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Problems of Buffon Type for Large Test Bodies

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Abstract

A needle of length l dropped at random on a lattice of parallel planes at a distance d apart can have multiple intersections if $l > d$. The distribution of the number of crossings and approximate moments for large l are derived.

Similar problems are discussed by using as test body a disk of radius $r > d/2$ as test body.

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1 Introduction

In the classical formulations of the problems of Buffon-Laplace type, a "small" convex test body \mathcal{T} is thrown at random onto a lattice of parallel and equidistant lines, or of rectangles, and one asks for the probability of an intersection. By "small" test body one means that the diameter of \mathcal{T} is not larger than the equidistance between the parallel lines, or than the shortest side of the rectangle. In other words, at each fall \mathcal{T} must either intersect the series of parallel lines only once, or not at all; and in the case of the lattice of rectangles, \mathcal{T} can never intersect two non-consecutive sides an elementary tile.

When these hypotheses are not satisfied, i.e. for "large" test bodies, there can be multiple intersections and problems connected with the probability of a specific number of crossings arise. These problems have been studied by various authors in different instances: we just mention the work of Diaconis [1], who considered the case of a long needle dropped at random onto a plane ruled by a grid of parallel and equidistant lines, and the results of Duma and Stoka, who considered a large ellipse in a grid of parallel and equidistant lines [3] or in a lattice of rectangles [4].

The purpose of this paper is to discuss the three dimensional situation: hence the lattice of parallel lines changes into a lattice of parallel and equidistant planes and the lattice of rectangles becomes a lattice of strips with rectangular section. We use as test bodies a segment S and a disk D .

2. Long and non small needles

Theorem 1. Let \mathcal{R} be a lattice of equally spaced parallel planes in E_3 (we shall take the common distance between them to be d) on which a segment S of length l , $nd \leq l \leq (n+1)d$ is allowed to fall at random. At each fall the segment can intersect the series of parallel planes $0, 1, 2, \dots, n$ or $n+1$ times, and the probability $p(h)$ of h intersections occurring at a fall is given by

$$\begin{aligned} p(0) &= \frac{d}{2l} & p(h) &= \frac{d}{l} & \text{for } 1 \leq h \leq n-1 & \quad (1) \\ p(n) &= \frac{(1-n^2-2n)d}{2l} + n + 1 - \frac{l}{2d} & p(n+1) &= \frac{d}{2l} \left(n - \frac{l}{d}\right)^2 \end{aligned}$$

Proof. Let us fix an orthonormal frame of reference $\{O; x, y, z\}$ such that one of the elementary tiles of the lattice is the strip S between the planes $z = 0$ and $z = d$. We denote by ϑ the angle between the z axis and the support line of S . For each fixed value of ϑ the number of planes intersected by S can be $[l \cos \vartheta]$ or $[l \cos \vartheta] + 1$, where $[x]$ means the greatest integer not larger than x . More precisely, if we let the angles ϑ_h , $0 < \vartheta_h \leq \frac{\pi}{2}$, be determined by

$$\cos \vartheta_h = \frac{hd}{l}$$

for $h = 0, \dots, n$, and if we define $\vartheta_{n+1} := 0$, then for $\vartheta_{h+1} \leq \vartheta \leq \vartheta_h$ the number of crossings is h or $h+1$, according to the value of the z coordinate of the middle point P of S . That is, we have h intersections if P belongs to the set

$$S_{h, [\vartheta_{h+1}, \vartheta_h]} = \begin{cases} \left\{ \frac{l}{2} \cos \vartheta - \frac{h}{2}d \leq z \leq \frac{h+2}{2}d - \frac{l}{2} \cos \vartheta \right\} & h \text{ even} \\ \left\{ 0 \leq z \leq \frac{h+1}{2}d - \frac{l}{2} \cos \vartheta \vee \frac{l}{2} \cos \vartheta - \frac{h-1}{2}d \leq z \leq d \right\} & h \text{ odd} \end{cases}$$

and $h+1$ intersections when P belongs to the set

$$S_{h+1, [\vartheta_{h+1}, \vartheta_h]} = \begin{cases} \left\{ 0 \leq z \leq \frac{l}{2} \cos \vartheta - \frac{h}{2}d \vee \frac{h+2}{2}d - \frac{l}{2} \cos \vartheta \leq z \leq d \right\} & h \text{ even} \\ \left\{ \frac{h+1}{2}d - \frac{l}{2} \cos \vartheta \leq z \leq \frac{l}{2} \cos \vartheta - \frac{h-1}{2}d \right\} & h \text{ odd} \end{cases}$$

The probability $p(h)$, $h = 0, \dots, n+1$, can then be obtained as the ratio of the measure $\mu(\mathcal{N}_h)$ of the set \mathcal{N}_h of all segments, of length l , with middle point in S and with h intersections with the planes of the lattice, to the measure $\mu(S)$ of the set of all segments with middle point in S .

