LINEAR AND NONLINEAR ANALYSIS OF THE VISCOUS RAYLEIGH-TAYLOR SYSTEM WITH NAVIER-SLIP BOUNDARY CONDITIONS

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Abstract. In this paper, we are interested in the linear and the nonlinear Rayleigh-Taylor instability for the gravity-driven incompressible Navier-Stokes equations with Navier-slip boundary conditions around a smooth increasing density profile $\rho_0(x_2)$ in a slab domain $2\pi L\mathbb{T}\times(-1,1)$ ($L>0$, $\mathbb{T}$ is the usual 1D torus). The linear instability study of the viscous Rayleigh-Taylor model amounts to the study of the following ordinary differential equation on the finite interval $(-1,1)$,

$$-\lambda^2[\rho_0 k^2\phi-(\rho_0\phi')']=\lambda\mu(\phi^{(4)}-2k^2\phi''+k^4\phi)-gk^2\rho_0'\phi, \tag{0.1}$$

with the boundary conditions

$$\begin{cases}\phi(-1)=\phi(1)=0,\\ \mu\phi''(1)=\xi_+\phi'(1),\\ \mu\phi''(-1)=-\xi_-\phi'(-1),\end{cases} \tag{0.2}$$

where $\lambda>0$ is the growth rate in time, $g>0$ is the gravity constant, k is the wave number and two Navier-slip coefficients $\xi_\pm$ are nonnegative constants. For each $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, we define a threshold of viscosity coefficient $\mu_c(k,\Xi)$ for the linear instability. So that, in the k-supercritical regime, i.e. $\mu>\mu_c(k,\Xi)$, we describe a spectral analysis adapting the operator method initiated by Lafitte-Nguyễn [13] and then prove that there are infinite nontrivial solutions $(\lambda_n,\phi_n)_{n\geqslant 1}$ of (0.1)-(0.2) with $\lambda_n\to 0$ as $n\to\infty$ and $\phi_n\in H^4((-1,1))$. Based on the existence of infinitely many normal modes of the linearized problem, we construct a wide class of initial data to the nonlinear equations, extending the previous framework of Guo-Strauss [5] and of Grenier [7], to prove the nonlinear Rayleigh-Taylor instability in a high regime of viscosity coefficient, namely $\mu>3\sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\mu_c(k,\Xi)$.

Contents

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1. Introduction

The Rayleigh–Taylor (RT) instability, studied first by Lord Rayleigh in [17] and then Taylor [18] is well known as a gravity-driven instability in two semi-infinite inviscid and incompressible fluids when the heavy one is on top of the light one. It has attracted much attention due to both its physical and mathematical importance. Two applications worth mentioning are implosion of inertial confinement fusion capsules [14] and core-collapse of supernovae [16]. For a detailed physical comprehension of the RT instability, we refer to three survey papers [11, 19, 20]. Mathematically speaking, for the inviscid and incompressible regime with a smooth density profile, the classical RT instability was investigated by Lafitte [12], by Guo and Hwang [4] and by Helffer and Lafitte [9].

Concerning the viscous RT instability, one of the first studies can be found in the book of Chandrasekhar [1, Chap. X]. He considered two uniform viscous fluid separated by a horizontal boundary and generalized the classical result of Rayleigh and Taylor. We refer the readers to mathematical viscous RT studies for two compressible channel flows by Guo and Tice [6], for incompressible fluid in the whole space $\mathbf{R}^3$ by Jiang et. al [8] and Lafitte and Nguyễn [13], respectively.

In this paper, we are concerned with the viscous RT of the gravity-driven incompressible Navier-Stokes equations in a 2D slab domain $\Omega = 2\pi L\mathbb{T} \times (-1,1)$ with $L > 0$ and $\mathbb{T}$ is the 1D-torus, that read as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho\vec{u}) = 0, \\ \partial_t(\rho\vec{u}) + \mathrm{div}(\rho\vec{u}\otimes\vec{u}) + \nabla P = \mu\Delta\vec{u} - \rho\vec{g}, \\ \mathrm{div}\vec{u} = 0, \end{cases} \tag{1.1}$$

where $t \geqslant 0, \vec{x} = (x_1, x_2) \in 2\pi L\mathbb{T} \times (-1, 1)$. The unknowns $\rho := \rho(t, \vec{x})$, $\vec{u} := \vec{u}(t, \vec{x})$ and $P := P(t, \vec{x})$ denote the density, the velocity and the pressure of the fluid, respectively, while μ is the viscosity coefficient and $\vec{g} := g\vec{e}_2$, $g > 0$ being the gravity constant. Let $\Sigma_\pm = 2\pi L\mathbb{T} \times \{\pm 1\}$, the Navier-slip boundary conditions proposed by Navier (see [15]) are given on $\Sigma_\pm$ as follows

$$\begin{aligned} \vec{u} \cdot \vec{n} &= 0 \quad \text{on } \Sigma_+ \cup \Sigma_-, \\ (\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n})_\tau &= \xi(\vec{x})\vec{u} \quad \text{on } \Sigma_+ \cup \Sigma_-. \end{aligned} \tag{1.2}$$

Here, $\vec{n}$ is the outward normal vector of the boundary, $(\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n})_\tau$ is the tangential component of $\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n}$ and $\xi(\vec{x})$ is a scalar function describing the slip effect on the boundary, only taking *nonnegative* constant values $\xi_\pm$ on $\Sigma_\pm$, respectively.

Let ρ_0 and P_0 be two C^1-functions on x_2 such that $P_0' = -g\rho_0$ with $' = d/dx_2$. Then, the laminar flow $(\rho_0(x_2), \vec{0}, P_0(x_2))$ is a steady-state solution of (1.1). Of interest of this paper is to study the nonlinear instability of the above laminar flow to Eq. (1.1)-(1.2) that satisfies

$$\rho_0 \in C^1([-1, 1]), \quad \rho_0' > 0 \text{ on } [-1, 1], \quad \rho_0(\pm 1) = \rho_\pm \in (0, +\infty), \tag{1.3}$$

i.e. to study the nonlinear Rayleigh–Taylor instability problem.

Linearizing (1.1) in the vicinity of $(\rho_0(x_2), \vec{0}, P_0(x_2))$ and then seeking a normal mode at a horizontal spatial frequency $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ of the form

$$e^{\lambda(k)t}\vec{U}(\vec{x}) = e^{\lambda(k)t}(\cos(kx_1)\omega(x_2), \sin(kx_1)\theta(x_2), \cos(kx_1)\phi(x_2), \cos(kx_1)q(x_2))^T,$$

the linear RT instability amounts to the investigation of the parameter $\lambda(k) \in \mathbb{C}$ ($\mathrm{Re}\lambda > 0$) such that there exists a nontrivial solution $\phi \in H^4((-1, 1))$ of the following ordinary differential equation for the second component of velocity

$$-\lambda^2(\rho_0 k^2\phi - (\rho_0\phi')') = \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) - gk^2\rho_0'\phi, \tag{1.4}$$

with the boundary conditions

$$\begin{cases} \phi(-1) = \phi(1) = 0, \\ \mu\phi''(1) = \xi_+\phi'(1), \\ \mu\phi''(-1) = -\xi_-\phi'(-1). \end{cases} \tag{1.5}$$

Note that the embedding $H^4((-1, 1)) \hookrightarrow C^3((-1, 1))$ allows us to write (1.5). In this case, such a λ is called a growth rate of the instability or a characteristic value of the linearized problem (see Eq. (2.5) below) as in [1, Sect. 92-93, Chap. X]). We will present the derivation of the physical model in Section 2.

As the density profile is increasing, we first show that λ is always real in Lemma 2.1. Since our goal is to study the instability, we are left to look for $\lambda > 0$. Hence, for the linear instability, we continue the spectral analysis of Helffer and Lafitte [9], Lafitte and Nguyễn [13] for Eq. (1.4)-(1.5).

For any horizontal spatial frequency $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, we then define a k-supercritical regime of the viscosity coefficient $\mu > \mu_c(k, \Xi)$ (see $\mu_c(k, \Xi)$ in Proposition 3.1), we prove that there exists an infinite sequence of characteristic values $(\lambda_n(k, \mu))_{n\geqslant 1}$, decreasing towards 0 as $n \to \infty$. This is stated in Theorem 2.1.

The second goal, described in Section 5 is to obtain a nonlinear instability result on more general initial data using the linear result of Theorem 2.1 (see (2.15)) and working in the regime $\mu > 3\sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}} \mu_c(k, \Xi)$. The classical way of proving the nonlinear instability is to estimate the difference between the solution to the nonlinear problem and the normal mode solution to the linearized problem. In order

to show that, in the spirit of Guo-Strauss [5] and Grenier [7], only the maximal normal mode $e^{\lambda_1 t}\vec{U}_1(\vec{x})$ was taken to derive a solution of the nonlinear equation whose initial datum is $\delta\vec{U}_1(\vec{x})$ with $0<\delta\ll 1$. Our nonlinear result, Theorem 2.2, generalizes the previous results of Guo-Strauss and of Grenier, by showing that a wide class of initial data (related to a linear combination of normal modes) to the nonlinear problem departing from the equilibrium gives rise to the nonlinear instability.

This paper is organized as follows. In Section 2, we present the governing equations and state the main results. Section 3 is devoted to some materials for the linear study. Then, in Section 4, we prove the linear instability, i.e. Theorem 2.1. Section 5 is to prove the nonlinear instability, i.e. Theorem 2.2.

2. Main results

2.1. **The governing equations.** Let us recall the steady state $(\rho_0(x_2),\vec{0},P_0(x_2))$ of (1.1), with ρ_0 satisfies (1.3) and $P_0'=-g\rho_0$. We now derive the linearization of Eq. (1.1) around the equilibrium state $(\rho_0(x_2),\vec{0},P_0(x_2))$. The perturbations

$$\sigma=\rho-\rho_0,\quad \vec{u}=\vec{u}-\vec{0},\quad p=P-P_0$$

thus satisfy

$$\begin{cases}\partial_t\sigma+\vec{u}\cdot\nabla(\rho_0+\sigma)=0,\\ (\rho_0+\sigma)\partial_t\vec{u}+(\rho_0+\sigma)\vec{u}\cdot\nabla\vec{u}+\nabla p=\mu\Delta\vec{u}-\sigma\vec{g},\\ \mathrm{div}\vec{v}=0.\end{cases}\tag{2.1}$$

Note that $(\mu(\nabla\vec{u}+\nabla\vec{u}^T)\cdot\vec{n})_\tau=\vec{n}\times(\mu(\nabla\vec{u}+\nabla\vec{u}^T)\cdot\vec{n})\times\vec{n}$ and that $\vec{n}=(0,\pm1)^T$. Hence, the boundary conditions are

$$\begin{cases}u_2=0, & \text{on } \Sigma_\pm,\\ \mu\partial_{x_2}u_1=\xi_+u_1 & \text{on } \Sigma_+,\\ \mu\partial_{x_2}u_1=-\xi_-u_1 & \text{on } \Sigma_-.\end{cases}\tag{2.2}$$

The linearized equations are

$$\begin{cases}\partial_t\sigma+\rho_0'u_2=0,\\ \rho_0\partial_t\vec{u}+\nabla p=\mu\Delta\vec{u}-\sigma\vec{g},\\ \mathrm{div}\vec{u}=0,\end{cases}\tag{2.3}$$

and the corresponding boundary conditions remaining (2.2).

The linear RT instability problem is to seek a normal mode of the form

$$\begin{cases}\sigma(t,\vec{x})=e^{\lambda t}\cos(kx_1)\omega(x_2),\\ u_1(t,\vec{x})=e^{\lambda t}\sin(kx_1)\theta(x_2),\\ u_2(t,\vec{x})=e^{\lambda t}\cos(kx_1)\phi(x_2),\\ q(t,\vec{x})=e^{\lambda t}\cos(kx_1)q(x_2).\end{cases}\tag{2.4}$$

where $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, $\lambda\in\mathbb{C}\backslash\{0\}$ and $\mathrm{Re}\lambda\geqslant 0$. It follows from (2.3) that

$$\begin{cases}\lambda\omega+\rho_0'\phi=0,\\ \lambda\rho_0\theta-kq+\mu(k^2\theta-\theta'')=0,\\ \lambda\rho_0\phi+q'+\mu(k^2\phi-\phi'')=-g\omega,\\ k\theta+\phi'=0\end{cases}\tag{2.5}$$

and from (2.2) that

$$\phi(\pm 1) = 0, \quad \mu\theta'(1) = \xi_+\theta(1), \quad \mu\theta'(-1) = -\xi_-\theta(-1). \tag{2.6}$$

We obtain

$$\omega = -\frac{\rho_0'}{\lambda}\phi, \quad \theta = -\frac{1}{k}\phi', \quad q = -\frac{1}{k^2}(\lambda\rho_0\phi' + \mu(k^2\phi' - \phi''')). \tag{2.7}$$

Then, we substitute q, ω into $(2.5)_3$ to get a fourth-order ordinary differential equation (1.4). We have the boundary conditions (1.5) deduced from (2.2), which are obtained by assuming the solution to be in $C^2([-1,1])$.

2.2. **Main results.** Before stating our main results, we present our material for the linearized equations.

When the density profile ρ_0 is increasing, we show that all characteristic values λ are real. Let L_0 be the characteristic length, such that $L_0^{-1} = \|\frac{\rho_0'}{\rho_0}\|_{L^\infty((-1,1))}$, we further obtain the uniform upper bound $\sqrt{\frac{g}{L_0}}$ of λ.

Lemma 2.1. *For any $k \in L^{-1}\mathbb{Z}\backslash\{0\}$,*

- *all characteristic values λ are always real,*
- *all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

Proof of Lemma 2.1 is given in Section 3.1. In view of Lemma 2.1, we look for functions ϕ being real and we only consider the vector space of real-valued functions in what follows in the linear analysis.

We now study the linearized problem, i.e. (1.4)-(1.5). Of importance is to construct a continuous and coercive bilinear form $\mathcal{B}_{k,\lambda,\mu}$ as $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$ (i.e. we do not restrict $\lambda \in (0, \sqrt{\frac{g}{L_0}})$ and $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ at this step) on the functional space

$$\tilde{H}^2((-1,1)) := \{\varphi \in H^2((-1,1)), \varphi(\pm 1) = 0\},$$

so that the finding of a solution $\phi \in H^4((-1,1))$ of Eq. (1.4)-(1.5) on $(-1,1)$ is equivalent to finding a weak solution $\phi \in \tilde{H}^2((-1,1))$ to the variational problem

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\theta) = gk^2\int_{-1}^{1}\rho_0'\phi\theta dx_2 \quad \text{for all } \theta \in \tilde{H}^2((-1,1)), \tag{2.8}$$

and thus improving the regularity of that weak solution ϕ.

