LINEAR AND NONLINEAR ANALYSIS OF THE RAYLEIGH-TAYLOR SYSTEM WITH NAVIER-SLIP BOUNDARY CONDITIONS

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Abstract. In this paper, we are interested in the linear and the nonlinear Rayleigh instability for the incompressible Navier-Stokes equations with Navier-slip boundary conditions around a laminar smooth density profile $\rho_0(x_2)$ being increasing in an infinite slab $2\pi L\mathbb{T}\times(-1,1)$ ($L>0$, $\mathbb{T}$ is the usual 1D torus). The linear instability study of the viscous Rayleigh-Taylor model amounts to the study of the following ordinary differential equation on the finite interval $(-1,1)$,

$$-\lambda^2[\rho_0 k^2\phi-(\rho_0\phi')']=\lambda\mu(\phi^{(4)}-2k^2\phi''+k^4\phi)-gk^2\rho_0'\phi, \tag{0.1}$$

with the boundary conditions

$$\begin{cases}\phi(-1)=\phi(1)=0,\\ \mu\phi''(1)=\xi_+\phi'(1),\\ \mu\phi''(-1)=-\xi_-\phi'(-1),\end{cases} \tag{0.2}$$

where λ is the growth rate in time, k is the wave number transverse to the density profile. For each $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, we define a threshold of viscosity coefficient $\mu_c(k,\Xi)$ for linear instability. So that, in the k-supercritical regime, i.e. $\mu>\mu_c(k,\Xi)$, we provide a spectral analysis adapting the operator method of Lafitte-Nguyễn in [12] and then prove that there are infinitely solutions of (0.1)-(0.2). Secondly, we will extend a result of Grenier [6], by considering a wider class of initial data to the nonlinear perturbation problem, based on infinitely unstable modes of the linearized problem and we will prove nonlinear Rayleigh-Taylor instability in a high regime of viscosity coefficient, namely $\mu>3\sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\mu_c(k,\Xi)$.

Contents

Date: **20th Apr, 2022 at 11:57.**
2010 *Mathematics Subject Classification.* 34B07, 47A10, 47B07, 76D05, 76E30.
Key words and phrases. Navier–Stokes equations, linear growth rate, spectral analysis, Rayleigh–Taylor instability.

1. Introduction

The Rayleigh–Taylor (RT) instability, studied first by Lord Rayleigh in [16] and then Taylor [17] is well known as a gravity-driven instability in two semi-infinite inviscid and incompressible fluids when the heavy one is on top of the light one. It has attracted much attention due to both its physical and mathematical importance. Two applications worth mentioning are implosion of inertial confinement fusion capsules [13] and core-collapse of supernovae [15]. For a detailed physical comprehension of the RT instability, we refer to three survey papers [10, 18, 19]. Mathematically speaking, for the inviscid and incompressible regime with smooth density profile, the classical RT instability was investigated by Lafitte [11], by Guo and Hwang [4] and by Helffer and Lafitte [8].

Concerning the viscous RT instability, one of the first studies can be found in the book of Chandrasekhar [1, Chap. X], considering two uniform viscous fluid separated by a horizontal boundary and generalizing the classical result of Rayleigh and Taylor. We refer the readers to mathematical viscous RT studies for two compressible channel flows by Guo and Tice [5], for incompressible fluid in the whole space $\mathbf{R}^3$ by Jiang et. al [7] and Lafitte and Nguyễn [12], respectively.

In this paper, we are concerned with the viscous RT of the nonhomogeneous incompressible Navier-Stokes equations with gravity in a 2D slab domain $\Omega = 2\pi L\mathbb{T} \times (-1, 1)$ with $L > 0$ and $\mathbb{T}$ is the 1D-torus, that read as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \mathrm{div}(\rho \vec{u} \otimes \vec{u}) + \nabla P = \mu \Delta \vec{u} - \rho \vec{g}, \\ \mathrm{div} \vec{u} = 0, \end{cases} \tag{1.1}$$

where $t \geqslant 0, \vec{x} = (x_1, x_2) \in 2\pi L\mathbb{T} \times (-1, 1)$. The unknowns $\rho := \rho(t, \vec{x})$, $\vec{u} := \vec{u}(t, \vec{x})$ and $P := P(t, \vec{x})$ denote the density, the velocity and the pressure of the fluid, respectively, while μ is the viscosity coefficient and $\vec{g} := g\vec{e}_2$, $g > 0$ being the gravitational constant. Let $\Sigma_\pm = 2\pi L\mathbb{T} \times \{\pm 1\}$, the Navier-slip boundary conditions proposed by Navier (see [14]) are given on $\Sigma_\pm$ as follows

$$\begin{aligned} \vec{u} \cdot \vec{n} &= 0, \\ (\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n})_\tau &= \xi(\vec{x})\vec{u}. \end{aligned} \tag{1.2}$$

Here, $\vec{n}$ is the outward normal vector of the boundary, $(\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n})_\tau$ is the tangential component of $\mu(\nabla \vec{u} + \nabla \vec{u}^T) \cdot \vec{n}$ and $\xi(\vec{x})$ is a scalar function describing the slip effect on the boundary, only taking *nonnegative* constant values $\xi_\pm$ on $\Sigma_\pm$, respectively.

Let ρ_0 and P_0 be two C^1-functions on x_2 such that $P_0' = -g\rho_0$ with $' = d/dx_2$. Then, the laminar flow $(\rho_0(x_2), \vec{0}, P_0(x_2))$ is a steady-state solution of (1.1). Of interest of this paper is to study the nonlinear instability of the above laminar flow to Eq. (1.1)-(1.2) that satisfies

$$\rho_0 \in C^1([-1, 1]), \quad \rho_0' > 0 \text{ on } [-1, 1], \quad \rho_0(\pm 1) = \rho_\pm \in (0, +\infty), \tag{1.3}$$

i.e. to study the nonlinear Rayleigh–Taylor instability problem.

Linearizing (1.1) in the vicinity of $(\rho_0(x_2), \vec{0}, P_0(x_2))$ and then seeking a growing normal mode at a horizontal spatial frequency $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ of the form

$$e^{\lambda(k)t}\vec{U}(\vec{x}) = e^{\lambda(k)t}(\cos(kx_1)\omega(x_2), \sin(kx_1)\theta(x_2), \cos(kx_1)\phi(x_2), \cos(kx_1)q(x_2))^T,$$

the linear RT instability amounts to the investigation of the parameter $\lambda(k) \in \mathbb{C}$ ($\mathrm{Re}\lambda > 0$) such that there exists a solution $\phi \in H^4((-1, 1))$ of the following ordinary differential equation for the third component of velocity

$$-\lambda^2(\rho_0 k^2\phi - (\rho_0\phi')') = \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) - gk^2\rho_0'\phi, \tag{1.4}$$

with the boundary conditions

$$\begin{cases} \phi(-1) = \phi(1) = 0, \\ \mu\phi''(1) = \xi_+\phi'(1), \\ \mu\phi''(-1) = -\xi_-\phi'(-1). \end{cases} \tag{1.5}$$

Note that $H^4((-1, 1)) \hookrightarrow C^3((-1, 1))$ allows us to write (1.5). In this case, such a λ is called a growth rate of the instability or a characteristic value of the linearized problem (see Eq. (2.5) below) as in [1, Sect. 92-93, Chap. X]). We will present the derivation of the physical model in Section 2.

As the density profile is increasing, we firstly show that λ is always real in Lemma 2.1. Since our goal is to study instability, we restrict our study to the case $\lambda > 0$. Hence, for linear instability, we continue the spectral analysis of Helffer and Lafitte [8], Lafitte and Nguyễn [12] for Eq. (1.4)-(1.5).

For any horizontal spatial frequency $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, we then define a k-supercritical regime of the viscosity coefficient $\mu > \mu_c(k, \Xi)$ (see $\mu_c(k, \Xi)$ in Proposition 3.1) , we first prove that there exist an infinite sequence of characteristic values $(\lambda_n(k, \mu))_{n\geqslant 1}$, decreasing towards 0 as $n \to \infty$. This is stated in Theorem 2.1.

The second goal, described in Section 5 is to obtain a nonlinear instability result on more general initial conditions using the linear result of Theorem 2.1 (see (2.13)) and working in the regime $\mu > 3\sup_{k\in L^{-1}\mathbb{Z}\backslash\{0\}} \mu_c(k, \Xi)$. The classical way of proving nonlinear instability is to estimate the difference between the solution to the nonlinear problem and the growing mode solution to the linearized problem. In

order to show that, we will apply the general framework of Grenier in the celebrated paper [6], that took the maximal growing mode $\delta e^{\lambda_1 t}\vec{U}_1(\vec{x})$ with $0<\delta\ll 1$ to be an approximate solution of the nonlinear perturbation equation. Our nonlinear result, Theorem 2.2, generalizes Grenier [6].

Organization of the paper. In Section 2, we present the governing equations and state the main results. Section 3 is devoted to some materials for the linear study. Then, in Section 4, we prove the linear instability, i.e. Theorem 2.1. The last Section 5 is to prove the nonlinear instability, i.e. Theorem 2.2.

2. Main results

2.1. **The governing equations.** Let us recall the steady state $(\rho_0(x_2),\vec{0},P_0(x_2))$ of (1.1), with ρ_0 satisfies (1.3) and $P_0'=-g\rho_0$. We now derive the linearization of Eq. (1.1) around the equilibrium state $(\rho_0(x_2),\vec{0},P_0(x_2))$. The perturbations

$$\sigma=\rho-\rho_0,\quad \vec{u}=\vec{u}-\vec{0},\quad p=P-P_0$$

thus satisfy

$$\begin{cases}\partial_t\sigma+\vec{u}\cdot\nabla(\rho_0+\sigma)=0,\\ (\rho_0+\sigma)\partial_t\vec{u}+(\rho_0+\sigma)\vec{u}\cdot\nabla\vec{u}+\nabla p=\mu\Delta\vec{u}-\sigma\vec{g},\\ \mathrm{div}\vec{v}=0.\end{cases}\tag{2.1}$$

Note that $(\mu(\nabla\vec{u}+\nabla\vec{u}^T)\cdot\vec{n})_\tau=\vec{n}\times(\mu(\nabla\vec{u}+\nabla\vec{u}^T)\cdot\vec{n})\times\vec{n}$ and that $\vec{n}=(0,\pm1)^T$. Hence, the boundary conditions are

$$\begin{cases}u_2=0, & \text{on } \Sigma_\pm,\\ \mu\partial_{x_2}u_1=\xi_+u_1 & \text{on } \Sigma_+,\\ \mu\partial_{x_2}u_1=-\xi_-u_1 & \text{on } \Sigma_-.\end{cases}\tag{2.2}$$

The linearized system reads as

$$\begin{cases}\partial_t\sigma+\rho_0'u_2=0,\\ \rho_0\partial_t\vec{u}+\nabla p=\mu\Delta\vec{u}-\sigma\vec{g},\\ \mathrm{div}\vec{u}=0,\end{cases}\tag{2.3}$$

with the corresponding boundary conditions remain the same as (2.2).

The linear RT instability problem is to seek a growing mode of the form

$$\begin{cases}\sigma(t,\vec{x})=e^{\lambda t}\cos(kx_1)\omega(x_2),\\ u_1(t,\vec{x})=e^{\lambda t}\sin(kx_1)\theta(x_2),\\ u_2(t,\vec{x})=e^{\lambda t}\cos(kx_1)\phi(x_2),\\ q(t,\vec{x})=e^{\lambda t}\cos(kx_1)q(x_2).\end{cases}\tag{2.4}$$

where $k\in L^{-1}\mathbb{Z}\backslash\{0\}$, $\lambda\in\mathbb{C}\backslash\{0\}$ and $\mathrm{Re}\lambda\geqslant 0$. It follows from (2.3) that

$$\begin{cases}\lambda\omega+\rho_0'\phi=0,\\ \lambda\rho_0\theta-kq+\mu(k^2\theta-\theta'')=0,\\ \lambda\rho_0\phi+q'+\mu(k^2\phi-\phi'')=-g\omega,\\ k\theta+\phi'=0\end{cases}\tag{2.5}$$

and from (2.2) that

$$\phi(\pm1)=0,\quad \mu\theta'(1)=\xi_+\theta(1),\quad \mu\theta'(-1)=-\xi_-\theta(-1).\tag{2.6}$$

We obtain

$$\omega = -\frac{\rho_0'}{\lambda}\phi, \quad \theta = -\frac{1}{k}\phi', \quad q = -\frac{1}{k^2}(\lambda\rho_0\phi' + \mu(k^2\phi' - \phi''')). \tag{2.7}$$

Then, we substitute q, ω into $(2.5)_3$ to get a fourth-order ordinary equation (1.4). We have the boundary conditions deduced from (2.2) that are obtained by assuming the solution to be in $C^2([-1,1])$, that are (1.5).

2.2. **Main results.** Before stating our main results, we present our materials for the linearized problem.

When the density profile ρ_0 is increasing, we firstly show that all characteristic values λ are real. Let $L_0^{-1} = \|\frac{\rho_0'}{\rho_0}\|_\infty$, we further obtain the following uniform upper bound $\sqrt{\frac{g}{L_0}}$ of λ.

Lemma 2.1. *For any $k \in L^{-1}\mathbb{Z}\backslash\{0\}$,*

- *all characteristic values λ are always real and non-negative,*
- *all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

Proof of Lemma 2.1 is given in Section 3.1. In view of Lemma 2.1, we seek for functions ϕ being real and we only consider vector space of real functions in what follows.