We compute $\mu(\mathcal{N}_h)$ and $\mu(S)$ by means of the kinematic density

$$dK_3 = |\sin \vartheta| d\vartheta \wedge d\varphi \wedge d\psi \wedge dx \wedge dy \wedge dz$$

(where $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ are the spherical coordinates on the unit sphere in E_3 and $\psi \in [0, 2\pi]$ is an angle of rotation). In order to do so, we must suppose

that P moves in a bounded region of the xy plane. However we can arbitrarily choose such a region, because when we translate S keeping fixed the quote of P , the number of intersections remains unchanged.

For $h = 0$ and $h = n + 1$ we have $\mathcal{N}_0 = \mathcal{S}_{0, [\vartheta_1, \pi/2]}$ and $\mathcal{N}_{n+1} = \mathcal{S}_{n+1, [0, \vartheta_n]}$, and we find

$$p(0) = \frac{1}{d} \int_{\vartheta_1}^{\frac{\pi}{2}} (d - l \cos \vartheta) \sin \vartheta d\vartheta$$

$$p(n+1) = \frac{1}{d} \int_0^{\vartheta_n} (l \cos \vartheta - nd) \sin \vartheta d\vartheta$$

while for $1 \leq h \leq n$ we have $\mathcal{N}_h = \mathcal{S}_{h, [\vartheta_h, \vartheta_{h-1}]} \cup \mathcal{S}_{h, [\vartheta_{h+1}, \vartheta_h]}$ and thus we get

$$p(h) = \frac{1}{d} \left\{ \int_{\vartheta_h}^{\vartheta_{h-1}} [l \cos \vartheta - (h-1)d] \sin \vartheta d\vartheta + \int_{\vartheta_{h+1}}^{\vartheta_h} [(h+1)d - l \cos \vartheta] \sin \vartheta d\vartheta \right\}$$

By solving the above integrals we find the probabilities (1). ■

Let us define now the random variable N as the number of crossings of the random "long" needle with the lattice \mathcal{R} .

An easy consequence of theorem 1 is the following

Corollary. *The mean and the second moment of N are*

$$E[N] = \frac{l}{2d} \quad E[N^2] = \frac{2n+1}{2} \frac{l}{d} + \frac{n(n+1)(2n+1)d}{6l} - n(n+1).$$

For higher moments we have the asymptotic approximation as $\frac{l}{d} \rightarrow \infty$

$$E[N^k] = \frac{k}{2} n^{k-1} \frac{l}{d} + \frac{k(k-1)}{2(k+1)} n^{k+1} \frac{d}{l} + (1-k)n^k + O(n^{k-1})$$

Theorem 2. *As $\frac{l}{d} \rightarrow \infty$, the random variable $\frac{dN}{l}$ converges in distribution to a uniform distribution in $[0, 1]$.*

Proof. As $\frac{l}{d} \rightarrow \infty$ the moments $E\left[\left(\frac{dN}{l}\right)^k\right]$ converge to the numbers $\frac{1}{k+1}$, that is to the moments of the uniform distribution on $[0, 1]$.

Since all the concerned distributions are constrained to the unit interval, the method of moments is in force and yields the desired result. ■

We suppose now that the space is ruled by two families, \mathcal{R}_1 and \mathcal{R}_2 , of parallel and equidistant planes (with equidistances a and b), the two families being orthogonal one to the other, and we prove

Theorem 3. *The probability that a random segment of constant length l , with $l > \sqrt{a^2 + b^2}$, intersects the lattice $\mathcal{R}_1 \cup \mathcal{R}_2$ is*

$$p = 2 + \frac{2 \left(\arcsin \frac{a}{\sqrt{a^2 + b^2}} + \arcsin \frac{b}{\sqrt{a^2 + b^2}} - \frac{\pi}{2} \right) \cos \vartheta_3 + \omega(a) + \omega(b)}{\pi} \quad (2)$$

$$\frac{(\cos \vartheta_3 - \cos \vartheta_1) [\min(a, b)]^2 + (\cos \vartheta_3 - \cos \vartheta_2) [\max(a, b)]^2}{3\pi ab}$$

$$\frac{\left(\frac{\pi}{2} - \vartheta_4 \right) \min(a, b) + \left(\frac{\pi}{2} - \vartheta_5 \right) \max(a, b)}{\pi l}$$

$$\frac{\left(\frac{\pi}{2} - \vartheta_1 - \vartheta_5 \right) \min(a, b) + \left(\frac{\pi}{2} - \vartheta_2 - \vartheta_4 \right) \max(a, b)}{\pi ab} l$$

$$\frac{2}{3} \frac{1 - \cos \vartheta_1 + \cos \vartheta_3 - \cos \vartheta_2}{\pi ab} l^2$$

where $\omega(x) = \arctan \left[\frac{x^2 - l^2 - l\sqrt{l^2 - a^2 - b^2}}{ab} \right] + \arctan \left[\frac{l^2 - x^2 - l\sqrt{l^2 - a^2 - b^2}}{ab} \right]$ and

$$\vartheta_1 = \arcsin \frac{\min(a, b)}{l} \quad \vartheta_2 = \arcsin \frac{\max(a, b)}{l} \quad \vartheta_3 = \arcsin \frac{\sqrt{a^2 + b^2}}{l}$$

$$\vartheta_4 = \arcsin \sqrt{1 - \frac{[\max(a, b)]^2}{l^2 - [\min(a, b)]^2}} \quad \vartheta_5 = \arcsin \sqrt{1 - \frac{[\min(a, b)]^2}{l^2 - [\max(a, b)]^2}}$$

Proof. If $a < b$, then we fix an orthonormal frame of reference such that the planes of \mathcal{R}_1 are parallel to the yz plane and those of \mathcal{R}_2 are parallel to the xz plane, and conversely if $b < a$.

