a_2 \ell_s \sqrt{k}\,\kappa}{2}\right) + \frac{\ell_s}{2}\left(\xi_x\,(p_L - a_3\kappa) + \sum_{i=1}^{4} \xi_i\, p_L^i\right)\right] V^{(\frac{1}{2})} \tag{6.67}$$

where we denoted

$$\begin{aligned} \hat\xi_+ &= (m+j)(m-1-j)\,\xi_+\,, & \hat\xi_- &= (m-j)(m+1+j)\,\xi_-\,, \\ \hat\xi'_+ &= (j'(j'+1) - m'(m'-1))\,\xi'_+\,, & \hat\xi'_- &= (j'(j'+1) - m'(m'+1))\,\xi'_-\,. \end{aligned} \tag{6.68}$$

Taking into account the BRST constraints (6.42), that we solve for κ, the Virasoro constraint following from (6.67) then reads

$$\begin{aligned} \hat\xi_+ + \hat\xi_- + \hat\xi'_+ + \hat\xi'_- + m'\,\xi'_3 + \frac{\ell_s}{2}\left(\xi_x\, p_L + \sum_{i=1}^{4} \xi_i\, p_L^i\right) \\ = \left(\xi'_3\, a_2\, \sqrt{k} + \xi_x\, a_3\right)\left(\frac{\ell_s\, p_L\, a_3}{2} + \frac{m\, a_1 + m'\, a_2}{\sqrt{k}}\right)\,. \end{aligned} \tag{6.69}$$

We can solve this for $\hat\xi_\pm$ simply as

$$\begin{aligned} \hat\xi_\pm = \pm\,\hat\xi - \frac{1}{2}\Bigg(\hat\xi'_+ + \hat\xi'_- + m'\,\xi'_3 + \frac{\ell_s}{2}\left(\xi_x\, p_L + \sum_{i=1}^{4} \xi_i\, p_L^i\right) \\ + \left(\xi'_3\, a_2\, \sqrt{k} + \xi_x\, a_3\right)\left(\frac{\ell_s\, p_L\, a_3}{2} + \frac{m\, a_1 + m'\, a_2}{\sqrt{k}}\right)\Bigg)\,. \end{aligned} \tag{6.70}$$

The most general massless holomorphic physical vertex operator is then given by (6.63) with polarizations satisfying (6.70). It is also defined up to the Virasoro exact massless null state, given by

$$G_{-\frac{1}{2}}\, V^{(0)} = \left[\frac{2}{k}\left(\psi^a_{-\frac{1}{2}}\, \mathtt{j}^a_0 + \psi'^a_{-\frac{1}{2}}\, \mathtt{j}'^a_0\right) + \psi^x_{-\frac{1}{2}}\, j^x_0 - \psi^w_{-\frac{1}{2}}\, j^w_0 + \sum_{i=1}^{4} \psi^i_{-\frac{1}{2}}\, j^i_0\right] V^{(0)}\,, \tag{6.71}$$

therefore rendering a massless string state with eight independent physical polarizations.

In preparation for discussion of holography in section 7, we are now going to consider (anti-)holomorphic massless vertex operators polarized along the direction of propagation

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of excitations, as well as along the $\mathrm{U}(1)_x$ circle. Without loss of generality, we can define the former to be the coordinate z^1 on the torus $\mathbb{T}^4$, $p^1_L = p^1_R = p$, setting the rest of the momenta on the torus to zero, $p^i_{L,R} = 0$, $i = 2,3,4$. We are also going to perform the KK reduction on the $\mathrm{U}(1)_x$ circle, setting the corresponding momenta to zero, $p_L = p_R = 0$. (Anti-)holomorphic vertex operators polarized along the $\mathrm{U}(1)_x$ circle will then be found as constituents of the gauge field vertex operators of type-II superstring theory in the nine-dimensional space-time.

In other words, we will be looking for the physical massless vertex operators of the form $\Psi^1_{-\frac{1}{2}} V^{(0)}$ and $\Psi^x_{-\frac{1}{2}} V^{(0)}$, with $\Psi^1_{-\frac{1}{2}} = \psi^1_{-\frac{1}{2}} + \dots$ and $\Psi^x_{-\frac{1}{2}} = \psi^x_{-\frac{1}{2}} + \dots$, where we will fill in the missing terms accordingly with the expression for the physical vertex operator (6.63), (6.70). Specifically, in the former case we set $\xi_1 = 1$, $\xi_x = 0$, while in the latter case we set $\xi_x = 1$, $\xi_1 = 0$. We also set $\xi_{2,3,4} = 0$, $\xi'_\pm = 0$, $\xi'_3 = 0$, $\hat{\xi} = 0$ everywhere. For convenience, we will denote

$$\Psi^i = \psi^i\,, \qquad \tilde{\Psi}^i = \tilde{\psi}^i\,, \qquad i = 2,3,4\,. \tag{6.72}$$

In general, physical massless states can then be defined as

$$V^{(\frac{1}{2})} = \xi_m\, \Psi^m_{-\frac{1}{2}} \cdot V^{(0)}\,, \qquad \tilde{V}^{(\frac{1}{2})} = \tilde{\xi}_m\, \tilde{\Psi}^n_{-\frac{1}{2}} \cdot V^{(0)}\,. \tag{6.73}$$

With such a convention, physical massless states are constructed using creation operators $\Psi^m_{-\frac{1}{2}}$, $\tilde{\Psi}^m_{-\frac{1}{2}}$, that automatically take into account the coset BRST and super-Virasoro constraints. Using (6.63), (6.70), we then obtain

$$\Psi^1_{-\frac{1}{2}} = \psi^1_{-\frac{1}{2}} - \frac{\ell_s\, p}{4(m+j)(m-1-j)}\, \mathtt{j}^-_0 \psi^+_{-\frac{1}{2}} - \frac{\ell_s\, p}{4(m-j)(m+1+j)}\, \mathtt{j}^+_0 \psi^-_{-\frac{1}{2}}\,, \tag{6.74}$$

$$\Psi^x_{-\frac{1}{2}} = \psi^x_{-\frac{1}{2}} - \frac{(m\,a_1 + m'\,a_2)\,a_3}{2\sqrt{k}\,(m+j)(m-1-j)}\, \mathtt{j}^-_0 \psi^+_{-\frac{1}{2}} - \frac{(m\,a_1 + m'\,a_2)\,a_3}{2\sqrt{k}\,(m-j)(m+1+j)}\, \mathtt{j}^+_0 \psi^-_{-\frac{1}{2}}\,, \tag{6.75}$$

and similarly for the anti-holomorphic components.

Using (6.66), (A.67), we obtain the two-point correlation functions

$$\begin{aligned}
\langle \Psi^1_{\frac{1}{2}} \Psi^1_{-\frac{1}{2}} \rangle_0 &= \left(\frac{1}{2} + \frac{k\,\ell_s^2\,p^2}{16}\, F_{j;m} \right) \langle V^{(0)} V^{(0)} \rangle\,, \\
\langle \Psi^x_{\frac{1}{2}} \Psi^x_{-\frac{1}{2}} \rangle_0 &= \left(\frac{1}{2} + \frac{(m\,a_1 + m'\,a_2)^2\,a_3^2}{4}\, F_{j;m} \right) \langle V^{(0)} V^{(0)} \rangle\,, \\
\langle \Psi^x_{\frac{1}{2}} \Psi^1_{-\frac{1}{2}} \rangle_0 &= \langle \Psi^1_{\frac{1}{2}} \Psi^x_{-\frac{1}{2}} \rangle_0 = \frac{(ma_1 + m'a_2)a_3 \ell_s p \sqrt{k}}{8}\, F_{j;m}\, \langle V^{(0)} V^{(0)} \rangle\,,
\end{aligned} \tag{6.76}$$

where we have denoted

$$F_{j;m} = \frac{1}{(m-j)(m+1+j)} + \frac{1}{(m+j)(m-1-j)}\,, \tag{6.77}$$

and used a short-hand notation $\langle \cdots \rangle_0$ for averages in the state $V^{(0)} \tilde{V}^{(0)} |0\rangle$. Analogous correlation functions can be written down in the anti-holomorphic sector. The tachyon two-point function $\langle V^{(0)} V^{(0)} \rangle$ is given by a product of the two-point functions of its constituent

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vertex operators (6.35), (6.37). For the full state, including both the anti-holomorphic and the holomorphic sectors, we obtain

$$\langle V^{(0)}V^{(0)}\rangle\langle\tilde{V}^{(0)}\tilde{V}^{(0)}\rangle = \langle V_{j;m,\bar{m}}V_{j;m,\bar{m}}\rangle\langle V'_{j';m',\bar{m}'}V'_{j';m',\bar{m}'}\rangle \times \text{flat space correlators}\,, \quad (6.78)$$

where one should substitute (2.24), (2.53) for the two-point functions of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ ground state vertex operators. Notice that zero of (2.24) at $m=-j$ cancels out the simple pole of (6.77) at $m=-j$. At the same time, $F_{j;m}$ has a simple pole at $m=j$, which is inherited by all the correlation functions (6.76). Similarly, the anti-holomorphic sector has a simple pole at $\bar{m}=-j$, while exhibiting regular behavior at $\bar{m}=j$.

6.7 Spectrum of single-trace $T\bar{T}$-deformed symmetric product CFT

We will end this section with a discussion of the spectrum of the long strings in the coset model (3.1), and its relation to the spectrum obtained by a deformation of CFT_2 by the $T\bar{T}$ operator. As it turns out, the spectrum of a single long string in the sector with the unit winding number, $\mathrm{w}=1$, corresponds to the spectrum of the double-trace $T\bar{T}$ deformed CFT [32]. For a detailed review of the single-trace $T\bar{T}$ deformation of string theory in AdS_3, and its relation to the long strings, we refer the reader to [37].

Let us begin by reviewing certain aspects of the theory on a single long string in AdS_3. Via the AdS/CFT correspondence, string theory in $AdS_3\times\mathcal{N}_7$ is dual to a CFT_2 living on the boundary of AdS_3.[41] Here $\mathcal{N}_7$ stands for a seven-dimensional compact space, *e.g.*, $\mathcal{N}_7=\mathbb{S}^3\times\mathbb{T}^4$. In the presence of pure NS-NS flux in the bulk, the space-time theory has an $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum. This is either an NS vacuum, that corresponds to global AdS_3 in the bulk, or an R vacuum, that corresponds to a massless BTZ (denoted by $BTZ_{M=J=0}$). The NS sector contains states that belong to the discrete as well as the continuous series representation of $\mathrm{SL}(2,\mathbb{R})$. The continuous series representation starts above a gap of order $k/2$. The R sector, on the other hand, contains a continuum of long strings above a gap of order $1/k$. In the rest of the discussion of the spectrum of long strings in this section, we will consider only the long string states in the R sector.