We now study the linearized problem, i.e. (1.4)-(1.5). Of importance is to construct a bilinear coercive form $\mathcal{B}_{k,\lambda,\mu}$ as $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$ (i.e. we do not restrict $\lambda \in (0, \sqrt{\frac{g}{L_0}})$ and $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ at this step) on the functional space

$$\tilde{H}^2((-1,1)) := \{\phi \in H^2((-1,1)), \phi(\pm 1) = 0\},$$

so that we can transform our problem into solving the variational problem

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\theta) = gk^2\int_{-1}^{1}\rho_0'\phi\theta dx_2 \quad \text{for all } \theta \in \tilde{H}^2((-1,1)), \tag{2.8}$$

for all ϕ, θ staying in the functional space $\tilde{H}^2((-1,1))$ associated with the norm $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. The desired bilinear form $\mathcal{B}_{k,\lambda,\mu}$ is

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) :=& \lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 \\ &- \xi_-\vartheta'(-1)\varrho'(-1) - \xi_+\vartheta'(1)\varrho'(1),\end{aligned} \tag{2.9}$$

For all $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$, we will place ourselves in a k-supercritical regime of the viscosity coefficient $\mu > \mu_c(k,\Xi)$ (see precise formula $\mu_c(k,\Xi)$ in Proposition 3.1) such that

$$\mathcal{B}_{k,0,\mu} \text{ is coercive if and only if } \mu > \mu_c(k,\Xi), \tag{2.10}$$

it yields $\mathcal{B}_{k,\lambda,\mu}$ is coercive for all $\lambda \geqslant 0$ and $\mu > \mu_c(k,\Xi)$. In view of the Riesz representation theorem, we thus obtain an abstract operator $Y_{k,\lambda,\mu}$ from $\tilde{H}^2((-1,1))$ to its dual, such that

$$\mathcal{B}_{k,\lambda,\mu}(\phi,\theta) = \langle Y_{k,\lambda,\mu}\phi, \theta\rangle \tag{2.11}$$

for all $\theta \in \tilde{H}^2((-1,1))$. It turns out that the existence of $H^4((-1,1))$ solutions of Eq. (1.4)-(1.5) on $(-1,1)$ is reduced to that one of weak solutions $\phi \in \tilde{H}^2((-1,1)$ of

$$\lambda Y_{k,\lambda,\mu}\phi = gk^2\rho_0'\phi$$

on $(-1,1)$. Owing to a bootstrap argument to solution ϕ, we obtain (see Proposition 3.3) $Y_{k,\lambda,\mu}$ explicitly, that is a fourth differential operator of the form

$$Y_{k,\lambda,\mu}\phi = \lambda(\rho_0 k^2\phi - (\rho_0\phi')') + \mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi)$$

and get back that $\phi \in H^4((-1,1))$ satisfies (1.4) on $(-1,1)$ and the boundary conditions (1.5).

Denoting by $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((-1,1))$, the problem of finding (λ,ϕ) is thus to find v such that

$$\frac{\lambda}{gk^2}v = \mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}v.$$

This is the aim of Section 4. The theory of self-adjoint and compact operators for a Sturm-Liouville problem on the functional space $H^2((-1,1))$ will play a key role here. Once it is proved that the operator $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ is compact and self-adjoint from $L^2((-1,1))$ to itself, then its discrete spectrum is a sequence of eigenvalues (denoted by $\gamma_n(k,\lambda,\mu)$) and the problem of finding characteristic values $\lambda(k,\mu)$ of (1.4) amounts to solving all the equations

$$\gamma_n(k,\lambda,\mu) = \frac{\lambda}{gk^2}. \tag{2.12}$$

For each n, we will show that (2.12) has a unique root $\lambda_n \in \mathbf{R}_+$ because of the decrease of γ_n in λ, which is an easy extension of Kato's perturbation theory of spectrum of operators [9]. In addition, thanks to Lemma 2.1(2), we then have λ_n is a characteristic value, hence $\lambda_n \leqslant \sqrt{\frac{g}{L_0}}$ for all $n \geqslant 1$. That implies that for any horizontal spatial frequency $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, there exists a sequence of characteristic values $(\lambda_n(k,\mu))_{n\geqslant 1}$, that is uniformly bounded and decreases towards 0 as $n \to \infty$.

We sum up the above arguments in our first theorem.

Theorem 2.1. *Let $k \in L^{-1}\mathbb{Z}\backslash\{0\}$ be fixed and let ρ_0 satisfy that (1.3), i.e.*

$$\rho_0 \in C^1([-1,1]), \quad \rho_0(\pm 1) = \rho_\pm \in (0,\infty), \quad \rho_0' > 0 \text{ everywhere on } [-1,1].$$

For all $\mu > \mu_c(k,\Xi)$, there exists an infinite sequence $(\lambda_n,\phi_n)_{n\geqslant 1}$ with $\lambda_n > 0$ decreasing towards 0 and $\phi_n \in H^4((-1,1))$, ϕ_n non trivial, satisfying (1.4)-(1.5).

Once Eq. (1.4)-(1.5) is solved, we go back to the linearized equations (2.3). For a fixed $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, we obtain a sequence of solutions to the linearized equations (2.3) as indicated in Proposition 4.2, that are $(e^{\lambda_j(k,\mu)t}\vec{U}_j(\vec{x}))_{j\geqslant 1}$, with $\vec{U}_j(\vec{x}) = (\sigma_j,\vec{u}_j,p_j)(\vec{x})$.

Let us choose a $k_0 \in L^{-1}\mathbb{Z}\backslash\{0\}$. In view of getting infinitely many characteristic values of the linearized problem, we introduce a linear combination of unstable growing modes

$$\vec{U}^M(t,\vec{x}) = \delta\sum_{j=1}^{M}\mathsf{c}_j e^{\lambda_j(k_0,\mu)t}\vec{U}_j(\vec{x}) \tag{2.13}$$

to construct an approximate solution to the nonlinear problem (1.1)-(1.2), with $\delta > 0$ and constants c_j being chosen such that

$$|\mathsf{c}_1|\|\vec{u}_1\|_{L^2(\Omega)} > \frac{1}{2}\sum_{j\geqslant 2}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)} > 0. \tag{2.14}$$

Assume (2.1)–(2.2) is supplemented with initial datum $\vec{U}^M(0,\vec{x})$, there is a unique local strong solution $(\sigma^\delta,\vec{u}^\delta)$ with an associated pressure q^δ to the nonlinear perturbed problems (2.1)–(2.2) on $[0,T_{\max})$ (see Proposition 5.1). We define the differences

$$(\sigma^d,\vec{u}^d,q^d)=(\sigma^\delta,\vec{u}^\delta,q^\delta)-(\sigma^M,\vec{u}^M,q^M)$$

and then estimate the bound of $\|(\sigma^d,\vec{u}^d)\|_{L^2(\Omega)}$ in time (see Proposition 5.2) in the regime

$$\mu>3\mu_c(\Xi),\quad \text{with } \mu_c(\Xi):=\sup_{k\in L^{-1}\mathbb{Z}\setminus\{0\}}\mu_c(k,\Xi). \tag{2.15}$$

Indeed, since $\mu>3\mu_c(\Xi)$, we choose a constant $\varpi_0>0$ such that

$$\mu>(3+\varpi_0)\mu_c(\Xi). \tag{2.16}$$

Hence, $\nu_0=\frac{3+\varpi_0}{2+\varpi_0}\in(1,\frac{3}{2})$. It follows from Theorem 2.1 and Lemma 2.1(2) that exists

$$0<\Lambda=\sup_{k\in L^{-1}\mathbb{Z}\setminus\{0\}}\lambda_1(k,\mu)\leqslant\sqrt{\frac{g}{L_0}}. \tag{2.17}$$

We further look for $k_0\in L^{-1}\mathbb{Z}\setminus\{0\}$ to have that

$$\Lambda\geqslant\lambda_1(k_0,\mu)>\lambda_2(k_0,\mu)>\cdots>\lambda_N(k_0,\mu)>\frac{2\nu_0}{3}\Lambda>\lambda_{N+1}(k_0,\mu)>\ldots. \tag{2.18}$$

For t small enough, we then deduce the bound of $\|(\sigma^d,\vec{u}^d)\|_{L^2(\Omega)}$ in Proposition 5.2, that is

$$\|(\sigma^d(t),\vec{u}^d(t))\|^2_{L^2(\Omega)}\leqslant C\delta^3\Big(\sum_{j=1}^N|\mathsf{c}_j|e^{\lambda_j t}+\max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda t}\Big)^3.$$

The nonlinear result then follows.

Theorem 2.2. *Let $\mu_c(\Xi)$ be defined as in (2.15) and $\mu>3\mu_c(\Xi)$. Let ρ_0 satisfies (1.3), i.e.*

$$\rho_0\in C^1([-1,1]),\quad \rho_0(\pm1)=\rho_\pm\in(0,\infty),\quad \rho_0'>0 \text{ everywhere on } [-1,1].$$

Let $M\in\mathbb{N}^\star$, there exist positive constants m_0,δ_0 and ϵ_0 such that for any $\delta\in(0,\delta_0)$, the nonlinear perturbation equation (2.1) with boundary conditions (2.2) and the initial data $\delta\sum_{j=1}^M\mathsf{c}_j\vec{U}_j(\vec{x})$ satisfying (2.14) admits a unique local strong solution $(\sigma^\delta,\vec{u}^\delta)$ with an associated pressure q^δ such that

$$\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)}\geqslant m_0\epsilon_0, \tag{2.19}$$

where $T^\delta\in(0,T_{\max})$ satisfies uniquely $\delta\sum_{j=1}^M|\mathsf{c}_j|e^{\lambda_j T^\delta}=\epsilon_0$.

3. Preliminaries

3.1. **Proof of Lemma 2.1.** Let $\phi\in H^4((-1,1))$ satisfy (1.4)-(1.5) Multiplying by ϕ on both sides of (1.4) and then using the integration by parts, we get that

$$-\int_{-1}^1(\rho_0\phi')'\phi dx_2=-\rho_0\phi'\phi\Big|_{-1}^1+\int_{-1}^1\rho_0\phi'\phi'dx_2$$

that

$$-\int_{-1}^1\phi''\phi dx_2=-\phi'\phi\Big|_{-1}^1+\int_{-1}^1\phi'\phi'dx_2$$

and that

$$\int_{-1}^1\phi^{(4)}\phi dx_2=\phi'''\phi\Big|_{-1}^1-\phi''\phi'\Big|_{-1}^1+\int_{-1}^1\phi''\phi''dx_2,$$

we obtain that

$$\begin{aligned}\lambda\Big(\mu\int_{-1}^{1}\Big(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2\Big)dx_2-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2\Big)\\+\lambda^2\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2=gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\end{aligned}\tag{3.1}$$

Suppose that $\lambda=\lambda_1+i\lambda_2$, then one deduces from (3.1) that

$$\begin{aligned}\lambda_1\Big(\mu\int_{-1}^{1}\Big(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2\Big)dx_2-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2\Big)\\+(\lambda_1^2-\lambda_2^2)\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2=gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2\end{aligned}\tag{3.2}$$

and that

$$\begin{aligned}\lambda_2\Big(\mu\int_{-1}^{1}\Big(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2\Big)dx_2-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2\Big)\\=-2\lambda_1\lambda_2\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2.\end{aligned}\tag{3.3}$$

If $\lambda_2\neq 0$, (3.3) leads us to

$$\begin{aligned}-2\lambda_1\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2=\mu\int_{-1}^{1}\Big(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2\Big)dx_2\\-\xi_-|\phi'(-1)|^2-\xi_+|\phi'(1)|^2,\end{aligned}$$

which yields

$$\begin{aligned}-(\lambda_1^2-\lambda_2^2)\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2=-2\lambda_1^2\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2\\-gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\end{aligned}$$

Equivalently,

$$(\lambda_1^2+\lambda_2^2)\int_{-1}^{1}\Big(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2\Big)dx_2=-gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\tag{3.4}$$

That implies

$$(\lambda_1^2+\lambda_2^2)k^2\rho_-\int_{-1}^{1}|\phi|^2dx_2\leqslant -gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

The positivity of ρ_0' yields a contradiction, then λ is real. Due to (3.2) again, we further get that

$$\lambda^2\int_{-1}^{1}\rho_0(k^2|\phi|^2+|\phi'|^2)dx_2\leqslant gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 2.1.

3.2. **The threshold of viscosity coefficient.** We begin with the precise formula of critical viscosity coefficients $\mu_c(k,\Xi)$ (see (2.10) above) for all $k\in\mathbf{R}\backslash\{0\}$. Note that $\mu_c(k,\Xi)=\mu_c(-k,\Xi)$ for all $k\in\mathbf{R}\backslash\{0\}$, it suffices to find $\mu_c(k,\Xi)$ for $k\in\mathbf{R}_+$.