The desired bilinear form $\mathcal{B}_{k,\lambda,\mu}$ is

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) :=& \lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 \\ &- \xi_-\vartheta'(-1)\varrho'(-1) - \xi_+\vartheta'(1)\varrho'(1),\end{aligned} \tag{2.9}$$

For all $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$, we will place ourselves in a k-supercritical regime of the viscosity coefficient $\mu > \mu_c(k,\Xi)$ with $\Xi = (\xi_-,\xi_+)$ (see the precise formula $\mu_c(k,\Xi)$ in Proposition 3.1) such that

$$\mathcal{B}_{k,0,\mu} \text{ is coercive if and only if } \mu > \mu_c(k,\Xi), \tag{2.10}$$

it yields that $\mathcal{B}_{k,\lambda,\mu}$ is coercive for all $\lambda \geqslant 0$ and $\mu > \mu_c(k,\Xi)$. As $\mathcal{B}_{k,\lambda,\mu}$ is a coercive form on $H^2((-1,1))$, we have that $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$ is a norm on $H^2((-1,1))$. Let $(\tilde{H}^2((-1,1)))'$ be the dual space of $\tilde{H}^2((-1,1))$ associated with the norm

$\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. In view of Riesz's representation theorem, we thus obtain an abstract operator $Y_{k,\lambda,\mu}$ from $\tilde{H}^2((-1,1))$ to $(\tilde{H}^2((-1,1)))'$) such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)=\langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle \quad \text{for all } \vartheta,\varrho\in\tilde{H}^2((-1,1)). \tag{2.11}$$

Owing to (2.8) and (2.11), it turns out that the existence of a solution $\phi\in H^4((-1,1))$ of Eq. (1.4)-(1.5) is reduced to the existence of a weak solution $\phi\in\tilde{H}^2((-1,1)$ of

$$\lambda Y_{k,\lambda,\mu}\phi=gk^2\rho_0'\phi \quad \text{in } (\tilde{H}^2((-1,1)))'. \tag{2.12}$$

Restricting $\varrho\in C_0^\infty((-1,1))$ in (2.11), we find that, for all $\vartheta\in H^2((-1,1))$ (see Proposition 3.3),

$$Y_{k,\lambda,\mu}\vartheta=\lambda(\rho_0k^2\vartheta-(\rho_0\vartheta')')+\mu(\vartheta^{(4)}-2k^2\vartheta''+k^4\vartheta) \quad \text{in } \mathcal{D}'((-1,1)).$$

Owing to a bootstrap argument, we further get that a weak solution ϕ of (2.11) actually belongs to $H^4((-1,1))$ and ϕ satisfies the boundary conditions (1.5). Hence, we are able to define the inverse operator of $Y_{k,\lambda,\mu}^{-1}$ of $Y_{k,\lambda,\mu}$ (see Proposition 3.4), from $L^2((-1,1))$ to a subspace of $H^4((-1,1))$ requiring all elements satisfy (1.5). Composing the above operator $Y_{k,\lambda,\mu}^{-1}$ with the continuous injection from $H^4((-1,1))$ to $L^2((-1,1))$, we obtain that $Y_{k,\lambda,\mu}^{-1}$ is a compact and self-adjoint operator from $L^2((-1,1))$ to itself.

Denoting by $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((-1,1))$. Note from (2.12) that, we thus find (λ,v) such that

$$\frac{\lambda}{gk^2}v=\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}v.$$

Once it is proven that the operator $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ is compact and self-adjoint from $L^2((-1,1))$ to itself, then

$$\text{the discrete spectrum of } \mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M} \text{ is an infinite sequence of eigenvalues}$$
$$\text{(denoted by } \{\gamma_n(k,\lambda,\mu)\}_{n\geqslant1}).$$

Let $v_{n,k,\lambda,\mu}$ be an eigenfunction of the operator $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ associated with the eigenvalue $\gamma_n(k,\lambda,\mu)$ and let $\phi_{n,k,\lambda,\mu}=Y_{k,\lambda,\mu}^{-1}\mathcal{M}v_{n,k,\lambda,\mu}$, we have

$$\gamma_n(k,\lambda,\mu)Y_{k,\lambda,\mu}\phi_{n,k,\lambda,\mu}=\mathcal{M}^2\phi_{n,k,\lambda,\mu}=\rho_0'\phi_{n,k,\lambda,\mu}. \tag{2.13}$$

From (2.13), it can be seen that, for each n, we have to solve the equation

$$\gamma_n(k,\lambda,\mu)=\frac{\lambda}{gk^2}. \tag{2.14}$$

We will show that Eq. (2.14) has a unique root $\lambda_n(k,\mu)\in\mathbf{R}_+$ because of the decrease of γ_n in λ (see Lemma 4.1), which is an extension of Kato's perturbation theory of the spectrum of operators [10]. In addition, when λ_n is a characteristic value, we have $\lambda_n\leqslant\sqrt{\frac{g}{L_0}}$ for all $n\geqslant1$.

This yields that for any horizontal spatial frequency $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, there exists a sequence of characteristic values $(\lambda_n(k,\mu))_{n\geqslant1}$, that is uniformly bounded and we further obtain that λ_n decreases towards 0 as $n\to\infty$. For each λ_n, we have that $\phi_{n,k,\lambda_n,\mu}=Y_{k,\lambda_n,\mu}^{-1}v_{n,k,\lambda_n,\mu}$ is a solution in $H^4((-1,1))$ of (1.4)-(1.5) associated with $\lambda=\lambda_n$.

We sum up the above arguments in our first theorem.

Theorem 2.1. *Let $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ be fixed and let ρ_0 satisfy that (1.3), i.e.*

$$\rho_0 \in C^1([-1,1]), \quad \rho_0(\pm 1) = \rho_\pm \in (0,\infty), \quad \rho_0' > 0 \text{ everywhere on } [-1,1].$$

For all $\mu > \mu_c(k,\Xi)$, there exists an infinite sequence $(\lambda_n, \phi_n)_{n\geqslant 1}$ with $\lambda_n > 0$ decreasing towards 0 and $\phi_n \in H^4((-1,1))$, ϕ_n non trivial, satisfying (1.4)-(1.5).

Once Eq. (1.4)-(1.5) is solved, we go back to the linearized equations (2.3). For a fixed $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, we obtain a sequence of solutions to the linearized equations (2.3) as indicated in Proposition 4.2, which are $(e^{\lambda_j(k,\mu)t}\vec{U}_j(k,\mu,\vec{x}))_{j\geqslant 1}$, with $\vec{U}_j(k,\mu,\vec{x}) = (\sigma_j, \vec{u}_j, p_j)^T(k,\mu,\vec{x})$.

Let us choose a $k_0 \in L^{-1}\mathbb{Z}\backslash\{0\}$ and fix $\mu > 3\mu_c(\Xi)$. In view of getting infinitely many characteristic values of the linearized problem, we introduce a linear combination of normal modes

$$\vec{U}^M(t,\vec{x}) = \sum_{j=1}^{M} \mathsf{c}_j e^{\lambda_j(k_0,\mu)t}\vec{U}_j(k_0,\mu,\vec{x}) \tag{2.15}$$

to construct an approximate solution to the nonlinear problem (1.1)-(1.2), with constants c_j being chosen such that

$$\text{at least one of } \mathsf{c}_j \ (1 \leqslant j \leqslant N) \text{ is non-zero} \tag{2.16}$$

and let $j_m := \min\{j : 1 \leqslant j \leqslant N, \mathsf{c}_j \neq 0\}$,

$$|\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)} > \frac{1}{2}\sum_{j\geqslant j_m+1} |\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}. \tag{2.17}$$

Eq. (2.1)–(2.2) is supplemented with the initial data $\delta U^M(0,x)$ $(0 < \delta \ll 1)$ has a unique local strong solution $(\sigma^\delta, u^\delta)$ with an associated pressure q^δ on $[0, T_{\max})$ (see Proposition 5.1). We define the differences

$$\sigma^d = \sigma^\delta - \delta\sigma^M, \quad \vec{u}^d = \vec{u}^\delta - \delta\vec{u}^M, \quad q^d = q^\delta - \delta q^M,$$

which satisfy Eq. (5.6), along with the initial condition (5.7) and the boundary conditions (5.8). For t small enough, we estimate the bound in time of $\|(\sigma^d, \vec{u}^d)(t)\|_{L^2(\Omega)}$ (see Proposition 5.2) in the regime

$$\mu > 3\mu_c(\Xi), \quad \text{with } \mu_c(\Xi) := \sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}} \mu_c(k,\Xi). \tag{2.18}$$

Indeed, since $\mu > 3\mu_c(\Xi)$, we can choose a constant $\varpi_0 > 0$ such that

$$\mu > (3+\varpi_0)\mu_c(\Xi). \tag{2.19}$$

Hence, $\nu_0 = \frac{3+\varpi_0}{2+\varpi_0} \in (1, \frac{3}{2})$. It follows from Theorem 2.1 and Lemma 2.1(2) that exists

$$0 < \Lambda = \sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}} \lambda_1(k,\mu) \leqslant \sqrt{\frac{g}{L_0}}. \tag{2.20}$$

We further look for $k_0 \in L^{-1}\mathbb{Z}\backslash\{0\}$ to have that

$$\Lambda \geqslant \lambda_1(k_0,\mu) > \lambda_2(k_0,\mu) > \cdots > \lambda_N(k_0,\mu) > \frac{2\nu_0}{3}\Lambda > \lambda_{N+1}(k_0,\mu) > \ldots. \tag{2.21}$$

For t small enough, we deduce the bound in time of $\|(\sigma^d, \vec{u}^d)(t)\|_{L^2(\Omega)}$ in Proposition 5.2, that is

$$\|(\sigma^d, \vec{u}^d)(t)\|^2_{L^2(\Omega)} \leqslant C\delta^3\Big(\sum_{j=j_m}^{N} |\mathsf{c}_j|e^{\lambda_j t} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M} |\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda t}\Big)^3.$$

The nonlinear result follows.

Theorem 2.2. *Let $\mu_c(\Xi)$ be defined as in (2.18) and $\mu > 3\mu_c(\Xi)$. Let ρ_0 satisfies (1.3), i.e.*

$$\rho_0 \in C^1([-1,1]), \quad \rho_0(\pm 1) = \rho_\pm \in (0,\infty), \quad \rho_0' > 0 \text{ everywhere on } [-1,1].$$

Let $M \in \mathbb{N}^\star$, there exist a positive constant m_0 and two positive constants δ_0 and ϵ_0 sufficiently small such that for any $\delta \in (0,\delta_0)$, the nonlinear equations (2.1) with boundary conditions (2.2) and the initial data

$$\delta \sum_{j=1}^{M} \mathsf{c}_j \vec{U}_j(\vec{x})$$

satisfying (2.16)-(2.17) has a unique local strong solution $(\sigma^\delta, \vec{u}^\delta)$ with an associated pressure q^δ such that

$$\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)} \geqslant m_0\epsilon_0, \tag{2.22}$$

where $T^\delta \in (0, T_{\max})$ is given by $\delta \sum_{j=j_m}^{M} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \epsilon_0$.

3. Preliminaries

The first aim is to prove Lemma 3.1, showing that all characteristic values λ are real for any increasing density profile ρ_0. In the second part, we find the exact formula of the k-critical viscosity coefficient $\mu_c(k,\Xi)$ (see (2.10) above) for all $k > 0$. The last goal is to study the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ in Section 3.3 to prepare for our linear study.

3.1. The positivity of characteristic values λ.

Lemma 3.1. *For any $k \in L^{-1}\mathbb{Z}\backslash\{0\}$,*

- *all characteristic values λ are always real,*
- *all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

Proof. Let $\phi \in H^4((-1,1))$ satisfy (1.4)-(1.5). Multiplying by $\overline{\phi}$ on both sides of (1.4) and using the integration by parts, we get that

$$-\int_{-1}^{1} (\rho_0\phi')'\overline{\phi} dx_2 = -\rho_0\phi'\overline{\phi}\Big|_{-1}^{1} + \int_{-1}^{1} \rho_0|\phi'|^2 dx_2$$

that

$$-\int_{-1}^{1} \phi''\overline{\phi} dx_2 = -\phi'\overline{\phi}\Big|_{-1}^{1} + \int_{-1}^{1} |\phi'|^2 dx_2$$

and that

$$\int_{-1}^{1} \phi^{(4)}\overline{\phi} dx_2 = \phi'''\overline{\phi}\Big|_{-1}^{1} - \phi''\overline{\phi}'\Big|_{-1}^{1} + \int_{-1}^{1} |\phi''|^2 dx_2,$$

we obtain that

$$\begin{aligned} &\lambda\Big(\mu\int_{-1}^{1}(|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_2 - \xi_-|\phi'(-1)|^2 - \xi_+|\phi'(1)|^2\Big) \\ &\quad + \lambda^2\int_{-1}^{1}(k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_2 = gk^2\int_{-1}^{1}\rho_0'|\phi|^2 dx_2. \end{aligned} \tag{3.1}$$

Suppose that $\lambda = \lambda_1 + i\lambda_2$, then one deduces from (3.1) that

$$\begin{aligned}\lambda_1\Big(\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2\Big)\\ +(\lambda_1^2-\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2\end{aligned}\tag{3.2}$$

and that

$$\begin{aligned}&\lambda_2\Big(\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2\Big)\\ &=-2\lambda_1\lambda_2\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2.\end{aligned}\tag{3.3}$$

If $\lambda_2 \neq 0$, Eq. (3.3) leads us to

$$\begin{aligned}-2\lambda_1\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2\\ -\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2,\end{aligned}$$

which yields

$$\begin{aligned}-(\lambda_1^2-\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=-2\lambda_1^2\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2\\ -gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\end{aligned}$$

Equivalently,

$$(\lambda_1^2+\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=-gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\tag{3.4}$$

That implies

$$(\lambda_1^2+\lambda_2^2)k^2\rho_-\int_{-1}^{1}|\phi|^2dx_2\leqslant -gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

The positivity of ρ_0' yields a contradiction, then λ is real. Due to (3.2) again, we further get that

$$\lambda^2\int_{-1}^{1}\rho_0(k^2|\phi|^2+|\phi'|^2)dx_2\leqslant gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 3.1. □

Note again that, thanks to Lemma 2.1, in what follows in this section, we only use real-valued functions for the linear analysis.

3.2. **The threshold of viscosity coefficient.** We obtain the precise formula of the critical viscosity coefficient $\mu_c(k,\Xi)$ for all $k \in \mathbf{R}\backslash\{0\}$. Note that $\mu_c(k,\Xi) = \mu_c(-k,\Xi)$ for all $k \in \mathbf{R}\backslash\{0\}$, it suffices to find $\mu_c(k,\Xi)$ for $k \in \mathbf{R}_+$.