On the $AdS_3\times\mathcal{N}_7$ with pure NS-NS flux, the theory on a single long string is given by the product of the $\mathcal{N}=2$ Liouville theory $\mathbb{R}_\phi$ and the compact SCFT on $\mathcal{N}_7$ [57]

$$\mathcal{M}_{6k}^{(L)} = \mathbb{R}_\phi\times\mathcal{N}_7\,, \quad (6.79)$$

where $c_{\mathcal{M}}=6k$ is the central charge of $\mathcal{M}_{6k}^{(L)}$ and the theory on $\mathbb{R}_\phi$ has a dilaton linear in the radial direction of AdS_3 denoted by ϕ with the slope

$$Q^{(L)} = (k-1)\sqrt{\frac{2}{k}}\,. \quad (6.80)$$

The effective coupling on the long strings is given by

$$g^{(L)} \simeq e^{Q^{(L)}\phi} \quad (6.81)$$

[41] In section 7 we will discuss holographic description of a $4+1$ dimensional boundary field theory, with the spatial coordinates spanned by the $\mathbb{T}^4$.

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implying that for $k > 1$, the long strings become strongly coupled as they move towards the boundary. But there is a wide range of positions along the radial direction where they are weakly coupled. The effective theory of N such long strings is described by the symmetric product [37, 58]

$$\frac{(\mathcal{M}_{6k}^{(L)})^N}{S_N}. \tag{6.82}$$

Let us now proceed to discussion of long strings on the coset (3.1). Recall that the value of κ is equal to the energy of a string excitation mode propagating in the coset background. The long strings in that background belong to the continuous representation of $\mathrm{SL}(2,\mathbb{R})$, for which we define, due to (2.12),

$$j = -\frac{1}{2} + is\,, \qquad s \in \mathbb{R}\,. \tag{6.83}$$

For a non-vanishing winding w around the $\mathrm{U}(1)_x$-circle, the energy E of such a long string in the spectrally un-flowed sector (*i.e.*, $\hat{\omega} = \hat{\omega}' = \hat{\bar{\omega}}' = 0$), above the BPS configuration is given by

$$E = \kappa - \frac{\mathrm{w}R_x}{\ell_s^2}\,, \tag{6.84}$$

where due to (6.57) we have

$$\kappa^2 = \left(\frac{n}{R_x}\right)^2 + \left(\frac{\mathrm{w}R_x}{\ell_s^2}\right)^2 + \frac{2}{\ell_s^2}\left(-2\,\frac{j(j+1) - j'(j'+1)}{k} + \Delta_{\mathbb{T}^4} + \bar{\Delta}_{\mathbb{T}^4} + N + \bar{N} - 1\right). \tag{6.85}$$

Due to the dispersion relation of the long strings in the winding w sector in $BTZ_{M=J=0} \times \mathbb{S}^3 \times \mathbb{T}^4$ that is given by [59]

$$\begin{aligned} E_L &= \frac{1}{\mathrm{w}}\left[-\frac{j(j+1)}{k} + \frac{j'(j'+1)}{k} + \Delta_{\mathbb{T}^4} + N - \frac{1}{2}\right], \\ E_R &= \frac{1}{\mathrm{w}}\left[-\frac{j(j+1)}{k} + \frac{j'(j'+1)}{k} + \bar{\Delta}_{\mathbb{T}^4} + \bar{N} - \frac{1}{2}\right], \end{aligned} \tag{6.86}$$

expression (6.85) can be written as

$$\kappa^2 = \left(\frac{n}{R_x}\right)^2 + \left(\frac{\mathrm{w}R_x}{\ell_s^2}\right)^2 + \frac{2\,\mathrm{w}}{\ell_s^2}\,(E_L + E_R)\,. \tag{6.87}$$

Expression (6.86) also gives the spectrum of the $\mathbb{Z}_\mathrm{w}$ twisted sector of the symmetric product CFT $\mathcal{M}^N/S_N$ where the block CFT $\mathcal{M}$ is the CFT with central charge [58]

$$c_\mathcal{M} = 6k \tag{6.88}$$

living on a single long string with winding one. This is not surprising because the effective theory on the long strings are well described by a symmetric product (6.82) at weak coupling [58]. Spectrum of operators of dimension h_w in the $\mathbb{Z}_\mathrm{w}$ twisted sector of the symmetric product CFT in the Ramond sector above the Ramond vacuum is given by [58, 60]

$$E_L = h_\mathrm{w} - \frac{k\mathrm{w}}{4}\,, \qquad E_R = \bar{h}_\mathrm{w} - \frac{k\mathrm{w}}{4}\,. \tag{6.89}$$

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For every operator of dimension h_1 in the untwisted sector there exists an operator in the $\mathbb{Z}_{\rm w}$ twisted sector whose dimension is related to h_1 as [60]

$$h_{\rm w} = \frac{h_1}{\rm w} + \frac{k}{4}\left({\rm w} - \frac{1}{\rm w}\right), \qquad \bar{h}_{\rm w} = \frac{\bar{h}_1}{\rm w} + \frac{k}{4}\left({\rm w} - \frac{1}{\rm w}\right). \tag{6.90}$$

Next we solve (6.84) for κ in terms of E, and we also substitute (6.90) into (6.89). Plugging the result into (6.87) one obtains [32, 35–37]

$$\left(E + \frac{{\rm w}R_x}{\ell_s^2}\right)^2 - \left(\frac{{\rm w}R_x}{\ell_s^2}\right)^2 = \frac{2}{\ell_s^2}\left(h_1 + \bar{h}_1 - \frac{k}{2}\right) + \left(\frac{n}{R_x}\right)^2, \tag{6.91}$$

and

$$\bar{h}_1 - h_1 = n. \tag{6.92}$$

In the untwisted sector corresponding to ${\rm w} = 1$, (6.91) takes the form [40, 41]

$$E = -\frac{R_x}{\ell_s^2} + \sqrt{\left(\frac{R_x}{\ell_s^2}\right)^2 + \frac{2}{\ell_s^2}\left(h_1 + \bar{h}_1 - \frac{c_{\mathcal{M}}}{2}\right) + \left(\frac{h_1 - \bar{h}_1}{R_x}\right)^2}, \tag{6.93}$$

which is the spectrum of a CFT_2 on a cylinder of radius R_x deformed by an irrelevant operator $\delta\mathcal{L} = -t\,T\bar{T}$ with $t = \ell_s^2 > 0$. For ${\rm w} > 1$, (6.91) is the spectrum of the $\mathbb{Z}_{\rm w}$ twisted sector of single trace $T\bar{T}$ deformation of a symmetric product CFT [61].

7 Holography

In this section, we are going to derive various two-point correlation functions of the field theory holographically dual to the bulk string theory on the coset (3.1). We are going to explore different regimes of the holographic correspondence, and calculate various correlation functions in the NS-NS sector of string theory in the bulk.

In subsection 7.1, we will consider the bulk configuration, derived in section 5, obtained by compactification of the $\mathrm{U}(1)_x$ circle of the coset (3.1). This background is characterized by a rotating black brane geometry with an event horizon, finite temperature, and two independent non-trivial U(1) gauge field profiles. We will study massless excitations modes described by the metric, B-field, dilaton, and gauge field, on top of this bulk configuration. The corresponding holographic dual $4+1$ dimensional field theory in the asymptotic region is a finite-temperature system, with two conserved U(1) charges having non-vanishing vacuum expectations. We are interested in gapless collective excitations in this system, that we determine via holographic world-sheet calculation of two-point functions of the stress-energy tensor and charge currents.

In subsection 7.2, we compactify the $\mathbb{T}^4$ sub-space of the coset (3.1), which allows us to study the $1+1$ dimensional boundary field theory. The resulting expressions that we obtain for the two-point correlation functions in momentum space interpolate between the well-known CFT_2 expressions in the IR, calculated within the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, and non-local expressions for LST in the UV.

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7.1 Collective excitation modes

In section 6.6, we constructed the (anti-)holomorphic vertex operators $(\tilde{V}^{(\frac{1}{2})})\ V^{(\frac{1}{2})}$ describing massless NS (right-) left-moving string states. The NS-NS states of closed type-II superstring are obtained by taking a direct product of the (anti-)holomorphic states. The corresponding vertex operator $V^{(\frac{1}{2})}\otimes\tilde{V}^{(\frac{1}{2})}$ can then be split into irreducible representations of the target space symmetry group, rendering the graviton, B-field, and dilaton vertex operators, and forming the NS-NS bosonic degrees of freedom of the type-II supergravity multiplet.

These operators describe excitation on top of the metric, B-field, and dilaton ten-dimensional background configuration (3.100), (3.103), (3.101) that we derived in section 3.2, as well as the nine-dimensional two-charge background (5.2) derived in section 5. An excitation mode is characterized by the energy κ, and momentum p. Without loss of generality, we can align the momentum with the direction z^1 on the torus $\mathbb{T}^4$, $p^1_L = p^1_R = p$, setting the rest of the momenta to zero, $p^i_{L,R} = 0$, $i = 2, 3, 4$.

In this section, we are going to discuss holographic interpretation of string excitations in the bulk in terms of the dual $4+1$ dimensional field theory on the boundary.[42] Recall that the field theory dual of a charged black brane describes matter at finite density.[43] In particular, we are interested in the corresponding collective excitation modes, that we can compare with the hydrodynamic theory. These appear as poles of two-point correlation functions of the stress-energy tensor. In general, these correlation functions also include conserved charge current, and form correlation matrices. Correspondingly, the bulk holographic dual of the stress-energy tensor is given by the fluctuating metric field, which is coupled to fluctuations of the gauge fields, B-field, and dilaton.[44]

For the space-like polarizations $m, n = x,\ i$, where $i = 1, 2, 3, 4$, along the circle $\mathrm{U}(1)_x$ and the torus sub-space $\mathbb{T}^4$, we write down the NS-NS type-II supergravity vertex operators

$$\begin{aligned} g^{mn} &= \left(\Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^n_{-\frac{1}{2}} + \Psi^n_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}}\right) V^{(0)}\,\tilde{V}^{(0)}\,, \\ b^{mn} &= \left(\Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^n_{-\frac{1}{2}} - \Psi^n_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}}\right) V^{(0)}\,\tilde{V}^{(0)}\,, \\ \varphi &= \Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}}\,V^{(0)}\,\tilde{V}^{(0)}\,. \end{aligned} \tag{7.1}$$

Performing the Kaluza-Klein reduction on the $\mathrm{U}(1)_x$ circle, we set $p_L = p_R = 0$, and derive the gauge fields vertex operators

$$\begin{aligned} a^m &= \left(\Psi^x_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}} + \Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^x_{-\frac{1}{2}}\right) V^{(0)}\,\tilde{V}^{(0)}\,, \\ b^m &= \left(\Psi^x_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}} - \Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^x_{-\frac{1}{2}}\right) V^{(0)}\,\tilde{V}^{(0)}\,. \end{aligned} \tag{7.2}$$

These vertex operators can be further split into three groups, defined by the spin rotation properties in the sub-space $\alpha,\ \beta = 2, 3, 4$ transverse to the direction z^1 of propagation

[42] This is to be contrasted with discussion in section 6.7, where we compactify $\mathbb{T}^4$ and study $1+1$ dimensional field theory on the boundary.

[43] See, e.g., [62, 63] and references therein.

[44] See [31, 33] for analogous calculation in coset theories involving $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)/\mathrm{U}(1)$.