Proposition 3.1. *The following results hold.*

(1) *For all* $k \in \mathbf{R}_+$, *we have*

$$\mu_c(k,\Xi) = \max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 (|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_2}. \tag{3.5}$$

Moreover,

$$\mu_c(k,\Xi) = \frac{1}{4k\sinh^2(2k)} \left(\begin{array}{l} (\sinh(2k)\cosh(2k) - 2k)(\xi_+ + \xi_-) \\ + \left(\begin{array}{l} (\sinh(2k) - 2k\cosh(2k))^2(\xi_+ + \xi_-)^2 \\ + \sinh^2(2k)(\sinh^2(2k) - 4k^2)(\xi_+ - \xi_-)^2 \end{array} \right)^{\frac{1}{2}} \end{array} \right). \tag{3.6}$$

(2) $\mu_c(k,\Xi)$ *is a decreasing function in* $k \in \mathbf{R}_+$ *and*

$$\lim_{k\to 0} \mu_c(k,\Xi) = \sup_{k\in\mathbf{R}\setminus\{0\}} \mu_c(k,\Xi) =: \mu_c^s(\Xi). \tag{3.7}$$

We have the asymptotic expansion of $\mu_c(k,\Xi)$ *as* $k \to 0^+$,

$$\begin{aligned} \mu_c^s(k,\Xi) = {} & \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big) \\ & - \frac{2}{15}\Big(4(\xi_+ + \xi_-) + \frac{4\xi_+^2 - \xi_+\xi_- + 4\xi_-^2}{\sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}}\Big)k^2 + O(k^3). \end{aligned} \tag{3.8}$$

That implies

$$\mu_c^s(\Xi) = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \tag{3.9}$$

As $k \gg 1$, *we obtain the limit*

$$\mu_c(k,\Xi) \leqslant \frac{\sqrt{2(\xi_+^2 + \xi_-^2)}}{k} \to 0. \tag{3.10}$$

(3) *We have*

$$\begin{aligned} \mu_c^s(\Xi) &= \max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} \\ &= \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \end{aligned} \tag{3.11}$$

The proof is postponed to the appendices A, B, C.

Remark 3.1. *We see that* $\mu_c(\Xi)$ *in* [3, Proposition 2.2] *is incorrect and redo the computation the critical viscosity coefficient in Appendix C.2. The authors in* [3] *consider* $2\pi L\mathbb{T} \times (0,1)$ *instead of* $2\pi L\mathbb{T} \times (-1,1)$ *and constant values* $k_{0,1}$ *instead of* $\xi_\pm$. *The formula of the critical viscosity defined as in* [3, (1.29)] *is*

$$\mu_c := \sup_{\phi\in\mathcal{Y}} Z(\phi),$$

where

$$\mathcal{Y} = \{\phi \in H_0^1((0,1)) \cap H^2((0,1)), \frac{1}{2}\int_0^1 |\phi''|^2 dx = 1\},$$

$$Z(\phi) = \frac{k_0}{2}|\phi'(0)|^2 + \frac{k_1}{2}|\phi'(1)|^2.$$

The authors in [3, Proposition 2.2] *claim that* $\mu_c = \frac{k}{6}$ *if* $k_0 = k_1 = k > 0$. *However, in that case, let us take* $\phi = \frac{1}{\sqrt{2}}(x^2 - x) \in \mathcal{Y}$, *then we have*

$$Z\Big(\frac{1}{\sqrt{2}}(x^2 - x)\Big) = \frac{k}{2} > \frac{k}{6}.$$

It is a contradiction.

3.3. **A bilinear form and a self-adjoint invertible operator.** In what follows in this section we have $\lambda \geqslant 0$. Let us recall the definition of $\mathcal{B}_{k,\lambda,\mu}$ from (2.9),

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) := \lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_2+\mu\int_{-1}^{1}(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_2 \\ -\xi_-\vartheta'(-1)\varrho'(-1)-\xi_+\vartheta'(1)\varrho'(1).$$

Lemma 3.1. *We have the followings.*

- *For all $\mu>0$, $\mathcal{B}_{k,\lambda,\mu}$ is a continuous bilinear form on $\tilde{H}^2((-1,1))$.*
- *For all $\mu>\mu_c(k,\Xi)$, $\mathcal{B}_{k,\lambda,\mu}$ is coercive.*

Proof of Lemma 3.1. Clearly, $\mathcal{B}_{k,\lambda,\mu}$ is a bilinear form on $\tilde{H}^2((-1,1))$. We then establish the boundedness of $\mathcal{B}_{k,\lambda,\mu}$. The integral terms of $\mathcal{B}_{k,\lambda,\mu}$ are bounded by

$$C^\star(\lambda+1)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}. \tag{3.12}$$

Meanwhile, it follows from the general Sobolev inequality that

$$\|\vartheta(y)\|_{C^{0,j}((-1,1))}\leqslant C^\star\|\vartheta(y)\|_{H^1((-1,1))}\quad\text{for all } j\in[0,\frac{1}{2}).$$

Therefore, we obtain

$$\max(|\vartheta'(-1)|^2,|\vartheta'(1)|^2)\leqslant C^\star\|\vartheta'\|^2_{H^1((-1,1))},$$

where $C^\star$ is a generic constant depending on ρ_0 and other physical parameters, independent of λ. Consequently, we get

$$|\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)|\leqslant C^\star(2+\lambda)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}, \tag{3.13}$$

i.e. $\mathcal{B}_{k,\lambda,\mu}$ is bounded.

We move to show the coercivity of $\mathcal{B}_{k,\lambda,\mu}$. We have that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta)=\lambda\int_{-1}^{1}\rho_0(k^2|\vartheta|^2+|\vartheta'|^2)dx_2+\mu\int_{-1}^{1}(|\vartheta''|^2+2k^2|\vartheta'|^2+k^4|\vartheta|^2)dx_2 \\ -\xi_-|\vartheta'(-1)|^2-\xi_+|\vartheta'(1)|^2.$$

As $\lambda\geqslant 0$ and $\mu>\mu_c(k,\Xi)$, we have

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta)&\geqslant\lambda\int_{-1}^{1}\rho_0(k^2|\vartheta|^2+|\vartheta'|^2)dx_2\\&\quad+(\mu-\mu_c(k,\Xi))\int_{-1}^{1}(|\vartheta''|^2+2k^2|\vartheta'|^2+k^4|\vartheta|^2)dx_2 \qquad (3.14)\\&\geqslant(\mu-\mu_c(k,\Xi))\int_{-1}^{1}(|\vartheta''|^2+2k^2|\vartheta'|^2+k^4|\vartheta|^2)dx_2.\end{aligned}$$

It then follows from (3.13) and (3.14) that $\mathcal{B}_{k,\lambda,\mu}$ is a continuous and coercive bilinear form on $\tilde{H}^2((-1,1))$. □

With the above property of $\mathcal{B}_{k,\lambda,\mu}$, we then establish that.

Proposition 3.2. *Let $\mu>\mu_c(k,\Xi)$ and $(\tilde{H}^2((-1,1)))'$ be the dual space of $\tilde{H}^2((-1,1))$, associated with the norm $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. There is a unique operator*

$$Y_{k,\lambda,\mu}\in\mathcal{L}(H^2((-1,1)),(\tilde{H}^2((-1,1)))'),$$

that is also bijective, such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)=\langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle \tag{3.15}$$

for all $\vartheta, \varrho \in \tilde{H}^2((-1,1))$.

Proof. It follows from Riesz representation theorem that there exists an operator $Y_{k,\lambda,\mu} \in \mathcal{L}(\tilde{H}^2((-1,1)), (\tilde{H}^2((-1,1)))')$ such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta, \varrho) = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle$$

for all $\varrho \in \tilde{H}^2((-1,1))$. Proof of Proposition 3.1 is complete. □

Proposition 3.3. *We have the following results.*

(1) *For all* $\vartheta \in \tilde{H}^2((-1,1))$,

$$Y_{k,\lambda,\mu}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta)$$

in $\mathcal{D}'((-1,1))$.

(2) *Let* $f \in L^2((-1,1))$ *be given, there exists a unique solution* $\vartheta \in \tilde{H}^2((-1,1))$ *of*

$$Y_{k,\lambda,\mu}\vartheta = f \text{ in } (\tilde{H}^2((-1,1)))', \tag{3.16}$$

then $\vartheta \in H^4((-1,1))$ *satisfies the boundary conditions* (1.5).

Proof. It follows from Proposition 3.2 that there is a unique $\vartheta \in \tilde{H}^2((-1,1))$ such that

$$\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.17}$$

for all $\varrho \in C_0^\infty((-1,1))$. We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'' which is in $L^2((-1,1))$. Hence, (3.17) is equivalent to

$$\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}((\vartheta'')''\varrho + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.18}$$

for all $\varrho \in C_0^\infty((-1,1))$. We deduce from (3.18) that

$$\lambda\int_{-1}^{1}(k^2\rho_0\vartheta - (\rho_0\vartheta')')\varrho dx_2 + \mu\langle(\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta, \varrho\rangle = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{3.19}$$

for all $\varrho \in C_0^\infty((-1,1))$. The resulting equation implies that

$$\mu((\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta) + \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') = Y_{k,\lambda,\mu}\vartheta \quad \text{in } \mathcal{D}'((-1,1)). \tag{3.20}$$

The first assertion holds.

Under the assumption $f \in L^2((-1,1))$, we then enhance the regularity of the weak solution $\vartheta \in \tilde{H}^2((-1,1))$ of (3.20). Indeed, we rewrite (3.20) as

$$\mu\int_{-1}^{1}(\vartheta'')''\varrho dx_2 = \int_{-1}^{1}(Y_{k,\lambda,\mu}\vartheta + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')\varrho dx_2$$

for all $\varrho \in C_0^\infty((-1,1))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-1,1))$, it then follows from (3.19) that $(\vartheta'')'' \in L^2((-1,1))$. Furthermore, by usual distribution theory, we define $\phi \in \mathcal{D}'((-1,1))$ such that

$$\langle\phi, \varrho\rangle = \langle(\vartheta'')'', \zeta_\varrho\rangle \tag{3.21}$$

for all $\varrho \in C_0^\infty((-1,1))$, where $\zeta_\varrho(x) = \int_{-a}^{x}(\varrho(y) - \int_{-1}^{1}\varrho(s)ds)dx_2$. Hence, it can be seen that

$$\langle\phi', \varrho\rangle = -\langle\phi, \varrho'\rangle = -\langle(\vartheta'')'', \zeta_{\varrho'}\rangle = -\langle(\vartheta'')'', \varrho\rangle$$

that implies $(\vartheta'')' - \phi \equiv$ constant. In view of $(\vartheta'')'' \in L^2((-1,1))$ and (3.21), we know that $(\vartheta'')' \in L^2((-1,1))$. Since $\vartheta \in \tilde{H}^2((-1,1))$ and $(\vartheta')', (\vartheta'')'' \in L^2((-1,1))$,

it tells us that ϑ belongs to $H^4((-1,1))$ and we can take their traces of derivatives of ϑ up to order 3.

By exploiting (3.19), we then show that ϑ satisfies (1.5). Indeed, for all $\varrho \in \tilde{H}^2((-1,1))$, we perform the integration by parts to obtain from (3.19) that

$$\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_2+\mu\int_{-1}^{1}(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_2$$
$$-\lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1}+\mu\Big(\vartheta'''\varrho\Big|_{-1}^{1}-\vartheta''\varrho'\Big|_{-1}^{1}-2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big)=\int_{-1}^{1}(Y_{k,\lambda,\mu}\vartheta)\varrho dx_2.$$

It then follows from the definition of the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ that

$$\lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1}-\mu\Big(\vartheta'''\varrho\Big|_{-1}^{1}-\vartheta''\varrho'\Big|_{-1}^{1}-2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big)=\xi_-\vartheta'(-1)\varrho'(-1)+\xi_+\vartheta'(1)\varrho'(1), \tag{3.22}$$

for all $\varrho\in\tilde{H}^2((-1,1))$. By collecting all terms corresponding to $\varrho'(\pm1)$ in (3.22), we deduce that

$$\mu\vartheta''(\pm1)=\pm\xi_\pm\vartheta'(\pm1).$$

We have just proved that ϑ satisfies (1.5). This ends the proof of Proposition 3.1. □

We obtain more information on the inverse operator $Y_{k,\lambda,\mu}^{-1}$.

Proposition 3.4. *The operator $Y_{k,\lambda,\mu}^{-1}:L^2((-1,1))\to L^2((-1,1))$ is compact and self-adjoint.*

Proof. It follows from Proposition 3.3 that $Y_{k,\lambda,\mu}$, being supplemented with (1.5), admits an inverse operator $Y_{k,\lambda,\mu}^{-1}$ from $L^2((-1,1))$ to a subspace of $H^4((-1,1))$ requiring all elements satisfy (1.5), which is symmetric due to Proposition 3.1. We compose $Y_{k,\lambda,\mu}^{-1}$ with the continuous injection from $H^4((-1,1))$ to $L^2((-1,1))$. Notice that the embedding $H^p((-1,1))\hookrightarrow H^q((-1,1))$ for $p>q\geqslant 0$ is compact. Therefore, $Y_{k,\lambda,\mu}^{-1}$ is compact and self-adjoint from $L^2((-1,1))$ to $L^2((-1,1))$. □

4. Linear instability

4.1. **A sequence of characteristic values.** We continue considering $\lambda\geqslant 0$ and study the operator $S_{k,\lambda,\mu}:=\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$.