Proposition 3.1. *The following results hold.*

(1) *For all $k \in \mathbf{R}_+$, we have*

$$\mu_c(k,\Xi)=\max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^{1}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_2}.\tag{3.5}$$

Moreover,

$$\mu_c(k,\Xi)=\frac{1}{4k\sinh^2(2k)}\left(\begin{array}{l}(\sinh(2k)\cosh(2k)-2k)(\xi_++\xi_-)\\ +\left(\begin{array}{l}(\sinh(2k)-2k\cosh(2k))^2(\xi_++\xi_-)^2\\ +\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+-\xi_-)^2\end{array}\right)^{\frac{1}{2}}\end{array}\right). \tag{3.6}$$

(2) $\mu_c(k,\Xi)$ *is a decreasing function in* $k\in\mathbf{R}_+$ *and*

$$\lim_{k\to 0}\mu_c(k,\Xi)=\sup_{k\in\mathbf{R}\setminus\{0\}}\mu_c(k,\Xi)=:\mu_c^s(\Xi). \tag{3.7}$$

We have the asymptotic expansion of $\mu_c(k,\Xi)$ *as* $k\to 0^+$*,*

$$\begin{aligned}\mu_c(k,\Xi)=&\frac{1}{3}\Big(\xi_++\xi_-+\sqrt{\xi_+^2-\xi_+\xi_-+\xi_-^2}\Big)\\&-\frac{2}{15}\Big(4(\xi_++\xi_-)+\frac{4\xi_+^2-\xi_+\xi_-+4\xi_-^2}{\sqrt{\xi_+^2-\xi_+\xi_-+\xi_-^2}}\Big)k^2+O(k^3).\end{aligned} \tag{3.8}$$

That implies

$$\mu_c^s(\Xi)=\frac{1}{3}\Big(\xi_++\xi_-+\sqrt{\xi_+^2-\xi_+\xi_-+\xi_-^2}\Big). \tag{3.9}$$

As $k\gg 1$*, we obtain the limit*

$$\mu_c(k,\Xi)\leqslant\frac{\sqrt{2(\xi_+^2+\xi_-^2)}}{k}\to 0. \tag{3.10}$$

(3) *We have*

$$\begin{aligned}\mu_c^s(\Xi)&=\max_{\phi\in\dot{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2dx_2}\\&=\frac{1}{3}\Big(\xi_++\xi_-+\sqrt{\xi_+^2-\xi_+\xi_-+\xi_-^2}\Big).\end{aligned} \tag{3.11}$$

The proof of Proposition 3.1 is postponed to Appendix A.

Remark 3.1. *We see that* $\mu_c^s(\Xi)$ *in* [3, Proposition 2.2] *should be revised and we redo the computation the critical viscosity coefficient. The authors in* [3] *consider* $2\pi L\mathbb{T}\times(0,1)$ *instead of* $2\pi L\mathbb{T}\times(-1,1)$ *and constant values* $k_{0,1}$ *instead of* $\xi_\pm$*. The formula of the critical viscosity defined as in* [3, (1.29)] *is*

$$\mu_c:=\sup_{\phi\in\mathcal{Y}}Z(\phi),$$

where

$$\begin{aligned}\mathcal{Y}&=\{\phi\in H_0^1((0,1))\cap H^2((0,1)),\frac{1}{2}\int_0^1|\phi''|^2dx=1\},\\Z(\phi)&=\frac{k_0}{2}|\phi'(0)|^2+\frac{k_1}{2}|\phi'(1)|^2.\end{aligned}$$

The authors in [3, Proposition 2.2] *claim that* $\mu_c=\frac{k}{6}$ *if* $k_0=k_1=k>0$*. However, in that case, let us take* $\phi=\frac{1}{\sqrt{2}}(x^2-x)\in\mathcal{Y}$*, then we have a contradiction that*

$$Z\Big(\frac{1}{\sqrt{2}}(x^2-x)\Big)=\frac{k}{2}>\frac{k}{6}.$$

3.3. **A bilinear form and a self-adjoint invertible operator.** In what follows in this section we have $\lambda \geqslant 0$ and $k \in \mathbf{R}_+$ being fixed. Let us recall the definition of $\mathcal{B}_{k,\lambda,\mu}$ from (2.9),

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) := \lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_2+\mu\int_{-1}^{1}(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_2 \\ -\xi_-\vartheta'(-1)\varrho'(-1)-\xi_+\vartheta'(1)\varrho'(1).$$

Lemma 3.2. *We have the followings.*

- *For all $\mu>0$, $\mathcal{B}_{k,\lambda,\mu}$ is a continuous bilinear form on $\tilde{H}^2((-1,1))$.*
- *For all $\mu>\mu_c(k,\Xi)$, the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ is coercive.*

Proof of Lemma 3.2. Clearly, $\mathcal{B}_{k,\lambda,\mu}$ is a bilinear form on $\tilde{H}^2((-1,1))$. We then establish the boundedness of $\mathcal{B}_{k,\lambda,\mu}$. The integral terms of $\mathcal{B}_{k,\lambda,\mu}$ are bounded by

$$C^\star(\lambda+1)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}, \tag{3.12}$$

where $C^\star$ is generic constant depending on physical parameters. Meanwhile, it follows from the general Sobolev inequality that

$$(\vartheta'(-1))^2+(\vartheta'(1))^2\leqslant C^\star\|\vartheta'\|^2_{H^1((-1,1))}.$$

Consequently, we get

$$|\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)|\leqslant C^\star(1+\lambda)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}, \tag{3.13}$$

i.e. $\mathcal{B}_{k,\lambda,\mu}$ is bounded.

We show the coercivity of $\mathcal{B}_{k,\lambda,\mu}$. We have that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta)=\lambda\int_{-1}^{1}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx_2+\mu\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2 \\ -\xi_-(\vartheta'(-1))^2-\xi_+(\vartheta'(1))^2.$$

As $\lambda\geqslant 0$ and $\mu>\mu_c(k,\Xi)$, we have

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta)&\geqslant\lambda\int_{-1}^{1}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx_2\\&\quad+(\mu-\mu_c(k,\Xi))\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2 \qquad (3.14)\\&\geqslant(\mu-\mu_c(k,\Xi))\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2.\end{aligned}$$

It then follows from (3.13) and (3.14) that $\mathcal{B}_{k,\lambda,\mu}$ is a continuous and coercive bilinear form on $\tilde{H}^2((-1,1))$. □

With the above property of $\mathcal{B}_{k,\lambda,\mu}$, we then establish:

Proposition 3.2. *Let $\mu>\mu_c(k,\Xi)$ and $(\tilde{H}^2((-1,1)))'$ be the dual space of $\tilde{H}^2((-1,1))$, associated with the norm $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. There is a unique operator*

$$Y_{k,\lambda,\mu}\in\mathcal{L}(H^2((-1,1)),(\tilde{H}^2((-1,1)))'),$$

which is also bijective, such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)=\langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle \tag{3.15}$$

for all $\vartheta,\varrho\in\tilde{H}^2((-1,1))$.

Proof. It follows from Riesz's representation theorem that there exists a unique operator $Y_{k,\lambda,\mu} \in \mathcal{L}(\tilde{H}^2((-1,1)), (\tilde{H}^2((-1,1)))')$ such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta, \varrho) = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle$$

for all $\varrho \in \tilde{H}^2((-1,1))$. Proof of Proposition 3.2 is complete. □

Proposition 3.3. *We have the following results.*

(1) *For all* $\vartheta \in \tilde{H}^2((-1,1))$,

$$Y_{k,\lambda,\mu}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta)$$

in $\mathcal{D}'((-1,1))$.

(2) *Let* $f \in L^2((-1,1))$ *be given, there exists a unique solution* $\vartheta \in \tilde{H}^2((-1,1))$ *of*

$$Y_{k,\lambda,\mu}\vartheta = f \text{ in } (\tilde{H}^2((-1,1)))'. \tag{3.16}$$

Moreover, we have that $\vartheta \in H^4((-1,1))$ *satisfies the boundary conditions* (1.5).

Proof. It follows from Proposition 3.2 that there is a unique $\vartheta \in \tilde{H}^2((-1,1))$ such that

$$\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.17}$$

for all $\varrho \in C_0^\infty((-1,1))$. We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'' which is in $L^2((-1,1))$. Hence, Eq. (3.17) is equivalent to

$$\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\langle(\vartheta'')'', \varrho\rangle + \mu\int_{-1}^{1}(2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.18}$$

for all $\varrho \in C_0^\infty((-1,1))$. We deduce from (3.18) that

$$\lambda\int_{-1}^{1}(k^2\rho_0\vartheta - (\rho_0\vartheta')')\varrho dx_2 + \mu\langle(\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta, \varrho\rangle = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.19}$$

for all $\varrho \in C_0^\infty((-1,1))$. The resulting equation implies that

$$\mu((\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta) + \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') = Y_{k,\lambda,\mu}\vartheta \quad \text{in } \mathcal{D}'((-1,1)). \tag{3.20}$$

The first assertion holds.

Under the assumption $f \in L^2((-1,1))$, we improve the regularity of the weak solution $\vartheta \in \tilde{H}^2((-1,1))$ of (3.20). Indeed, we rewrite (3.20) as

$$\mu\langle(\vartheta'')'', \varrho\rangle = \int_{-1}^{1}(Y_{k,\lambda,\mu}\vartheta + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')\varrho dx_2$$

for all $\varrho \in C_0^\infty((-1,1))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-1,1))$, it then follows from (3.19) that $(\vartheta'')'' \in L^2((-1,1))$. Let $\chi \in C_0^\infty((-1,1))$ satisfy $\int_{-1}^{1}\chi(y)dy = 1$. Using the distribution theory, we define $\Sigma \in \mathcal{D}'((-1,1))$ such that

$$\langle\Sigma, \theta\rangle = \langle(\vartheta'')'', \zeta_\theta\rangle \tag{3.21}$$

for all $\theta \in C_0^\infty((-1,1))$, where

$$\zeta_\theta(x_2) = \int_{-1}^{x_2}\Big(\theta(y) - \chi(y)\int_{-1}^{1}\theta(s)ds\Big)dy$$

for all $-1 < x_2 < 1$. We obtain

$$\langle\Sigma', \theta\rangle = -\langle\Sigma, \theta'\rangle = -\langle(\vartheta'')'', \zeta_{\theta'}\rangle.$$

Note that

$$\langle(\vartheta'')'', \zeta_{\theta'}\rangle = \langle(\vartheta'')'', \theta(x_2) - \int_{-1}^{x_2}\chi(y)\int_{-1}^{1}\theta'(s)dsdy\rangle = \langle(\vartheta'')'', \theta\rangle,$$

this yields $\langle\Sigma', \theta\rangle = -\langle(\vartheta'')'', \theta\rangle$. Hence, we have that $(\vartheta'')' + \Sigma \equiv$ constant. In view of $(\vartheta'')'' \in L^2((-1,1))$ and (3.21), we know that $(\vartheta'')' \in L^2((-1,1))$. Since $\vartheta \in \tilde{H}^2((-1,1))$ and $(\vartheta'')', (\vartheta'')'' \in L^2((-1,1))$, it tells us that ϑ belongs to $H^4((-1,1))$ and we can take their traces of derivatives of ϑ up to order 3.

By performing (3.19), we then show that ϑ satisfies (1.5). Indeed, for all $\varrho \in \tilde{H}^2((-1,1))$, we perform the integration by parts to obtain from (3.19) that

$$\begin{aligned}&\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2\\ &\quad - \lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1} + \mu\Big(\vartheta'''\varrho\Big|_{-1}^{1} - \vartheta''\varrho'\Big|_{-1}^{1} - 2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big) = \int_{-1}^{1}(Y_{k,\lambda,\mu}\vartheta)\varrho dx_2.\end{aligned}$$

It then follows from the definition of the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ that

$$\lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1} - \mu\Big(\vartheta'''\varrho\Big|_{-1}^{1} - \vartheta''\varrho'\Big|_{-1}^{1} - 2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big) = \xi_-\vartheta'(-1)\varrho'(-1) + \xi_+\vartheta'(1)\varrho'(1), \tag{3.22}$$

for all $\varrho \in \tilde{H}^2((-1,1))$. By collecting all terms corresponding to $\varrho'(\pm1)$ in (3.22), we deduce that

$$\mu\vartheta''(\pm1) = \pm\xi_\pm\vartheta'(\pm1).$$

This yields that ϑ satisfies (1.5). The proof of Proposition 3.3 is complete. □

We obtain more information on the inverse operator $Y_{k,\lambda,\mu}^{-1}$.

Proposition 3.4. *The operator $Y_{k,\lambda,\mu}^{-1} : L^2((-1,1)) \to L^2((-1,1))$ is compact and self-adjoint.*

Proof. It follows from Proposition 3.3 that $Y_{k,\lambda,\mu}$, being supplemented with (1.5), admits an inverse operator $Y_{k,\lambda,\mu}^{-1}$ from $L^2((-1,1))$ to a subspace of $H^4((-1,1))$ requiring all elements satisfy (1.5), which is symmetric due to Proposition 3.2. We compose $Y_{k,\lambda,\mu}^{-1}$ with the continuous injection from $H^4((-1,1))$ to $L^2((-1,1))$. Notice that the embedding $H^p((-1,1)) \hookrightarrow H^q((-1,1))$ for $p > q \geqslant 0$ is compact. Therefore, the operator $Y_{k,\lambda,\mu}^{-1}$ is compact and self-adjoint from $L^2((-1,1))$ to $L^2((-1,1))$. □

4. Linear instability

4.1. **A sequence of characteristic values.** We continue considering $\lambda \geqslant 0$ and $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ being fixed. We study the operator $S_{k,\lambda,\mu} := \mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$.

Proposition 4.1. *Under the hypothesis (1.3), the operator $S_{k,\lambda,\mu} : L^2((-1,1)) \to L^2((-1,1))$ is compact and self-adjoint.*

Proof. Due to the assumption on ρ_0 (1.3), the operator $S_{k,\lambda,\mu}$ is well-defined and bounded from $L^2((-1,1))$ to itself. $Y_{k,\lambda,\mu}^{-1}$ is compact, so is $S_{k,\lambda,\mu}$. Moreover, because both the inverse $Y_{k,\lambda,\mu}^{-1}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{k,\lambda,\mu}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{k,\lambda,\mu}$ is discrete, i.e. is a sequence $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant 1}$ of eigenvalues of $S_{k,\lambda,\mu}$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-1,1))$. That means

$$S_{k,\lambda,\mu}\varpi_n = \mathcal{M}Y^{-1}_{k,\lambda,\mu}\mathcal{M}\varpi_n = \gamma_n(k,\lambda,\mu)\varpi_n.$$

So that $\phi_n = Y^{-1}_{k,\lambda,\mu}\mathcal{M}\varpi_n$ belongs to $H^4((-1,1))$ and satisfies (1.5). One thus has

$$\gamma_n(k,\lambda,\mu)Y_{k,\lambda,\mu}\phi_n = \rho_0'\phi_n \tag{4.1}$$

and ϕ_n satisfies (1.5). Eq. (4.1) also tells us that $\gamma_n(k,\lambda,\mu) > 0$ for all n. Indeed, we obtain

$$\gamma_n(k,\lambda,\mu)\int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)\phi_n dx_2 = \int_{-1}^{1}\rho_0'\phi_n^2 dx_2.$$

That implies

$$\gamma_n(k,\lambda,\mu)\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) = \int_{-1}^{1}\rho_0'\phi_n^2 dx_2. \tag{4.2}$$

Since $\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) > 0$ and $\rho_0' > 0$ on $(-1,1)$, we know that $\gamma_n(k,\lambda,\mu)$ is positive for all $n \geqslant 1$. Hence, by reordering and using the spectral theory of compact and self-adjoint operators again, we have that $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant 1}$ is a positive sequence decreasing towards 0 as $n \to \infty$.

For each n, we have that ϕ_n is a solution of (1.4)-(1.5) if and only if there are positive λ_n such that (2.14) holds. To solve (2.14), we use the two following lemmas.

Lemma 4.1. *For each n,*

- *$\gamma_n(k,\lambda,\mu)$ and ϕ_n are differentiable in λ.*
- *$\gamma_n(k,\lambda,\mu)$ is decreasing in λ.*

Proof. The proof of Lemma 4.1(1) is the same as [13, Lemma 3.3], we omit the details here. We now prove that $\gamma_n(k,\lambda,\mu)$ is decreasing in λ.

Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (4.1) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{k,\lambda,\mu}z_n = \frac{1}{\gamma_n(k,\lambda,\mu)}\rho_0'z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\rho_0'\phi_n \tag{4.3}$$

on $(-1,1)$. At $x_2 = \pm 1$, we have

$$\begin{cases} z_n(-1) = z_n(1) = 0, \\ \mu z_n''(1) = \xi_+ z_n'(1), \\ \mu z_n''(-1) = -\xi_- z_n'(-1). \end{cases} \tag{4.4}$$

Multiplying by ϕ_n on both sides of (4.3), we obtain that

$$\begin{aligned} &\int_{-1}^{1}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_2 + \int_{-1}^{1}(Y_{k,\lambda,\mu}z_n)\phi_n dx_2 \\ &\qquad = \frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'z_n\phi_n dx_2 + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'\phi_n^2 dx_2. \end{aligned} \tag{4.5}$$

Note that z_n enjoys (4.4), then

$$\int_{-1}^{1}(Y_{k,\lambda,\mu}z_n)\phi_n dx_2 = \int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)z_n dx_2 = \frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'z_n\phi_n dx_2.$$

That implies

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'\phi_n^2 dx_2 = \int_{-1}^{1}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_2. \tag{4.6}$$

Using the integration by parts, we obtain from (4.6) that

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'\phi_n^2dx_2=\int_{-1}^{1}\rho_0(k^2\phi_n^2+(\phi_n')^2)dx_2>0.$$

Consequently, $\gamma_n(k,\lambda,\mu)$ is decreasing in $\lambda>0$. □

4.2. **Proof of Theorem 2.1 and normal modes of the linearized equations.** In view of Lemma 4.1, we are able to prove Theorem 2.1.

Proof of Theorem 2.1. For each n, there is only one solution λ_n of (2.14). Indeed, using (4.2), we know that

$$\frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'\phi_n^2dx_2=\int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)\phi_ndx_2=\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n).$$

Hence, it follows from (3.14) that

$$\begin{aligned}\frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'\phi_n^2dx_2&\geqslant\lambda\int_{-1}^{1}\rho_0(k^2\phi_n^2+(\phi_n')^2)dx_2\\&\quad+(\mu-\mu_c(k,\Xi))\int_{-1}^{1}((\phi_n'')^2+2k^2(\phi_n')^2+k^4\phi_n^2)dx_2\\&\geqslant\lambda k^2\int_{-1}^{1}\rho_0\phi_n^2dx_2+(\mu-\mu_c(k,\Xi))k^4\int_{-1}^{1}\phi_n^2dx_2.\end{aligned}$$

That implies

$$\frac{1}{L_0\gamma_n(k,\lambda,\mu)}\geqslant\lambda k^2+\frac{(\mu-\mu_c(k,\Xi))k^4}{\rho_+}.$$

Consequently, for all $n\geqslant1$,

$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)}>gk^2\text{ for }\lambda\text{ large.}\tag{4.7}$$

Meanwhile, for all $n\geqslant1$ and $\lambda\leqslant\frac{1}{2}\sqrt{\frac{g}{L_0}}$,

$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)}\leqslant\frac{\lambda}{\gamma_n(k,\frac{1}{2}\sqrt{\frac{g}{L_0}},\mu)}\to0\text{ as }\lambda\to0.\tag{4.8}$$

In view of (4.7), (4.8) and Lemma 4.1, we obtain only one solution λ_n of (2.14) and (λ_n,ϕ_n) satisfies (1.4)-(1.5). That means for all n, λ_n is a characteristic value, hence it is bounded by $\sqrt{\frac{g}{L_0}}$.

We now prove that $(\lambda_n)_{n\geqslant1}$ decreases towards 0 as $n\to\infty$. If $\lambda_m<\lambda_{m+1}$ for some $m\geqslant1$, we have

$$\gamma_m(k,\lambda_m,\mu)>\gamma_m(k,\lambda_{m+1},\mu).$$

Meanwhile, we also have

$$\gamma_m(k,\lambda_{m+1},\mu)>\gamma_{m+1}(k,\lambda_{m+1},\mu).$$

That implies

$$\frac{\lambda_m}{gk^2}=\gamma_m(k,\lambda_m,\mu)>\gamma_{m+1}(k,\lambda_{m+1},\mu)=\frac{\lambda_{m+1}}{gk^2}.$$

That contradiction tells us that $(\lambda_n)_{n\geqslant1}$ is a decreasing sequence. Suppose that

$$\lim_{n\to\infty}\lambda_n=d_0>0.$$

Note that for all n, $\gamma_n(k,\lambda_n,\mu)=\frac{\lambda_n}{gk^2}$, then

$$\gamma_n(k,d_0,\mu)\geqslant\gamma_n(k,\lambda_n,\mu)=\frac{\lambda_n}{gk^2}.$$

Let $n\to\infty$, we get that $0\geqslant d_0$, a contradiction. Hence λ_n decreases towards 0 as $n\to\infty$. The proof of Theorem 2.1 is complete. □

We derive the following property for the largest characteristic value λ_1 found in Theorem 2.1.

Lemma 4.2. *Let us recall the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ on $H^2((-1,1))$ (2.9) and (λ_1,ϕ_1) from Theorem 2.1. We have that*

$$\frac{1}{gk^2}=\max_{\phi\in H^2((-1,1))}\frac{\int_{-1}^1\rho_0'\phi^2dx_2}{\lambda_1\mathcal{B}_{k,\lambda_1,\mu}(\phi,\phi)},\tag{4.9}$$

and the variational problem (4.9) is attained by the function ϕ_1.

Proof. For all $\lambda>0$, we solve the variational problem

$$\beta(k,\lambda,\mu)=\max\Big(\int_{-1}^1\rho_0'\phi^2dx_2\Big|\phi\in\tilde{H}^2((-1,1)),\quad\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi)=1\Big).$$

Let us define the Lagrangian functional

$$\mathcal{L}_{\mathcal{B}}(\phi,\beta)=\int_{-1}^1\rho_0'\phi^2dx_2-\beta(\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi)-1).$$

Thanks to the Lagrange multiplier theorem, the extrema of the quotient

$$\frac{\int_{-1}^1\rho_0'\phi^2dx_2}{\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi)}$$

are necessarily the stationary points $(\beta_\star,\phi_\star)$ of $\mathcal{L}_{\mathcal{B}}$, which satisfy

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\phi_\star,\phi_\star)=1\tag{4.10}$$

and

$$\int_{-1}^1\rho_0'\phi_\star\theta dx_2-\beta_\star\lambda\mathcal{B}_{k,\lambda,\mu}(\phi_\star,\theta)=0,\tag{4.11}$$

for all $\theta\in\tilde{H}^2((-1,1))$. Restricting $\theta\in C_0^\infty((-1,1))$ and following the line of the proof of Proposition 3.3, one deduces from (4.11) that $\phi_\star$ has to satisfy

$$\beta_\star\lambda Y_{k,\lambda,\mu}\phi_\star=\rho_0'\phi_\star\tag{4.12}$$

in a weak sense. We further get that $\phi_\star\in H^4((-1,1))$ and satisfies (4.10) and the boundary conditions (1.5). Hence, $\lambda\beta_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{k,\lambda,\mu}$ from $L^2((-1,1))$ to itself, with $\mathcal{M}^{-1}Y_{k,\lambda,\mu}\phi_\star\in L^2((-1,1))$ being an associated eigenfunction. That implies

$$\beta(k,\lambda,\mu)\leqslant\lambda^{-1}\gamma_1(k,\lambda,\mu).\tag{4.13}$$

Meanwhile, since the operator $S_{k,\lambda,\mu}$ is self-adjoint and positive, we thus obtain that

$$\gamma_1(k,\lambda,\mu)=\sup_{\omega\in L^2((-1,1))}\frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle}{\|\omega\|^2_{L^2((-1,1))}}.$$

Hence, for all $\omega\in L^2((-1,1))$ and for $\phi=Y^{-1}_{k,\lambda,\mu}\mathcal{M}\omega\in H^4((-1,1))$, we have

$$\gamma_1(k,\lambda,\mu)\langle Y_{k,\lambda,\mu}\phi,\phi\rangle\leqslant\frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle^2}{\|\omega\|^2_{L^2((-1,1))}}\leqslant\|S_{k,\lambda,\mu}\omega\|^2_{L^2((-1,1))}.$$

Equivalently,

$$\gamma_1(k,\lambda,\mu) \leqslant \sup\Big\{ \frac{\|\mathcal{M}\phi\|^2_{L^2((-1,1))}}{\langle Y_{k,\lambda,\mu}\phi,\phi\rangle} | \phi \in H^4((-1,1)) \text{ and } \mathcal{M}^{-1}Y_{k,\lambda,\mu}\phi \in L^2((-1,1))\Big\},$$

it yields

$$\lambda^{-1}\gamma_1(k,\lambda,\mu) \leqslant \beta(k,\lambda,\mu). \tag{4.14}$$

Two inequalities (4.13) and (4.14) tell us that $\beta(k,\lambda,\mu) = \lambda^{-1}\gamma_1(k,\lambda,\mu)$ for all $\lambda > 0$. We thus obtain $\beta(k,\lambda_1,\mu) = \frac{1}{gk^2}$ and the variational problem (4.9) is attained by the function ϕ_1. Proof of Lemma 4.2 is complete. □

We now solve the linearized equations (2.3) to prepare for our nonlinear part.

Proposition 4.2. *For each $k \in L^{-1}\mathbb{Z}\backslash\{0\}\backslash\{0\}$ and for all $\mu > \mu_c(k,\Xi)$, there exists an infinite sequence of solutions ($n \geqslant 1$)*

$$\begin{aligned} e^{\lambda_n(k,\mu)t}\vec{U}_n(k,\mu,\vec{x}) &= e^{\lambda_n(k,\mu)t}(\sigma_n,\vec{u}_n,p_n)^T(k,\mu,\vec{x}) \\ &= e^{\lambda_n(k,\mu)t}\begin{pmatrix} \cos(kx_1)\omega_n(k,\mu,x_2) \\ \sin(kx_1)\theta_n(k,\mu,x_2) \\ \cos(kx_1)\phi_n(k,\mu,x_2) \\ \cos(kx_1)q_n(k,\mu,x_2) \end{pmatrix} \end{aligned}$$

to the linearized equation (2.3)-(2.2), such that

$$\sigma_n \in H^2(\Omega), \vec{u}_n \in (H^3(\Omega))^2 \text{ and } p_n \in H^1(\Omega).$$

Proof. For each solution $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ of (2.14), we have that $\phi_n = Y^{-1}_{k,\lambda_n,\mu}\mathcal{M}\varpi_n$ is a solution of (1.4)-(1.5) in $H^4((-1,1))$ as $\lambda = \lambda_n$. We now find a solution to the system (2.5) as $\lambda = \lambda_n$. First, we obtain $\theta_n = -\frac{\phi_n'}{k}$ and $\omega_n = -\frac{\rho_0'\phi_n}{\lambda_n}$. Due to (2.7), we get

$$q_n = -\frac{1}{k^2}(\lambda_n\rho_0\phi_n' + \mu(k^2\phi_n' - \phi_n''')) \in H^1((-1,1)).$$

With a solution $(\omega_n,\theta_n,\phi_n,q_n)$ of (2.5), we then conclude that

$$e^{\lambda_n(k,\mu)t}(\sigma_n,u_{n,1},u_{n,2},p_n)^T(k,\mu,\vec{x}) = e^{\lambda_n(k,\mu)t}\begin{pmatrix} \cos(kx_1)\omega_n(k,\mu,x_2) \\ \sin(kx_1)\theta_n(k,\mu,x_2) \\ \cos(kx_1)\phi_n(k,\mu,x_2) \\ \cos(kx_1)q_n(k,\mu,x_2) \end{pmatrix}$$

is a solution to the linearized equations (2.3)-(2.2). □

5. Nonlinear instability

5.1. **The local existence.** The first important things are the local existence of strong solutions to the nonlinear equations and *a priori* energy estimates to those solutions. We restate Proposition 4.1 of [2].

Proposition 5.1. *Suppose that the steady state satisfies (1.3). For any given initial data $(\sigma_0,\vec{u}_0) \in (H^1(\Omega)\cap L^\infty(\Omega)) \times (H^2(\Omega))^2$ satisfying $div\vec{u}_0 = 0$, and also being compatible with the boundary conditions (1.2), the nonlinear equations (2.1) has a local strong solution*

$$(\sigma,\vec{u},\nabla q) \in C([0,T^{\max}), H^1(\Omega) \times (H^2(\Omega))^2 \times (L^2(\Omega)))^2. \tag{5.1}$$

Let $\mathcal{E}(t) := \sqrt{\|\sigma(t)\|^2_{H^1(\Omega)} + \|\vec{u}(t)\|^2_{H^2(\Omega)}}$ *and* $\delta_0 > 0$ *be sufficiently small, we further get that if* $\sup_{0\leqslant s\leqslant t}\mathcal{E}(t) \leqslant \delta_0$, *there holds*

$$\begin{aligned}\mathcal{E}^2(t) + \|(\nabla q, \partial_t\vec{u})\|^2_{L^2(\Omega)} + \int_0^t (\|\partial_t\vec{u}(s)\|^2_{H^1(\Omega)} + \|\vec{u}(s)\|^2_{H^2(\Omega)})ds \\ \leqslant C_0\Big(\mathcal{E}^2(0) + \int_0^t \|(\sigma,\vec{u})(s)\|^2_{L^2(\Omega)}ds\Big).\end{aligned} \tag{5.2}$$

Thanks to Proposition 4.2, we will formulate a sequence of approximate solutions $e^{\lambda_n(k,\mu)}\vec{U}_n(k,\mu,\vec{x})$ to the nonlinear equations (2.1)-(2.2), which are solutions to the linearized equations (2.3)-(2.2). Let us fix a $k = k_0 \in L^{-1}\mathbb{Z}\backslash\{0\}$ such that (2.21) holds and $\mu > 3\mu_c(\Xi)$. We recall (2.15),

$$(\sigma^M, \vec{u}^M, q^M)(t,\vec{x}) := \sum_{j=1}^M \mathsf{c}_j e^{\lambda_j(k,\mu)t}\vec{U}_j(k,\mu,\vec{x}).$$

Keeping in mind that $\min_{[-1,1]}\rho_0 > 0$, then due to the embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$, there exists a constant $\delta_0 > 0$ such that

$$\delta_0\|\sum_{j\geqslant 1}\sigma_j(0,\vec{x})\|_{L^\infty(\Omega)} > \frac{1}{2}\min_{[-1,1]}\rho_0(x_2). \tag{5.3}$$

Hence, for $\delta \leqslant \delta_0$,

$$\frac{1}{2}\min_{[-1,1]}\rho_0(x_2) \leqslant \min_{\Omega}(\rho_0(x_2) + \delta\sigma^M(0,\vec{x})).$$

By virtue of Proposition 5.1, the nonlinear equations (2.1)-(2.2) with the initial data $\delta(\sigma^M, \vec{u}^M)(0)$ admits a local solution

$$(\sigma^\delta, \vec{u}^\delta) \in C^0([0,T^{\max}), H^1(\Omega)\times(H^2(\Omega))^2)$$

with an associated pressure $q^\delta \in C^0([0,T^{\max}), L^2(\Omega))$. Furthermore, we have

$$\frac{1}{2}\min_{[-1,1]}\rho_0(x_2) \leqslant \inf_{\Omega}(\rho_0(x_2) + \sigma^\delta(t,\vec{x}))$$

for all $t \in [0, T^{\max})$.