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of excitations:

$$\begin{aligned} &\text{spin-0 sound channel:} && g^{11},\ a^1,\ b^1\,, \\ &\text{spin-1 shear channel:} && g^{1\alpha},\ b^{1\alpha},\ a^\alpha,\ b^\alpha\,, \\ &\text{spin-2 scalar channel:} && g^{\alpha\beta},\ b^{\alpha\beta},\ \varphi\,. \end{aligned} \tag{7.3}$$

Notice that due to rotation symmetry, operators from different groups do not couple to each other.

Furthermore, the B-field excitations and the b gauge field mix with the graviton excitations and the a gauge field. These can be decoupled from each other into the left-moving and right-moving sectors as follows:

$$\begin{aligned} &\text{spin-0 sound channel:} && \begin{cases} W^1 = \frac{1}{2}\,(a^1 + b^1),\ g^{11} \\ U^1 = \frac{1}{2}\,(a^1 - b^1) \end{cases} \\ &\text{spin-1 shear channel:} && \begin{cases} W^\alpha = \frac{1}{2}\,(a^\alpha + b^\alpha),\ S^{1\alpha} = \frac{1}{2}(g^{1\alpha} + b^{1\alpha}) \\ U^\alpha = \frac{1}{2}\,(a^\alpha - b^\alpha),\ R^{1\alpha} = \frac{1}{2}(g^{1\alpha} - b^{1\alpha}) \end{cases} \\ &\text{spin-2 scalar channel:} && \begin{cases} S^{\alpha\beta} = \frac{1}{2}(g^{\alpha\beta} + b^{\alpha\beta}) \\ R^{\alpha\beta} = \frac{1}{2}(g^{\alpha\beta} - b^{\alpha\beta}) \end{cases} \end{aligned} \tag{7.4}$$

Here we have (recall that $m,\, n = 1, 2, 3, 4$, and $\alpha,\, \beta = 2, 3, 4$)

$$W^m = \Psi^x_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}}\,V^{(0)}\,\tilde{V}^{(0)}\,, \qquad U^m = \Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^x_{-\frac{1}{2}}\,V^{(0)}\,\tilde{V}^{(0)}\,, \tag{7.5}$$

$$S^{mn} = \Psi^m_{-\frac{1}{2}}\,\tilde{\Psi}^n_{-\frac{1}{2}}\,V^{(0)}\,\tilde{V}^{(0)}\,, \qquad R^{mn} = \Psi^n_{-\frac{1}{2}}\,\tilde{\Psi}^m_{-\frac{1}{2}}\,V^{(0)}\,\tilde{V}^{(0)}\,. \tag{7.6}$$

Non-trivial two-point correlation functions in the sound channel are then given by

$$\begin{aligned} \langle W^1 W^1\rangle_0 &= \langle \tilde{\Psi}^1_{\frac{1}{2}}\tilde{\Psi}^1_{-\frac{1}{2}}\rangle_0\,\langle \Psi^x_{\frac{1}{2}}\Psi^x_{-\frac{1}{2}}\rangle_0\,, \\ \langle g^{11} W^1\rangle_0 &= \langle \Psi^1_{\frac{1}{2}}\Psi^x_{-\frac{1}{2}}\rangle_0\,\langle \tilde{\Psi}^1_{\frac{1}{2}}\tilde{\Psi}^1_{-\frac{1}{2}}\rangle_0\,, \\ \langle g^{11} g^{11}\rangle_0 &= \langle \Psi^1_{\frac{1}{2}}\Psi^1_{-\frac{1}{2}}\rangle_0\,\langle \tilde{\Psi}^1_{\frac{1}{2}}\tilde{\Psi}^1_{-\frac{1}{2}}\rangle_0\,, \end{aligned} \tag{7.7}$$

and

$$\langle U^1 U^1\rangle_0 = \langle \tilde{\Psi}^x_{\frac{1}{2}}\tilde{\Psi}^x_{-\frac{1}{2}}\rangle_0\,\langle \Psi^1_{\frac{1}{2}}\Psi^1_{-\frac{1}{2}}\rangle_0\,. \tag{7.8}$$

Similarly, the shear channel non-vanishing correlation functions are given by

$$\begin{aligned} \langle W^\alpha W^\beta\rangle_0 &= \langle \tilde{\Psi}^\alpha_{\frac{1}{2}}\tilde{\Psi}^\beta_{-\frac{1}{2}}\rangle_0\,\langle \Psi^x_{\frac{1}{2}}\Psi^x_{-\frac{1}{2}}\rangle_0\,, \\ \langle S^{1\alpha} S^{1\beta}\rangle_0 &= \langle \Psi^1_{\frac{1}{2}}\Psi^1_{-\frac{1}{2}}\rangle_0\,\langle \tilde{\Psi}^\alpha_{\frac{1}{2}}\tilde{\Psi}^\beta_{-\frac{1}{2}}\rangle_0\,, \\ \langle W^\alpha S^{1\beta}\rangle_0 &= \langle \tilde{\Psi}^\alpha_{\frac{1}{2}}\tilde{\Psi}^\beta_{-\frac{1}{2}}\rangle_0\,\langle \Psi^x_{\frac{1}{2}}\Psi^1_{-\frac{1}{2}}\rangle_0\,, \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} \langle U^\alpha U^\beta\rangle_0 &= \langle \Psi^\alpha_{\frac{1}{2}}\Psi^\beta_{-\frac{1}{2}}\rangle_0\,\langle \tilde{\Psi}^x_{\frac{1}{2}}\tilde{\Psi}^x_{-\frac{1}{2}}\rangle_0\,, \\ \langle R^{1\alpha} R^{1\beta}\rangle_0 &= \langle \tilde{\Psi}^1_{\frac{1}{2}}\tilde{\Psi}^1_{-\frac{1}{2}}\rangle_0\,\langle \Psi^\alpha_{\frac{1}{2}}\Psi^\beta_{-\frac{1}{2}}\rangle_0\,, \\ \langle U^\alpha R^{1\beta}\rangle_0 &= \langle \Psi^\alpha_{\frac{1}{2}}\Psi^\beta_{-\frac{1}{2}}\rangle_0\,\langle \tilde{\Psi}^x_{\frac{1}{2}}\tilde{\Psi}^1_{-\frac{1}{2}}\rangle_0\,. \end{aligned} \tag{7.10}$$

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In these expressions we have $\langle \Psi^{\alpha}_{\frac{1}{2}} \Psi^{\beta}_{-\frac{1}{2}} \rangle_0 = \frac{1}{2}\,\delta^{\alpha\beta}$, $\langle \tilde{\Psi}^{\alpha}_{\frac{1}{2}} \tilde{\Psi}^{\beta}_{-\frac{1}{2}} \rangle_0 = \frac{1}{2}\,\delta^{\alpha\beta}$, while the rest of the two-point functions are given by (6.76) and its anti-holomorphic counterparts. These correlation functions were shown in section 6.6 to possess simple poles at $j = m$ and $j = -\bar{m}$.

Let us locate the gapless shear diffusion mode. At zero propagation velocity, for the mass M, pressure P, volume V, and viscosity η, we have (for the dimensionless ω, p, measured in units of string length)

$$\omega = -\frac{i\eta}{(M+P)/V}\,p^2\,. \tag{7.11}$$

Defining entropy density $s = \mathcal{S}/V$, taking into account that due to (4.24) the pressure vanishes, $P = \mathcal{F} = 0$, and using expressions for the mass $M = \mathcal{E}$ given by (3.111), and entropy $\mathcal{S}$ given by (4.10), we obtain[45]

$$\frac{\eta}{s} = \frac{\mathcal{E}}{\mathcal{S}}\,\frac{\omega}{-ip^2} = \frac{1}{\pi\sqrt{k}\,(a_1+b_1)}\,\frac{\omega}{-i\,\ell_s\,p^2}\,. \tag{7.12}$$

At the same time, the mass-shell condition (6.56) at $N = \bar{N} = 1/2$ gives

$$\frac{\ell_s^2\,\kappa^2}{4} = \frac{j(j+1) - j'(j'+1)}{k} + \frac{\ell_s^2\,p^2}{4} \tag{7.13}$$

while the BRST constraints (6.42) impose

$$\frac{\ell_s\,\kappa}{2} = \frac{m\,a_1 + m'\,a_2}{\sqrt{k}} = -\frac{\bar{m}\,b_1 + \bar{m}'\,b_2}{\sqrt{k}}\,. \tag{7.14}$$

Using results of section 6.4 and performing Wick rotation, we obtain $\kappa = -i\omega$. From (7.14) we can therefore express

$$m = -\frac{1}{a_1}\left(i\,\frac{\ell_s\,\sqrt{k}\,\omega}{2} + m'\,a_2\right)\,, \qquad \bar{m} = \frac{1}{b_1}\left(i\,\frac{\ell_s\,\sqrt{k}\,\omega}{2} + \bar{m}'\,b_2\right)\,, \tag{7.15}$$

while from (7.13) we derive

$$j = \frac{1}{2}\left(-1 + \sqrt{1 + 4j'(j'+1) - k\,\ell_s^2\,(p^2+\omega^2)}\right)\,. \tag{7.16}$$

As discussed above, excitation modes are located at $j = m$ and $j = -\bar{m}$, coming from the holomorphic and the anti-holomorphic sector respectively. At $j' = 0$, $m' = 0$, these give rise to the modes

$$\omega_L = -i\,\frac{\sqrt{k}\,a_1}{2}\,\ell_s\,p^2\,, \qquad \omega_R = -i\,\frac{\sqrt{k}\,b_1}{2}\,\ell_s\,p^2\,, \tag{7.17}$$

Using this in (7.12) we obtain

$$\frac{\eta_L}{s} = \frac{a_1}{2(a_1+b_1)\pi}\,, \qquad \frac{\eta_R}{s} = \frac{b_1}{2(a_1+b_1)\pi}\,. \tag{7.18}$$

Analogous systems of two decoupled liquids has been found in [31].

[45] We hope that recycling the same letter η, used before as a parameter of BRST transformation (6.12), does not cause a confusion.

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7.2 Correlation functions on two-dimensional boundary

In this subsection, we will compute correlation functions of operators in momentum space of the space-time theory dual to string theory on the coset (3.1) in the pure NS-NS sector. Unlike the previous section, we will work with the setup where the sub-space $\mathbb{T}^4$ has been compactified. The dual boundary theory is therefore two-dimensional.

Since the space-time theory is not a local field theory,[46] it is not immediately clear if there exist local operators on the boundary theory. To get around this issue, we restrict ourselves to correlation functions of operators in momentum space. One can, of course, Fourier-transform the momentum space correlation function to position space but the interpretation of such position space two-point correlation function is not clear.