Let ϑ be the angle between the support line of S and the z axis and let $P \equiv (x, y, z)$ be the middle point of S . A similar reasoning to that one employed in the proof of theorem 1 shows that we can suppose $|z| \leq M$, where M is an arbitrarily fixed constant. Hence $p = 1 - \frac{\mu(\mathcal{N}_P)}{\mu(\mathcal{M}_P)}$, where now \mathcal{M}_P and \mathcal{N}_P are respectively the set of segments with middle point in

$$\mathcal{P} = \{0 \leq x \leq \min(a, b), 0 \leq y \leq \max(a, b), |z| \leq M\}$$

and the set of segments with middle point in \mathcal{P} and with no intersections with the planes of \mathcal{R}_1 and \mathcal{R}_2 .

Thus we get

$$p = 1 - \frac{\int_0^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_0^{\frac{\pi}{2}} d\varphi \int_{\{(x,y) \in \mathcal{P}: S \cap (\mathcal{R}_1 \cup \mathcal{R}_2) = \emptyset\}} dx dy}{\pi^2 ab}$$

$$= 1 - \frac{1}{\pi^2 ab} \left[\int_0^{\arcsin \frac{\min(a,b)}{l}} \sin \vartheta d\vartheta \int_0^{\frac{\pi}{2}} f(\vartheta, \varphi) d\varphi + \right.$$

$$\begin{aligned}
 & + \int_{\arcsin \frac{\min(a,b)}{l}}^{\arcsin \frac{\max(a,b)}{l}} \sin \vartheta d\vartheta \int_{\arccos \frac{\min(a,b)}{l \sin \vartheta}}^{\frac{\pi}{2}} f(\vartheta, \varphi) d\varphi \\
 & + \int_{\arcsin \frac{\max(a,b)}{l}}^{\arcsin \frac{\sqrt{a^2+b^2}}{l}} \sin \vartheta d\vartheta \int_{\arccos \frac{\min(a,b)}{l \sin \vartheta}}^{\arcsin \frac{\max(a,b)}{l \sin \vartheta}} f(\vartheta, \varphi) d\varphi \Big]
 \end{aligned}$$

where the function $f(\vartheta, \varphi)$ is defined as

$$f(\vartheta, \varphi) := ab - [\min(a, b) \sin \varphi + \max(a, b) \cos \varphi] l \sin \vartheta + l^2 \sin^2 \vartheta \sin \varphi \cos \varphi$$

The above expression of p yields the final value (2). Note that $\lim_{l \rightarrow \infty} p = 1$. ■

The hypothesis $l > \sqrt{a^2 + b^2}$ in theorem 2 means that the segment S is really "long" when compared to the lattice. As a matter of fact, if S is dropped in such a way that its angle ϑ to the z axis is sufficiently large, namely $\vartheta > \arcsin \frac{\sqrt{a^2 + b^2}}{l}$, then S certainly intersects $\mathcal{R}_1 \cup \mathcal{R}_2$. Hence, as l grows, for almost all directions of fall the probability of intersection is 1. When $\max(a, b) < l < \sqrt{a^2 + b^2}$ we can speak of "non-small" needle in the sense of A. Duma (see [2]). We study now this kind of problem in the special case when $a = b$, since for general values of a and b there are not interesting modifications to the method of solution, except for the formalism that becomes too heavy.

Theorem 4. Consider the lattice $\mathcal{R}_1 \cup \mathcal{R}_2$ when $a = b$, and a random segment S of constant length l , $a < l < a\sqrt{2}$. Let N_1 and N_2 be the random variables "number of crossings of S with the planes of the families \mathcal{R}_1 and \mathcal{R}_2 ". Then (N_1, N_2) has the following distribution