Proposition 4.1. *Under the hypothesis (1.3), the operator $S_{k,\lambda,\mu}:L^2((-1,1))\to L^2((-1,1))$ is compact and self-adjoint.*

Proof. Due to the assumption of ρ_0 (1.3), the operator $S_{k,\lambda,\mu}$ is well-defined and bounded from $L^2((-1,1))$ to itself. $Y_{k,\lambda,\mu}^{-1}$ is compact, so is $S_{k,\lambda,mu}$. Moreover, because both the inverse $Y_{k,\lambda,\mu}^{-1}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{k,\lambda,\mu}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{k,\lambda,\mu}$ is discrete, i.e. is a decreasing sequence $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant1}$ of positive eigenvalues of $S_{k,\lambda,\mu}$ that tends to 0 as $n\to\infty$, associated with normalized orthogonal eigenvectors $\{\varpi_n\}_{n\geqslant1}$ in $L^2((-1,1))$. That means

$$S_{k,\lambda,\mu}\varpi_n=\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}\varpi_n=\gamma_n(k,\lambda,\mu)\varpi_n.$$

So that $\phi_n = Y^{-1}_{k,\lambda,\mu}\mathcal{M}\varpi_n$ belongs to $H^4((-1,1))$ and satisfies (1.5). One thus has

$$\gamma_n(k,\lambda,\mu)Y_{k,\lambda,\mu}\phi_n = \rho_0'\phi_n \tag{4.1}$$

and ϕ_n satisfies (1.5). (4.1) also tells us that $\gamma_n(k,\lambda,\mu) > 0$ for all n. Indeed, we obtain

$$\gamma_n(k,\lambda,\mu)\int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)\phi_n dx_2 = \int_{-1}^{1}\rho_0'|\phi_n|^2 dx_2.$$

That implies

$$\gamma_n(k,\lambda,\mu)\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) = \int_{-1}^{1}\rho_0'|\phi_n|^2 dx_2. \tag{4.2}$$

Since $\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) > 0$ and $\rho_0' > 0$ on $(-1,1)$, we know that $\gamma_n(k,\lambda,\mu)$ is positive.

For each n, ϕ_n is a solution of (1.4)-(1.5) if and only if there are positive λ_n such that (2.12). To solve (2.12), we use two following lemmas.

Lemma 4.1. *For each n,*

- *$\gamma_n(k,\lambda,\mu)$ and ϕ_n is differentiable in λ.*
- *$\gamma_n(k,\lambda,\mu)$ is decreasing in λ.*

Proof. The proof of Lemma 4.1(1) is as same as [12, Lemma 3.3], we omit detail here. We now prove that $\gamma_n(k,\lambda,\mu)$ is decreasing in λ.

Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (4.1) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{k,\lambda,\mu}z_n = \frac{1}{\gamma_n(k,\lambda,\mu)}\rho_0'z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\rho_0'\phi_n \tag{4.3}$$

on $(-1,1)$. At $x_2 = \pm 1$, we have

$$\begin{cases} z_n(-1) = z_n(1) = 0, \\ \mu z_n''(1) = \xi_+ z_n'(1), \\ \mu z_n''(-1) = -\xi_- z_n'(-1). \end{cases} \tag{4.4}$$

Multiplying by ϕ_n on both sides of (4.3), we obtain that

$$\begin{aligned} &\int_{-1}^{1}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_2 + \int_{-1}^{1}(Y_{k,\lambda,\mu}z_n)\phi_n dx_2 \\ &\qquad = \frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'z_n\phi_n dx_2 + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'|\phi_n|^2 dx_2. \end{aligned} \tag{4.5}$$

Note that z_n enjoys (4.4), then

$$\int_{-1}^{1}(Y_{k,\lambda,\mu}z_n)\phi_n dx_2 = \int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)z_n dx_2 = \frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'z_n\phi_n dx_2.$$

That implies

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'|\phi_n|^2 dx_2 = \int_{-1}^{1}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_2. \tag{4.6}$$

Using the integration by parts, we obtain from (4.6) that

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\int_{-1}^{1}\rho_0'|\phi_n|^2 dx_2 = \int_{-1}^{1}\rho_0(k^2|\phi_n|^2 + |\phi_n'|^2)dx_2 > 0.$$

Consequently, $\gamma_n(k,\lambda,\mu)$ is decreasing in $\lambda > 0$. □

4.2. **Proof of Theorem 2.1.** In view of Lemma 4.1, we are able to prove Theorem 2.1.

Proof of Theorem 2.1. For each n, there is only one solution λ_n of (2.12). Indeed, using (4.2), we know that

$$\frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'|\phi_n|^2dx_2=\int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)\phi_n dx_2=\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n).$$

Hence, it follows from (3.14) that

$$\begin{aligned}\frac{1}{\gamma_n(k,\lambda,\mu)}\int_{-1}^{1}\rho_0'|\phi_n|^2dx_2&\geqslant\lambda\int_{-1}^{1}\rho_0(k^2|\phi_n|^2+|\phi_n'|^2)dx_2\\&\quad+(\mu-\mu_c(k,\Xi))\int_{-1}^{1}(|\phi_n''|^2+2k^2|\phi_n'|^2+k^4|\phi_n|^2)dx_2\\&\geqslant\lambda k^2\int_{-1}^{1}\rho_0|\phi_n|^2dx_2+(\mu-\mu_c(k,\Xi))k^4\int_{-1}^{1}|\phi_n|^2dx_2.\end{aligned}$$

That implies

$$\frac{1}{L_0\gamma_n(k,\lambda,\mu)}\geqslant\lambda k^2+\frac{(\mu-\mu_c(k,\Xi))k^4}{\rho_+}.$$

Consequently, for all $n\geqslant 1$,

$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)}>gk^2\text{ for }\lambda\text{ large.}\tag{4.7}$$

Meanwhile, for all $n\geqslant 1$ and $\lambda\leqslant\frac{1}{2}\sqrt{\frac{g}{L_0}}$,

$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)}\leqslant\frac{\lambda}{\gamma_n(k,\frac{1}{2}\sqrt{\frac{g}{L_0}},\mu)}\to 0\text{ as }\lambda\to 0.\tag{4.8}$$

In view of (4.7), (4.8) and Lemma 4.1, we obtain only one solution λ_n of (2.12) and (λ_n,ϕ_n) satisfies (1.4)-(1.5). That means for all n, λ_n is a characteristic value, hence it is bounded by $\sqrt{\frac{g}{L_0}}$.

We now prove that $(\lambda_n)_{n\geqslant 1}$ decreases towards 0 as $n\to\infty$. If $\lambda_m<\lambda_{m+1}$ for some $m\geqslant 1$, we have

$$\gamma_m(k,\lambda_m,\mu)>\gamma_m(k,\lambda_{m+1},\mu).$$

Meanwhile, we also have

$$\gamma_m(k,\lambda_{m+1},\mu)>\gamma_{m+1}(k,\lambda_{m+1},\mu).$$

That implies

$$\frac{\lambda_m}{gk^2}=\gamma_m(k,\lambda_m,\mu)>\gamma_{m+1}(k,\lambda_{m+1},\mu)=\frac{\lambda_{m+1}}{gk^2}.$$

That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence. Suppose that

$$\lim_{n\to\infty}\lambda_n=d_0>0.$$

Note that for all n, $\gamma_n(k,\lambda_n,\mu)=\frac{\lambda_n}{gk^2}$, then

$$\gamma_n(k,d_0,\mu)\geqslant\gamma_n(k,\lambda_n,\mu)=\frac{\lambda_n}{gk^2}.$$

Let $n\to\infty$, we get that $0\geqslant d_0$, a contradiction, i.e. λ_n decreases towards 0 as $n\to\infty$. The proof of Theorem 2.1 is complete. □

4.3. **Growing mode solutions of the linearized system.** We now solve the linearized equations (2.3) to prepare for our nonlinear part.

Proposition 4.2. *For each $k \in L^{-1}\mathbb{Z}\backslash\{0\}\backslash\{0\}$ and for all $\mu > \mu_c(k, \Xi)$, there exists an infinite sequence of solutions $(n \geqslant 1)$*

$$e^{\lambda_n(k,\mu)t}\vec{U}_n(\vec{x}) = e^{\lambda_n(k,\mu)t}(\sigma_n, \vec{u}_n, p_n)^T(\vec{x})$$
$$= e^{\lambda_n(k,\mu)t}\begin{pmatrix} \cos(kx_1)\omega_n(x_2) \\ \sin(kx_1)\theta_n(x_2) \\ \cos(kx_1)\phi_n(x_2) \\ \cos(kx_1)q_n(x_2) \end{pmatrix}$$

to the linearized equation (2.3)-(2.2), such that

$$\sigma_n \in H^2(\Omega), \vec{u}_n \in (H^3(\Omega))^2 \text{ and } p_n \in H^1(\Omega).$$

Proof. For each solution $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ of (2.12), we then have a solution $\phi_n = Y^{-1}_{k,\lambda_n,\mu}\mathcal{M}\varpi_n \in H^4((-1,1))$ of (1.4)-(1.5) as $\lambda = \lambda_n$. We then find a solution to the system (2.5) as $\lambda = \lambda_n$. Firstly, we obtain $\theta_n = -\frac{\phi_n'}{k}$ and $\omega_n = -\frac{\rho_0'\phi_n}{\lambda_n}$. Due to (2.7), we get

$$q_n = -\frac{1}{k^2}(\lambda_n\rho_0\phi_n' + \mu(k^2\phi_n' - \phi_n''')) \in H^1((-1,1)).$$

With a solution $(\omega_n, \theta_n, \phi_n, q_n)$ of (2.5), we then conclude that

$$e^{\lambda_n(k,\mu)t}(\sigma_n, u_{n,1}, u_{n,2}, p_n)^T(\vec{x})$$
$$= e^{\lambda_n(k,\mu)t}(\cos(kx_1)\omega_n(x_2), \sin(kx_1)\theta_n(x_2), \cos(kx_1)\phi_n(x_2), \cos(kx_1)q_n(x_2))^T$$

is a solution to the linearized equation (2.3)-(2.2). □

5. Nonlinear instability

5.1. **The local existence.** The first important things are the local existence of strong solutions to the nonlinear perturbed equation and a prior nonlinear energy estimates to those solutions. We restate Proposition 4.1 of [2].

Proposition 5.1. *Suppose that the steady state satisfies (1.3). Then for any given initial data $(\sigma_0, \vec{u}_0) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times (H^2(\Omega))^2$ satisfying $div\vec{u}_0 = 0$, and also being compatible with the boundary conditions (1.2), the nonlinear perturbed problem (2.1) has a local strong solution*

$$(\sigma, \vec{u}, \nabla q) \in C([0, T^{\max}), H^1(\Omega) \times (H^2(\Omega))^2 \times (L^2(\Omega)))^2. \tag{5.1}$$

Let $\mathcal{E}(t) := \sqrt{\|\sigma(t)\|^2_{H^1(\Omega)} + \|\vec{u}(t)\|^2_{H^2(\Omega)}}$ and $\delta_0 > 0$ be sufficiently small, we further get that if $\mathcal{E}(t) \leqslant \delta_0$, there holds

$$\begin{aligned} &\mathcal{E}^2(t) + \|(\nabla q, \vec{u}_t)\|^2_{L^2(\Omega)} + \int_0^t (\|\vec{u}_t(s)\|^2_{H^1(\Omega)} + \|\vec{u}(s)\|^2_{H^2(\Omega)})ds \\ &\qquad \leqslant C_0\Big(\mathcal{E}^2(0) + \int_0^t \|(\sigma, \vec{u})(s)\|^2_{L^2(\Omega)}ds\Big). \end{aligned} \tag{5.2}$$

Thanks to Proposition 4.2, we will formulate a sequence of approximate solutions $e^{\lambda_n(k,\mu)}\vec{U}_n(\vec{x})$ to the nonlinear perturbed problem (2.1)-(2.2), that are solutions to

the linearized one (2.3). Let us fix a $k = k_0$ such that (2.18) holds. For $\delta > 0$, we define

$$(\sigma^M, \vec{u}^M, q^M)(t, \vec{x}) := \delta \sum_{j=1}^{M} e^{\lambda_j(k,\mu)t} \vec{U}_j(\vec{x}). \tag{5.3}$$

Keeping in mind that $\min_{[-1,1]} \rho_0 > 0$, then due to the embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$, there exists a constant $\delta_0 > 0$ such that

$$\delta_0 \| \sum_{j \geqslant 1} \sigma_j(0, \vec{x}) \|_{L^\infty(\Omega)} > \frac{1}{2} \min_{[-1,1]} \rho_0(x_2). \tag{5.4}$$

Hence,

$$\frac{1}{2} \min_{[-1,1]} \rho_0(x_2) \leqslant \min_{\Omega} (\rho_0(x_2) + \sigma^M(0, \vec{x}))$$

for $\delta \leqslant \delta_0$. By virtue of Proposition 5.1, the perturbed problem (2.1)-(2.2) with initial data $(\sigma^M, \vec{u}^M, q^M)(0)$ admits a strong solution

$$(\sigma^\delta, \vec{u}^\delta) \in C^0([0, T^{\max}), H^1(\Omega) \times (H^2(\Omega))^2)$$

with an associated pressure $q^\delta \in C^0([0, T^{\max}), L^2(\Omega))$. Furthermore, we have

$$\frac{1}{2} \min_{[-1,1]} \rho_0(x_2) \leqslant \inf_{\Omega} (\rho_0(x_2) + \sigma^\delta(t, \vec{x}))$$

for all $t \in [0, T^{\max})$.

In what follows, the constants $C_i (i \geqslant 1)$ are universal ones depending only on physical parameters, M and $\mathsf{c}_j (j \geqslant 1)$.