To start with, let us review the computation of two-point correlation function of local operators in CFT_2 dual to superstring theory in $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$. A large class of scalar observables of string theory in $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ in the $(-1,-1)$ picture is given by the expression [8]

$$\mathcal{O}(x,\bar{x}) = \int_\Sigma d^2z\, e^{-\varphi-\bar{\varphi}}\, \Phi_h(z,\bar{z};x,\bar{x})\, V_{su}(z,\bar{z})\, V_{\mathbb{T}^4}(z,\bar{z})\,, \tag{7.19}$$

where $x,\bar{x}$ are coordinates of the space-time theory, $z,\bar{z}$ are coordinates of the world-sheet Riemann surface Σ, $\varphi,\bar{\varphi}$ are the world-sheet superconformal ghosts, Φ_h is the usual non-normalizable vertex operator of the world-sheet theory on AdS_3 with world-sheet dimensions $(\Delta_h,\bar{\Delta}_h)$ where

$$\Delta_h = \bar{\Delta}_h = -\frac{h(h-1)}{k}\,, \tag{7.20}$$

V_{su} is the $\mathcal{N}=1$ superconformal primary of the CFT on $\mathbb{S}^3$ with dimensions $(\Delta_{su},\bar{\Delta}_{su})$, and $V_{\mathbb{T}^4}$ is the $\mathcal{N}=1$ superconformal primary of the CFT on $\mathbb{T}^4$ with dimensions $(\Delta_{\mathbb{T}^4},\bar{\Delta}_{\mathbb{T}^4})$. The local operator $\mathcal{O}(x,\bar{x})$, given by (7.19), of the space-time theory dual to string theory in AdS_3 has dimensions (h,h). For simplicity let us assume $\Delta_{su}=\bar{\Delta}_{su}$ and $\Delta_{\mathbb{T}^4}=\bar{\Delta}_{\mathbb{T}^4}$.

The onshell condition of the operator (7.19) reads

$$-\frac{h(h-1)}{k} + \Delta_{su} + \Delta_{\mathbb{T}^4} = \frac{1}{2}\,. \tag{7.21}$$

The vertex operators V_{su} and $V_{\mathbb{T}^4}$ are normalized so that

$$\langle V_{su}(z_1)V_{su}(z_2)\rangle = \frac{1}{|z_1-z_2|^{4\Delta_{su}}}\,, \quad \langle V_{\mathbb{T}^4}(z_1)V_{\mathbb{T}^4}(z_2)\rangle = \frac{1}{|z_1-z_2|^{4\Delta_{\mathbb{T}^4}}}\,, \tag{7.22}$$

while the world-sheet operators $\Phi_h(z;x)$ are normalized so that

$$\langle \Phi_h(z_1;x_1)\Phi_{h'}(z_2;x_2)\rangle = \delta(h-h')\frac{B(h)}{|z_1-z_2|^{4\Delta_h}|x_1-x_2|^{4h}}\,, \tag{7.23}$$

where the coefficient $B(h)$ is given in (2.23). Taking the Fourier transform in $x,\bar{x}$ one recovers the correlation function in momentum space $p,\bar{p}$,

$$\langle \Phi_h(z_1;p)\Phi_{h'}(z_2;-p)\rangle = \delta(h-h')\pi B(h)\gamma(1-2h)\left(\frac{p^2}{4}\right)^{2h-1}\frac{1}{|z_1-z_2|^{4\Delta_h}}\,, \tag{7.24}$$

[46] The space-time theory is non-local (LST) in the sense that it asymptotes to a linear dilaton background and the UV behavior of the boundary theory is not described by a fixed point.

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where $\gamma(x)$ is defined in (2.23) and $p^2 = p\bar{p}$. Thus the two point function of the operator $\mathcal{O}(p)$ of the space-time theory takes the form

$$\langle \mathcal{O}(p,\bar{p})\mathcal{O}(-p,-\bar{p})\rangle = \pi(2h-1)B(h)\gamma(1-2h)\left(\frac{p^2}{4}\right)^{2h-1} . \tag{7.25}$$

Fourier transforming (7.25) back to position space, one obtains

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \simeq \frac{1}{|x_1-x_2|^{4h}} . \tag{7.26}$$

Next let us focus on computation of two-point function of operators in momentum space of the boundary theory dual to string theory in the background (3.1). As stated earlier, the boundary field theory is not conformal because the bulk space-time does not asymptote to AdS_3. One can write the $(-1,-1)$ picture operators in momentum space in the coset construction as

$$\mathcal{O}(p) = \int d^2z\, e^{-\varphi-\bar{\varphi}}\, \Phi_{h_p}(z;p)\, V_{su}(z)\, e^{il_s p_x x}\, e^{i\ell_s\kappa w}\, V_{\mathbb{T}^4}(z) . \tag{7.27}$$

The onshell condition is given by

$$-\frac{h_p(h_p-1)}{k} + \Delta_{su} + \frac{\ell_s^2(p_x^2-\kappa^2)}{4} + \Delta_{\mathbb{T}^4} = \frac{1}{2} , \tag{7.28}$$

where (h_p, h_p)[47] is the space-time dimension of Φ_{h_p}, and $h = \lim_{|p|\to 0} h_p$. (Additionally, the BRST constraints (6.42) need to be satisfied.) Then from (7.28) we derive

$$2h_p - 1 = \sqrt{(2h-1)^2 + \ell_s^2 k p^2} , \tag{7.29}$$

where the momentum of the space-time theory is given by

$$p^2 = p_x^2 - \kappa^2 . \tag{7.30}$$

Notice that, as before, the Φ_{h_p}, V_{su} and $V_{\mathbb{T}^4}$ are vertex operators of the CFT on AdS_3, SU(2), and $\mathbb{T}^4$ with the two-point functions given by (7.24) (with h replaced by h_p), and (7.22). Thus, for string theory on (3.1), the two-point function of the dual space-time theory is given by

$$\langle \mathcal{O}(p,\bar{p})\mathcal{O}(-p,-\bar{p})\rangle = \pi(2h_p-1)B(h)\gamma(1-2h_p)\left(\frac{p^2}{4}\right)^{2h_p-1} . \tag{7.31}$$

This matches exactly with the two-point function computed in similar coset backgrounds that asymptotes to a linear-dilaton background [32, 64]. As expected, in the deep IR, *i.e.*, $|\ell_s p| \ll 1$, one recovers the CFT_2 correlation function (7.25). In the deep UV,

[47] One can interpret (h_p, h_p) as the dimension of the operator $\mathcal{O}(p)$ in the space-time theory dual to string theory in the coset (3.1) (see [42, 43] on related discussion on the momentum dependent dimension of operators in non-local CFTs).

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i.e., $|\ell_s p| \gg 1$, the two-point function (7.31), however, behaves as $|p|^{2\sqrt{k}\ell_s|p|}$ which is a highly non-local behavior.

The normalization of the two-point function of primary operators in a CFT (7.25) can be changed by rescaling the operators by an arbitrary analytic function of their dimensions. Physics remains invariant due to such a rescaling of the operators. In the coset theory, however, it seems that the two-point function depends on the normalization of the operators due the momentum dependence of h_p. At this point, it is unclear how to fix the normalization of the operators of the space-time theory.

Let us end this section by giving a brief sketch of the general structure of the three-point correlation function of the operator (7.27) on the boundary theory in momentum space, in the un-spectrally flowed sector. Analogously to calculation of the two-point function, we determine

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = \tilde{C}(h_{p_1}, h_{p_2}, h_{p_3}) C_{sl}(h_{p_1}, h_{p_2}, h_{p_3}) C_{su}(j'_i, m'_i, \bar{m}'_i) \tag{7.32}$$

where OPE coefficients due to $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ factors are given by (2.26), (2.56), and

$$\tilde{C}(h_1,h_2,h_3)=(2\pi)^2\delta^2(p_1+p_2+p_3)\int d^2x_1 d^2x_2 \frac{e^{-ip_1.x_1}e^{-ip_3.x_3}}{|x_{12}|^{2(h_1+h_2-h_3)}|x_2|^{2(h_2+h_3-h_1)}|x_1|^{2(h_3+h_1-h_2)}}, \tag{7.33}$$

where

$$2h_{p_i} - 1 = \sqrt{(2h_i-1)^2 + \ell_s^2 k p_i^2}\,. \tag{7.34}$$

8 Discussion

In this paper, we have investigated type-II superstring theory on the coset space (3.1). It is described by the superconformal gWZW model at level k, such that $R = \sqrt{k}\,\ell_s$ is the characteristic length scale of the target space-time. We have derived an exact (in k) ten-dimensional target space-time background configuration of the NS-NS fields, including the metric, B-field, and dilaton. This background interpolates between local $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ in the IR and linear dilaton times $\mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{T}^4$ in the UV. It possesses an event horizon, with non-vanishing angular velocities along two of the $\mathbb{S}^3$ directions, as well as along the $\mathbb{S}^1$ circle. We have calculated the corresponding temperature, entropy, energy, conserved angular momenta, and free energy. Whenever applicable, our results reproduce the known expressions for thermodynamics of (charged) black brane backgrounds obtained in the near-horizon limit of the NS5-F1 systems, and described by coset models that can be found as particular cases of (3.1).

We have also constructed a nine-dimensional background obtained by performing the KK reduction of the ten-dimensional background on the $\mathrm{U}(1)_x$ circle $\mathbb{S}^1$. Such a background is particularly relevant in the context of holographic description of strongly-coupled systems at finite density. This is due to the fact that KK reduction on $\mathbb{S}^1$ generates non-trivial profiles of gauge fields, that furnish holographic dual of non-vanishing densities of conserved $\mathrm{U}(1)$ charges of the boundary field theory. In our case, the role of the boundary theory is played by LST found in the asymptotic linear-dilaton region of the bulk space-time, and characterized by non-vanishing vacuum expectations of two independent $\mathrm{U}(1)$ charges.

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Due to exact solvability of the corresponding gWZW model, the bulk theory enjoys an exact in k world-sheet control, going beyond the effective supergravity approximation $R/\ell_s \gg 1$. In particular, it allows one to access the regime of dual field theory characterized by a finite number of degrees of freedom, determined by k. We take advantage of this power of exact world-sheet theory to calculate two-point correlation functions of massless NS-NS vertex operators describing the graviton, B-field, and dilaton excitations on top of the constructed background. Holographically, these give us correlation matrices of the stress-energy tensor and two U(1) currents of the dual field theory. These correlation matrices manifest a rich structure of collective density excitations, that include gapless diffusion modes. We derive the corresponding dispersion relations and comment on their hydrodynamic interpretation.

We also derived the two- and three-point correlation functions of the scalars in the $1+1$-dimensional boundary theory in momentum space. The expressions for the correlation functions take the form of CFT_2 correlation functions with the dimensions h being replaced by h_p (7.29). (In fact, such a prescription seem to hold for higher-order correlation functions as well.) At low energies we recover the CFT_2 correlators, but at high energies the correlators seem to develop highly non-local behavior. This is in harmony with the fact the short-distance physics is governed by LST rather than a local fixed point. However, the correlation functions at an arbitrary point on the RG flow seem to depend on the normalization of the operators at the IR fixed point. In this paper we do not provide a recipe to fix the normalization ambiguity in studying holography in non-AdS backgrounds. It would be nice to see if the Ward identities of the theory at a generic point on the RG flow can impose constraints in the normalization. It would also be interesting to perform an LSZ reduction on the correlators and keep only contributions that give rise to physical onshell particles. This may shed some light on the normalization of the operators of the boundary theory.