$\mathcal{R}_1 \backslash \mathcal{R}_2$	$N_2 = 0$	$N_2 = 1$	$N_2 = 2$
$N_1 = 0$	$ \begin{aligned} p_{00} = & -1 - \frac{2}{3\pi} \cos \vartheta_1 + \frac{a}{l} \\ & + (\pi - 2\vartheta_1) \frac{l}{\pi a} \\ & + \frac{2}{3} \left(1 - \cos \vartheta_1 \frac{l^2}{\pi a^2} \right) \end{aligned} $	$p_{01} = p_{10}$	$p_{02} = p_{20}$
$N_1 = 1$	$ \begin{aligned} p_{10} = & 2 + \frac{3}{\pi} \cos \vartheta_1 - \frac{a}{l} \\ & + \frac{(-2\pi - 3 \sin \vartheta_1 \cos \vartheta_1 + 3\vartheta_1)l}{\pi a} \\ & - \frac{(\frac{2}{3} - 3 \cos \vartheta_1 + \cos^3 \vartheta_1)l^2}{\pi a^2} \end{aligned} $	$ \begin{aligned} p_{11} = & -\frac{4}{\pi} \cos \vartheta_1 \\ & + \frac{2(\pi - 2\vartheta_1 + 2 \sin \vartheta_1 \cos \vartheta_1)l}{\pi a} \\ & + \frac{2(1 - 6 \cos \vartheta_1 + 2 \cos^3 \vartheta_1)l^2}{3\pi a^2} \end{aligned} $	$p_{12} = p_{21}$
$N_1 = 2$	$ \begin{aligned} p_{20} = & -1 - \frac{1}{\pi} \cos \vartheta_1 + \frac{a}{2l} \\ & + \frac{(\pi - \vartheta_1 + \cos \vartheta_1 \sin \vartheta_1)l}{\pi a} \\ & - \frac{(\cos \vartheta_1 - \frac{1}{3} \cos^3 \vartheta_1)l^2}{\pi a^2} \end{aligned} $	$ \begin{aligned} p_{21} = & \frac{1}{\pi} \cos \vartheta_1 \\ & + \frac{(\vartheta_1 - \frac{\pi}{2} - \cos \vartheta_1 \sin \vartheta_1)l}{\pi a} \\ & + \frac{(\cos \vartheta_1 - \frac{1}{3} \cos^3 \vartheta_1)l^2}{\pi a^2} \end{aligned} $	0

where $\vartheta_1 = \arcsin \frac{a}{l}$.

Proof. Let $\mathcal{D} = \{0 \leq x \leq a, 0 \leq y \leq a\}$ be the elementary tile of the lattice, and let us denote by \mathcal{N}_{ij} the set of segments, of length l , with middle point P in \mathcal{D} and with respectively i and j intersections with the planes of \mathcal{R}_1 and \mathcal{R}_2 . According to theorem 1, the set \mathcal{N}_{ij} is not empty if and only if $0 \leq i, j \leq 2$. Let us partition \mathcal{D} into four sets

$$\mathcal{D}_1 = \mathcal{D} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2) \quad \mathcal{D}_2 = \mathcal{D} \cap (\mathcal{S}_2 \setminus \mathcal{S}_1) \quad \mathcal{D}_3 = \mathcal{D} \cap (\mathcal{S}_1 \setminus \mathcal{S}_2) \quad \mathcal{D}_4 = \mathcal{D} \cap \mathcal{S}_1 \cap \mathcal{S}_2$$

where

$$\begin{aligned} \mathcal{S}_1 &= \left\{ 0 \leq x \leq \frac{l \sin \vartheta \cos \varphi}{2} \vee a - \frac{l \sin \vartheta \cos \varphi}{2} \leq x \leq a \right\} \\ \mathcal{S}_2 &= \left\{ 0 \leq y \leq \frac{l \sin \vartheta \sin \varphi}{2} \vee a - \frac{l \sin \vartheta \sin \varphi}{2} \leq y \leq a \right\} \end{aligned}$$

Some geometrical considerations show that, for any fixed direction of S , we find a different value of the number of intersections (N_1, N_2) for each one of the sets \mathcal{D}_i . More precisely, the situation can be summed up as follows

$(\vartheta, \varphi) \in [0, \arcsin \frac{a}{l}] \times [0, \frac{\pi}{2}]$	$(\vartheta, \varphi) \in [\arcsin \frac{a}{l}, \frac{\pi}{2}] \times [0, \arccos \frac{a}{l \sin \vartheta}]$
$S_{(\vartheta, \varphi)} \in \begin{cases} \mathcal{N}_{00} & \text{if } P \in \mathcal{D}_1 \\ \mathcal{N}_{01} & \text{if } P \in \mathcal{D}_2 \\ \mathcal{N}_{10} & \text{if } P \in \mathcal{D}_3 \\ \mathcal{N}_{11} & \text{if } P \in \mathcal{D}_4 \end{cases}$	$S_{(\vartheta, \varphi)} \in \begin{cases} \mathcal{N}_{20} & \text{if } P \in \mathcal{D}_1 \\ \mathcal{N}_{21} & \text{if } P \in \mathcal{D}_2 \\ \mathcal{N}_{10} & \text{if } P \in \mathcal{D}_3 \\ \mathcal{N}_{11} & \text{if } P \in \mathcal{D}_4 \end{cases}$

$(\vartheta, \varphi) \in [\arcsin \frac{a}{l}, \frac{\pi}{2}] \times [\arccos \frac{a}{l \sin \vartheta}, \frac{\pi}{2} - \arccos \frac{a}{l \sin \vartheta}]$
$S_{(\vartheta, \varphi)} \in \begin{cases} \mathcal{N}_{00} & \text{if } P \in \mathcal{D}_1 \\ \mathcal{N}_{01} & \text{if } P \in \mathcal{D}_2 \\ \mathcal{N}_{10} & \text{if } P \in \mathcal{D}_3 \\ \mathcal{N}_{11} & \text{if } P \in \mathcal{D}_4 \end{cases}$

where $S_{(\vartheta, \varphi)}$ means that the direction of S is given by $(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ (the number of intersections for $(\vartheta, \varphi) \in [\arcsin \frac{a}{l}, \frac{\pi}{2}] \times [\frac{\pi}{2} - \arccos \frac{a}{l \sin \vartheta}, \frac{\pi}{2}]$ can be easily obtained just by taking into account the symmetry of the problem).

