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# Une version Maker-Breaker du jeu du plus grand sous-graphe connexe <sup>†</sup>

Julien Bensmail<sup>1</sup> et Foivos Fioravantes<sup>1</sup> et Fionn Mc Inerney<sup>2</sup> et Nicolas Nisse<sup>1</sup> et Nacim Oijid<sup>3</sup>

<sup>1</sup> Université Côte d’Azur, Inria, CNRS, I3S, Biot, France

<sup>2</sup> CISPA Helmholtz Center for Information Security, Saarbrücken, Germany

<sup>3</sup> Univ. Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France

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Dans le jeu du plus grand sous-graphe connexe, deux joueurs, Anaïs et Gilles, s’affrontent en colorant les sommets d’un graphe. À chaque tour, Anaïs colore un sommet non coloré en rouge puis Gilles colore un sommet non coloré en bleu. Lorsque tous les sommets sont colorés, le joueur dont la couleur induit la plus grande composante connexe gagne. Ce jeu a été défini dans [Bensmail *et al*, WG’21 et AlgoTel’21] où il a été montré entre autres que Gilles n’a jamais de stratégie gagnante (au mieux, il peut espérer une égalité). Nous étudions la version de ce jeu où on se donne aussi un entier  $k \geq 1$ . Dans ce cas, Anaïs gagne si elle crée une composante connexe rouge de taille au moins  $k$  et Gilles gagne sinon. Étant donné un graphe  $G$ , nous étudions  $c_g(G)$ , le plus grand entier  $k$  qui garantit la victoire d’Anaïs.

Outre le fait que cela donne une chance à Gilles de gagner, cette variante fait partie de la famille des jeux combinatoires *Maker-Breaker*, qui ont été très étudiés. De plus, cette variante offre des outils différents pour mieux comprendre la version *Maker-Maker* initiale. En particulier, nous étudions les graphes *A-parfaits* pour lesquels Anaïs peut gagner en créant une unique composante rouge, *i.e.*, les graphes  $G$  à  $n$  sommets tels que  $c_g(G) = \lceil \frac{n}{2} \rceil$ .

Nous montrons que le calcul de  $c_g$  est PSPACE-complet dans les graphes bipartis, scindés (*split*) ou planaires, et que  $c_g$  et une stratégie correspondante peuvent être calculés en temps linéaire dans les graphes  $P_4$ -clairsemés (généralisant les cographes). Nous donnons ensuite des conditions suffisantes (liées aux degrés ou au nombre d’arêtes) pour qu’un graphe soit *A-parfait*. Un résultat surprenant est qu’il n’existe aucun graphe 3-régulier *A-parfait* avec plus de 132 sommets.

**Mots-clefs :** Maker-Breaker game, connection game, largest connected subgraph game, PSPACE-complete.

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## 1 Introduction

In the largest connected subgraph game [BFMN21], two players, Anaïs and Gilles, play on a graph  $G$  whose vertices are initially uncoloured. Each round, Anaïs colours an uncoloured vertex of  $G$  red, and then Gilles colours one blue (if any remain). The game ends when every vertex is coloured. If the largest connected red (blue, resp.) component is larger than the largest connected blue (red, resp.) one, then Anaïs (Gilles, resp.) wins. If both components have the same order, then the game ends in a draw. In [BFMN21], it was shown that Anaïs can always avoid losing this game. For more fairness, we study the following variant where, in addition to  $G$ , an integer  $k \geq 1$  is given as input. In this paper, Anaïs wins the game if she can create a connected red component of order at least  $k$  in  $G$ , and Gilles wins otherwise. Let  $c_g(G)$  denote the largest  $k$  such that Anaïs has a strategy to create a connected red component of order at least  $k$  in  $G$ , regardless of how Gilles plays.

Besides the fact that it offers Gilles the opportunity to win (if  $k > c_g(G)$ ), the version of the largest connected subgraph game studied in this paper is part of the family of *Maker-Breaker* games that have been widely studied. Since their introduction with famous games like *Hex* [Gar59] and the *Shannon switching game* [Gar61], *Maker-Breaker* games arguably drew more attention after the 1973 paper of Erdős and

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Selfridge on *positional* games [ES73] which, in particular, gives sufficient conditions for Breaker to win. In general, the outcome of a Maker-Breaker game is PSPACE-complete [RW21]. Among other examples, Chvátal and Erdős introduced Maker-Breaker games played on the complete graph  $K_n$ : the *Hamiltonicity game*, the *Connectivity game*, and the *Clique game* [CE78]. In each of these games, Anaïs and Gilles alternate picking edges of the complete graph  $K_n$  on  $n$  vertices, and Anaïs wins if she creates a Hamiltonian cycle for the former, spanning tree for the second, and clique of a given size for the latter.