One of the questions left unanswered in this paper concerns microscopic interpretation of the obtained background in terms of extended string theory objects. Particularly, we expect that the background configuration that we constructed in this paper can be found in the near-horizon limit of a certain system involving separated rotating NS5-brane in the double-scaling limit [65, 66] and fundamental F1-strings with non-trivial momentum quantum numbers. We believe this question can be resolved in a manner analogous to [32] that recently provided an embedding of the $(\text{SL}(2,\mathbb{R})\times\text{U}(1))/\text{U}(1)$ target-space background as a near-horizon limit of a certain NS5-F1 system. We would like to have an exact map between the gauge-invariant operators of the boundary theory and the world-sheet theory and identify the operators in the chiral ring algebra of the boundary theory.

The supergravity background obtained in this paper is characterized by asymptotically linear dilaton geometry. In order to describe this background as a near-horizon limit of NS5-F1 system, it would be helpful to embed it into asymptotically flat space-time. We expect that such an embedding can be done in the spirit of [67] which considered D1-D5 system. We leave this for future work.

Our work paves the avenue for various future research. In section 7.2 we computed the correlation function of scalar operators in the momentum space. It would be an interesting to check if the same correlation functions are recoverable from the supergravity analysis.

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As stated earlier, the boundary field theory is a certain vacua of LST that flows to a CFT_2 in the IR. At a generic point on the RG flow, the boundary field theory is not local. It has been shown before [68, 69] that in similar non-local setups, the Ryu-Takayanagi curve develops non-local features below a certain scale (non-locality scale proportional to ℓ_s). It would be interesting to understand the non-local features of boundary theory through the lenses of holographic entanglement entropy. In the same spirit it is worth computing the holographic complexity where we expect to observe exotic divergences reflecting the non-local features of LST. It would also be nice to set up a computation of holographic Wilson-loop to understand the potential energy between a probe quark-anti-quark pair in the boundary theory [70]. From the analytic structure of the quark-anti-quark potential energy, one may expect to understand the various phases of the theory.

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A Conventions and review

In this appendix, we review some well-known facts about the WZW models that are relevant for this paper. We parametrize the two-dimensional Euclidean string world-sheet by complex coordinates $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$. The world-sheet metric in the $\sigma^{1,2}$ coordinate frame is given by $h_{ab} = \delta_{ab}$, $a, b = 1, 2$, while in the complex coordinates $z, \bar{z}$ frame we have $h_{z\bar{z}} = h_{\bar{z}z} = 1/2$, $h^{z\bar{z}} = h^{\bar{z}z} = 2$. Consider the field $g(z, \bar{z})$, taking values on the group manifold $\mathbb{G}$. The WZW action for this field, at the level $k = 1$, is given by the expression

$$\mathtt{S}[g] = S_\sigma[g] + S_{\text{WZ}}[g]\,, \tag{A.1}$$

where the sigma-model action is given by

$$S_\sigma[g] = \frac{1}{4\pi}\int d^2z\,\text{Tr}\left(g^{-1}\partial g g^{-1}\bar{\partial}g\right)\,, \tag{A.2}$$

while the Wess-Zumino (WZ) term is defined by

$$S_{\text{WZ}}[g] = \frac{i}{24\pi}\int_{\mathcal{B}}\text{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)\,, \tag{A.3}$$

and the trace Tr is taken in the given representation of $\mathbb{G}$. We define the measure of integration as $d^2z \equiv d\sigma^1 d\sigma^2$. The integral in (A.3) is taken over a three-dimensional compact space $\mathcal{B}$ with the boundary given by the string world-sheet.

By taking a variation δg of the field g we derive from (A.2)

$$\delta S_\sigma = -\frac{1}{4\pi}\int d^2z\,\text{Tr}\left(g^{-1}\delta g\left(\partial(g^{-1}\bar{\partial}g\right) + \bar{\partial}\left(g^{-1}\partial g\right)\right)\,. \tag{A.4}$$

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At the same time, from the WZ action (A.3) we derive

$$\delta S_{\rm WZ} = \frac{i}{24\pi}\int_{\mathcal{B}} 3\,{\rm Tr}\left((-g^{-1}\delta g g^{-1}dg + g^{-1}d\delta g)\wedge g^{-1}dg\wedge g^{-1}dg\right)\,, \tag{A.5}$$

Performing cyclical permutation under the trace, while taking into account anti-symmetric property of the wedge product, we can re-arrange the first term in the r.h.s. of (A.5), obtaining as a result

$$\delta S_{\rm WZ} = \frac{i}{8\pi}\int_{\mathcal{B}} {\rm Tr}\left(-g^{-1}dg\wedge g^{-1}\delta g g^{-1}dg\wedge g^{-1}dg + g^{-1}d\delta g\wedge g^{-1}dg\wedge g^{-1}dg\right)\,. \tag{A.6}$$

Using

$$d(g^{-1}dg\wedge g^{-1}dg) = 0\,, \tag{A.7}$$

we can subsequently re-write (A.6) as

$$\delta S_{\rm WZ} = \frac{i}{8\pi}\int_{\mathcal{B}} d\,{\rm Tr}\left(g^{-1}\delta g\, g^{-1}dg\wedge g^{-1}dg\right) = \frac{i}{8\pi}\int {\rm Tr}\left(g^{-1}\delta g\, g^{-1}dg\wedge g^{-1}dg\right)\,. \tag{A.8}$$

Substituting here $dg = \partial g\, dz + \bar\partial g\, d\bar z$, and using

$$dz\wedge d\bar z = -2i\, d\sigma^1\wedge d\sigma^2 \equiv -2i\, d^2\sigma \equiv -2i\, d^2 z\,, \tag{A.9}$$

we obtain

$$\delta S_{\rm WZ} = \frac{1}{4\pi}\int d^2 z\,{\rm Tr}\left(g^{-1}\delta g\,(g^{-1}\partial g g^{-1}\bar\partial g - g^{-1}\bar\partial g g^{-1}\partial g)\right)\,, \tag{A.10}$$

that we finally re-write as

$$\delta S_{\rm WZ} = \frac{1}{4\pi}\int d^2 z\,{\rm Tr}\left(g^{-1}\delta g\,(\bar\partial(g^{-1}\partial g) - \partial(g^{-1}\bar\partial g)\right)\,. \tag{A.11}$$

Combining (A.4), (A.11), we obtain variation of the WZW action (A.1) determined by the expression

$$\delta\mathtt{S} = -\frac{1}{2\pi}\int d^2 z\,{\rm Tr}\left(g^{-1}\delta g\,\partial(g^{-1}\bar\partial g)\right)\,. \tag{A.12}$$

In particular, for the variation determined by a single local parameter $\xi(z,\bar z)$ and the generators $T_{L,R}$ belonging to a $u(1)$ sub-algebra of the group algebra of $\mathbb{G}$,

$$\delta g(z,\bar z) = \xi(z,\bar z)\,(T_L g + g T_R)\,, \tag{A.13}$$

we obtain

$$\delta\mathcal{S} = -\frac{1}{2\pi}\int d^2 z\,\xi\,\left(\partial\tilde{\mathcal{J}} + \bar\partial\mathcal{J}\right)\,, \tag{A.14}$$

where we have defined the (anti-)holomorphic U(1) currents at the level $k=1$ as

$$\mathcal{J} = {\rm Tr}\left(T_L\,\partial g g^{-1}\right) \qquad \tilde{\mathcal{J}} = {\rm Tr}\left(T_R\, g^{-1}\bar\partial g\right)\,. \tag{A.15}$$

Using (A.13) we derive the following gauge transformations of these currents,

$$\delta\mathcal{J} = \left({\rm Tr}(T_L^2) + {\rm Tr}(T_L g T_R g^{-1})\right)\,\partial\xi\,, \qquad \delta\tilde{\mathcal{J}} = \left({\rm Tr}(T_R^2) + {\rm Tr}(T_L g T_R g^{-1})\right)\,\bar\partial\xi\,. \tag{A.16}$$

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From expression (A.12) one can see that it is possible to gauge the corresponding U(1) sub-group of $\mathbb{G}$ parametrized by ξ. To that end, introduce auxiliary (non-dynamical) gauge fields A, $\tilde{A}$, that transform as

$$\delta A = -\partial\xi\,, \qquad \delta\tilde{A} = -\bar{\partial}\xi \tag{A.17}$$

under the infinitezimal gauge transformation, and define the gauged WZW action as

$$\mathcal{S}_g = \mathtt{S} + \frac{1}{2\pi}\int d^2z\,\left(A\tilde{\mathcal{J}} + \tilde{A}\mathcal{J} + \mathcal{M}'\,A\tilde{A}\right)\,, \tag{A.18}$$

where $\mathcal{M}'$ is a gauge-invariant function of g to be determined below. Algebraic equations of motion for the gauge fields A, $\tilde{A}$ that follow from the action (A.18) are given by

$$\mathcal{J} + \mathcal{M}'\,A = 0\,, \qquad \tilde{\mathcal{J}} + \mathcal{M}'\,\tilde{A} = 0\,. \tag{A.19}$$

Demanding that the equations (A.19) are invariant w.r.t. the gauge transformation, and using (A.16), (A.17), we recover the anomaly cancellation condition $\mathrm{Tr}(T_L^2) = \mathrm{Tr}(T_R^2)$, as well as the gauge-invariant object $\mathcal{M}' = \mathrm{Tr}(T_L^2) + \mathrm{Tr}(T_R g^{-1} T_L g)$. It is convenient to use the following normalization of the $u(1)$ generators

$$\mathrm{Tr}(T_L^2) = \mathrm{Tr}(T_R^2) = 2\,, \tag{A.20}$$

and define

$$\mathcal{M}' = \mathcal{M} + 2\,, \qquad \mathcal{M} = \mathrm{Tr}(T_R g^{-1} T_L g)\,. \tag{A.21}$$

Gauge transformations (A.16) can then be re-written as

$$\delta\mathcal{J} = (2+\mathcal{M})\,\partial\xi\,, \qquad \delta\tilde{\mathcal{J}} = (2+\mathcal{M})\,\bar{\partial}\xi\,. \tag{A.22}$$

Since we have explicitly established that e.o.m. (A.19) are gauge-invariant, variation of non-WZW terms in the action (A.18) can be obtained either before or after integrating out non-dynamical fields A, $\tilde{A}$. Let us integrate these out, arriving at

$$\mathcal{S}_g = \mathtt{S} - \frac{1}{2\pi}\int d^2z\,\frac{\mathcal{J}\tilde{\mathcal{J}}}{\mathcal{M}+2}\,. \tag{A.23}$$

Using (A.14), (A.22), we then observe that the action (A.23) is indeed gauge-invariant.

A.1 WZW model on SU(2)

Consider the following parametrization of SU(2) group element,[48]

$$g = e^{i\alpha\sigma_3}e^{i\theta\sigma_1}e^{i\beta\sigma_3}\,, \tag{A.24}$$

where $\alpha\in[0,\pi]$, $\theta\in[0,\frac{\pi}{2}]$, $\beta\in[0,2\pi]$ are real-valued parameters, and the Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,. \tag{A.25}$$

[48] Performing a constant shift of parameters $\alpha \to \alpha - \pi/4$, $\beta \to \beta + \pi/4$ in (A.24) we obtain $g = e^{i\alpha\sigma_3}e^{i\theta\sigma_2}e^{i\beta\sigma_3}$ as another parametrization of SU(2).