In what follows, the constants $C_i (i \geqslant 1)$ are universal ones depending only on physical parameters, M and $\mathsf{c}_j (j \geqslant 1)$.

Let $F_M(t) = \sum_{j=j_m}^M |\mathsf{c}_j|e^{\lambda_j t}$ and $0 < \epsilon_0 \ll 1$ be fixed later (5.41). There exists a unique T^δ such that $\delta F_M(T^\delta) = \epsilon_0$. Let

$$C_1 = \sqrt{\|\sigma^M(0)\|^2_{H^1(\Omega)} + \|\vec{u}^M(0)\|^2_{H^2(\Omega)}}, \quad C_2 = \sqrt{\|\sigma^M(0)\|^2_{L^2(\Omega)} + \|\vec{u}^M(0)\|^2_{L^2(\Omega)}}.$$

We define

$$\begin{aligned}T^\star &:= \sup\Big\{t \in (0,T^{\max})|\mathcal{E}(\sigma^\delta(t),\vec{u}^\delta(t)) \leqslant C_1\delta_0\} > 0,\\ T^{\star\star} &:= \sup\{t \in (0,T^{\max})|\|(\sigma^\delta,\vec{u}^\delta)(t)\|_{L^2(\Omega)} \leqslant 2C_2\delta F_M(t)\Big\} > 0.\end{aligned} \tag{5.4}$$

Note that $\mathcal{E}(\sigma^\delta(0),\vec{u}^\delta(0)) = C_1\delta < C_1\delta_0$ and because of (5.1), we then have $T^\star > 0$. Similarly, we have $T^{\star\star} > 0$. Then for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, it follows from the *a priori* energy estimate (5.2) that

$$\mathcal{E}^2(\sigma^\delta(t),\vec{u}^\delta(t)) + \|\partial_t\vec{u}^\delta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\partial_t\vec{u}^\delta(\tau)\|^2_{L^2(\Omega)}d\tau \leqslant C_3\delta^2F_M^2(t). \tag{5.5}$$

5.2. **The difference functions.** Let us recall

$$\sigma^d = \sigma^\delta - \delta\sigma^M, \quad \vec{u}^d = \vec{u}^\delta - \delta\vec{u}^M, \quad q^d = q^\delta - \delta q^M.$$

Hence, $(\sigma^d, \vec{u}^d, q^d)$ satisfies

$$\begin{cases} \partial_t\sigma^d + \rho_0' u_2^d = -\vec{u}^\delta \cdot \nabla\sigma^\delta, \\ \rho_0\partial_t\vec{u}^d - \mu\Delta\vec{u}^d + \nabla q^d = -\sigma^\delta\partial_t\vec{u}^\delta - (\rho_0 + \sigma^\delta)\vec{u}^\delta \cdot \nabla\vec{u}^\delta - g\sigma^d\vec{e}_2, \\ \mathrm{div}\vec{u}^d = 0, \end{cases} \tag{5.6}$$

along with the initial condition,

$$(\sigma^d, \vec{u}^d)(0) = 0 \tag{5.7}$$

and the boundary conditions,

$$\begin{cases} u_2^d = 0, & \text{on } \Sigma_\pm, \\ \mu\partial_{x_2}u_1^d = \xi_+ u_1^d & \text{on } \Sigma_+, \\ \mu\partial_{x_2}u_1^d = -\xi_- u_1^d & \text{on } \Sigma_-. \end{cases} \tag{5.8}$$

The compatibility conditions read as

$$u_1^d(0, x_1, -1) = u_1^d(0, x_1, 1), \quad \mathrm{div}\vec{u}^d(0) = 0. \tag{5.9}$$

We now establish the error estimate for $\|(\sigma^d, \vec{u}^d)\|_{L^2(\Omega)}$.

Proposition 5.2. *For all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$, there holds*

$$\|(\sigma^d, \vec{u}^d)(t)\|_{L^2(\Omega)}^2 \leqslant C_4\delta^3\Big(\sum_{j=1}^N |\mathsf{c}_j|e^{\lambda_j t} + \max(0, M-N)\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|e^{\frac{2}{3}\nu_0\Lambda t}\Big)^3. \tag{5.10}$$

The proof of Proposition 5.2 relies on Lemmas 5.1, 5.2, 5.3, 5.4 below.

Lemma 5.1. *We have the following inequalities*

$$\sum_{0\leqslant s\leqslant 2, 0\leqslant\tau\leqslant 1} \|\partial_t^\tau\vec{u}^d(t)\|_{H^s(\Omega)} \leqslant C_5\delta F_M(t), \tag{5.11}$$

and

$$\|\sigma^d(t)\|_{H^1(\Omega)} + \|\partial_t\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_6\delta F_M(t). \tag{5.12}$$

Proof. For $\tau \in \{0, 1\}$,

$$\partial_t^\tau\vec{u}^M(t) = \sum_{j=1}^M \lambda_j^\tau \mathsf{c}_j e^{\lambda_j t}\vec{U}_j(k_0, \vec{x}),$$

it yields, for all $s \in \{0, 1, 2\}$,

$$\|\partial_t^\tau\vec{u}^M(t)\|_{H^s(\Omega)} \leqslant C_7 F_M(t).$$

In view of (5.5), we then obtain that for $s \in \{0, 1, 2\}$ and $\tau \in \{0, 1\}$,

$$\|\partial_t^\tau\vec{u}^d(t)\|_{H^s(\Omega)} \leqslant \delta\|\partial_t^\tau\vec{u}^M(t)\|_{H^s(\Omega)} + \|\partial_t^\tau\vec{u}^\delta(t)\|_{H^s(\Omega)} \leqslant C_8\delta F_M(t).$$

To prove (5.12), we use $(5.6)_1$ and (5.5) again,

$$\begin{aligned} \|\sigma^d(t)\|_{H^1(\Omega)} + \|\partial_t\sigma^d(t)\|_{L^2(\Omega)} &\leqslant \|\sigma^\delta(t)\|_{H^1(\Omega)} + \delta\|\sigma^M(t)\|_{H^1(\Omega)} \\ &\quad + C_9\|u_2^d(t)\|_{L^2(\Omega)} + \|\vec{u}^\delta(t)\|_{L^2(\Omega)}\|\nabla\sigma^\delta\|_{L^2(\Omega)} \\ &\leqslant C_{10}\delta F_M(t). \end{aligned}$$

Lemma 5.1 is proven. □

Lemma 5.2. *There holds*

$$\|\partial_t \vec{u}^d(0)\|^2_{L^2(\Omega)} \leqslant C_{11}\delta^3. \tag{5.13}$$

Proof. From $(5.6)_{2,3}$ and the boundary conditions (5.8), we use the integration by parts to obtain that

$$\int_\Omega \rho_0|\partial_t \vec{u}^d|^2 d\vec{x} = \int_\Omega \mu\Delta\vec{u}^d \cdot \partial_t \vec{u}^d d\vec{x} - \int_\Omega (\sigma^\delta \partial_t \vec{u}^\delta + (\rho_0+\sigma^\delta)\vec{u}^\delta \cdot \nabla \vec{u}^\delta) \cdot \partial_t \vec{u}^d d\vec{x} - \int_\Omega g\sigma^d \partial_t u_2^d d\vec{x}.$$

Thanks to Lemma 5.1, one has

$$-\int_\Omega (\sigma^\delta \partial_t \vec{u}^\delta + (\rho_0+\sigma^\delta)\vec{u}^\delta \cdot \nabla \vec{u}^\delta) \cdot \partial_t \vec{u}^d d\vec{x} \leqslant C_{12}\delta^3 F_M^3(t). \tag{5.14}$$

That implies

$$\|\partial_t \vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{13}\Big((\|\vec{u}^d(t)\|_{H^2(\Omega)} + \|\sigma^d(t)\|_{L^2(\Omega)})\|\partial_t \vec{u}^d(t)\|_{L^2(\Omega)} + \delta^3 F_M^3(t)\Big).$$

Using Young's inequality, we further get

$$\|\partial_t \vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant \frac{1}{2}\|\partial_t \vec{u}^d(t)\|^2_{L^2(\Omega)} + C_{14}(\|\vec{u}^d(t)\|^2_{H^2(\Omega)} + \|\sigma^d(t)\|^2_{L^2(\Omega)}) + C_{13}\delta^3 F_M^3(t).$$

That implies

$$\|\partial_t \vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{15}\Big(\|\vec{u}^d(t)\|^2_{H^2(\Omega)} + \|\sigma^d(t)\|^2_{L^2(\Omega)} + \delta^3 F_M^3(t)\Big). \tag{5.15}$$

Letting $t \to 0$ in (5.15), we complete the proof Lemma 5.2. □

Lemma 5.3. *Let*

$$\mathsf{X} := \{\vec{w} \in (H^2(\Omega))^2, \vec{w} \text{ satisfies } (1.2) \text{ and } div\vec{w} = 0\}.$$

There holds for all $\vec{w} \in H^2_\star(\Omega)$,

$$\begin{aligned}\int_\Omega g\rho_0'|w_2|^2 d\vec{x} + \Lambda\int_{(2\pi L\mathbb{T})^2} (\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2)dx_1 \\ \leqslant \Lambda^2\int_\Omega \rho_0|\vec{w}|^2 d\vec{x} + \Lambda\mu\int_\Omega |\nabla\vec{w}|^2 d\vec{x}.\end{aligned} \tag{5.16}$$

The proof of Lemma 5.3 is due to the definition of Λ (2.20) and Lemma 4.2, that is similar to [2, Lemma 5.1], hence we omit the details here.

Lemma 5.4. *There holds for all* $\vec{w} \in \mathsf{X}\backslash\{\vec{0}\}$,

$$\sup_{\vec{w}\in\mathsf{X}} \frac{\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2)dx_1}{\|\nabla\vec{w}\|^2_{L^2(\Omega)}} \leqslant \mu_c(\Xi). \tag{5.17}$$

Proof. Let us fix a horizontal frequency $k \in L^{-1}\mathbb{Z}$ and introduce the horizontal Fourier transform

$$\hat{f}(k,x_2) = \int_{2\pi L\mathbb{T}} f(\vec{x})e^{-ikx_1}dx_1.$$

For $\vec{w} \in \mathsf{X}$, we write

$$\hat{w}_1(k,x_2) = -i\theta(k,x_2), \quad \hat{w}_2(k,x_2) = \phi(k,x_2).$$

Then, $k\theta+\phi'=0$ and (θ,ϕ) enjoys (2.6). Following Fubini's and Parseval's theorem, one thus deduces

$$\begin{aligned}\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ =\frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}}(\xi_+(|\theta(k,1)|^2+\xi_-|\theta(k,-1)|^2)\end{aligned}\tag{5.18}$$

and

$$\|\nabla\vec{w}\|^2_{L^2(\Omega)}=\frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}}\int_{-1}^{1}(k^2(|\theta|^2+|\phi|^2)+|\theta'|^2+|\phi'|^2)(k,x_2)dx_2.\tag{5.19}$$

We may reduce to estimate (5.17) when θ and ϕ are real-valued and continue the estimate to the real and imaginary part of θ and ϕ. For any $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, we have from $k\theta+\phi'=0$ that

$$\xi_+(\theta(k,1))^2+\xi_-(\theta(k,-1))^2=\frac{1}{k^2}(\xi_+((\phi'(k,1))^2+\xi_-(\phi'(k,-1))^2)\tag{5.20}$$

and that

$$\begin{aligned}\int_{-1}^{1}\Big(k^2(\theta^2+\phi^2)+(\theta')^2+(\phi')^2\Big)(k,x_2)dx_2\\ =\frac{1}{k^2}\int_{-1}^{1}(k^4\phi^2+2k^2(\phi')^2+(\phi'')^2)(k,x_2)dx_2.\end{aligned}\tag{5.21}$$

Owing to (5.18), (5.20) and the definition of $\mu_c(k,\Xi)$ (see (3.5)), we get

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &\leqslant\frac{1}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to0}\dfrac{1}{k^2}(\xi_+(\phi'(k,1))^2+\xi_-(\phi'(k,-1))^2)\\ +\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{1}{k^2}(\xi_+(\phi'(k,1))^2+\xi_-(\phi'(k,-1))^2)\end{array}\right)\\ &\leqslant\frac{1}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to0}\dfrac{\mu_c(k,\Xi)}{k^2}\displaystyle\int_{-1}^{1}(k^4\phi^2+2k^2\phi'^2+\phi''^2)(k,x_2)dx_2\\ +\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{\mu_c(k,\Xi)}{k^2}\int_{-1}^{1}(k^4\phi^2+2k^2\phi'^2+\phi''^2)(k,x_2)dx_2\end{array}\right).\end{aligned}$$

Thanks to Proposition 3.1, we obtain

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &\leqslant\frac{\mu_c(\Xi)}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to0}\dfrac{1}{k^2}\displaystyle\int_{-1}^{1}(k^4\phi^2+2k^2(\phi')^2+(\phi'')^2)(k,x_2)dx_2\\ +\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{1}{k^2}\int_{-1}^{1}(k^4\phi^2+2k^2(\phi')^2+(\phi'')^2)(k,x_2)dx_2\end{array}\right).\end{aligned}\tag{5.22}$$

Combining (5.19), (5.21) and (5.22), it gives

$$\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\leqslant\mu_c(\Xi)\|\nabla\vec{w}\|^2_{L^2(\Omega)}.$$

Lemma 5.4 is proven. □

We now prove Proposition 5.2.