Let $F_M(t) = \sum_{j=1}^{M} |\mathsf{c}_j| e^{\lambda_j t}$ and $0 < \epsilon_0 \ll 1$ be fixed later (5.45). There exists a unique T^δ such that $\delta F_M(T^\delta) = \epsilon_0$. Let

$$C_1 = \sqrt{\|\sigma^M(0)\|^2_{H^1(\Omega)} + \|\vec{u}^M(0)\|^2_{H^2(\Omega)}}, \quad C_2 = \sqrt{\|\sigma^M(0)\|^2_{L^2(\Omega)} + \|\vec{u}^M(0)\|^2_{L^2(\Omega)}}.$$

We define

$$\begin{aligned} T^\star &:= \sup \Big\{ t \in (0, T^{\max}) | \mathcal{E}(\sigma^\delta(t), \vec{u}^\delta(t)) \leqslant C_1 \delta_0 \} > 0, \\ T^{\star\star} &:= \sup \{ t \in (0, T^{\max}) | \|(\sigma^\delta, \vec{u}^\delta)(t)\|_{L^2(\Omega)} \leqslant 2 C_2 \delta F_M(t) \Big\} > 0. \end{aligned} \tag{5.5}$$

Note that $\mathcal{E}(\sigma^\delta(0), \vec{u}^\delta(0)) = C_1 \delta < C_1 \delta_0$ and because of (5.1), we then have $T^\star > 0$. Similarly, we have $T^{\star\star} > 0$. Then for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, it follows from (5.2) that

$$\mathcal{E}^2(\sigma^\delta(t), \vec{u}^\delta(t)) + \|\vec{u}^\delta_t(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla \vec{u}^\delta_t(\tau)\|^2_{L^2(\Omega)} d\tau \leqslant C_3 \delta^2 F_M^2(t). \tag{5.6}$$

5.2. **The difference functions.** Let

$$\sigma^d = \sigma^\delta - \sigma^M, \quad \vec{u}^d = \vec{u}^\delta - \vec{u}^M, \quad q^d = q^\delta - q^M.$$

Then $(\sigma^d, \vec{u}^d, q^d)$ satisfies

$$\begin{cases} \partial_t \sigma^d + \rho_0' u_2^d = -\vec{u}^\delta \cdot \nabla \sigma^\delta, \\ \rho_0 \partial_t \vec{u}^d - \mu \Delta \vec{u}^d + \nabla q^d = -\sigma^\delta \partial_t \vec{u}^\delta - (\rho_0 + \sigma^\delta) \vec{u}^\delta \cdot \nabla \vec{u}^\delta - g \sigma^d \vec{e}_2, \\ \mathrm{div} \vec{u}^d = 0. \end{cases} \tag{5.7}$$

The initial condition is

$$(\sigma^d, \vec{u}^d)(0) = 0 \tag{5.8}$$

and the boundary conditions are

$$\begin{cases} u_2^d = 0, & \text{on } \Sigma_{\pm}, \\ \mu\partial_{x_2} u_1^d = \xi_+ u_1^d & \text{on } \Sigma_+, \\ \mu\partial_{x_2} u_1^d = -\xi_- u_1^d & \text{on } \Sigma_-. \end{cases} \tag{5.9}$$

with compatibility conditions read as

$$u_1^d(0, x_1, -1) = u_1^d(0, x_1, 1), \quad \mathrm{div}\vec{u}^d(0) = 0. \tag{5.10}$$

We now establish the error estimate for $\|\vec{u}^d\|_{L^2(\Omega)}$.

Proposition 5.2. *For all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$, there holds*

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)} \leqslant C_4\delta^3\Big(\sum_{j=1}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \max_{N+1\leqslant j\leqslant M} |\mathsf{c}_j| e^{\frac{2}{3}\nu_0\Lambda t}\Big)^3. \tag{5.11}$$

The proof of Proposition 5.2 relies on Lemmas 5.1, 5.2, 5.3, 5.4 below.

Lemma 5.1. *We have the following inequalities*

$$\sum_{0\leqslant s\leqslant 2, 0\leqslant\tau\leqslant 1} \Big(\|\partial_t^\tau \vec{u}^M(t)\|_{H^s(\Omega)} + \|\partial_t^\tau \vec{u}^d(t)\|_{H^s(\Omega)}\Big) \leqslant C_5\delta F_M(t), \tag{5.12}$$

and

$$\|\sigma^d(t)\|_{H^1(\Omega)} + \|\sigma_t^d(t)\|_{L^2(\Omega)} \leqslant C_6\delta F_M(t). \tag{5.13}$$

Proof. For $\tau \in \{0, 1\}$,

$$\partial_t^\tau \vec{u}^M(t) = \sum_{j=1}^{M} \lambda_j^\tau \mathsf{c}_j e^{\lambda_j t} \vec{U}_j(\vec{x}),$$

it yields, for all $s \in \{0, 1, 2\}$,

$$\|\partial_t^\tau \vec{u}^M(t)\|_{H^s(\Omega)} \leqslant C_7\delta F_M(t).$$

In view of (5.6), we then obtain that for $s \in \{0, 1, 2\}$ and $\tau \in \{0, 1\}$,

$$\|\partial_t^\tau \vec{u}^d(t)\|_{H^s(\Omega)} \leqslant \|\partial_t^\tau \vec{u}^M(t)\|_{H^s(\Omega)} + \|\partial_t^\tau \vec{u}^\delta(t)\|_{H^s(\Omega)} \leqslant C_8\delta F_M(t).$$

To prove (5.13), we use $(5.7)_1$ and (5.6) again,

$$\begin{aligned} \|\sigma^d(t)\|_{H^1(\Omega)} + \|\sigma_t^d(t)\|_{L^2(\Omega)} &\leqslant \|\sigma^\delta(t)\|_{H^1(\Omega)} + \|\sigma^M(t)\|_{H^1(\Omega)} \\ &\quad + C_9\|u_2^d(t)\|_{L^2(\Omega)} + \|\vec{u}^\delta(t)\|_{L^2(\Omega)}\|\nabla\sigma^\delta\|_{L^2(\Omega)} \\ &\leqslant C_{10}\delta F_M(t). \end{aligned}$$

Lemma 5.1 follows. □

We derive the following lemma for λ_1.

Lemma 5.2. *The largest characteristic value λ_1 is the solution of the variational problem*

$$\frac{1}{gk^2} = \max_{\phi\in H^2((-1,1))} \frac{\int_{-1}^{1} \rho_0'|\phi|^2 dx_2}{\lambda_1 \mathcal{B}_{k,\lambda_1,\mu}(\phi, \phi)}, \tag{5.14}$$

where the bilinear form $\mathcal{B}_{k,\lambda_1,\mu}$ is defined as in (2.9) and ϕ_1 is an extremal function.

Proof. For all $\lambda > 0$, we solve the variational problem

$$\beta(k,\lambda,\mu) = \max\Big(\int_{-1}^{1} \rho_0'|\phi|^2 dx_2 \Big| \phi \in \tilde{H}^2((-1,1)), \quad \lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi) = 1\Big). \tag{5.15}$$

Let us define the Lagrangian functional

$$\mathcal{L}_{\mathcal{B}}(\phi,\beta(k,\lambda,\mu)) = \int_{-1}^{1} \rho_0'|\phi|^2 dx_2 - \beta(k,\lambda,\mu)(\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi) - 1). \tag{5.16}$$

Thanks to the Lagrange multiplier theorem, it suffices to find a stationary point $(\beta,\tilde{\phi})$ of $\mathcal{L}_{\mathcal{B}}$, that satisfies

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\tilde{\phi},\tilde{\phi}) = 1 \tag{5.17}$$

and

$$\int_{-1}^{1} \rho_0'\tilde{\phi}\theta dx_2 - \beta(k,\lambda,\mu)\lambda\mathcal{B}_{k,\lambda,\mu}(\tilde{\phi},\theta) = 0, \tag{5.18}$$

for all $\theta \in \tilde{H}^2((-1,1))$. Restricting $\theta \in C_0^\infty((-1,1))$ and following the line of Proposition 3.3, one deduces from (5.18) that $\tilde{\phi}$ has to satisfy

$$\beta(k,\lambda,\mu)\lambda Y_{k,\lambda,\mu}\tilde{\phi} = \rho_0'\tilde{\phi} \tag{5.19}$$

in a weak sense. We further get that $\tilde{\phi} \in H^4((-1,1))$ and satisfies (5.17) and the boundary conditions (1.5). Hence, if $(\beta(k,\lambda,\mu),\tilde{\phi})$ exists, $\lambda\beta(k,\lambda,\mu)$ is an eigenvalue of the operator $S_{k,\lambda,\mu}$ with $\tilde{\phi}$ being an associated eigenvector. That implies

$$\beta(k,\lambda,\mu) \leqslant \lambda^{-1}\gamma_1(k,\lambda,\mu). \tag{5.20}$$

Meanwhile, since the operator $S_{k,\lambda}$ is self-adjoint and positive, we thus obtain that

$$\gamma_1(k,\lambda,\mu) = \sup_{\omega\in L^2((-1,1))} \frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle}{\|\omega\|^2_{L^2((-1,1))}}.$$

Hence, for all $\omega \in L^2((-1,1))$ and for $\phi = Y^{-1}_{k,\lambda,\mu}\mathcal{M}\omega \in H^4((-1,1))$, we have

$$\gamma_1(k,\lambda,\mu)\langle Y_{k,\lambda,\mu}\phi,\phi\rangle \leqslant \frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle^2}{\|\omega\|^2_{L^2((-1,1))}} \leqslant \|S_{k,\lambda,\mu}\omega\|^2_{L^2((-1,1))}.$$

Equivalently,

$$\gamma_1(k,\lambda,\mu) \leqslant \sup\Big\{\frac{\|\mathcal{M}\phi\|^2_{L^2((-1,1))}}{\langle Y_{k,\lambda,\mu}\phi,\phi\rangle} | \phi \in H^4((-1,1)) \text{ and } \mathcal{M}^{-1}Y_{k,\lambda,\mu}\phi \in L^2((-1,1))\Big\},$$

it yields

$$\lambda^{-1}\gamma_1(k,\lambda,\mu) \leqslant \beta(k,\lambda,\mu). \tag{5.21}$$

(5.20) and (5.21) tell us that $\beta(k,\lambda,\mu) = \lambda^{-1}\gamma_1(k,\lambda,\mu)$ for all $\lambda > 0$, then the proof of Lemma 5.2 finishes. □

Lemma 5.3. *Let*

$$\mathsf{X} := \{\vec{w} \in (H^2(\Omega))^2, \vec{w} \text{ satisfies (1.2) and } div\vec{w} = 0\}.$$

There holds for all $\vec{w} \in H^2_\star(\Omega)$,

$$\begin{aligned}\int_\Omega g\rho_0'|w_2|^2 d\vec{x} + \Lambda\int_{(2\pi L\mathbb{T})^2}(\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2)dx_1 \\ \leqslant \Lambda^2\int_\Omega \rho_0|\vec{w}|^2 d\vec{x} + \Lambda\mu\int_\Omega |\nabla\vec{w}|^2 d\vec{x}.\end{aligned} \tag{5.22}$$

The proof of Lemma 5.3 is due to the definition of Λ (2.17) and Lemma 5.2, that is similar to [2, Lemma 5.1], hence we omit here.

Lemma 5.4. *There holds for all* $\vec{w} \in \mathsf{X}\backslash\{\vec{0}\}$,

$$\sup_{\vec{w}\in\mathsf{X}} \frac{\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1}{\|\nabla\vec{w}\|^2_{L^2(\Omega)}} \leqslant \mu_c(\Xi). \tag{5.23}$$

Proof. Let us fix a horizontal frequency $k \in L^{-1}\mathbb{Z}$ and introduce the horizontal Fourier transform

$$\hat{f}(k,x_2) = \int_{2\pi L\mathbb{T}} f(\vec{x})e^{-ikx_1}dx_1.$$

For $\vec{w} \in \mathsf{X}$, we write

$$\hat{w}_1(k,x_2) = -i\theta(k,x_2), \quad \hat{w}_2(k,x_2) = \phi(k,x_2).$$

Then, $k\theta+\phi' = 0$ and (θ,ϕ) enjoy (2.6). Following Fubini's and Parseval's theorem, one thus deduces

$$\begin{aligned}\int_{2\pi L\mathbb{T}}&(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &= \frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}}(\xi_+(|\theta(k,1)|^2+\xi_-|\theta(k,-1)|^2)\end{aligned} \tag{5.24}$$

and

$$\|\nabla\vec{w}\|^2_{L^2(\Omega)} = \frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}}\int_{-1}^{1}(k^2(|\theta|^2+|\phi|^2)+|\theta'|^2+|\phi'|^2)(k,x_2)dx_2. \tag{5.25}$$

For any $k \in L^{-1}\mathbb{Z}\backslash\{0\}$, we have from $k\theta+\phi' = 0$ that

$$\xi_+|\theta(k,1)|^2+\xi_-|\theta(k,-1)|^2 = \frac{1}{k^2}(\xi_+(|\phi'(k,1)|^2+\xi_-|\phi'(k,-1)|^2) \tag{5.26}$$

and that

$$\begin{aligned}\int_{-1}^{1}&(k^2(|\theta|^2+|\phi|^2)+|\theta'|^2+|\phi'|^2\Big)(k,x_2)dx_2\\ &= \frac{1}{k^2}\int_{-1}^{1}(k^4|\phi|^2+2k^2|\phi'|^2+|\phi''|^2)(k,x_2)dx_2.\end{aligned} \tag{5.27}$$

Owing to (5.24), (5.26) and the definition of $\mu_c(k,\Xi)$, we get

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &\quad\leqslant \frac{1}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to 0}\dfrac{1}{k^2}(\xi_+|\phi'(k,1)|^2+\xi_-|\phi'(k,-1)|^2)\\ +\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{1}{k^2}(\xi_+|\phi'(k,1)|^2+\xi_-|\phi'(k,-1)|^2)\end{array}\right)\\ &\quad\leqslant \frac{1}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to 0}\dfrac{\mu_c(k,\Xi)}{k^2}\displaystyle\int_{-1}^{1}(k^4|\phi|^2+2k^2|\phi'|^2+|\phi''|^2)(k,x_2)dx_2\\ +\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{\mu_c(k,\Xi)}{k^2}\int_{-1}^{1}(k^4|\phi|^2+2k^2|\phi'|^2+|\phi''|^2)(k,x_2)dx_2\end{array}\right).\end{aligned} \tag{5.28}$$

Thanks to Proposition 3.1, we obtain

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\&\leqslant\frac{\mu_c(\Xi)}{2\pi L}\left(\begin{array}{l}\limsup\limits_{k\to 0}\dfrac{1}{k^2}\displaystyle\int_{-1}^{1}(k^4|\phi|^2+2k^2|\phi'|^2+|\phi''|^2)(k,x_2)dx_2\\+\displaystyle\sum_{k\in L^{-1}\mathbb{Z}\setminus\{0\}}\frac{1}{k^2}\int_{-1}^{1}(k^4|\phi|^2+2k^2|\phi'|^2+|\phi''|^2)(k,x_2)dx_2\end{array}\right).\end{aligned}$$

Combining the resulting inequality and (5.25), (5.27) gives

$$\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\leqslant\mu_c(\Xi)\|\nabla\vec{w}\|^2_{L^2(\Omega)}.$$

This ends the proof. □

We now prove Proposition 5.2.