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This can also be written as

$$g = \begin{pmatrix} a+ib & ic-d \\ ic+d & a-ib \end{pmatrix} , \tag{A.26}$$

where the $\mathrm{SU}(2) \sim S^3$ manifold $a^2+b^2+c^2+d^2=1$ is parametrized by

$$a = \cos\theta\,\cos\gamma_+ , \quad b = \cos\theta\,\sin\gamma_+ , \quad c = \sin\theta\,\cos\gamma_- , \quad d = \sin\theta\,\sin\gamma_- , \tag{A.27}$$

for $\gamma_\pm = \alpha \pm \beta$. After a straightforward calculation we obtain

$$S_\sigma[g] = \frac{1}{2\pi}\int d^2z\, G_{\mu\nu}\partial X^\mu \bar\partial X^\nu , \tag{A.28}$$

where we denoted the target-space coordinate as $X^\mu = (\alpha, \theta, \beta)$, and defined the corresponding background metric as

$$G_{\mu\nu} = k \begin{pmatrix} -1 & 0 & -\cos(2\theta) \\ 0 & -1 & 0 \\ -\cos(2\theta) & 0 & -1 \end{pmatrix} \tag{A.29}$$

For the representation (A.24) we also obtain

$$\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) = 6d\,\left(\cos(2\theta)\, d\alpha \wedge d\beta\right) , \tag{A.30}$$

and therefore the Wess-Zumino term induces the background B-field via

$$S_{\mathrm{WZ}}[g] = \frac{1}{2\pi}\int d^2z\, B_{\mu\nu}\partial X^\mu \bar\partial X^\nu , \tag{A.31}$$

where we denoted

$$B_{\mu\nu} = k \begin{pmatrix} 0 & 0 & \cos(2\theta) \\ 0 & 0 & 0 \\ -\cos(2\theta) & 0 & 0 \end{pmatrix} \tag{A.32}$$

We conclude this section by writing down expression for the volume of the unit 3-sphere. Using negative of the metric (A.29), which is achieved by selecting the level $k=-1$, we obtain

$$V_{\mathbb{S}^3} = \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{\frac{\pi}{2}} d\theta\, \sqrt{-\det G} = \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{\frac{\pi}{2}} d\theta\, \sin(2\theta) = 2\pi^2 . \tag{A.33}$$

A.2 WZW model on $\mathbf{SL(2,\mathbb{R})}$

Consider the following parametrization of $\mathrm{SL}(2,\mathbb{R})$ group element,

$$g = e^{\alpha\sigma_3} e^{\theta\sigma_1} e^{\beta\sigma_3} , \tag{A.34}$$

where α, θ, β are real-valued parameters, and Pauli matrices are as in (A.25). After a straightforward calculation we obtain

$$S_\sigma[g] = \frac{1}{2\pi}\int d^2z\, G_{\mu\nu}\partial X^\mu \bar\partial X^\nu , \tag{A.35}$$

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where we denoted the target-space coordinate as $X^\mu = (\alpha, \theta, \beta)$, and defined the corresponding background metric as

$$G_{\mu\nu} = k \begin{pmatrix} 1 & 0 & \cosh(2\theta) \\ 0 & 1 & 0 \\ \cosh(2\theta) & 0 & 1 \end{pmatrix} \tag{A.36}$$

For the representation (A.34) we also obtain

$$\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) = -6d\,\left(\cosh(2\theta)\, d\alpha \wedge d\beta\right)\,, \tag{A.37}$$

and therefore the Wess-Zumino term induces the background B-field via

$$S_{\mathrm{WZ}}[g] = \frac{1}{2\pi}\int d^2z\, B_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu\,, \tag{A.38}$$

where we denoted

$$B_{\mu\nu} = k \begin{pmatrix} 0 & 0 & -\cosh(2\theta) \\ 0 & 0 & 0 \\ \cosh(2\theta) & 0 & 0 \end{pmatrix} \tag{A.39}$$

A.3 Effective action and superconformal WZW model

We have reviewed the classical (gauged) WZW models describing bosonic string theory with the target space given by the group manifold $\mathbb{G}$. In particular, we considered examples of $\mathbb{G} = \mathrm{SL}(2,\mathbb{R})$ and $\mathbb{G} = \mathrm{SU}(2)$, and derived the corresponding classical target space background. The latter is generally specified by the metric, dilaton, and B-field.

The WZW model is characterized by the level k, such that the classical action of the WZW model is given by $S = k\,\mathtt{S}$, where the $k = 1$ action $\mathtt{S}$ is given by (A.1). Perturbative corrections to the classical theory, due to the world-sheet path integral expansion, therefore behave as $1/k$. The semi-classical limit is then achieved in the limit of large k.

In terms of the target space theory, the semi-classical limit translates into the small curvature or, equivalently, the large length scale limit. Indeed, from our discussion in sections A.1, A.2, one can see that the target space metric scales as k, that sets the characteristic length scale of the semi-classical geometry to be $\sqrt{k}\,\ell_s$.

Similar observations can be made when one considers gauged WZW models. In particular, integrating out auxiliary gauge fields, via their equations of motion, in the classical WZW action on a coset space renders a classical sigma-model. Classical background fields, including the metric, dilaton and B-field, can be read off from the sigma-model action.

We now proceed to reviewing how classical results get modified when $1/k$ corrections, due to expansion of the world-sheet path integral, are taken into account. It is illustrative to follow two different perspectives, that ultimately are shown to render consistent results. The first perspective involves the (anti-) holomorphic symmetry argument and the affine algebra relations for the corresponding currents, while the second one involves calculation of the effective action for the WZW model [71–73].

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Recall that the WZW model at level k on the group manifold $\mathbb{G}$ enjoys the $\mathbb{G}_L \times \mathbb{G}_R$ Kac-Moody symmetry. The corresponding affine currents j^a, $\tilde{j}^a$ satisfy the OPE (similar OPE can be written down for the anti-holomorphic current components $\tilde{j}^a$)

$$j^a(z)j^b(w) \simeq \frac{k}{2}\frac{k^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c\, j^c(w)}{z-w}\,, \tag{A.40}$$

where the index a takes values in the adjoint representation of $\mathbb{G}$, k_{ab} is the Cartan metric in the algebra space, and f^{abc} are the algebra structure constants. Indices are raised and lowered with the tensor k_{ab}.

The WZW model also enjoys conformal symmetry, with the generators given by the (anti-)holomorphic stress-energy tensor components T, $\tilde{T}$. These can be determined in terms of the Kac-Moody currents via the Sugawara expression as

$$T = \frac{1}{k+c_G}\, j^a\, j^a\,, \tag{A.41}$$

where the dual Coxeter number c_G is defined via[49]

$$f_{abc}f_d{}^{bc} = c_G\,\eta_{ad}\,. \tag{A.42}$$

In quantum theory the product of currents in the r.h.s. of (A.41) is normally-ordered. The overall prefactor of the r.h.s. is determitned from the requirement that j^a is a primary field of dimension one,

$$T(z)j^a(w) \simeq \frac{j^a(w)}{(z-w)^2} + \frac{\partial j^a(w)}{z-w}\,, \tag{A.43}$$

or, equivalently, that the stress-energy tensor satisfies the Virasoro OPE

$$T(z)T(w) \simeq \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \tag{A.44}$$

Expression (A.41) for the stress-energy tensor is exact in quantum world-sheet theory for all values of the level k. All of the $1/k$ corrections of the world-sheet perturbation theory are accounted for in the difference between the semi-classical expression $T = \frac{1}{k}\, j^a j^a$, and the full quantum expression (A.41). The same conclusion can be arrived at from the perspective of the effective WZW action in quantum theory [72, 73]. A non-trivial functional determinant appearing in the Legendre transform calculation of the effective action via the world-sheet path integral can be reformulated as $e^{-c_G\,\mathtt{S}[g]}$. The effective action of the WZW model is then given by

$$\Gamma[g] = (k+c_G)\,\mathtt{S}\,. \tag{A.45}$$

This is simply the original classical WZW action with the shifted level, $k \to k + c_G$. The level $k + c_G$ in the effective action results in the prefactor $1/(k+c_G)$ in the expression for

[49] Recall that $c_{\mathrm{SL}(2,\mathbb{R})} = -2$ and $c_{\mathrm{SU}(2)} = 2$. While the structure constants $f^{abc} = \epsilon^{abc}$, given by the Levi-Civita symbol, are the same for both algebras, the difference between the values of the dual Coxeter numbers can be seen to originate from the $k_{ab} = \mathrm{diag}\{1,\,1,\,-1\}$ Lorentzian metric of the $sl(2,\mathbb{R})$, versus the $k_{ab} = \mathrm{diag}\{1,\,1,\,1\}$ Euclidean metric of the $su(2)$.

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the stress-energy tensor, in agreement with (A.41), manifesting consistency of the effective action and the symmetry perspectives [72, 73].

Let us now proceed to discussion of supersymmetric WZW models. Analogously to the case of bosonic WZW models, the symmetry and the effective action perspectives can be considered [72–74].

The (anti-) holomorphic Kac-Moody symmetry and the conformal symmetry of the bosonic WZW model get upgraded to the super Kac-Moody algebra and the superconformal symmetry in the supersymmetric WZW model. To start with, the bosonic degrees of freedom represented by the affine currents j^a, $\tilde{j}^a$ acquire superpartners, given by the Majorana-Weyl fermions ψ^a, $\tilde{\psi}^a$, where index $a = 1, \ldots, \dim(\mathbb{G})$ takes values in the adjoint representation of the group $\mathbb{G}$. The fermions are free, in particular, their OPEs with the bosonic currents j^a, $\tilde{j}^a$ are non-singular. At the same time, we have

$$\psi^a(z)\psi^b(w) \simeq \frac{k}{2}\frac{k^{ab}}{z-w}\,, \tag{A.46}$$

and similarly for the anti-holomorphic sector fermions $\tilde{\psi}^a$.

The Kac-Moody currents of supersymmetric WZW model then acquire contributions both from the bosonic degrees of freedom $g \in \mathbb{G}$, via the purely bosonic currents j^a, $\tilde{j}^a$, as well as from the fermions ψ^a, $\tilde{\psi}^a$, resulting in the total affine currents

$$\mathtt{j}^a = j^a - \frac{i}{k} f^a{}_{bc}\,\psi^b\psi^c\,, \tag{A.47}$$

where the product of fermions in the second term in the r.h.s. is normally-ordered. The total bosonic currents $\mathtt{j}^a$ satisfy the Kac-Moody OPE at level k,

$$\mathtt{j}^a(z)\mathtt{j}^b(w) \simeq \frac{k}{2}\frac{k^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c\,\mathtt{j}^c(w)}{z-w}\,. \tag{A.48}$$

This implies that the purely bosonic Kac-Moody current components j^a satisfy the OPE at level $\mathtt{k} = k - c_G$,

$$j^a(z)j^b(w) \simeq \frac{k-c_G}{2}\frac{k^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c\,j^c(w)}{z-w}\,. \tag{A.49}$$

The currents $\mathtt{j}^a$, $\mathtt{j}^a$ are primary operators of dimension 1,

$$T(z)\mathtt{j}^a(w) \simeq \frac{\mathtt{j}^a(w)}{(z-w)^2} + \frac{\partial\mathtt{j}^a(w)}{z-w}\,, \tag{A.50}$$

While the purely bosonic currents j^a are decoupled from the fermions ψ^a,

$$j^a(z)\psi^b(w) = \mathcal{O}(z-w)\,, \tag{A.51}$$

the total bosonic currents $\mathtt{j}^a$ satisfy non-trivial OPEs with the fermions ψ^a,

$$\mathtt{j}^a(z)\psi^b(w) \simeq \psi^a(z)\mathtt{j}^b(w) \simeq \frac{if^{ab}{}_c\,j^c(w)}{z-w}\,. \tag{A.52}$$

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Combining everything together, relations (A.46), (A.49), (A.52) define the super Kac-Moody affine algebra.