Proof of Proposition 5.2. We rewrite $(5.6)_2$ as

$$(\rho_0+\sigma^\delta)\partial_t\vec{u}^d-\mu\Delta\vec{u}^d+\nabla q^d=\vec{f^\delta}-g\sigma^d\vec{e}_2,$$

where $\vec{f^\delta}=-\sigma^\delta\partial_t\vec{u}^M-(\rho_0+\sigma^\delta)\vec{u}^\delta\cdot\nabla\vec{u}^\delta$. Differentiate the resulting equation with respect to t and then multiply by $\partial_t\vec{u}^d$, we obtain after integration that

$$\begin{aligned}&\int_\Omega\partial_t\sigma^\delta|\partial_t\vec{u}^d|^2d\vec{x}+\int_\Omega(\rho_0+\sigma^\delta)\partial_t^2\vec{u}^d\cdot\partial_t\vec{u}^d d\vec{x}\\&=\int_\Omega\mu\Delta\partial_t\vec{u}^d\cdot\partial_t\vec{u}^d d\vec{x}-\int_\Omega\nabla\partial_t q^d\cdot\partial_t\vec{u}^d d\vec{x}+\int_\Omega(\partial_t\vec{f^\delta}-g\partial_t\sigma^d\vec{e}_2)\cdot\partial_t\vec{u}^d d\vec{x}.\end{aligned}$$

Since $\mathrm{div}\partial_t\vec{u}^d=0$, we use the integration by parts to further obtain

$$\begin{aligned}&\int_\Omega\partial_t\sigma^\delta(t)|\partial_t\vec{u}^d(t)|^2d\vec{x}+\int_\Omega(\rho_0+\sigma^\delta(t))\partial_t^2\vec{u}^d(t)\cdot\partial_t\vec{u}^d(t)d\vec{x}\\&=\int_\Omega(\partial_t\vec{f^\delta}(t)-g\partial_t\sigma^d(t)\vec{e}_2)\cdot\partial_t\vec{u}^d(t)d\vec{x}-\mu\int_\Omega|\nabla\partial_t\vec{u}^d(t)|^2d\vec{x}\\&\qquad+\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(t,x_1,1)|^2+\xi_-|\partial_t u_1^d(t,x_1,-1)|^2)dx_1.\end{aligned}$$

That means,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho_0+\sigma^\delta(t))|\partial_t\vec{u}^d(t)|^2d\vec{x}\\&=-\frac{1}{2}\int_\Omega\partial_t\sigma^\delta(t)|\partial_t\vec{u}^d(t)|^2d\vec{x}+\int_\Omega(\partial_t\vec{f^\delta}(t)-g\partial_t\sigma^d(t)\vec{e}_2)\cdot\partial_t\vec{u}^d(t)d\vec{x}\\&\quad-\mu\int_\Omega|\nabla\partial_t\vec{u}^d(t)|^2d\vec{x}+\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(t,x_1,1)|^2+\xi_-|\partial_t u_1^d(t,x_1,-1)|^2)dx_1.\end{aligned}$$

Using $(5.6)_1$, we then get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big((\rho_0+\sigma^\delta(t))|\partial_t\vec{u}^d(t)|^2-g\rho_0'|u_2^d(t)|^2\Big)d\vec{x}\\&\quad+2\mu\int_\Omega|\nabla\partial_t\vec{u}^d(t)|^2d\vec{x}-2\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(t,x_1,1)|^2+\xi_-|\partial_t u_1^d(t,x_1,-1)|^2)dx_1\\&=-\int_\Omega\partial_t\sigma^\delta(t)|\partial_t\vec{u}^d(t)|^2d\vec{x}+2\int_\Omega(\partial_t\vec{f^\delta}(t)+g\vec{u}^\delta(t)\cdot\nabla\sigma^\delta(t)\vec{e}_2)\cdot\partial_t\vec{u}^d(t)d\vec{x}.\end{aligned}$$

Integrating in time variable, we get

$$\begin{aligned}&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\&\quad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\&=\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+\Big(\int_\Omega(\rho_0+\sigma^\delta(t))|\partial_t\vec{u}^d(t)|^2d\vec{x}\Big)\Big|_{t=0}\\&\quad+\int_0^t\int_\Omega(2\partial_t f^\delta(s)+2g\vec{u}^\delta(s)\cdot\nabla\sigma^\delta(s)\vec{e}_2-\partial_t\sigma^\delta(s)\partial_t\vec{u}^d(s))\cdot\partial_t\vec{u}^d(s)ds.\end{aligned}\tag{5.23}$$

We continue using (5.11), (5.12) and (5.13) to estimate each term of the r.h.s of (5.23). This yields

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\qquad-2\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(t,x_1,1)|^2+\xi_-|\partial_t u_1^d(t,x_1,-1)|^2)dx_1 \qquad (5.24)\\
&\leqslant\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+C_{16}\delta^3F_M^3(t).
\end{aligned}$$

Due to (5.16), we further get that

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\qquad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\leqslant\Lambda^2\int_\Omega\rho_0|\vec{u}^d(t)|^2d\vec{x}+\Lambda\mu\int_\Omega|\nabla\vec{u}^d(t)|^2d\vec{x}\\
&\qquad-\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1+C_{16}\delta^3F_M^3(t)\\
&\leqslant\Lambda^2\int_\Omega(\rho_0+\sigma^\delta(t))|\vec{u}^d(t)|^2d\vec{x}+\Lambda\mu\int_\Omega|\nabla\vec{u}^d(t)|^2d\vec{x}\\
&\qquad-\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1+C_{17}\delta^3F_M^3(t). \qquad (5.25)
\end{aligned}$$

On the other hand, we have

$$\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}=2\int_\Omega(\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\partial_t\vec{u}^d(t)d\vec{x}+\int_\Omega\partial_t\sigma^\delta(t)|\vec{u}^d(t)|^2d\vec{x}.$$

Let us recall ϖ_0 from (2.19) and $\nu_0=\frac{3+\varpi_0}{2+\varpi_0}\in(1,\frac{3}{2})$. We fix two positive constants $m_{1,2}$ such that

$$m_1=\nu_0+\sqrt{\nu_0^2-1} \qquad (5.26)$$

and that

$$m_2=\mu(m_1^2-m_1+1)-\mu_c(\Xi)(m_1^2+1)+\sqrt{(\mu(m_1^2-m_1+1)-\mu_c(\Xi)(m_1^2+1))^2-\mu^2m_1^2}. \qquad (5.27)$$

With $m_1>0$ from (5.26), we use Young's inequality to observe

$$\begin{aligned}
&2\int_\Omega(\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\partial_t\vec{u}^d(t)d\vec{x}\\
&\leqslant\frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

That will imply

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}&\leqslant\frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\quad+\Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C_{18}\delta^3F_M^3(t). \qquad (5.28)
\end{aligned}$$

With $m_2 > 0$ defined as in (5.27), we obtain from (5.25) and (5.28) that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant \Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + \Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d\|^2_{L^2(\Omega)}\\
&\quad + \frac{2}{\Lambda m_1}\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2 + \xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\quad - \frac{2\mu}{\Lambda m_1}\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds + C_{19}\delta^3F_M^3(t).
\end{aligned}$$

Together with (5.17), we deduce

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant \Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + \Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\quad - \frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds + C_{19}\delta^3F_M^3(t).
\end{aligned}\tag{5.29}$$

We use Young's inequality to get that

$$\begin{aligned}
\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)} &= 2\Big(\frac{\mu}{m_1}+m_2\Big)\int_0^t\int_\Omega\nabla\vec{u}^d(s)\cdot\nabla\partial_t\vec{u}^d(s)d\vec{x}ds\\
&\leqslant \frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\quad + \frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds.
\end{aligned}\tag{5.30}$$

Combining (5.29) and (5.30) gives us

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant \Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\quad + \frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds + C_{19}\delta^3F_M^3(t).
\end{aligned}\tag{5.31}$$

It follows from (5.26) and (5.27) that

$$\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))} = \Lambda\Big(m_1+\frac{1}{m_1}\Big)m_2 = 2\nu_0\Lambda m_2.$$

Therefore, (5.31) becomes

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant 2\nu_0\Lambda\Big(\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\Big) + C_{19}\delta^3F_M^3(t).
\end{aligned}\tag{5.32}$$

Recalling that $\vec{u}^d(0) = \vec{0}$, thus, applying Gronwall's inequality to (5.32), one obtains

$$\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)} + m_2\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds \leqslant C_{19}\delta^3e^{2\nu_0\Lambda t}\int_0^t e^{-2\nu_0\Lambda s}F_M^3(s)ds.\tag{5.33}$$

Since $F_M^3(t) \leqslant M^2 \max_{j_m \leqslant j \leqslant M} |\mathsf{c}_j|^2 F_M(3t)$, we then have from (5.33) that

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{20}\delta^3 e^{2\nu_0\Lambda t} \sum_{j=j_m}^{M} \int_0^t |\mathsf{c}_j| e^{(3\lambda_j - 2\nu_0\Lambda)s} ds. \tag{5.34}$$

Because of (2.21), we have $\lambda_j > \frac{2}{3}\nu_0\Lambda$ for $j_m \leqslant j \leqslant N$ and $\lambda_j < \frac{2}{3}\nu_0\Lambda$ for $j \geqslant N+1$. It yields that for $j_m \leqslant j \leqslant N$,

$$\int_0^t e^{(3\lambda_j - 2\nu_0\Lambda)s} ds = \frac{1}{3\lambda_j - 2\nu_0\Lambda}(e^{(3\lambda_j - 2\nu_0\Lambda)t} - 1) \leqslant \frac{1}{3\lambda_j - 2\nu_0\Lambda} e^{(3\lambda_j - 2\nu_0\Lambda)t} \tag{5.35}$$

and that for $j \geqslant N+1$,

$$\int_0^t e^{(3\lambda_j - 2\nu_0\Lambda)s} ds = \frac{1}{3\lambda_j - 2\nu_0\Lambda}(e^{(3\lambda_j - 2\nu_0\Lambda)t} - 1) \leqslant \frac{1}{2\nu_0\Lambda - 3\lambda_j}. \tag{5.36}$$

In view of (5.35) and (5.36), we obtain from (5.34) that if $M \leqslant N$,

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{20}\delta^3 \Big(\sum_{j=j_m}^{M} \frac{|\mathsf{c}_j|}{3\lambda_j - 2\nu_0\Lambda} e^{3\lambda_j t} \Big)$$

and if $M \geqslant N+1$,

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{20}\delta^3 \Big(\sum_{j=j_m}^{M} \frac{|\mathsf{c}_j|}{3\lambda_j - 2\nu_0\Lambda} e^{3\lambda_j t} + \sum_{j=N+1}^{M} \frac{|\mathsf{c}_j|}{2\nu_0\Lambda - 3\lambda_j} e^{2\nu_0\Lambda t} \Big).$$

That means

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_{21}\delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \Big(\max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| \Big) e^{\frac{2}{3}\nu_0\Lambda t} \Big)^3. \tag{5.37}$$

To show the bound of $\|\sigma^d(t)\|_{L^2(\Omega)}$, we use Cauchy-Schwarz's inequality to deduce from $(5.6)_1$ that

$$\frac{d}{dt}\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant \|\sigma^d(t)\|_{L^2(\Omega)} \leqslant (\max \rho_0')\|u_2^d(t)\|_{L^2(\Omega)} + \|\vec{u}^\delta(t)\|_{L^2(\Omega)}\|\sigma^\delta(t)\|_{H^1(\Omega)}.$$

Using (5.5), we obtain

$$\frac{d}{dt}\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_{22}(\|u_2^d(t)\|_{L^2(\Omega)} + \delta^2 F_M^2(t)).$$

Note that $\sigma^d(0) = 0$, integrating the resulting inequality in time, we have

$$\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_{22} \int_0^t (\|u_2^d(s)\|_{L^2(\Omega)} + \delta^2 F_M^2(s)) ds.$$

Together with (5.37), we have

$$\|\sigma^d(t)\|^2_{L^2(\Omega)} \leqslant C_{23}\delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \Big(\max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| \Big) e^{\frac{2}{3}\nu_0\Lambda t} \Big)^3. \tag{5.38}$$

The inequality (5.10) follows from (5.37) and (5.38). Proof of Proposition 5.2 is complete. □

5.3. **Proof of Theorem 2.2.** Note that

$$\|\vec{u}^M(t)\|^2_{L^2(\Omega)} = \sum_{i=j_m}^{M} \mathsf{c}_i^2 e^{2\lambda_i t}\|\vec{u}_i\|^2_{L^2(\Omega)} + 2\sum_{j_m\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega \vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}. \tag{5.39}$$

It can be seen that

$$\begin{aligned}\|\vec{u}^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|^2_{L^2(\Omega)} + 2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega \vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}\\ - |\mathsf{c}_{j_m}|\|\vec{u}_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}.\end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega \vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}\\ \geqslant -\sum_{j_m+1\leqslant i<j\leqslant M}|\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_i+\lambda_j)t}\|\vec{u}_i\|_{L^2(\Omega)}\|\vec{u}_j\|_{L^2(\Omega)}\\ \geqslant -e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)^2.\end{aligned}$$

This yields

$$\begin{aligned}\|\vec{u}^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|^2_{L^2(\Omega)} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)^2\\ - |\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|\vec{u}_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big).\end{aligned}$$

Due to the assumption (2.17), we deduce that

$$\begin{aligned}\|\vec{u}^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|^2_{L^2(\Omega)} - \frac{1}{2}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|\vec{u}_{j_m}\|^2_{L^2(\Omega)}\\ - \frac{1}{4}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|\vec{u}_{j_m}\|^2_{L^2(\Omega)}.\end{aligned}$$

This yields

$$\begin{aligned}\|\vec{u}^M(t)\|^2_{L^2(\Omega)} \geqslant \mathsf{c}_{j_m}^2\Big(e^{2\lambda_{j_m}t} - \frac{1}{2}e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - \frac{1}{4}e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big)\|\vec{u}_{j_m}\|^2_{L^2(\Omega)}\\ + \sum_{j=j_m+1}^{M}\mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|^2_{L^2(\Omega)}.\end{aligned}$$

Notice that for all $t\geqslant 0$,

$$e^{2\lambda_{j_m}t} - \frac{1}{2}e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - \frac{1}{4}e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} \geqslant \frac{1}{4}e^{2\lambda_{j_m}t}.$$

Hence, we have

$$\|\vec{u}^M(t)\|_{L^2(\Omega)} \geqslant C_{24}F_M(t), \tag{5.40}$$

for all $t\leqslant \min(T^\delta, T^\star, T^{\star\star})$.

Let

$$\tilde{\mathsf{c}}(M) = \max_{N+1\leqslant j\leqslant M}\frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\geqslant 0.$$

We recall the definition of $T^\star$ and $T^{\star\star}$ from (5.4) and the fact that T^δ satisfies uniquely $\delta F_M(T^\delta)=\epsilon_0$, provided that ϵ_0 is taken to be

$$\epsilon_0<\min\Big(\frac{C_2\delta_0}{C_3},\frac{C_2^2}{2C_4(1+M\tilde{\mathsf{c}}(M))^3},\frac{C_{24}^2}{4C_4(1+M\tilde{\mathsf{c}}(M))^3}\Big). \tag{5.41}$$

We then prove that

$$T^\delta\leqslant\min\{T^\star,T^{\star\star}\}. \tag{5.42}$$

In fact, if $T^\star<T^\delta$, we have from (5.5) that

$$\mathcal{E}((\sigma^\delta,\vec{u}^\delta)(T^\star))\leqslant C_3\delta F_M(T^\star)\leqslant C_3\delta F_M(T^\delta)=C_3\epsilon_0<C_2\delta_0.$$

And if $T^{\star\star}<T^\delta$, we have by (5.10) and the definition of T^δ that

$$\begin{aligned}&\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)}\\&\leqslant\delta\|(\sigma^M,\vec{u}^M)(T^\delta)\|_{L^2(\Omega)}+\|(\sigma^d,\vec{u}^d)(T^\delta)\|_{L^2(\Omega)}\\&\leqslant C_2\delta F_M(T^\delta)+\sqrt{C_4}\delta^{\frac32}\Big(\sum_{j=1}^N|\mathsf{c}_j|e^{\lambda_jT^\delta}+\max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac23\nu_0\Lambda T^\delta}\Big)^{\frac32}.\end{aligned} \tag{5.43}$$

Notice from (2.21) that for $N+1\leqslant j\leqslant M$,

$$|\mathsf{c}_j|\delta e^{\frac23\nu_0\Lambda T^\delta}<\frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}(\delta|\mathsf{c}_{j_m}|e^{\lambda_1T^\delta})<\frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\delta F_M(T^\delta)=\frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\epsilon_0.$$

Then, it follows from (5.43) that

$$\begin{aligned}\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)}&\leqslant C_2\delta F_M(T^\delta)+\sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac32}\delta^{\frac32}F_M^{\frac32}(T^\delta)\\&\leqslant C_2\epsilon_0+\sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac32}\epsilon_0^{\frac32}.\end{aligned}$$

Using (5.41) again, we deduce

$$\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)}<2C_2\epsilon_0=2C_2\delta F_M(T^\delta).$$

which also contradicts the definition of $T^{\star\star}$.