Hence we have

$$p_{00} = \frac{2}{\pi a^2} \left[\int_0^{\vartheta_1} \sin \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{D}_1) d\varphi + \int_{\vartheta_1}^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_{\arccos \frac{a}{l \sin \vartheta}}^{\frac{\pi}{2} - \arccos \frac{a}{l \sin \vartheta}} \text{vol}(\mathcal{D}_1) d\varphi \right]$$

$$\begin{aligned} p_{10} &= \frac{2}{\pi a^2} \left[\int_0^{\vartheta_1} \sin \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{D}_3) d\varphi + \int_{\vartheta_1}^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_0^{\arccos \frac{a}{l \sin \vartheta}} \text{vol}(\mathcal{D}_3) d\varphi \right. \\ &\quad \left. + \int_{\vartheta_1}^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_{\arccos \frac{a}{l \sin \vartheta}}^{\frac{\pi}{2} - \arccos \frac{a}{l \sin \vartheta}} \text{vol}(\mathcal{D}_3) d\varphi \right] \end{aligned}$$

$$p_{11} = \frac{2}{\pi a^2} \int_0^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{D}_4) d\varphi$$

$$p_{20} = \frac{2}{\pi a^2} \int_{\vartheta_1}^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_0^{\arccos \frac{a}{l \sin \vartheta}} \text{vol}(\mathcal{D}_1) d\varphi$$

$$p_{21} = \frac{2}{\pi a^2} \int_{\vartheta_1}^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_0^{\arccos \frac{a}{l \sin \vartheta}} \text{vol}(\mathcal{D}_2) d\varphi$$

where $\vartheta_1 = \arcsin \frac{a}{l}$. To get the final values of the probabilities p_{ij} , we make use of the following equalities

$$\int_{\vartheta_1}^{\frac{\pi}{2}} \arcsin \left(\frac{a}{l \sin \vartheta} \right) \sin \vartheta d\vartheta = \frac{\pi}{2} \left(\cos \vartheta_1 + \frac{a}{l} - 1 \right)$$

$$\int_{\vartheta_1}^{\frac{\pi}{2}} \sin^2 \vartheta \sin \left[\arccos \frac{a}{l \sin \vartheta} \right] d\vartheta = \frac{\pi}{4} \left(1 - \frac{a^2}{l^2} \right)$$

The symmetry of the problem yields $p_{01} = p_{10}$, $p_{02} = p_{20}$ and $p_{12} = p_{21}$. ■

Remark. If we consider the marginal distribution of N_1 (or N_2) we find again the results of theorem 1 in the special case $n = 1$.

3. Large disks

We discuss now some results about the problem of multiple intersections for the test body "disk".

Theorem 5. A disk D , of radius r , is dropped at random in E_3 , which we suppose ruled by a family of parallel and equidistant planes, with equidistance d . The number of planes intersected by D can range between 0 and $n + 1$, where $n = [2r/d]$ is the greatest integer less than or equal to $2r/d$. For $0 < k \leq n$, the probability $p(k)$ of exactly k intersections is:

$$p(k) = -\frac{k-1}{2} \cos \vartheta_{k-1} + k \cos \vartheta_k - \frac{k+1}{2} \cos \vartheta_{k+1} + (-\vartheta_{k-1} + 2\vartheta_k - \vartheta_{k+1}) \frac{r}{d} \quad (3)$$

while for $k = 0$ and $k = n + 1$ we have

$$p(0) = 1 - \frac{1}{2} \cos \vartheta_1 - \vartheta_1 \frac{r}{d} \quad p(n+1) = -\frac{n}{2} \cos \vartheta_n + \left(\frac{\pi}{2} - \vartheta_n \right) \frac{r}{d} \quad (4)$$

where $\vartheta_k = \arcsin \left(\frac{kd}{2r} \right)$, $k = 0, \dots, n$, and $\vartheta_{n+1} = \frac{\pi}{2}$.

Proof. As in the proof of theorem 1, we fix an orthonormal frame of reference where the planes of the lattice have equations $\{z = nd, n \in \mathbb{Z}\}$ and we denote by S the elementary tile $\{0 \leq z \leq d\}$.