Proof of Proposition 5.2. Let us recall ϖ_0 from (2.16) and $\nu_0=\frac{3+\varpi_0}{2+\varpi_0}\in(1,\frac{3}{2})$. We first fix two positive constants $m_{1,2}$ such that

$$m_1=\nu_0+\sqrt{\nu_0^2-1}\tag{5.29}$$

and that

$$m_2=\mu(m_1^2-m_1+1)-\mu_c(\Xi)(m_1^2+1)+\sqrt{(\mu(m_1^2-m_1+1)-\mu_c(\Xi)(m_1^2+1))^2-\mu^2m_1^2}.\tag{5.30}$$

and then derive nonlinear energy estimates.

We now rewrite $(5.7)_2$ as

$$(\rho_0+\sigma^\delta)\partial_t\vec{u}^d-\mu\Delta\vec{u}^d+\nabla q^d=\vec{f}^\delta-g\sigma^d\vec{e}_2,$$

where $\vec{f}^\delta=-\sigma^\delta\partial_t\vec{u}^M-(\rho_0+\sigma^\delta)\vec{u}^\delta\cdot\nabla\vec{u}^\delta$. Differentiate the resulting equation with respect to t and then multiply by $\vec{u}_t^d$, we obtain after integration that

$$\begin{aligned}\int_\Omega\sigma_t^\delta|\vec{u}_t^d|^2d\vec{x}+\int_\Omega(\rho_0+\sigma^\delta)\vec{u}_{tt}^d\cdot\vec{u}_t^dd\vec{x}=\int_\Omega\mu\Delta\vec{u}_t^d\cdot\vec{u}_t^dd\vec{x}-\int_\Omega\nabla q_t^d\cdot\vec{u}_t^dd\vec{x}\\+\int_\Omega(\vec{f}_t^\delta-g\sigma_t^d\vec{e}_2)\cdot\vec{u}_t^dd\vec{x}.\end{aligned}$$

Since $\mathrm{div}\vec{u}_t^d=0$, we use the integration by parts to further obtain

$$\begin{aligned}&\int_\Omega\sigma_t^\delta(t)|\vec{u}_t^d(t)|^2d\vec{x}+\int_\Omega(\rho_0+\sigma^\delta(t))\vec{u}_{tt}^d(t)\cdot\vec{u}_t^d(t)d\vec{x}\\&=\int_\Omega(\vec{f}_t^\delta(t)-g\sigma_t^d(t)\vec{e}_2)\cdot\vec{u}_t^d(t)d\vec{x}-\mu\int_\Omega|\nabla\vec{u}_t^d(t)|^2d\vec{x}\\&\quad+\int_{2\pi L\mathbb{T}}(\xi_+|u_{1,t}^d(t,x_1,1)|^2+\xi_-|u_{1,t}^d(t,x_1,-1)|^2)dx_1.\end{aligned}$$

That means,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho_0+\sigma^\delta(t))|\vec{u}_t^d(t)|^2d\vec{x}\\&=-\frac{1}{2}\int_\Omega\sigma_t^\delta|\vec{u}_t^d|^2d\vec{x}+\int_\Omega(\vec{f}_t^\delta(t)-g\sigma_t^d(t)\vec{e}_2)\cdot\vec{u}_t^d(t)d\vec{x}-\mu\int_\Omega|\nabla\vec{u}_t^d(t)|^2d\vec{x}\\&\quad+\int_{2\pi L\mathbb{T}}(\xi_+|u_{1,t}^d(t,x_1,1)|^2+\xi_-|u_{1,t}^d(t,x_1,-1)|^2)dx_1.\end{aligned}$$

Using $(5.7)_1$, we then get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big((\rho_0+\sigma^\delta(t))|\vec{u}_t^d(t)|^2 - g\rho_0'|u_2^d(t)|^2\Big)d\vec{x}\\
&\quad + 2\mu\int_\Omega |\nabla\vec{u}_t^d(t)|^2 d\vec{x} - 2\int_{2\pi L\mathbb{T}} (\xi_+|u_{1,t}^d(t,x_1,1)|^2 + \xi_-|u_{1,t}^d(t,x_1,-1)|^2)dx_1\\
&= -\int_\Omega \sigma_t^\delta|\vec{u}_t^d|^2 d\vec{x} + 2\int_\Omega (\vec{f}_t^\delta(t) + g\vec{u}^\delta(t)\cdot\nabla\sigma^\delta(t)\vec{e}_2)\cdot\vec{u}_t^d(t)d\vec{x}.
\end{aligned}$$

Integrating in time variable, we get

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\nabla\vec{u}_t^d(s)\|_{L^2(\Omega)}^2 ds\\
&\quad - 2\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(s,x_1,1)|^2 + \xi_-|u_1^d(s,x_1,-1)|^2)dx_1 ds\\
&= \int_\Omega g\rho_0'|u_2^d(t)|^2 d\vec{x} + \Big(\int_\Omega(\rho_0+\sigma^\delta(t))|u_t^d(t)|^2 d\vec{x}\Big)\Big|_{t=0}\\
&\quad + \int_0^t\int_\Omega (2f_t^\delta(s) + 2g\vec{u}^\delta(s)\cdot\nabla\sigma^\delta(s)\vec{e}_2 - \sigma_t^\delta(s)\vec{u}_t^d(s))\cdot\vec{u}_t^d(s)ds.
\end{aligned} \tag{5.31}$$

We continue using (5.12), (5.13) and we can estimate each term of the r.h.s of (5.31) to obtain that

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\vec{u}_t^d(s)\|_{L^2(\Omega)}^2 ds\\
&\quad - 2\int_{2\pi L\mathbb{T}} (\xi_+|u_{1,t}^d(t,x_1,1)|^2 + \xi_-|u_{1,t}^d(t,x_1,-1)|^2)dx_1\\
&\leqslant \int_\Omega g\rho_0'|u_2^d(t)|^2 d\vec{x} + C_{11}\delta^3 F_M^3(t).
\end{aligned} \tag{5.32}$$

Due to (5.22), we further get that

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\nabla\vec{u}_t^d(s)\|_{L^2(\Omega)}^2 ds\\
&\quad - 2\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|u_{1,t}^d(s,x_1,1)|^2 + \xi_-|u_{1,t}^d(s,x_1,-1)|^2)dx_1 ds\\
&\leqslant \Lambda^2\int_\Omega \rho_0|\vec{u}^d(t)|^2 d\vec{x} + \Lambda\mu\int_\Omega |\nabla\vec{u}^d(t)|^2 d\vec{x}\\
&\quad - \Lambda\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1 + C_{11}\delta^3 F_M^3(t)\\
&\leqslant \Lambda^2\int_\Omega (\rho_0+\sigma^\delta(t))|\vec{u}^d(t)|^2 d\vec{x} + \Lambda\mu\int_\Omega |\nabla\vec{u}^d(t)|^2 d\vec{x}\\
&\quad - \Lambda\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1 + C_{12}\delta^3 F_M^3(t).
\end{aligned} \tag{5.33}$$

On the other hand,

$$\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2}^2 = 2\int_\Omega (\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\vec{u}_t^d(t)d\vec{x} + \int_\Omega \sigma_t^\delta(t)|\vec{u}^d(t)|^2 d\vec{x}.$$

With $m_1 > 0$ from (5.29), we use Young's inequality to observe

$$\begin{aligned}
&2\int_\Omega (\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\vec{u}^d_t(t)d\vec{x}\\
&\leqslant \frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d_t(t)\|^2_{L^2(\Omega)}+\Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

That will imply

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2}&\leqslant \frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d_t(t)\|^2_{L^2(\Omega)}\\
&\quad+\Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C_{13}\delta^3F^3_M(t).
\end{aligned}\tag{5.34}$$

With $m_2 > 0$ from (5.30), we obtain from (5.33) and (5.34) that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant\Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d\|^2_{L^2(\Omega)}\\
&\quad+\frac{2}{\Lambda m_1}\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u^d_{1,t}(s,x_1,1)|^2+\xi_-|u^d_{1,t}(s,x_1,-1)|^2)dx_1ds\\
&\quad-\frac{2\mu}{\Lambda m_1}\int_0^t\|\nabla\vec{u}^d_t(s)\|^2_{L^2(\Omega)}ds+C_{14}\delta^3F^3_M(t).
\end{aligned}$$

Thanks to Lemma 5.4, we deduce

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant\Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\quad-\frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\vec{u}^d_t(s)\|^2_{L^2(\Omega)}ds+C_{14}\delta^3F^3_M(t).
\end{aligned}\tag{5.35}$$

We use Young's inequality to get that

$$\begin{aligned}
\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}&=2\Big(\frac{\mu}{m_1}+m_2\Big)\int_0^t\int_\Omega\nabla\vec{u}^d(s)\cdot\nabla\vec{u}^d_t(s)d\vec{x}ds\\
&\leqslant\frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\vec{u}^d_t(s)\|^2_{L^2(\Omega)}ds\\
&\quad+\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds.
\end{aligned}\tag{5.36}$$

Combining (5.35) and (5.36) gives us

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\leqslant\Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}\\
&\quad+\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds+C_{14}\delta^3F^3_M(t).
\end{aligned}\tag{5.37}$$

It follows from (5.29) and (5.30) that

$$\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}=\Lambda\Big(m_1+\frac{1}{m_1}\Big)m_2=2\nu_0\Lambda m_2.$$

Therefore, (5.37) becomes

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\\&\leqslant 2\nu_0\Lambda\Big(\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\Big)+C_{14}\delta^3F_M^3(t).\end{aligned}\tag{5.38}$$

Recalling that $\vec{u}^d(0)=\vec{0}$, thus, applying Gronwall's inequality to (5.38), one obtains

$$\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+m_2\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds\leqslant C_{14}\delta^3e^{2\nu_0\Lambda t}\int_0^t e^{-2\nu_0\Lambda s}F_M^3(s)ds.\tag{5.39}$$

Since $F_M^3(t)\leqslant M^2\max_{1\leqslant j\leqslant M}|\mathsf{c}_j|^2F_M(3t)$, we then have from (5.39) that

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)}\leqslant C_{15}\delta^3e^{2\nu_0\Lambda t}\sum_{j=1}^M\int_0^t|\mathsf{c}_j|e^{(3\lambda_j-2\nu_0\Lambda)s}ds.\tag{5.40}$$

Because of (2.18), we have $\lambda_j>\frac{2}{3}\nu_0\Lambda$ for $1\leqslant j\leqslant N$ and $\lambda_j<\frac{2}{3}\nu_0\Lambda$ for $j\geqslant N+1$. It yields that for $1\leqslant j\leqslant N$,

$$\int_0^t e^{(3\lambda_j-2\nu_0\Lambda)s}ds=\frac{1}{3\lambda_j-2\nu_0\Lambda}(e^{(3\lambda_j-2\nu_0\Lambda)t}-1)\leqslant\frac{1}{3\lambda_j-2\nu_0\Lambda}e^{(3\lambda_j-2\nu_0\Lambda)t}\tag{5.41}$$

and that for $j\geqslant N+1$,

$$\int_0^t e^{(3\lambda_j-2\nu_0\Lambda)s}ds=\frac{1}{3\lambda_j-2\nu_0\Lambda}(e^{(3\lambda_j-2\nu_0\Lambda)t}-1)\leqslant\frac{1}{2\nu_0\Lambda-3\lambda_j}.\tag{5.42}$$