The game we introduce also takes place in connection game theory, *i.e.*, games in which players aim to create connected structures. Several of these games are well known, such as HEX. Connection games are known to be, in general, difficult complexity-wise. Indeed, the generalised Maker-Breaker version of HEX was proven to be PSPACE-complete in 1981 by Reisch [Rei81], even if the graph is planar. This version, called PLANAR GENERALISED HEX, is played on an arbitrary planar graph in which two special vertices, denoted by  $s$  and  $t$ , are initially coloured red. Anaïs and Gilles alternate colouring vertices, and Anaïs wins if she manages to connect  $s$  and  $t$  by a red path. Otherwise, Gilles wins.

Finally, studying the Maker-Breaker variant of the largest connected subgraph game offers new tools to obtain further results on the Maker-Maker version. In particular, the class of *A-perfect* graphs, *i.e.*, the  $n$ -node graphs  $G$  for which  $c_g(G) = \lceil \frac{n}{2} \rceil$ , is particularly interesting since, in any odd order *A-perfect* graph, Anaïs has a winning strategy in the initial Maker-Maker variant.

**Our contributions.** We first prove that computing  $c_g$  is PSPACE-complete, even for restricted families of graphs such as bipartite graphs of diameter at most 4, split graphs, and planar graphs. On the positive side,  $c_g(G) \leq 2$  for any path or cycle  $G$ , and  $c_g$  can be computed in linear time in  $(q, q-4)$ -graphs (a super-class of cographs). We then give sufficient conditions, depending on the degrees and on the number of edges, for a graph to be *A-perfect*. Finally, we prove that there exist arbitrarily large  $d$ -regular *A-perfect* graphs if and only if  $d \geq 4$ . In particular, surprisingly, there are no 3-regular *A-perfect* graphs of order more than 132.

## 2 PSPACE-Hardness and some polynomial cases

In this section, we study the complexity of computing  $c_g$ . Let us start with basic results. First, since Anaïs and Gilles colour vertices turn-by-turn, Anaïs cannot colour more than half of the vertices of the graph, making *A-perfect* graphs “ideal” for Anaïs. Moreover, if there exists a vertex of degree  $d$ , Anaïs can first colour it, and then colour as much of its neighbours as possible ensuring a red component of size at least  $1 + \lfloor \frac{d}{2} \rfloor$ . Hence, this gives the following tight bounds:

**Lemma 2.1** *For every  $n$ -node graph  $G$  with maximum degree  $\Delta(G)$ ,  $\lfloor \frac{\Delta(G)}{2} \rfloor + 1 \leq c_g(G) \leq \lceil \frac{n}{2} \rceil$ .*

In [BFMN21], paths, cycles, and cographs (graphs not containing the path  $P_4$  on four vertices as an induced subgraph), where Anaïs has a winning strategy (in the Maker-Maker variant) have been fully characterised. From these results, it follows that  $c_g(G) = 2$  for any path or cycle  $G$  with at least 3 vertices. The case of cographs in [BFMN21] is not trivial due to the fact that the Maker-Maker version seems difficult to handle in disconnected graphs. Here (Maker-Breaker version), the following lemma makes the study easier:

**Lemma 2.2** *Let  $G$  be any graph with connected components  $G_1, \dots, G_k$ . Then,  $c_g(G) = \max_{1 \leq i \leq k} c_g(G_i)$ .*

Taking advantage of Lemma 2.2 (in the Maker-Breaker variant, we can restrict our study to connected graphs), we can extend the result of [BFMN21] to the class of  $(q, q-4)$ -graphs [BO95], *i.e.*, graphs where any  $q$  vertices induce at most  $q-4$  induced paths  $P_4$  (cographs are  $(4, 0)$ -graphs):

**Theorem 2.3** *For any  $q \geq 0$ ,  $c_g(G)$  can be computed in linear time in the class of  $(q, q-4)$ -graphs  $G$ .*

However, computing  $c_g(G)$  is still a hard computational problem, even for restricted families of graphs.

**Theorem 2.4** *Given a graph  $G$  and an integer  $k \geq 1$ , it is PSPACE-complete to decide whether  $c_g(G) \geq k$ , even if  $G$  is a bipartite, split or planar graph.*

*Sketch of the proof.* The case of bipartite and split graphs follows from a typical reduction from the classical PSPACE-complete problem POS CNF. Due to lack of space, we only focus on the PSPACE-hardness for the case of planar graphs. The proof is via a reduction from PLANAR GENERALISED HEX. Let  $(H, s, t)$  be

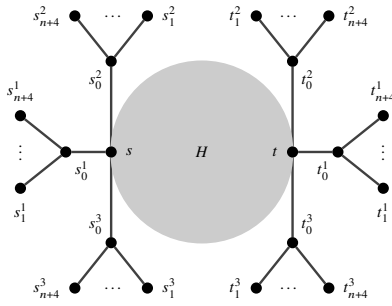


FIGURE 1: Construction in the proof of Theorem 2.4.