Let us consider the line s passing through the center C of D and orthogonal to the support plane of D and let ϑ be the angle between s and the z axis. The number of planes intersected by D is completely determined by ϑ and by the z coordinate of C . Precisely, if we let $\vartheta_k = \arcsin \left(\frac{kd}{2r} \right)$ for $k = 0, \dots, n$ and

$\vartheta_{n+1} = \frac{\pi}{2}$, then when $\vartheta_k \leq \vartheta \leq \vartheta_{k+1}$ we have k intersections if C belongs to the set

$$\mathcal{S}_{k, [\vartheta_k, \vartheta_{k+1}]} = \begin{cases} \left\{ r \sin \vartheta - \frac{k}{2}d \leq z \leq \frac{k+2}{2}d - r \sin \vartheta \right\} & k \text{ even} \\ \left\{ 0 \leq z \leq \frac{k+1}{2}d - r \sin \vartheta \quad \vee \quad r \sin \vartheta - \frac{k-1}{2}d \leq z \leq d \right\} & k \text{ odd} \end{cases}$$

whereas when C belongs to the set

$$\mathcal{S}_{k+1, [\vartheta_k, \vartheta_{k+1}]} = \begin{cases} \left\{ 0 \leq z \leq r \sin \vartheta - \frac{k}{2}d \quad \vee \quad \frac{k+2}{2}d - r \sin \vartheta \leq z \leq d \right\} & k \text{ even} \\ \left\{ \frac{k+1}{2}d - r \sin \vartheta \leq z \leq r \sin \vartheta - \frac{k-1}{2}d \right\} & k \text{ odd} \end{cases}$$

the number of intersected planes is $k+1$.

The measures of the sets \mathcal{S} , $\mathcal{S}_{k, [\vartheta_k, \vartheta_{k+1}]}$ and $\mathcal{S}_{k+1, [\vartheta_k, \vartheta_{k+1}]}$ are obviously infinite. Anyway, once it has fallen, translations of the disk, which preserve the angle ϑ and the z coordinate of C , cannot change the number of intersections, thus we can arbitrarily fix any bound to the x and y coordinates, say $\{|x| \leq M_1, |y| \leq M_2\}$, and we define $\mu(\mathcal{S})$ as the measure of the set $\mathcal{S} \cap \{|x| \leq M_1, |y| \leq M_2\}$, and likewise for $\mu(\mathcal{S}_{k, [\vartheta_k, \vartheta_{k+1}]})$ and $\mu(\mathcal{S}_{k+1, [\vartheta_k, \vartheta_{k+1}]})$.

Hence we have

$$p(0) = \frac{\mu(\mathcal{S}_{0, [\vartheta_0, \vartheta_1]})}{\mu(\mathcal{S})} = \frac{1}{d} \int_0^{\vartheta_1} (d - 2r \sin \vartheta) \sin \vartheta d\vartheta$$

$$p(n+1) = \frac{\mu(\mathcal{S}_{n+1, [\vartheta_n, \frac{\pi}{2}]})}{\mu(\mathcal{S})} = \frac{1}{d} \int_{\vartheta_n}^{\frac{\pi}{2}} (2r \sin \vartheta - nd) \sin \vartheta d\vartheta$$

and

$$p(k) = \frac{\mu(\mathcal{S}_{k, [\vartheta_{k-1}, \vartheta_k]} \cup \mathcal{S}_{k, [\vartheta_k, \vartheta_{k+1}]})}{\mu(\mathcal{S})} = \frac{1}{d} \int_{\vartheta_{k-1}}^{\vartheta_k} [2r \sin \vartheta - (k-1)d] \sin \vartheta d\vartheta + \frac{1}{d} \int_{\vartheta_k}^{\vartheta_{k+1}} [(k+1)d - 2r \sin \vartheta] \sin \vartheta d\vartheta$$

Straightforward computations yield the probabilities (3) and (4). ■

In order to get the values of the moments and of the distribution function of the random variable N , "number of planes intersected by the disk D ", a slight rewriting of the probabilities (3) and (4) is needed. If we let

$$\delta_k := -\frac{k}{2} \cos \vartheta_k - \vartheta_k \frac{r}{d} \quad \text{for } k = 0, 2, \dots, n$$

we find that

$$p(0) = 1 + \delta_1 \quad p(n) = \delta_{n-1} - 2\delta_n - \frac{\pi r}{2d} \quad p(n+1) = \delta_n + \frac{\pi r}{2d}$$

$$p(k) = \delta_{k-1} + \delta_{k+1} - 2\delta_k \quad 1 \leq k \leq n-1$$

With this notation we can easily prove the following

Theorem 6. A disk D of radius r falls at random on a lattice of parallel and equidistant planes, with equidistance d . Let us suppose that $nd \leq 2r \leq (n+1)r$ and let N be the number of planes of the lattice intersected by D . The mean value and the variance of N are

$$E[N] = \frac{\pi r}{2d} \quad \text{Var}[N] = \frac{r}{d} \left[\pi(2n+1) - 2 \sum_{i=1}^n \vartheta_i \right] - \sum_{i=1}^n i \cos \vartheta_i - \frac{\pi^2 r^2}{4d^2}$$

the moments of order $k \geq 2$ are given by

$$\mu_k = E[N^k] = 2 \sum_{i=1}^n \sum_{h=1}^{[k/2]} \binom{k}{2h} i^{k-2h} \delta_i + \frac{\pi r}{2d} [(n+1)^k - n^k]$$

and the distribution function is

$$F(k) = P(N \leq k) = \begin{cases} 1 - \delta_k + \delta_{k+1} & k = 0, 1, \dots, n-1 \\ 1 - \delta_n - \frac{\pi r}{2d} & k = n \end{cases}$$