The stress-energy tensor in supersymmetric WZW model is given by

$$T(z) = \frac{1}{k}\, j^a j^a - \frac{1}{k}\, \psi^a \partial \psi^a \,, \tag{A.53}$$

where products of fields in r.h.s. are normally-ordered. A couple of comments are in order here. First of all, we have contributions to (A.53) coming from the free fermionic sector, such that the field ψ^a is a primary of dimension 1/2,

$$T(z)\psi^a(w) \simeq \frac{1}{2}\, \frac{\psi^a(w)}{(z-w)^2} + \frac{\partial \psi^a(w)}{z-w} \,. \tag{A.54}$$

At the same time, the first term in r.h.s. of (A.53), coming from the bosonic sector, is similar to the stress-energy tensor (A.41) of the bosonic WZW model, except that the overall prefactor is $1/k$. This is due to the bosonic currents j^a in the supersymmetric theory satisfying the OPE relation (A.49) at the level $\mathtt{k} = k - c_G$; to be contrasted with the OPE relation (A.40) at level k for such currents in the bosonic WZW model. The rest of the calculation is the same as in the bosonic WZW model: the classical prefactor $1/\mathtt{k}$ in expression for the stress-energy tensor in supersymmetric WZW model changes as $1/\mathtt{k} \to 1/(\mathtt{k} + c_G) = 1/k$ in quantum theory.

Conformal symmetry of the bosonic WZW model gets upgraded to superconformal symmetry. The conformal symmetry generators T, $\tilde{T}$ become supplemented with the supersymmetry generators G, $\tilde{G}$. The latter are conformal primaries of dimension 3/2,

$$T(z)G(w) \simeq G(z)T(w) \simeq \frac{3}{2}\, \frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \,. \tag{A.55}$$

The other non-trivial OPE is satisfied by the supersymmetric generators G themselves,

$$G(z)G(w) \simeq \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} \,. \tag{A.56}$$

Together, (A.44), (A.55), (A.56) define the $N = 1$ superconformal algebra.

As a supercurrent that determines supersymmetry transformations, the operator G transforms bosonic degrees of freedom via fermionic ones, and vice versa. This is expressed in the OPEs

$$\begin{aligned} G(z)\mathtt{j}^a(w) &\simeq \frac{\psi^a(w)}{(z-w)^2} + \frac{\partial \psi^a(w)}{z-w} \,, \\ G(z)\psi^a(w) &\simeq \frac{\mathtt{j}^a(w)}{z-w} \,. \end{aligned} \tag{A.57}$$

Notice that the bosonic superpartner of the fermion ψ^a is given by the full current $\mathtt{j}^a$ rather than by its bosonic-sector component j^a. In components, the supercurrent is represented by the expression

$$G(z) = \frac{2}{k}\left(\psi^a\, j^a - \frac{i}{3k}\, f_{abc}\, \psi^a\, \psi^b\, \psi^c\right) = \frac{2}{k}\left(\psi^a\, \mathtt{j}^a + \frac{2i}{3k}\, f_{abc}\, \psi^a\, \psi^b\, \psi^c\right) \,, \tag{A.58}$$

where the products in r.h.s. are normally-ordered.

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Let us now turn to the discussion of effective action in supersymmetric WZW models. The starting point is supersymmetric action $k\, S_{\rm susy}[g, \chi^a, \tilde{\chi}^a]$ at level k. By performing a change of fermionic degrees of freedom, we can reformulate the theory in terms of decoupled free fermions ψ^a, $\tilde{\psi}^a$, discussed above. This is a chiral transformation, that produces a non-trivial functional determinant that can be reformulated as $e^{c_G\, \mathcal{S}[g]}$. The total action is then given by $\mathtt{k}\, \mathtt{S}[g] + k\, S_f[\psi, \tilde{\psi}]$, where $\mathcal{S}_f[\psi, \tilde{\psi}]$ is the action of free two-dimensional Majorana-Weyl fermions ψ^a, $\tilde{\psi}^a$ at level $k = 1$ [72, 73]. Consistently with our discussion above, we recognize the shifted level $\mathtt{k} = k - c_G$, that appeared in the OPE (A.49) of the bosonic sector Kac-Moody currents j^a.

Since the fermions ψ^a, $\tilde{\psi}^a$ are free and decoupled, the rest of derivation of the effective action proceeds similarly to the bosonic WZW case. Analogously to the effective action (A.45), going from classical supersymmetric WZW action to the effective action of supersymmetric WZW model, the level $\mathtt{k}$ gets shifted as $\mathtt{k} \to \mathtt{k} + c_G = k$. In other words, starting with the action $k\, S_{\rm susy}[g, \chi^a, \tilde{\chi}^a]$ at level k, first the fermionic determinant due to the change of variables $\chi, \tilde{\chi} \to \psi, \tilde{\psi}$ shifted the level k prefactor of the bosonic sector action by $-c_G$, that subsequently was offset by the $+c_G$ shift in calculation of the effective action, yielding at the end the total effective action

$$\Gamma_{\rm susy}[g, \psi, \tilde{\psi}] = k\, \mathtt{S}[g] + k\, S_f[\psi, \tilde{\psi}]\,. \tag{A.59}$$

Notice that the prefactor k in front of $\mathtt{S}[g]$ in r.h.s. of (A.59) is consistent with the normalization $1/k$ of the $j^a j^a$ terms in the stress-energy tensor expression (A.53). Of course the fermionic action $k\, S_f[\psi, \tilde{\psi}]$ at the level k reproduces the fermionic sector contribution to (A.53).

In the main text, whenever we write down action of (gauged) WZW model, we usually have in mind supersymmetric model, considered at level k, and write down the exact world-sheet effective action thereof. This becomes important, for instance, when we write down the action on the coset (3.1), that combines contributions from the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ sectors. Due to the $\pm c_G$ cancellations between the level k shifts, the effective action of supersymmetric model on (3.1) is at level k both in the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ sectors. While such $1/k$ corrections are unimportant in the semiclassical limit $k \to \infty$, our results are therefore in fact exact at finite k in the full quantum world-sheet theory.

A.4 Some (anti-)commutation relations

In this appendix we collect various (anti-)commutation relations satisfied by the amplitudes of the superconformal and the super-Kac-Moody currents. Specifically, we are interested in the NS sector of superstring states, and therefore all the fermionic operators are expanded in the half-integer modes.

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From the $N=1$ superconformal algebra OPEs (A.44), (A.55), (A.56) one obtains the (anti-)commutation algebra relations,

$$[L_m,\ L_n] = (m-n)\,L_{m+n} + \frac{c}{12}\,m(m^2-1)\,\delta_{m+n,0}\,, \tag{A.60}$$

$$\{G_r,\ G_s\} = 2L_{r+s} + \frac{c}{12}\,(4r^2-1)\,\delta_{r+s,0}\,, \tag{A.61}$$

$$[L_m,\ G_r] = \frac{m-2r}{2}\,G_{m+r}\,. \tag{A.62}$$

Expanding the total Kac-Moody currents as[50]

$$\mathtt{j}^M = \sum_n \frac{\mathtt{j}^M_n}{z^{n+1}}\,, \qquad \mathtt{j}^w = \sum_n \frac{\mathtt{j}^w_n}{z^{n+1}}\,, \tag{A.63}$$

and the fermions as

$$\psi^M = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\psi^M_r}{z^{r+\frac{1}{2}}}\,, \qquad \psi^w = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\psi^w_r}{z^{r+\frac{1}{2}}}\,, \tag{A.64}$$

where $M=a,\,a',\,i$ labels the $\mathrm{SL}(2,\mathbb{R})$ polarizations $a=1,2,3$, $\mathrm{SU}(2)$ polarizations $a'=1,2,3$, and $\mathbb{T}^4$ polarizations z^i, $i=1,2,3,4$, one obtains from the OPEs (A.50), (A.54)

$$\begin{aligned} [L_m,\,\mathtt{j}^M_n] &= -n\,\mathtt{j}^M_{m+n}\,, & [L_m,\,\mathtt{j}^w_n] &= -n\,\mathtt{j}^w_{m+n}\,,\\ [L_m,\,\psi^M_r] &= -\frac{m+r}{2}\,\psi^M_{m+r}\,, & [L_m,\,\psi^w_r] &= -\frac{m+r}{2}\,\mathtt{j}^w_{m+n}\,, \end{aligned} \tag{A.65}$$

and from the OPEs (A.57) we obtain

$$\begin{aligned} \{G_r,\,\mathtt{j}^M_n\} &= -n\,\psi^M_{n+r}\,, & \{G_r,\,\mathtt{j}^w_n\} &= -n\,\psi^w_{n+r}\,,\\ \{G_r,\,\psi^M_s\} &= \mathtt{j}^M_{r+s}\,, & \{G_r,\,\psi^w_s\} &= \mathtt{j}^w_{r+s}\,. \end{aligned} \tag{A.66}$$

Using the OPE expressions (3.45), (3.50), (6.24), we obtain (here $\delta^{+-}=\delta^{-+}=2$)

$$\begin{aligned} \{\psi^a_r,\,\psi^b_s\} &= \frac{k}{2}\delta^{ab}\delta_{r+s,0}\,, & \{\psi'^a_r,\,\psi'^b_s\} &= \frac{k}{2}\delta^{ab}\delta_{r+s,0}\,,\\ \{\psi^x_r,\,\psi^x_s\} &= \frac{1}{2}\delta_{r+s,0}\,, & \{\psi^w_r,\,\psi^w_s\} &= -\frac{1}{2}\delta_{r+s,0}\,, \end{aligned} \tag{A.67}$$

and similarly for the anti-holomorphic sector.

B GHY term in string frame

In this appendix, we are going to derive the GHY boundary term in string frame, that needs to be added to the action (3.60). To derive it, we first re-write the action (3.60) in Einstein frame, and identify the boundary terms necessary to allow for a well-defined variational principal. We will identify two contributing boundary terms: the usual GHY action in Einstein frame, and the extra term necessary to offset the boundary contributions due to

[50] For abelian currents the total Kac-Moody current coincides with its purely bosonic theory version, $\mathtt{j}^x = j^x$, $\mathtt{j}^w = j^w$, $\mathtt{j}^i = j^i$, $i=1,2,3,4$.