Once we have (5.42), we then get from (5.10) and (5.40) that

$$\begin{aligned}&\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)}\\&\geqslant\delta\|\vec{u}^M(T^\delta)\|_{L^2(\Omega)}-\|\vec{u}^d(T^\delta)\|_{L^2(\Omega)}\\&\geqslant C_{24}\delta F_M(T^\delta)-\sqrt{C_4}\delta^{\frac32}\Big(\sum_{j=1}^N|\mathsf{c}_j|e^{\lambda_jT^\delta}+\max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac23\nu_0\Lambda T^\delta}\Big)^{\frac32}.\end{aligned}$$

Therefore,

$$\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)}\geqslant C_{24}\epsilon_0-\sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac32}\epsilon_0^{\frac32}\geqslant\frac{C_{24}\epsilon_0}{2}>0. \tag{5.44}$$

The inequality (2.22) is proven by taking δ_0 satisfying (5.3), ϵ_0 satisfying (5.41) and $m_0=\frac12C_{24}$. This ends the proof of Theorem 2.2.

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Appendix A. The precise value of $\mu_c(k,\Xi)$

In this appendix, we prove Proposition 3.1(1). The equality (3.5) can be seen immediately from the definition of $\mathcal{B}_{k,0,\mu}$.

Note that the quotient

$$\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^1((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_2}$$

is bounded because of the embedding $H^2((-1,1))\hookrightarrow C^1((-1,1))$. To prove (3.6), let us consider the Lagrangian functional

$$\mathcal{L}_k(\phi,\beta)=\beta\Big(\int_{-1}^1((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_2-1\Big)-(\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2) \tag{A.1}$$

for any $\phi\in\tilde{H}^2((-1,1))$ and $\beta\neq 0$. Using Lagrange multiplier theorem, the extrema of the quotient

$$\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^1((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_2}$$

are necessarily the stationary points of (ϕ_k,β_k) of $\mathcal{L}_k$, which satisfy

$$\int_{-1}^1((\phi_k'')^2+2k^2(\phi_k')^2+k^4\phi_k^2)dx_2=1, \tag{A.2}$$

and

$$\beta_k\int_{-1}^1(\phi_k''\omega''+2k^2\phi_k'\omega'+k^4\phi_k\omega)dx_2=\xi_-\phi_k'(-1)\omega'(-1)+\xi_+\phi_k'(1)\omega'(1) \tag{A.3}$$

for all $\omega\in\tilde{H}^2((-1,1))$. We obtain from (A.3) after taking integration by parts that

$$\phi_k^{(4)}-2k^2\phi_k''+k^4\phi_k=0,$$

and

$$\begin{cases}\beta_k(\phi_k'''(1)+2k^2\phi_k'(1))\omega(1)=0,\\ (\beta_k\phi_k''(1)-\xi_+\phi_k'(1))\omega'(1)=0,\\ \beta_k(\phi_k'''(-1)+2k^2\phi_k'(-1))\omega(-1)=0,\\ (\beta_k\phi_k''(-1)+\xi_-\phi_k'(-1))\omega'(-1)=0,\end{cases}$$

for all $\omega\in\tilde{H}^2((-1,1))$. This yields

$$\begin{cases}\beta_k\phi_k''(1)-\xi_+\phi_k'(1)=0,\\ \beta_k\phi_k''(-1)+\xi_-\phi_k'(-1)=0.\end{cases} \tag{A.4}$$

Hence, ϕ_k is of the form

$$\phi_k(x_2)=(Ax_2+B)\sinh(kx_2)+(Cx_2+D)\cosh(kx_2),$$

with A,B,C,D are four constants such that $A^2+B^2+C^2+D^2>0$. Since $\phi_k\in\tilde{H}^2((-1,1))$, we get

$$\begin{cases}(A+B)\sinh k+(C+D)\cosh k=0,\\ (-A+B)\sinh(-k)+(-C+D)\cosh(-k)=0.\end{cases}$$

It yields

$$C=-B\tanh k\quad\text{and}\quad D=-A\tanh k. \tag{A.5}$$

We then compute

$$\phi_k'(x_2) = (A + kD + kCx_2)\sinh(kx_2) + (C + kB + kAx_2)\cosh(kx_2)$$

and

$$\phi_k''(x_2) = (2kC + k^2B + k^2Ax_2)\sinh(kx_2) + (2kA + k^2D + k^2Cx_2)\cosh(kx_2).$$

Substituting these formulas into (A.4), we have

$$\begin{cases} \beta_k\Big((2kC + k^2(B + A))\sinh k + (2kA + k^2(D + C))\cosh k\Big) \\ \qquad = \xi_+\Big((A + k(D + C))\sinh k + (C + k(B + A))\cosh k\Big), \\ \beta_k\Big((2kC + k^2(B - A))\sinh(-k) + (2kA + k^2(D - C))\cosh(-k)\Big) \\ \qquad = -\xi_-\Big((A + k(D - C))\sinh(-k) + (C + k(B - A))\cosh(-k)\Big). \end{cases}$$

Thanks to (A.5), that reduces to

$$\begin{cases} 2k\beta_k(C\sinh k + A\cosh k) \\ \qquad = \xi_+\Big(A + k(D + C)\sinh k + (C + k(B + A))\cosh k\Big), \\ 2k\beta_k(-C\sinh k + A\cosh k) \\ \qquad = \xi_-\Big((A + k(D - C))\sinh(k) - (C + k(B - A))\cosh(k)\Big). \end{cases}$$

Equivalently,

$$\begin{cases} 2k\beta_k\Big(A + B + (A - B)\cosh(2k)\Big) = \xi_+\Big((A - B)\sinh(2k) + 2k(A + B)\Big), \\ 2k\beta_k\Big(A - B + (A + B)\cosh(2k)\Big) = \xi_-\Big((A + B)\sinh(2k) + 2k(A - B)\Big). \end{cases} \tag{A.6}$$

Then, (A, B) is a solution of the following system

$$\begin{cases} A\Big(2k(1 + \cosh(2k))\beta_k - \xi_+(2k + \sinh(2k))\Big) \\ \qquad = B\Big(2k(\cosh(2k) - 1)\beta_k - \xi_+(\sinh(2k) - 2k)\Big), \\ A\Big(2k(1 + \cosh(2k))\beta_k - \xi_-(2k + \sinh(2k))\Big) \\ \qquad = -B\Big(2k(\cosh(2k) - 1)\beta_k - \xi_-(\sinh(2k) - 2k)\Big). \end{cases} \tag{A.7}$$

System (A.7) admits a nontrivial solution if and only

$$\begin{aligned} &\Big(2k(1 + \cosh(2k))\beta_k - \xi_+(2k + \sinh(2k))\Big) \\ &\qquad \times \Big(2k(\cosh(2k) - 1)\beta_k - \xi_-(\sinh(2k) - 2k)\Big) \\ &= -\Big(2k(1 + \cosh(2k))\beta_k - \xi_-(2k + \sinh(2k))\Big) \\ &\qquad \times \Big(2k(\cosh(2k) - 1)\beta_k - \xi_+(\sinh(2k) - 2k)\Big). \end{aligned} \tag{A.8}$$

We rewrite Eq. (A.8) as a quadratic equation of β_k, that is

$$\begin{aligned} 4k^2(\cosh^2(2k) - 1)\beta_k^2 - 2k(\sinh(2k)\cosh(2k) - 2k)(\xi_+ + \xi_-)\beta_k \\ + (\sinh^2(2k) - 4k^2)\xi_+\xi_- = 0. \end{aligned} \tag{A.9}$$

The discriminant is

$$\begin{aligned} \Delta_{k,\Xi} &= k^2(\sinh(2k)\cosh(2k) - 2k)^2(\xi_+ + \xi_-)^2 \\ &\quad - 4k^2(\cosh^2(2k) - 1)(\sinh^2(2k) - 4k^2)\xi_+\xi_- \\ &= k^2(\sinh(2k) - 2k\cosh(2k))^2(\xi_+ + \xi_-)^2 \\ &\quad + k^2\sinh^2(2k)(\sinh^2(2k) - 4k^2)(\xi_+ - \xi_-)^2. \end{aligned}$$

Because $\tanh(2k) < 2k$ for all $k > 0$ and $\xi_+^2 + \xi_-^2 > 0$, we have $\Delta_{k,\xi}$ is always positive. Hence, we have that (A.9) has two roots

$$\beta_{k,\pm} = \frac{k(\sinh(2k)\cosh(2k) - 2k)(\xi_+ + \xi_-) \pm \sqrt{\Delta_{k,\Xi}}}{4k^2\sinh^2(2k)}.$$

We take the higher value $\beta_{k,+} > 0$ and then solve the system (A.7) as $\beta_k = \beta_{k,+}$.

If $\xi_- \geqslant \xi_+$, we have

$$\begin{aligned}
&4\sinh^2 k\Big(2k(\cosh(2k)-1)\beta_{k,+} - \xi_+(\sinh(2k)-2k)\Big)\\
&= (\sinh(2k)\cosh(2k)-2k)(\xi_- - \xi_+) - 2(\sinh(2k) - 2k\cosh(2k))\xi_+ + \frac{1}{k}\sqrt{\Delta_{k,\Xi}}\\
&> 0.
\end{aligned}$$

Then, we obtain from $(\text{A.7})_1$ that

$$B = A\frac{2k(1+\cosh(2k))\beta_{k,+} - \xi_+(2k+\sinh(2k))}{2k(\cosh(2k)-1)\beta_{k,+} - \xi_+(\sinh(2k)-2k)} =: Aa_{k,\Xi}. \tag{A.10}$$

So that

$$(A,B,C,D) = A(1, a_{k,\Xi}, -a_{k,\Xi}\tanh k, -\tanh k)$$

with $A \neq 0$ and $\phi_k(x_2) = Az_k(x_2)$, with

$$z_k(x_2) = (x_2 + a_{k,\Xi})\sinh(kx_2) - \tanh k(a_{k,\Xi}x_2 + 1)\cosh(kx_2).$$

We find A from (A.2), such that

$$A^2\int_{-1}^{1}((z_k'')^2 + 2k^2(z_k')^2 + k^4z_k^2)dx_2 = 1. \tag{A.11}$$

If $0 < \xi_- < \xi_+$, that will imply

$$2k(\cosh(2k)-1)\beta_{k,+} - \xi_-(\sinh(2k)-2k) > 0.$$

We further get from $(\text{A.7})_2$ that

$$B = -A\frac{2k(1+\cosh(2k))\beta_{k,+} - \xi_-(2k+\sinh(2k))}{2k(\cosh(2k)-1)\beta_{k,+} - \xi_-(\sinh(2k)-2k)} =: -Ab_{k,\Xi}.$$

So that, we have

$$(A,B,C,D) = A(1, -b_{k,\Xi}, b_{k,\Xi}\tanh k, -\tanh k)$$

with $A \neq 0$ and $\phi_k(x_2) = Aw_k(x_2)$, with

$$w_k(x_2) = (x_2 - b_{k,\Xi})\sinh(kx_2) + (b_{k,\Xi}x_2 - 1)\tanh k\cosh(kx_2).$$

We still find A from (A.2),

$$A^2\int_{-1}^{1}((w_k'')^2 + 2k^2(w_k')^2 + k^4w_k^2)dx_2 = 1. \tag{A.12}$$

We have just shown that

$$\begin{aligned}
\mu_c(k,\Xi) &= \max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^{1}((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_2}\\
&= \frac{1}{4k\sinh^2(2k)}\left(\begin{array}{l}(\sinh(2k)\cosh(2k)-2k)(\xi_+ + \xi_-)\\ + \left(\begin{array}{l}(\sinh(2k)-2k\cosh(2k))^2(\xi_+ + \xi_-)^2\\ + \sinh^2(2k)(\sinh^2(2k) - 4k^2)(\xi_+ - \xi_-)^2\end{array}\right)^{\frac{1}{2}}\end{array}\right).
\end{aligned}$$

That variational problem is attained by the function $\phi_k(x_2) = Az_k(x_2)$, where A satisfies (A.11) or $\phi_k(x_2) = Aw_k(x_2)$, where A satisfies (A.12). The equality (3.6) is shown and the first part of Proposition 3.1 then follows.

Appendix B. Asymptotic behavior of $\mu_c(k,\Xi)$ in low/high regime of wave number

Let us prove Proposition 3.1(2). Clearly, we have that $\mu_c(k,\Xi)$ is a decreasing function in $k>0$. It yields (3.7).

We first consider $k \to 0$. Let us recall the Taylor's expansion of $\sinh(2k)$ and $\cosh(2k)$. We have

$$\sinh(2k) = 2k + \frac{4}{3}k^3 + \frac{4}{15}k^5 + O(k^6), \quad \text{and} \quad \cosh(2k) = 1 + 2k^2 + \frac{2}{3}k^4 + O(k^5).$$

We deduce that

$$\frac{\sinh(2k)\cosh(2k) - 2k}{4k\sinh^2(2k)} = \frac{\frac{1}{6} + \frac{2}{15}k^2 + O(k^3)}{\frac{1}{2} + \frac{2}{3}k^2 + O(k^3)} = \frac{1}{3} - \frac{8}{15}k^2 + O(k^3),$$

that

$$\frac{\sinh(2k) - 2k\cosh(2k)}{4k\sinh^2(2k)} = \frac{-\frac{8}{3} - \frac{16}{15}k^2 + O(k^3)}{16 + \frac{64}{3}k^2 + O(k^3)} = -\frac{1}{6} + \frac{7}{45}k^2 + O(k^3)$$

and that

$$\frac{\sinh^2(2k)(\sinh^2(2k) - 4k^2)}{16k^2\sinh^4(2k)} = \frac{\frac{16}{3} + \frac{128}{45}k^2 + O(k^2)}{64 + \frac{256}{3}k^2 + O(k^2)} = \frac{1}{12} - \frac{1}{15}k^2 + O(k^3).$$

We deduce that

$$\lim_{k\to 0}\mu_c(k,\Xi) = \frac{1}{3}(\xi_+ + \xi_-) + \sqrt{\frac{1}{36}(\xi_+ + \xi_-)^2 + \frac{1}{12}(\xi_+ - \xi_-)^2}. \tag{B.1}$$

That will imply (3.9), i.e.

$$\mu_c^s(\Xi) = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

Furthermore, we have that

$$\lim_{k\to 0}\frac{\mu_c(k,\Xi) - \mu_c^s(\Xi)}{k^2} = -\frac{2}{15}\Big(4(\xi_+ + \xi_-) + \frac{4\xi_+^2 - \xi_+\xi_- + 4\xi_-^2}{\sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-}}\Big). \tag{B.2}$$

Two limits (B.1) and (B.2) help us to get (3.8).