Proof. Let $\{\epsilon_i\}_{i=0, \dots, n+1}$ be any real numbers. We have

$$\begin{aligned} \sum_{i=0}^{n+1} \epsilon_i p(i) &= \epsilon_0(1 + \delta_1) + \epsilon_1(\delta_0 + \delta_2 - 2\delta_1) + \dots + \epsilon_{i-1}(\delta_{i-2} + \delta_i - 2\delta_{i-1}) \\ &\quad + \epsilon_i(\delta_{i-1} + \delta_{i+1} - 2\delta_i) + \epsilon_{i+1}(\delta_i + \delta_{i+2} - 2\delta_{i+1}) + \dots \\ &\quad \dots + \epsilon_n \left(\delta_{n-1} - 2\delta_n - \frac{\pi r}{2d} \right) + \epsilon_n \left(\delta_n + \frac{\pi r}{2d} \right) \end{aligned}$$

and summation by parts shows that

$$\sum_{i=0}^{n+1} \epsilon_i p(i) = \epsilon_0 + \sum_{i=1}^n (\epsilon_{i-1} - 2\epsilon_i + \epsilon_{i+1}) \delta_i + (\epsilon_{n+1} - \epsilon_n) \frac{\pi r}{2d} \quad (5)$$

For $\{\epsilon_i \equiv i^k\}_{i=0, \dots, n+1}$, $k = 1, 2, \dots$ formula (5) gives the values of $E[N]$ and μ_k . To get the value of $F(k)$ we substitute in (5) $\epsilon_0 = \dots = \epsilon_k = 1$ and $\epsilon_{k+1} = \dots = \epsilon_n = 0$. ■

Remark. Formula (5) also easily yields $\sum_{i=0}^{n+1} p(i) = 1$.

Finally we study a random variable that is closely related to the geometrical situation we are analysing:

Theorem 7. Let us consider a lattice of parallel planes, at a distance d apart, and a disk D of radius r , $nd \leq 2r \leq (n+1)d$, with random position in E_3 . We denote by L the random variable "sum of the lengths of the chords of D resulting as intersection between D and the planes of the lattice". The mean value of L is

$$E[L] = \frac{\pi^2 r^2}{4d}$$

Proof. With the notation of theorem 5, we suppose that the planes of the lattice have equations $\{z = nd, n \in \mathbb{Z}\}$ and we denote by ϑ the angle between a line orthogonal to the support plane of D and the z axis.

Let $L_{hd} = L_{hd}(\vartheta, z)$ be the length of the chord $D \cap \{z = hd\}$ for fixed values of the angle ϑ and of the z coordinate of the center C of D ; when D does not intersect the plane $z = hd$ we simply assume $L_{hd} = 0$, so that we can write

$$L = \sum_{h \in \mathbb{Z}} L_{hd}.$$

Figure 1 illustrates the problem in the special case when D intersects the planes $z = 0$, $z = d$ and $z = -d$ and therefore the value of the random variable L is given by $L_{-d} + L_0 + L_d$.

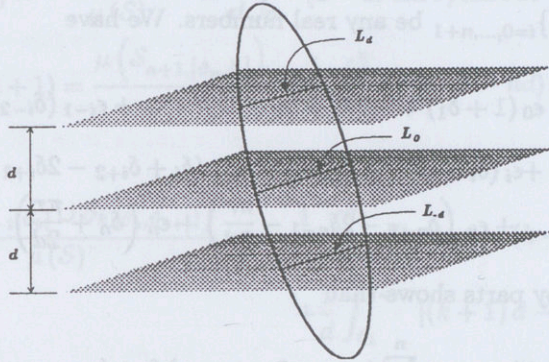


Figure 3.1:

It is easy to check that if $D \cap \{z = hd\} \neq \emptyset$ then

$$L_{hd} = L_{hd}(\vartheta, z) = \frac{2\sqrt{r^2 \sin^2 \vartheta - (z - hd)^2}}{\sin \vartheta}.$$

For $0 \leq \vartheta \leq \vartheta_1 = \arcsin\left(\frac{d}{2r}\right)$ and $\vartheta_1 \leq \vartheta \leq \vartheta_2 = \arcsin\left(\frac{d}{r}\right)$ all the functions L_{hd} vanish, except L_0 , when $0 \leq z \leq r \sin \vartheta$, and L_d , when $d - r \sin \vartheta \leq z \leq d$. For

$\vartheta_2 \leq \vartheta \leq \pi/2$, the list of the functions L_{hd} which are not identically zero is given by the following tables

$$k \geq 2 \text{ EVEN} \quad \vartheta_k = \arcsin\left(\frac{kd}{2r}\right) \leq \vartheta \leq \arcsin\left(\frac{(k+1)d}{2r}\right) = \vartheta_{k+1}$$

z RANGE	CHORDS
$0 \leq z \leq r \sin \vartheta - \frac{k}{2}d$	$L_{-\frac{k}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k}{2}d}$
$r \sin \vartheta - \frac{k}{2}d \leq z \leq \frac{k+2}{2}d - r \sin \vartheta$	$L_{-\frac{k-2}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k}{2}d}$
$\frac{k+2}{2}d - r \sin \vartheta \leq z \leq d$	$L_{-\frac{k-2}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k+2}{2}d}$