In view of (5.41) and (5.42), we obtain from (5.40) that if $M\leqslant N$,

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)}\leqslant C_{15}\delta^3\Big(\sum_{j=1}^M\frac{|\mathsf{c}_j|}{3\lambda_j-2\nu_0\Lambda}e^{3\lambda_jt}\Big)$$

and if $M\geqslant N+1$,

$$\|\vec{u}^d(t)\|^2_{L^2(\Omega)}\leqslant C_{15}\delta^3\Big(\sum_{j=1}^M\frac{|\mathsf{c}_j|}{3\lambda_j-2\nu_0\Lambda}e^{3\lambda_jt}+\sum_{j=N+1}^M|\mathsf{c}_j|e^{2\nu_0\Lambda}\Big).$$

Then, (5.11) follows. Proof of Proposition 5.2 is complete. □

5.3. **Proof of Theorem 2.2.** Note that

$$\|\vec{u}^M(t)\|^2_{L^2(\Omega)}=\delta^2\Big(\mathsf{c}_1^2\sum_{i=1}^M e^{2\lambda_it}\|\vec{u}_i\|^2_{L^2(\Omega)}+2\sum_{1\leqslant i<j\leqslant M}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega\vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}\Big).\tag{5.43}$$

It can be seen that

$$\|\vec{u}^M(t)\|^2_{L^2(\Omega)}\geqslant\delta^2\left(\begin{array}{l}\displaystyle\sum_{j=1}^M\mathsf{c}_j^2e^{2\lambda_jt}\|\vec{u}_j\|^2_{L^2(\Omega)}-|\mathsf{c}_1|\|\vec{u}_1\|_{L^2(\Omega)}\Big(\sum_{j=2}^M|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)e^{(\lambda_1+\lambda_2)t}\\\qquad\displaystyle+2\sum_{2\leqslant i<j\leqslant M}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega\vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}\end{array}\right).$$

By Cauchy-Schwartz inequality, we obtain

$$2\sum_{2\leqslant i<j\leqslant M}\mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega \vec{u}_i(\vec{x})\cdot\vec{u}_j(\vec{x})d\vec{x}\geqslant \sum_{2\leqslant i<j\leqslant M}|\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_i+\lambda_j)t}\|\vec{u}_i\|_{L^2(\Omega)}\|\vec{u}_j\|_{L^2(\Omega)}$$
$$\geqslant -e^{(\lambda_2+\lambda_3)t}\Big(\sum_{j=2}^M|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)^2.$$

That implies

$$\|\vec{u}^M(t)\|^2_{L^2(\Omega)}\geqslant\delta^2\left(\begin{array}{l}\displaystyle\sum_{j=1}^M\mathsf{c}_j^2e^{2\lambda_jt}\|\vec{u}_j\|^2_{L^2(\Omega)}-|\mathsf{c}_1|e^{(\lambda_1+\lambda_2)t}\|\vec{u}_1\|_{L^2(\Omega)}\Big(\displaystyle\sum_{j=2}^M|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)\\ \qquad\displaystyle-e^{(\lambda_2+\lambda_3)t}\Big(\sum_{j=2}^M|\mathsf{c}_j|\|\vec{u}_j\|_{L^2(\Omega)}\Big)^2\end{array}\right).$$

Due to the assumption (2.14), we deduce that

$$\begin{aligned}\|\vec{u}^M(t)\|^2_{L^2(\Omega)}&\geqslant\delta^2\left(\begin{array}{l}\displaystyle\sum_{j=1}^M\mathsf{c}_j^2e^{2\lambda_jt}\|\vec{u}_j\|^2_{L^2(\Omega)}-\frac{1}{2}\mathsf{c}_1^2e^{(\lambda_1+\lambda_2)t}\|\vec{u}_1\|^2_{L^2(\Omega)}\\ \qquad\displaystyle-\frac{1}{4}\mathsf{c}_1^2e^{(\lambda_2+\lambda_3)t}\|\vec{u}_1\|^2_{L^2(\Omega)}\end{array}\right)\\&\geqslant\delta^2\mathsf{c}_1^2\Big(e^{2\lambda_1t}-\frac{1}{2}e^{(\lambda_1+\lambda_2)t}-\frac{1}{4}e^{(\lambda_2+\lambda_3)t}\Big)\|\vec{u}_1\|^2_{L^2(\Omega)}\\&\quad+\delta^2\sum_{j=2}^M\mathsf{c}_j^2e^{2\lambda_jt}\|\vec{u}_j\|^2_{L^2(\Omega)}.\end{aligned}$$

Notice that for all $t\geqslant 0$,

$$e^{2\lambda_1t}-\frac{1}{2}e^{(\lambda_1+\lambda_2)t}-\frac{1}{4}e^{(\lambda_2+\lambda_3)t}\geqslant\frac{1}{4}e^{2\lambda_1t}.$$

Hence, we have

$$\|\vec{u}^M(t)\|_{L^2(\Omega)}\geqslant C_{16}\delta F_M(t),\tag{5.44}$$

for all $t\leqslant\min(T^\delta,T^\star,T^{\star\star})$.

Let

$$\tilde{\mathsf{c}}(M)=\max_{N+1\leqslant j\leqslant M}\frac{|\mathsf{c}_j|}{|\mathsf{c}_1|}\geqslant 0.$$

We recall the definition of $T^\star$ and $T^{\star\star}$ from (5.5) and the fact that T^δ satisfies uniquely $\delta F_M(T^\delta)=\epsilon_0$, provided that ϵ_0 is taken to be

$$\epsilon_0<\min\Big(\frac{C_2\delta_0}{C_3},\frac{C_2^2}{2C_4(1+M\tilde{\mathsf{c}}(M))^3},\frac{C_{16}^2}{4C_4(1+M\tilde{\mathsf{c}}(M))^3}\Big).\tag{5.45}$$

We then prove that

$$T^\delta\leqslant\min\{T^\star,T^{\star\star}\}.\tag{5.46}$$

In fact, if $T^\star<T^\delta$, we have by (5.6)

$$\mathcal{E}((\sigma^\delta,\vec{u}^\delta)(T^\star))\leqslant C_3\delta F_M(T^\star)\leqslant C_3\delta F_M(T^\delta)=C_3\epsilon_0<C_2\delta_0.$$

And if $T^{\star\star}<T^\delta$, we have by (5.11) and the definition of T^δ

$$\begin{aligned}&\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)}\\&\leqslant\|(\sigma^M,\vec{u}^M)(T^\delta)\|_{L^2(\Omega)}+\|(\sigma^d,\vec{u}^d)(T^\delta)\|_{L^2(\Omega)}\\&\leqslant C_2\delta F_M(T^\delta)+\sqrt{C_4}\delta^{\frac{3}{2}}\Big(\sum_{j=1}^N|\mathsf{c}_j|e^{\lambda_jT^\delta}+\max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda T^\delta}\Big)^{\frac{3}{2}}.\end{aligned}\tag{5.47}$$

Notice from (2.18) that for $N+1 \leqslant j \leqslant M$,

$$|\mathsf{c}_j|\delta e^{\frac{2}{3}\nu_0\Lambda T^\delta} < \frac{|\mathsf{c}_j|}{|\mathsf{c}_1|}(\delta|\mathsf{c}_1|e^{\lambda_1 T^\delta}) < \frac{|\mathsf{c}_j|}{|\mathsf{c}_1|}\delta F_M(T^\delta) = \frac{|\mathsf{c}_j|}{|\mathsf{c}_1|}\epsilon_0.$$

Then, it follows from (5.47) that

$$\begin{aligned}\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)} &\leqslant C_2\delta F_M(T^\delta) + \sqrt{C_4}\delta^{\frac{3}{2}}(1+M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}F_M^{\frac{3}{2}}(T^\delta)\\ &\leqslant C_2\epsilon_0 + \sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}\epsilon_0^{\frac{3}{2}}.\end{aligned}$$

Using (5.45) again, we deduce

$$\|(\sigma^\delta,\vec{u}^\delta)(T^\delta)\|_{L^2(\Omega)} < 2C_2\epsilon_0 = 2C_2\delta F_M(T^\delta).$$

which also contradicts the definition of $T^{\star\star}$.

Once we have (5.46), we then get from (5.11) and (5.44) that

$$\begin{aligned}&\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)}\\ &\geqslant \|\vec{u}^M(T^\delta)\|_{L^2(\Omega)} - \|\vec{u}^d(T^\delta)\|_{L^2(\Omega)}\\ &\geqslant C_{16}\delta F_M(T^\delta) - \sqrt{C_4}\delta^{\frac{3}{2}}\Big(\sum_{j=1}^{N}|\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda T^\delta}\Big)^{\frac{3}{2}}.\end{aligned}$$

Therefore,

$$\|\vec{u}^\delta(T^\delta)\|_{L^2(\Omega)} \geqslant C_{16}\epsilon_0 - \sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}\epsilon_0^{\frac{3}{2}} \geqslant \frac{C_{16}\epsilon_0}{2} > 0. \tag{5.48}$$

(2.19) is proven by taking δ_0 satisfying (5.4), ϵ_0 satisfying (5.45) and $m_0 = \frac{C_{16}}{2}$. This ends the proof of Theorem 2.2.

Acknowledgments

The author is deeply grateful to Prof. Jean-Marc Delort and Prof. Olivier Lafitte for their fruitful discussions on this paper. Thanks also go to Prof. Jeffrey Rauch for his advice on this study. The author wishes to thank Assoc. Prof. Quốc-Anh Ngô for his encouragement. The author would also thank the hospitality of Centre International de Rencontre Mathématique and Université de Montréal during the visit where parts of this work were accomplished. This work is supported by a grant from Région Île-de-France.

Appendix A. The precise value of $\mu_c(k,\Xi)$

In this appendix, we prove Proposition 3.1(1). (3.5) can be seen immediately from the definition of $\mathcal{B}_{k,0,\mu}$.

To prove (3.6), let us consider the Lagrangian functional

$$\mathcal{L}_k(\phi,\beta) = \int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2 - \beta(\xi_-|\phi'(-1)|^2+\xi_+|\phi'(1)|^2-1). \tag{A.1}$$

for any $\phi \in \tilde{H}^2((-1,1))$ and $\beta \neq 0$. Using Lagrange multiplier theorem again, we then find a stationary point of (ϕ_k,β_k) of $\mathcal{L}_k$, that satisfies

$$\xi_-|\phi_k'(-1)|^2 + \xi_+|\phi_k'(1)|^2 = 1, \tag{A.2}$$

and

$$\int_{-1}^{1}(\phi_k''\omega''+2k^2\phi_k'\omega'+k^4\phi_k\omega)dx_2 - \beta_k(\xi_-\phi_k'(-1)\omega'(-1)+\xi_+\phi_k'(1)\omega'(1)) = 0 \tag{A.3}$$

for all $\omega \in \tilde{H}^2((-1,1))$. We obtain from (A.3) after restricting $\omega \in C_0^\infty((-1,1))$ and taking integration by parts that

$$\phi_k^{(4)} - 2k^2\phi_k'' + k^4\phi_k = 0$$

on $(-1,1)$. Hence, ϕ_k is of the form

$$\phi_k(x) = (Ax+B)\sinh(kx) + (Cx+D)\cosh(kx),$$

with A, B, C, D are four constants. Since $\phi_k \in \tilde{H}^2((-1,1))$, we get

$$\begin{cases} (A+B)\sinh k + (C+D)\cosh k = 0, \\ (-A+B)\sinh(-k) + (-C+D)\cosh(-k) = 0. \end{cases}$$

It yields

$$C = -B\tanh(k) \quad \text{and} \quad D = -A\tanh(k). \tag{A.4}$$

We then compute

$$\phi_k'(x) = (A + kD + kCx)\sinh(kx) + (C + kB + kAx)\cosh(kx)$$

and

$$\phi_k''(x) = (2kC + k^2B + k^2Ax)\sinh(kx) + (2kA + k^2D + k^2Cx)\cosh(kx).$$

Substituting this form of ϕ_k into (A.3), we have that for all $\omega \in \tilde{H}^2((-1,1))$,

$$\begin{aligned} &\int_{-1}^1 \Big((2kC + k^2B + k^2Ax)\sinh(kx) + (2kA + k^2D + k^2Cx)\cosh(kx)\Big)\omega''(x)dx_2 \\ &\quad + 2k^2\int_{-1}^1 \Big((A + kD + kCx)\sinh(kx) + (C + kB + kAx)\cosh(kx)\Big)\omega'(x)dx_2 \\ &\quad + k^4\int_{-1}^1 \Big((Ax+B)\sinh(kx) + (Cx+D)\cosh(kx)\Big)\omega(x)dx_2 \\ &= \beta_k \begin{pmatrix} \xi_-\Big((A + k(D-C))\sinh k + (C + k(B-A))\cosh k\Big)\omega'(-1) \\ + \xi_+\Big((A + k(D+C))\sinh k + (C + k(B+A))\cosh k\Big)\omega'(1) \end{pmatrix}. \end{aligned}$$

Integrating by parts, we observe

$$\begin{cases} (2kC + k^2(B+A))\sinh k + (2kA + k^2(D+C))\cosh k \\ \qquad = \beta_k\xi_+\Big((A + k(D+C))\sinh k + (C + k(B+A))\cosh k\Big), \\ (2kC + k^2(B-A))\sinh(-k) + (2kA + k^2(D-C))\cosh(-k) \\ \qquad = -\beta_k\xi_-\Big((A + k(D-C))\sinh(-k) + (C + k(B-A))\cosh(-k)\Big). \end{cases}$$

With the help of (A.4), that reduces to

$$\begin{cases} 2k(C\sinh k + A\cosh k) \\ \quad = \beta_k\xi_+\Big(A + k(D+C)\sinh k + (C + k(B+A))\cosh k\Big), \\ 2k(-C\sinh k + A\cosh k) \\ \quad = \beta_k\xi_-\Big((A + k(D-C))\sinh(k) - (C + k(B-A))\cosh(k)\Big). \end{cases}$$

Equivalently,

$$\begin{cases} 2k\Big(A + B + (A-B)\cosh(2k)\Big) = \beta_k\xi_+\Big((A-B)\sinh(2k) + 2k(A+B)\Big), \\ 2k\Big(A - B + (A+B)\cosh(2k)\Big) = \beta_k\xi_-\Big((A+B)\sinh(2k) + 2k(A-B)\Big). \end{cases} \tag{A.5}$$

Then, $(A-B, A+B)$ is a solution of the system

$$\begin{pmatrix} \beta_k\xi_+\sinh(2k)-2k\cosh(2k) & 2k(\beta_k\xi_+-1) \\ 2k(\beta_k\xi_--1) & \beta_k\xi_-\sinh(2k)-2k\cosh(2k) \end{pmatrix}\begin{pmatrix} A-B \\ A+B \end{pmatrix}=0.$$

That admits a nontrivial solution if and only if

$$\begin{aligned}&\Big(\beta_k\xi_+\sinh(2k)-2k\cosh(2k)\Big)\Big(\beta_k\xi_-\sinh(2k)-2k\cosh(2k)\Big)\\ &\qquad=4k^2(\beta_k\xi_--1)(\beta_k\xi_+-1).\end{aligned}\tag{A.6}$$

We then consider three following cases

(1) $\xi_+>0=\xi_-$,
(2) $\xi_->0=\xi_+$,
(3) $\xi_\pm>0$.