For high wave number, i.e. $k \to +\infty$, we can see that

$$\frac{\sinh(2k)\cosh(2k) - 2k}{\sinh^2(2k)} = \frac{1 - e^{-8k} - 8ke^{-4k}}{1 + e^{-8k} - 2e^{-4k}} \leqslant 2,$$

that

$$\frac{\sinh(2k) - 2k\cosh(2k)}{\sinh^2(2k)} = \frac{1}{2}\frac{1 - 2k - (1+2k)e^{-4k}}{e^{2k} + e^{-6k} - 2e^{-2k}} \leqslant 1.$$

Hence,

$$\mu_c(k,\Xi) = \frac{1}{4k}\left(\begin{array}{l}\dfrac{\sinh(2k)\cosh(2k)-2k}{\sinh^2(2k)}(\xi_+ + \xi_-) \\ + \left(\begin{array}{l}\Big(\dfrac{\sinh(2k)-2k\cosh(2k)}{\sinh^2(2k)}\Big)^2(\xi_+ + \xi_-)^2 \\ + \Big(1 - \dfrac{4k^2}{\sinh^2(2k)}\Big)(\xi_+ - \xi_-)^2\end{array}\right)^{\frac{1}{2}}\end{array}\right)$$
$$\leqslant \frac{1}{4k}\Big(2(\xi_+ + \xi_-) + \sqrt{2(\xi_+^2 + \xi_-^2)}\Big).$$

That implies (3.10). The proof of the second assertion of Proposition 3.1 is complete.

Appendix C. Proof of Proposition 3.1(3)

In this appendix, we prove Proposition 3.1(3). We first show that

$$\mu_c^s(\Xi) = \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2} \tag{C.1}$$

Indeed, let

$$\tilde{\mu}_c(\Xi) := \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2}$$

and then prove that $\mu_c^s(\Xi) = \tilde{\mu}_c(\Xi)$. Clearly, we have $\mu_c(k,\Xi) \leqslant \tilde{\mu}_c(\Xi)$ for all $k \in \mathbf{R}\backslash\{0\}$. It yields $\mu_c^s(\Xi) \leqslant \tilde{\mu}_c(\Xi)$. It suffices to show that $\tilde{\mu}_c(\Xi) \geqslant \mu_c^s(\Xi)$. For any $\varepsilon > 0$, from the definition of $\mu_c(\Xi)$, we fix a function $\phi_\varepsilon \in \tilde{H}^2((-1,1))$ such that

$$\frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1(\phi_\varepsilon'')^2 dx_2} \geqslant \tilde{\mu}_c(\Xi) - \varepsilon.$$

Let $k \neq 0$ be small enough, we then obtain

$$\frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1((\phi_\varepsilon'')^2 + 2k^2(\phi_\varepsilon')^2 + k^4\phi_\varepsilon^2)dx_2} > \frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1(\phi_\varepsilon'')^2 dx_2} - \varepsilon.$$

That implies

$$\mu_c(k,\Xi) > \tilde{\mu}_c(\Xi) - 2\varepsilon.$$

We deduce that $\tilde{\mu}_c(\Xi) = \sup_{k\in\mathbf{R}\backslash\{0\}} \mu_c(k,\Xi)$, i.e. (C.1).

Then, we show that

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2} = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \tag{C.2}$$

Let us consider the Lagrangian functional

$$\mathcal{L}_0(\phi,\beta) = \beta\Big(\int_{-1}^1 (\phi'')^2 dx_2 - 1\Big) - \xi_-(\phi'(-1))^2 - \xi_+(\phi'(1))^2. \tag{C.3}$$

for any $\phi \in \tilde{H}^2((-1,1))$ and $\beta \neq 0$. Owing to Lagrange multiplier theorem, the extrema of the quotient

$$\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2}$$

are necessarily the stationary points (ϕ_0, β_0) of $\mathcal{L}_0$, which satisfy

$$\int_{-1}^{1} (\phi_0'')^2 dx_2 = 1, \tag{C.4}$$

and

$$\beta_0 \int_{-1}^{1} \phi_0'' \omega'' dx_2 - (\xi_- \phi_0'(-1)\omega'(-1) + \xi_+ \phi_0'(1)\omega'(1)) = 0 \tag{C.5}$$

for all $\omega \in \tilde{H}^2((-1,1))$. We obtain from (C.5) after taking integration by parts that

$$\phi_0^{(4)} = 0 \quad \text{on } (-1,1).$$

and

$$\begin{cases} \phi_0''(1) = \xi_+ \phi_0'(1), \\ \phi_0''(-1) = -\xi_- \phi_0'(-1). \end{cases} \tag{C.6}$$

Hence, ϕ_0 is of the form

$$\phi_0(x_2) = (x_2^2 - 1)(Ax_2 + B).$$

Substituting this form of ϕ_0 into (C.6), we have that

$$\begin{cases} \beta_0(3A + B) = \xi_+(A + B), \\ \beta_0(3A - B) = \xi_-(A - B). \end{cases}$$

Hence,

$$\begin{cases} A(3\beta_0 - \xi_+) + B(\beta_0 - \xi_+) = 0, \\ A(3\beta_0 - \xi_-) - B(\beta_0 - \xi_-) = 0. \end{cases} \tag{C.7}$$

System (C.7) admits a nontrivial solution (A, B) if and only if

$$(3\beta_0 - \xi_+)(\beta_0 - \xi_-) + (3\beta_0 - \xi_-)(\beta_0 - \xi_+) = 0.$$

It yields

$$3\beta_0^2 - 2(\xi_+ + \xi_-)\beta_0 + \xi_-\xi_+ = 0. \tag{C.8}$$

The discriminant of (C.8) is

$$\Delta_{0,\xi} = (\xi_+ + \xi_-)^2 - 3\xi_-\xi_+ = \xi_+^2 - \xi_+\xi_- + \xi_-^2 > 0.$$

Then, Eq. (C.8) has two roots

$$\beta_{0,\pm} = \frac{1}{3}\Big(\xi_+ + \xi_- \pm \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

We take the higher value $\beta_{0,+}$. As $\beta_0 = \beta_{0,+}$, we have from $(\text{C.7})_2$ that

$$A(3\beta_{0,+} - \xi_-) = B(\beta_{0,+} - \xi_-).$$

It is obvious that

$$3\beta_{0,+} - \xi_- = \xi_+ + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2} > 0.$$

Then we have

$$A = B\frac{\beta_{0,+} - \xi_-}{3\beta_{0,+} - \xi_-}$$

and

$$\phi_0(x_2) = Bz_0(x_2), \quad \text{with } z_0(x_2) = \Big(\frac{\beta_{0,+} - \xi_-}{3\beta_{0,+} - \xi_-}x_2 + 1\Big)(x_2^2 - 1).$$

We continue using (C.4) to find a non-zero B. This yields

$$B^2 \int_{-1}^{1} (z_2''(x_2))^2 dx_2 = 1.$$

That is equivalent to

$$8B^2\Big(3\frac{(\beta_{0,+}-\xi_-)^2}{(3\beta_{0,+}-\xi_-)^2}+4\Big)=1,$$

this yields

$$B=\pm\frac{1}{2\sqrt{2}}\frac{3\beta_{0,+}-\xi_-}{\sqrt{39\beta_{0,+}^2-30\beta_{0,+}\xi_-+7\xi_-^2}}.$$

That means, we observe

$$\max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2dx_2}=\frac{1}{3}\Big(\xi_++\xi_-+\sqrt{\xi_+^2-\xi_+\xi_-+\xi_-^2}\Big).$$

That variational problem is attained by the function

$$\phi_0(x_2)=\pm\frac{1}{2\sqrt{2}}\frac{3\beta_{0,+}-\xi_-}{\sqrt{39\beta_{0,+}^2-30\beta_{0,+}\xi_-+7\xi_-^2}}\Big(\frac{\beta_{0,+}-\xi_-}{3\beta_{0,+}-\xi_-}x_2+1\Big)(x_2^2-1).$$

Combining (C.1) and (C.2), we obtain Proposition 3.1(3).

Appendix D. Comments on the paper of Ding, Zi and Li

In [2], the authors Ding, Zi and Li construct an approximate solution generated by the maximal normal mode, $(\sigma^a,\vec{u}^a,q^a)(t,x)=\delta e^{\lambda_1(k)t}\vec{U}_1(\vec{x})$ with k being fixed such that $\frac{2\Lambda}{3}<\lambda_1(k)<\Lambda$. Applying Proposition 5.1, the nonlinear equations (2.1)-(2.2) with the initial data

$$(\sigma^\delta,\vec{u}^\delta,q^\delta)(0)=(\sigma^a,\vec{u}^a,q^a)(0).$$

admits a strong solution $(\sigma^\delta,\vec{u}^\delta)\in C^0([0,T^{\max}),H^1\times H^2)$ with an associated pressure $q^\delta\in C^0([0,T^{\max}),L^2)$. Let T^δ such that $\delta e^{\lambda_1 T^\delta}=\epsilon_0\ll 1$. We define

$$T^\star:=\sup\Big\{t\in(0,T^{\max})|\mathcal{E}(\sigma^\delta(t),\vec{u}^\delta(t))\leqslant C\delta_0\}>0,$$
$$T^{\star\star}:=\sup\{t\in(0,T^{\max})|\|(\sigma^\delta,\vec{u}^\delta)(t)\|_{L^2(\Omega)}\leqslant C\delta e^{\lambda_1 t}\Big\}>0.$$

Then for all $t\leqslant\min\{T^\delta,T^\star,T^{\star\star}\}$, we have

$$\mathcal{E}^2(\sigma^\delta(t),\vec{u}^\delta(t))+\|\partial_t\vec{u}^\delta(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla\partial_t\vec{u}^\delta(\tau)\|_{L^2(\Omega)}^2d\tau\leqslant C\delta^2e^{2\lambda_1 t}.$$

In [2, Proposition 5.2], they claim that the difference functions

$$(\sigma^d,\vec{u}^d,q^d)=(\sigma^\delta,\vec{u}^\delta,q^\delta)-(\sigma^a,\vec{u}^a,q^a)$$

enjoy

$$\|(\sigma^d,\vec{u}^d)\|_{L^2(\Omega)}^2\leqslant C\delta^3e^{3\lambda_1 t}\tag{D.1}$$

for all $\mu>0$. We believe that (D.1) needs to be corrected, not for all $\mu>0$. Precisely, we are in doubt about inequality **(137)** in that paper, that is for all $t\leqslant\min\{T^\delta,T^\star,T^{\star\star}\}$,

$$\begin{aligned}&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|_{L^2(\Omega)}^2+\Lambda\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2+\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2ds\\&\leqslant\Lambda\Big(\int_0^t\|\sqrt{\rho_0+\sigma^\delta(s)}\vec{u}^d(s)\|_{L^2(\Omega)}^2+\Lambda\mu\|\nabla\vec{u}^d(s)\|_{L^2(\Omega)}^2\Big)\\&\qquad+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2(\Omega)}^2+C\delta^3e^{3\lambda_1 t}.\end{aligned}\tag{D.2}$$

Due to (D.2) and the following inequality

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&=2\int_\Omega(\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\partial_t\vec{u}^d(t)d\vec{x}+\int_\Omega\partial_t\sigma^\delta(t)|\vec{u}^d(t)|^2d\vec{x}\\
&\leqslant\frac{1}{\Lambda}\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C\delta^3e^{3\lambda_1 t},
\end{aligned}\tag{D.3}$$

it is claimed in [2, **(138)**] that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Big(\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\Big)\\
&\leqslant\Lambda\int_0^t\Big(\|\sqrt{\rho_0+\sigma^\delta(s)}\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}\Big)ds\\
&\qquad+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.4}$$

The inequality (D.1) is followed by applying Gronwall's inequality to (D.4).

We shall explain the arguments of (D.2) in [2]. First, we still have

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\qquad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&=\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+\Big(\int_\Omega(\rho_0+\sigma^\delta(t))|\partial_t\vec{u}^d(t)|^2d\vec{x}\Big)\Big|_{t=0}\\
&\qquad+\int_0^t\int_\Omega(2\partial_t f^\delta(s)+2g\vec{u}^\delta(s)\cdot\nabla\sigma^\delta(s)e_2-\partial_t\sigma^\delta(s)\partial_t\vec{u}^d(s))\cdot\partial_t\vec{u}^d(s)ds.
\end{aligned}\tag{D.5}$$

We estimate

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\qquad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\leqslant\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.6}$$

That implies

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\\
&\qquad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\leqslant\Lambda^2\int_\Omega(\rho_0+\sigma^\delta(t))|\vec{u}^d(t)|^2d\vec{x}+\Lambda\mu\int_\Omega|\nabla\vec{u}^d(t)|^2d\vec{x}\\
&\qquad-\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.7}$$

By using the inequality

$$\Lambda\mu\|\nabla\vec{u}^d\|^2_{L^2(\Omega)}\leqslant\Lambda^2\mu\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds+\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|^2_{L^2(\Omega)}ds$$

and the identity

$$\begin{aligned}
&\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\\
&=\Lambda^2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\quad+\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\quad-\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,-1))dx_1ds,
\end{aligned}$$

it is obtained from (D.7) that (see **(134)** in [2])

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|_{L^2(\Omega)}^2\\
&\quad+\frac{1}{2}\Lambda\Big(\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\Big)\\
&\quad+\frac{1}{2}\int_0^t\Big(\mu\|\nabla\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1\Big)ds\\
&\leqslant\Lambda^2\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2(\Omega)}^2+C\delta^3e^{\lambda_1 t}\\
&\quad+\frac{3}{2}\Lambda^2\int_0^t\Big(\mu\|\nabla\vec{u}^d(s)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1\Big)ds\\
&\quad+\frac{3}{2}\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,-1))dx_1ds.
\end{aligned}\tag{D.8}$$

Integrating (D.3) in time from 0 to t and using (D.8) and Young's inequality, the authors deduce (D.2) without providing any detailed explanations.

However, we observe by integrating (D.3) in time that

$$\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2(\Omega)}^2\leqslant\frac{1}{\Lambda}\int_0^t e^{\Lambda(t-s)}\|\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2ds+Ce^{3\lambda_1 t}.$$

Then the l.h.s of (D.2) will be bounded by

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t\vec{u}^d(t)\|_{L^2(\Omega)}^2+\Lambda\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2+\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2ds\\
&\leqslant\Lambda\int_0^t e^{\Lambda(t-s)}\|\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2ds+\frac{1}{2}\Lambda\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2\\
&\quad+\frac{1}{2}\mu\int_0^t\|\nabla\partial_t\vec{u}^d(s)\|_{L^2(\Omega)}^2ds+\frac{3}{2}\Lambda^2\mu\int_0^t\|\nabla\vec{u}^d(s)\|_{L^2(\Omega)}^2ds\\
&\quad+\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\\
&\quad-\Lambda^2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\quad+\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,-1))dx_1+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.9}$$

We are not clear about the way in [2] to remove all integral terms over $2\pi L\mathbb{T}$ in the r.h.s of (D.9) to get (D.2) for all $\mu > 0$, especially the following term

$$\int_0^t \int_{2\pi L\mathbb{T}} (\xi_+ |\Lambda u_1^d - \partial_t u_1^d|^2(s, x_1, 1) + \xi_- |\Lambda u_1^d - \partial_t u_1^d|^2(s, x_1, -1)) dx_1.$$

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