$$k \geq 3 \text{ ODD} \quad \vartheta_k = \arcsin\left(\frac{kd}{2r}\right) \leq \vartheta \leq \arcsin\left(\frac{(k+1)d}{2r}\right) = \vartheta_{k+1}$$

z RANGE	CHORDS
$0 \leq z \leq \frac{k+1}{2}d - r \sin \vartheta$	$L_{-\frac{k-1}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k-1}{2}d}$
$\frac{k+1}{2}d - r \sin \vartheta \leq z \leq r \sin \vartheta - \frac{k-1}{2}d$	$L_{-\frac{k-1}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k+1}{2}d}$
$r \sin \vartheta - \frac{k-1}{2}d \leq z \leq d$	$L_{-\frac{k-3}{2}d} \dots L_{-d} L_0 L_d \dots L_{\frac{k+1}{2}d}$

as already done before, we let $\vartheta_{n+1} = \frac{\pi}{2}$.

Applying Stoka's formula ([5], pag.78) we have

$$E[L] = \frac{\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin \vartheta d\vartheta \int_{\{(x,y,z) \in S\}} \sum_{h \in Z} L_{hd}(\vartheta, z) dx dy dz}{\mu(S)}$$

where, as in the proof of theorem 5, we let $S = \{0 \leq z \leq d, |x| \leq M_1, |y| \leq M_2\}$.

When $n \geq 2$ is even the previous formula becomes

$$\begin{aligned}
 E[L] = & \frac{2}{d} \left\{ \sum_{k=0}^{\frac{n}{2}-1} \left[\int_{\vartheta_{2k}}^{\vartheta_{2k+2}} d\vartheta \int_0^{r \sin \vartheta - kd} \sqrt{r^2 \sin^2 \vartheta - (z + kd)^2} dz \right. \right. \\
 & + \int_{\vartheta_{2k+2}}^{\frac{\pi}{2}} d\vartheta \int_0^d \sqrt{r^2 \sin^2 \vartheta - (z + kd)^2} dz \\
 & + \int_{\vartheta_{2k}}^{\vartheta_{2k+2}} d\vartheta \int_{(k+1)d - r \sin \vartheta}^d \sqrt{r^2 \sin^2 \vartheta - ((k+1)d - z)^2} dz \\
 & + \left. \int_{\vartheta_{2k+2}}^{\frac{\pi}{2}} d\vartheta \int_0^d \sqrt{r^2 \sin^2 \vartheta - ((k+1)d - z)^2} dz \right] \\
 & + \int_{\vartheta_n}^{\frac{\pi}{2}} d\vartheta \int_0^{r \sin \vartheta - \frac{n}{2}d} \sqrt{r^2 \sin^2 \vartheta - \left(z + \frac{n}{2}d\right)^2} dz \\
 & + \left. \int_{\vartheta_n}^{\frac{\pi}{2}} d\vartheta \int_{\frac{n+2}{2}d - r \sin \vartheta}^d \sqrt{r^2 \sin^2 \vartheta - \left(\frac{n+2}{2}d - z\right)^2} dz \right\} \quad (6)
 \end{aligned}$$

whereas for $n \geq 3$ odd we find

$$\begin{aligned}
 E[L] = \frac{2}{d} & \left\{ \sum_{k=0}^{\frac{n-3}{2}} \left[\int_{\vartheta_{2k}}^{\vartheta_{2k+2}} d\vartheta \int_0^{r \sin \vartheta - kd} \sqrt{r^2 \sin^2 \vartheta - (z + kd)^2} dz \right. \right. \\
 & + \int_{\vartheta_{2k+2}}^{\frac{\pi}{2}} d\vartheta \int_0^d \sqrt{r^2 \sin^2 \vartheta - (z + kd)^2} dz \\
 & + \int_{\vartheta_{2k}}^{\vartheta_{2k+2}} d\vartheta \int_{(k+1)d - r \sin \vartheta}^d \sqrt{r^2 \sin^2 \vartheta - ((k+1)d - z)^2} dz \\
 & \left. + \int_{\vartheta_{2k+2}}^{\frac{\pi}{2}} d\vartheta \int_0^d \sqrt{r^2 \sin^2 \vartheta - ((k+1)d - z)^2} dz \right] \\
 & + \int_{\vartheta_{n-1}}^{\frac{\pi}{2}} d\vartheta \int_0^{r \sin \vartheta - \frac{n-1}{2}d} \sqrt{r^2 \sin^2 \vartheta - \left(z + \frac{n-1}{2}d\right)^2} dz \\
 & \left. + \int_{\vartheta_{n-1}}^{\frac{\pi}{2}} d\vartheta \int_{\frac{n+1}{2}d - r \sin \vartheta}^d \sqrt{r^2 \sin^2 \vartheta - \left(\frac{n+1}{2}d - z\right)^2} dz \right\} \quad (7)
 \end{aligned}$$

Both (6) and (7) yield the final value $E[L] = \frac{\pi^2 r^2}{4d}$.

The proof for $n = 1$ requires some changes in the expression of (7), but the modifications are straightforward and therefore omitted. ■

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