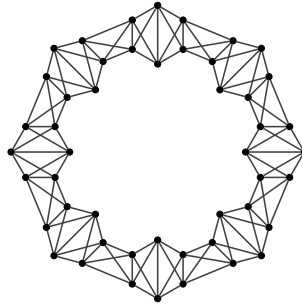


FIGURE 2: A 5-regular  $A$ -perfect graph.

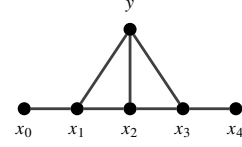


FIGURE 3: Aqueduct graph in the proof of Theorem 3.3.

an instance of PLANAR GENERALISED HEX such that  $H$  is an  $n$ -node planar graph. Let  $G$  be the graph obtained from  $H$  as follows (illustrated in Figure 1). Add to  $H$  three vertices  $s_0^1, s_0^2, s_0^3$  adjacent to  $s$ , and another three vertices  $t_0^1, t_0^2, t_0^3$  adjacent to  $t$ . To each of these six vertices we just added, attach  $n + 4$  new degree-1 vertices, so that a total of  $6(n + 4)$  degree-1 vertices (leaves) are added to  $G$ . Clearly,  $G$  is planar. We prove that  $c_g(G) \geq n + 5$  if and only if Anaïs wins PLANAR GENERALISED HEX in  $(H, s, t)$ .

Intuitively, to ensure a sufficiently large connected red component, Anaïs must colour “enough” vertices in the two subtrees rooted in  $s$  and  $t$  (that were added to  $H$  to obtain  $G$ ). To prevent Anaïs from winning, Gilles must also colour “enough” vertices in these subtrees. The result is that Anaïs will colour  $s$  and  $t$  before any other vertex of  $H$  is coloured. Finally, Anaïs creates a large enough connected red component if and only if she can connect  $s$  and  $t$  with a red path, *i.e.*, if and only if she wins PLANAR GENERALISED HEX in  $(H, s, t)$ .  $\triangleleft$

### 3 When Anaïs can win with a single red component

The rules of the game and Lemma 2.1 suggest that dense graphs clearly favour Anaïs. We now focus on  $A$ -perfect graphs, *i.e.*, graphs where Anaïs can ensure a unique red component at the end of the game (*i.e.*,  $n$ -node graphs  $G$  with  $c_g(G) = \lceil \frac{n}{2} \rceil$ ). First, we give two sufficient conditions, related to its degrees or number of edges, for a graph to be  $A$ -perfect. Note that we can prove that the next two bounds are tight. Recall that  $\Delta(G)$  ( $\delta(G)$ , resp.) denotes the maximum (minimum, resp.) degree of a vertex in a graph  $G$ .

**Theorem 3.1** *If  $G$  is a connected  $n$ -node graph with  $\Delta(G) + \delta(G) \geq n$ , then  $G$  is  $A$ -perfect.*

*Sketch of the proof.* First, Anaïs colours a vertex of degree  $\Delta(G)$ , forming a connected component of order 1 that she wants to extend into one of order  $\lceil \frac{n}{2} \rceil$ . Due to the degree conditions, we can prove by induction on the number of turns that, regardless of what Gilles does, Alice can always colour a vertex adjacent to her connected component from the previous turn. Thus, each turn, Alice colours such a vertex and increases the order of her connected component by 1, until the game ends, at which point her connected component has order  $\lceil \frac{n}{2} \rceil$ .  $\triangleleft$

**Theorem 3.2** *If  $G$  is a connected  $n$ -node graph with  $|E(G)| - 3 \geq \frac{(n-2)(n-3)}{2}$ , then  $G$  is  $A$ -perfect.*

*Sketch of the proof.* If  $\Delta(G) = n - 1$  or  $\Delta(G) \leq n - 4$ , the result holds (a simple counting argument ensures the conditions from Theorem 3.1). Hence, we may assume that  $n - 3 \leq \Delta(G) \leq n - 2$  and the proof follows by a tedious case analysis. Roughly, let  $v$  be a vertex of degree  $\Delta(G)$ , and let  $S$  be the set of vertices at distance at least 2 from  $v$  (so,  $|S| \leq 2$ ). Anaïs first colours  $v$ . Intuitively, to guarantee a connected red component of order  $\lceil \frac{n}{2} \rceil$ , Anaïs must colour a vertex  $u$  of  $S$ , and one of its neighbours (which is also a neighbour of  $v$ ) in order for  $u$  to be in the same connected red component as  $v$ . Then, she can colour as many neighbours of  $v$  as possible. We provide such strategies depending on the degree of the vertices in  $S$  and on the maximum degree of a neighbour of  $v$ .  $\triangleleft$