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the dilaton total derivative term. Finally, we will combine these two terms and re-write them in terms of metric in string frame.

Expressing the Ricci scalar $\tilde{R}$ in the Einstein frame in terms of its counterpart R in the string frame for the general transformation $\mathcal{G}_{\mu\nu} = e^{2\Omega}\,\tilde{\mathcal{G}}_{\mu\nu}$ we obtain in D-dimensional space-time

$$R = e^{-2\Omega}\left(\tilde{R} - 2(D-1)\nabla^2\Omega - (D-2)(D-1)(\partial\Omega)^2\right). \tag{B.1}$$

Fixing $D = 10$, $\Omega = (\Phi - \Phi_0)/4$, we obtain

$$S = \frac{1}{2\kappa^2}\int d^6x\,\tilde{V}_{\mathbb{T}^4}\sqrt{-\det\tilde{\mathcal{G}}}\left(\tilde{R} - \frac{9}{2}(\partial\Phi)^2 - \frac{9}{2}\tilde{\nabla}^2\Phi - \frac{1}{12}e^{\Phi-\Phi_0}\,\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}\right), \tag{B.2}$$

where now contraction of indices is done with the Einstein metric tensor, and tilde on top of the covariant derivative indicates that it is calculated in Einstein frame. Here we can extract the pure boundary term, at $\chi = \Lambda$, using the Stokes's theorem,

$$S \supset -\frac{9}{4\kappa^2}\int d^5x\,\sqrt{-\det\tilde{\gamma}}\,\tilde{n}_M\,\partial^M\Phi\,|_{\chi=\Lambda}\,, \tag{B.3}$$

where $\tilde{\gamma}_{\mu\nu}$ is the boundary metric in Einstein frame, and

$$\tilde{n}_M = \left(0,\ \sqrt{\tilde{\mathcal{G}}_{\chi\chi}},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right) \tag{B.4}$$

is an outward-pointing vector normal to the boundary. The gravity action in string frame is to be supplemented with the GHY boundary term,

$$S_{\rm GHY} = \frac{1}{\kappa^2}\int d^5x\,\tilde{V}_{\mathbb{T}^4}\sqrt{-\det\tilde{\gamma}}\,\tilde{\nabla}_M\tilde{n}^M\,, \tag{B.5}$$

The GHY term in Einstein frame needs to be supplemented with an extra contribution, to compensate for (B.3). In total, we obtain the boundary term in Einstein frame

$$S_b = \frac{1}{\kappa^2}\int d^5x\,\sqrt{-\det\tilde{\gamma}}\,\tilde{V}_{\mathbb{T}^4}\left(\tilde{\nabla}_M\tilde{n}^M + \frac{9}{4}\tilde{n}^\chi\,\partial_\chi\Phi\right)|_{\chi=\Lambda}\,. \tag{B.6}$$

On the other hand, in string frame we the outward-point unit normal to the boundary

$$n^M = \left(0,\ \sqrt{\mathcal{G}^{\chi\chi}},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right) = e^{\frac{\Phi_0-\Phi}{4}}\,\tilde{n}^M\,. \tag{B.7}$$

Using the relation between Christoffel symbols,

$$\Gamma^M_{M\chi} = \tilde{\Gamma}^M_{M\chi} + \frac{5}{2}\,\partial_\chi\Phi\,, \tag{B.8}$$

we obtain

$$\nabla_M n^M = e^{\frac{\Phi_0-\Phi}{4}}\left(\tilde{\nabla}_M\tilde{n}^M + \frac{9}{4}\,\tilde{n}^\chi\partial_\chi\Phi\right) \tag{B.9}$$

Introducing the boundary metric in string frame, $\gamma_{\mu\nu} = e^{\frac{\Phi-\Phi_0}{2}}\,\tilde{\gamma}_{\mu\nu}$, and combining everything together, we can re-write the total boundary term (B.6) as

$$S_b = \frac{1}{\kappa_0^2}\int d^5x\,e^{-2\Phi}\,\sqrt{-\det\gamma}\,V_{\mathbb{T}^4}\,\nabla_M n^M\,. \tag{B.10}$$

The action (B.10) therefore gives the GHY term in string frame. In the main text we will denote (B.10) as $S_{\rm GHY}$, having in mind that it is the full boundary term in the string frame.

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C Thermodynamics of $\mathbf{SL(2,\mathbb{R}) \times U(1)/U(1)}$

In this appendix, we are going to demonstrate that our new results for thermodynamics of the coset geometry (3.1) reproduce the known results for the

$$\frac{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)}{\mathrm{U}(1)} \times \mathrm{SU}(2) \times \mathbb{T}^4 \tag{C.1}$$

background, that corresponds to the limit of un-gauged SU(2) sub-group.[51] According to (3.12), this limit can be obtained by setting $a_2 = b_2 = 0$. The anomaly-free conditions (3.24) then impose two constraints $a_1^2 + a_3^2 = 1$, $b_1^2 + b_3^2 = 1$ on the four parameters of the model, $a_{1,3}$, $b_{1,3}$. The remaining two independent parameters can be expressed in terms of the angles $\psi \in [0, \pi/2]$, $\chi \in [0, \pi/2]$, as follows [51]

$$a_1 = \cos(\chi - \psi)\,, \quad b_1 = \cos\chi\,, \quad a_3 = \sin(\chi - \psi)\,, \quad b_3 = \sin\chi\,. \tag{C.2}$$

The same parameters ψ, χ are also used in the conventions of [32]. In the particular case of [30, 31, 50] one sets $\chi = 0$.[52]

We begin by writing down expression for the temperature (4.4) for the choice of parameters (C.2),

$$T = \frac{1}{\pi\sqrt{k}\,\ell_s} \frac{\cos\chi\cos(\chi - \psi)}{\cos\chi + \cos(\chi - \psi)}\,. \tag{C.3}$$

In particular, setting $\chi = 0$, (C.3) is seen to agree with the known result for the coset (C.1), that can be found, e.g., in eq. (2.26) of [50].

The entropy of (C.1) can be found in eq. (3.31) of [51][53]

$$\mathcal{S} = \pi\sqrt{k}\,\ell_s \left(\sqrt{M^2 - Q_L^2} + \sqrt{M^2 - Q_R^2}\right)\,, \tag{C.4}$$

where M is the ADM mass, and the $\mathrm{U}(1)_{L,R}$ charges are given by (C.16) of [51],[54]

$$Q_L = M\sin(\psi - \chi)\,, \qquad Q_R = M\sin\chi\,. \tag{C.5}$$

Combining (C.3), (C.4), (C.5), we then obtain

$$T\,\mathcal{S} = M\cos\chi\cos(\chi - \psi)\,. \tag{C.6}$$

This agrees with (4.10) for the choice of parameters (C.2), provided $M = \frac{4\pi^2}{\kappa^2}\, k^{\frac{3}{2}} \ell_s^2\, V_{\mathbb{S}^1}\, V_{\mathbb{T}^4}$. The latter indeed matches our result for the energy, (3.111).

[51] See [38, 75] for discussion of related thermodynamic systems with microscopic description of NS5-F1 system and single-trace $T\bar{T}$ deformation. See also the seven-parameter black hole solution in [76] with $\lambda_\pm = 0$.

[52] In conventions of [50] one also sets $\alpha \to \psi$.

[53] The extra factor of $1/\sqrt{2}$ relative to that equation is necessary once we relax the convention of $\alpha' = \ell_s^2 = 2$.

[54] In particular, in case of $\chi = 0$, we recover (2.30) of [50], with $Q_L = Q$, $Q_R = 0$. In the latter, we substitute the mass M and charge Q of the black hole, determined by eqs. (34), (36) of [30]. Specifically, one plugs $M = me^{-2\phi_0}$, $Q = qe^{-2\phi_0}$ into $\mathcal{S} = \pi\sqrt{k}\,\ell_s(M + \sqrt{M^2 - Q^2})$, where $m = \frac{1}{2}\frac{1+p^2}{1-p^2}$, $q = \frac{p}{1-p^2}$, $p = \tan(\psi/2)$, and $\phi_0 = \Phi_0 + \frac{1}{2}\log\frac{1+p^2}{2(1-p^2)}$, expressed in terms of asymptotic value of the flat space dilaton $\Phi_0 = \log(g_s)$ via (29), (35) of [30].

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The free energy is determined by

$$\mathcal{F} = M - T\mathcal{S} - \mu_L Q_L - \mu_R Q_R \,, \tag{C.7}$$

where $\mu_{L,R}$ are the left- and right-moving components of the chemical potential. As pointed out in [50], the free energy of the theory (C.1) vanishes, which is in agreement with our result (4.24) for (3.1) obtained independently of the choice of the anomaly-free gauging parameters a_i, b_i. We can in fact match expressions for the free energy term by term. We have already matched M and $T\mathcal{S}$ with our results in the limit of un-gauged SU(2). Now let us match $\mu_L Q_L + \mu_R Q_R$. The chemical potential can be found from the asymptotic behavior of the gauge field components[55]

$$A_t^{G,B} = \pm\sqrt{2}\mu^{G,B} + \dots \,, \tag{C.8}$$

where the gauge fields can be found in eq. (C.9) of [51]. Using eq. (C.9) of [51], we first determine the relation $y = \frac{t\sqrt{2}}{\sec\chi + \sec(\chi-\psi)}$ between the time coordinate y, used in that equation, and the time coordinate t, featuring asymptotic behavior of the metric $g_{tt} = -1$. We then obtain $\mu_G = \sec\left(\chi - \frac{\psi}{2}\right)\sin\left(\frac{\psi}{2}\right)$, $\mu_B = \cos\chi\tan\left(\frac{\psi}{2}\right) - \sin\chi$. Consequently

$$\begin{aligned} \mu_L &= \frac{\mu_G + \mu_B}{2} = -\frac{1}{2}\cos(\chi)\sec\left(\frac{\psi}{2}\right)\sin(\chi-\psi)\sec\left(\chi - \frac{\psi}{2}\right) \,, \\ \mu_R &= \frac{\mu_G - \mu_B}{2} = \frac{1}{2}\sin(\chi)\sec\left(\frac{\psi}{2}\right)\cos(\chi-\psi)\sec\left(\chi - \frac{\psi}{2}\right) \,. \end{aligned} \tag{C.9}$$

We then obtain

$$\mu_L Q_L + \mu_R Q_R = \frac{M}{2}\left(2 - \cos(2\chi - \psi) - \cos\psi\right) . \tag{C.10}$$

At the same time, from (4.27), using (3.111), (C.2) we obtain

$$\Omega \cdot \mathcal{J} = \mathcal{E}(1 - \cos(\chi-\psi)\cos\chi) \,, \tag{C.11}$$

that indeed agrees with (C.10) for $\mathcal{E} = M$. This completes the term-by-term match of our result for the free energy for the choice of parameters (C.2) with the known result for the (C.1).

[55] Again, the factor of $1/\sqrt{2}$ appears when we relax the convention of $\alpha' = \ell_s^2 = 2 \to \ell_s = \sqrt{2}$ used in [51]. This in turn rescales the time coordinate measured in units of ℓ_s by the factor of $1/\sqrt{2}$.

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