From the above paragraphs, the degrees of the vertices of a graph are important in determining whether it is  $A$ -perfect. Therefore, we consider the class of regular graphs (where all vertices have the same degree). From the results above, we already know that  $c_g$  is 2 for 2-regular graphs (disjoint union of cycles), and so they are not  $A$ -perfect (if they have at least 5 vertices). In contrast, for any  $d \geq 4$ , there are arbitrarily large  $d$ -regular  $A$ -perfect graphs. Surprisingly, there is no  $A$ -perfect 3-regular graph of order more than 132.

**Theorem 3.3** There exist arbitrarily large  $d$ -regular  $A$ -perfect graphs if and only if  $d \geq 4$ .

*Sketch of the proof.* For  $d \geq 5$  and  $h \geq 3$ , consider the graph  $G$  obtained from  $h$  disjoint copies  $K^0, \dots, K^{h-1}$  of the complete graph  $K_{d+1}$  on  $d+1$  vertices. For every  $0 \leq i < h$ , choose 4 vertices of  $K^i$ , and remove the edges between these 4 vertices. Finally, for any  $0 \leq i < h$ , add 4 (well-chosen) edges between  $K^i$  and  $K^{i+1 \bmod h}$  in order to build a  $d$ -regular graph  $G$  (see Figure 2 for an example for  $d = 5$ ). It can be proved that  $G$  (which has  $(d+1)h$  vertices) is  $A$ -perfect (see [BFM<sup>+</sup>21] for more details). A similar (but slightly different) construction allows to show that there are arbitrarily large  $A$ -perfect 4-regular graphs [BFM<sup>+</sup>21].

We now show that any 3-regular  $A$ -perfect graph has order at most 132. For this purpose, let us consider an optimal strategy for Anaïs in some 3-regular  $n$ -node graph  $G$ , *i.e.*, that allows her to make a connected red component of order  $c_g(G)$ . Let  $u_0 \in V(G)$  be the first vertex coloured by Anaïs by this strategy. The proof consists of two cases depending on whether  $u_0$  belongs to a cycle  $C$  of length at most  $\frac{n}{2} - 1$  or not.

In the case that such a short cycle  $C$  exists, we show that  $V(G)$  can be partitioned into two non-empty parts  $A$  and  $B$  such that  $V(C) \subseteq A$ , and there exists a matching  $M$  separating  $A$  and  $B$ . By colouring at least one vertex of each edge of  $M$ , Gilles can ensure there are at least two (disjoint) connected red components, and so,  $G$  is not  $A$ -perfect.

If every cycle containing  $u_0$  has size larger than  $\frac{n}{2} - 1$ . Then, by considering maximum disjoint paths starting from the three neighbours of  $u_0$ , we prove that, if  $n > 132$ ,  $G$  contains an *aqueduct* (see Figure 3) as an induced subgraph. It is then easy to see that any graph with an aqueduct as an induced subgraph is not  $A$ -perfect. Indeed, in that case, Gilles can always guarantee that one vertex in  $\{x_1, x_2, x_3, y\}$  will be red and not connected to the largest connected red component (by considering the matching  $\{\{x_0, x_1\}, \{x_2, y\}, \{x_3, x_4\}\}$ ). Hence, Anaïs cannot create a single connected red component, and  $G$  is not  $A$ -perfect.  $\triangleleft$

**Further work.** Several directions for further work on the Maker-Breaker largest connected subgraph game are appealing. First, it would be interesting to study it in other standard graph classes such as trees, for which determining  $c_g$  seems non-trivial. Another interesting case is that of grids, which are natural structures to play on in several types of games, as illustrated by HEX, and for which we only have partial results:

**Proposition 1** If  $G$  is any finite subgraph of the infinite hexagonal grid, then  $c_g(G) \leq 6$ .

**Proposition 2** For  $n \leq m$ , let  $P_n \square P_m$  be the  $n \times m$  Cartesian grid. Then,  $c_g(P_n \square P_m) \leq 2n$ .

One issue we ran into while considering grids (and to some extent trees) is that Anaïs can play in a non-connected way, and it is not clear how Gilles should answer to that. Precisely, it would be interesting to study the cost of connectedness, *i.e.*, how much less will Anaïs score if she is always (except on her first turn) constrained to colour a neighbour of another red vertex, and the game ends when she cannot.

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