The case $\xi_+>0=\xi_-$. We then have

$$\beta_k\xi_+\sinh(2k)-2k\cosh(2k)=\frac{2k}{\cosh(2k)}(\beta_k\xi_+-1).$$

Since $\sinh(4k)>4k$ for all $k>0$, we obtain

$$\frac{1}{\beta_k}=\xi_+\frac{\sinh(4k)-4k}{4k\sinh^2(2k)}.$$

Then we find a nontrivial (A,B,C,D) satisfying (A.5) and (A.4). From $(\text{A.5})_2$, we know that

$$A(1+\cosh(2k))=B(1-\cosh(2k)),\ \text{i.e.}\quad A=-B\tanh^2 k.$$

Then, $(A,B,C,D)=(-B\tanh^2 k, B, -B\tanh k, B\tanh^3 k)$ with $B\neq 0$. The constant B will be defined by (A.2), that satisfies

$$\xi_+B^2\Big((A+kD+kC)\sinh k+(C+kB+kA)\cosh k\Big)^2=1.$$

It yields

$$\xi_+B^2\cosh^2 k\Big(k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1)\Big)^2=1.$$

The direct computation shows that

$$\frac{d}{dk}\Big(k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1)\Big)=-\frac{4}{\cosh^4 k}(\sinh^2 k+k\tanh k)<0.$$

That yields $k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1)$ is a decreasing function on $k>0$. Hence,

$$k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1)<\lim_{k\searrow 0}\Big(k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1)\Big)=0.$$

for all $k>0$. Then

$$B=\pm\frac{1}{\sqrt{\xi_+}\cosh k(k(\tanh^2 k-1)^2-\tanh k(\tanh^2 k+1))}.$$

We have just shown that $\mu_c(k,\Xi)=\frac{1}{\beta_k}$, i.e.

$$\mu_c(k,\Xi)=\max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_+|\phi'(1)|^2}{\int_{-1}^1(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2}=\xi_+\frac{\sinh(4k)-4k}{4k\sinh^2(2k)}$$

and this variational problem is attained by functions

$$\phi_k(x) = \pm \frac{1}{\sqrt{\xi_+}\cosh k(k(\tanh^2 k-1)^2 - \tanh k(\tanh^2 k+1))} \times \Big((1-x\tanh^2 k)\sinh(kx) - \tanh k(1-x\tanh^2 k)\cosh(kx)\Big).$$

The case $\xi_- > 0 = \xi_+$.

We have the same arguments, that imply

$$\mu_c(k,\Xi) = \max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2}{\int_{-1}^1 (|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_2} = \xi_- \frac{\sinh(4k)-4k}{4k\sinh^2(2k)}$$

and that variational problem is attained by functions

$$\phi_k(x) = \pm \frac{1}{\sqrt{\xi_-}\cosh k(k(\tanh^2 k-1)^2 - \tanh k(\tanh^2 k+1))} \times \Big((1+x\tanh^2 k)\sinh(kx) - \tanh k(1+x\tanh^2 k)\cosh(kx)\Big).$$

The case $\xi_+\xi_- > 0$.

Clearly $\beta_k = 0$ is not a solution of (A.6). Eq. (C.7) rewrites as a quadratic equation of $\frac{1}{\beta_k}$, that is

$$\begin{aligned}(\sinh^2(2k) - 4k^2)\xi_+\xi_- - 2k(\sinh(2k)\cosh(2k) - 2k)(\xi_+ + \xi_-)\frac{1}{\beta_k} \\ + 4k^2(\cosh^2(2k)-1)\frac{1}{\beta_k^2} = 0.\end{aligned} \tag{A.7}$$

The discriminant is

$$\begin{aligned}\Delta_{k,\Xi} &= k^2(\sinh(2k)\cosh(2k)-2k)^2(\xi_+ + \xi_-)^2 \\ &\quad - 4k^2(\cosh^2(2k)-1)(\sinh^2(2k)-4k^2)\xi_+\xi_- \\ &= k^2(\sinh(2k) - 2k\cosh(2k))^2(\xi_+ + \xi_-)^2 \\ &\quad + k^2\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+ - \xi_-)^2.\end{aligned}$$

Because $\tanh(2k) < 2k$ for all $k > 0$ and $\xi_+\xi_- \neq 0$, $\Delta_{k,\xi}$ is always positive. Hence, (A.7) has two roots

$$\frac{1}{\beta_{k,\pm}} = \frac{k(\sinh(2k)\cosh(2k)-2k)(\xi_+ + \xi_-) \pm \sqrt{\Delta_{k,\Xi}}}{4k^2\sinh^2(2k)}.$$

We take the higher value $\frac{1}{\beta_{k,+}}$, that is always positive and then solve the system (A.5) as $\beta_k = \beta_{k,+}$. (A.5) implies that

$$\begin{cases} A\Big(2k(1+\cosh(2k)) - \beta_{k,+}\xi_+(2k+\sinh(2k))\Big) \\ \quad = B\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)\Big), \\ A\Big(2k(1+\cosh(2k)) - \beta_{k,+}\xi_-(2k+\sinh(2k))\Big) \\ \quad = -B\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_-(\sinh(2k)-2k)\Big), \end{cases} \tag{A.8}$$

If $0 < \xi_+ \leqslant \xi_-$,

$$\begin{aligned}
&4\cosh^2 k\Big(\frac{2k(\cosh(2k)-1)}{\beta_{k,+}} - \xi_+(\sinh(2k)-2k)\Big)\\
&= \frac{1}{2}(\sinh(4k)-4k)(\xi_- - \xi_+) - 2(\sinh(2k)-2k\cosh(2k))\xi_+ + \frac{1}{k}\sqrt{\Delta_{k,\Xi}}\\
&> 0,
\end{aligned}$$

Then, we obtain from $(A.8)_1$ that

$$B = A\frac{2k(1+\cosh(2k)) - \beta_{k,+}\xi_+(2k+\sinh(2k))}{2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)} =: Aa_{k,\Xi}. \tag{A.9}$$

So that $(A,B,C,D) = A(1, a_{k,\Xi}, -a_{k,\Xi}\tanh k, -\tanh k)$ with $A \neq 0$. We compute that

$$\begin{aligned}
\phi_k'(-1) &= -A(1-k\tanh k + ka_{k,\Xi}\tanh k)\sinh k + A(-a_{k,\Xi}\tanh k + ka_{k,\Xi} - k)\cosh k\\
&= -\frac{A}{2\cosh k}\Big(a_{k,\Xi}(\sinh(2k)-2k) + \sinh(2k) + 2k\Big)\\
&= -\frac{A}{\cosh k}\times\frac{k(\sinh(4k)-4k) - \beta_{k,+}\xi_+(\sinh^2(2k)-4k^2)}{2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)}
\end{aligned} \tag{A.10}$$

and that

$$\begin{aligned}
\phi_k'(1) &= A(1-k\tanh k - ka_{k,\Xi}\tanh k)\sinh k + (-a_{k,\Xi}\tanh k + ka_{k,\Xi} + k)\cosh k\\
&= \frac{A}{2\cosh k}\Big(-a_{k,\Xi}(\sinh(2k)-2k) + \sinh(2k) + 2k\Big)\\
&= -\frac{A}{\cosh k}\times\frac{2k(\sinh(2k)-2k\cosh(2k))}{2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)}.
\end{aligned} \tag{A.11}$$

Substituting (A.10) and (A.11) into (A.2), we obtain

$$\begin{aligned}
&A^2\xi_-\Big(k(\sinh(4k)-4k)\beta_{k,+}\xi_+ - 4k^2\sinh^2(2k)\Big)^2 + 4A^2\xi_+k^2(\sinh(2k)-2k\cosh(2k))^2\\
&= \cosh^2 k\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)\Big)^2.
\end{aligned} \tag{A.12}$$

Owing to (A.7), (A.12) becomes

$$\begin{aligned}
&A^2\Big(k(\sinh(4k)-4k)\beta_{k,+}\xi_+ - 4k^2\sinh^2(2k)\Big)\Big((\sinh^2(2k)-4k^2)\beta_{k,+}\xi_+ - k(\sinh(4k)-4k)\Big)\\
&\quad + 4A^2\beta_{k,+}\xi_+k^2(\sinh(2k)-2k\cosh(2k))^2\\
&= \beta_{k,+}\cosh^2 k\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)\Big)^2.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&kA^2\begin{pmatrix}(\sinh(4k)-4k)(\sinh^2(2k)-4k^2)(\beta_{k,+}\xi_+)^2\\ -8k\sinh^2(2k)(\sinh^2(2k)-4k^2)\beta_{k,+}\xi_+ + 4k^2\sinh^2(2k)(\sinh(4k)-4k)\end{pmatrix}\\
&= \beta_{k,+}\cosh^2 k\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)\Big)^2.
\end{aligned}$$

Notice that

$$\begin{aligned}
&(\sinh(4k)-4k)(\sinh^2(2k)-4k^2)(\beta_{k,+}\xi_+)^2 - 8k\sinh^2(2k)(\sinh^2(2k)-4k^2)\beta_{k,+}\xi_+ \\
&\quad + 4k^2\sinh^2(2k)(\sinh(4k)-4k) \\
&= (\sinh^2(2k)-4k^2)\Big(\sqrt{\sinh(4k)-4k}\beta_{k,+}\xi_+ - \frac{4k\sinh^2(2k)}{\sqrt{\sinh(4k)-4k}}\Big)^2 \\
&\quad + \frac{16k^2\sinh^2(2k)(\sinh^2(2k)-4k^2)(\sinh(2k)-2k\cosh(2k))^2}{\sinh(4k)-4k} \\
&> 0.
\end{aligned}$$

Then,

$$\begin{aligned}
A = \pm\sqrt{\beta_{k,+}}\cosh k\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_+(\sinh(2k)-2k)\Big) \\
\times k^{-\frac{1}{2}}\begin{pmatrix}(\sinh(4k)-4k)(\sinh^2(2k)-4k^2)(\beta_{k,+}\xi_+)^2 \\ -8k\sinh^2(2k)(\sinh^2(2k)-4k^2)\beta_{k,+}\xi_+ \\ +4k^2\sinh^2(2k)(\sinh(4k)-4k)\end{pmatrix}^{-\frac{1}{2}}.
\end{aligned} \tag{A.13}$$

We have just shown that as $0<\xi_+\leqslant\xi_-$,

$$\begin{aligned}
\mu_c(k,\Xi) &= \max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-|\phi'(-1)|^2+\xi_+|\phi'(1)|^2}{\int_{-1}^1(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2} \\
&= \frac{1}{4k\sinh^2(2k)}\begin{pmatrix}(\sinh(2k)\cosh(2k)-2k)(\xi_++\xi_-) \\ +\begin{pmatrix}(\sinh(2k)-2k\cosh(2k))^2(\xi_++\xi_-)^2 \\ +\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+-\xi_-)^2\end{pmatrix}^{\frac{1}{2}}\end{pmatrix}.
\end{aligned}$$

That variational problem is attained by functions

$$\phi_k(x) = A(x+a_{k,\Xi})\sinh(kx) - A(a_{k,\Xi}x+1)\tanh k\cosh(kx),$$

where $a_{k,\Xi}$ and A are from (A.9) and (A.13), respectively.

If $0<\xi_-<\xi_+$, that will imply

$$\frac{2k(\cosh(2k)-1)}{\beta_{k,+}} - \xi_-(\sinh(2k)-2k) > 0.$$

We further get from $(\text{A.8})_2$ that

$$B = -A\frac{2k(1+\cosh(2k)) - \beta_{k,+}\xi_-(2k+\sinh(2k))}{2k(\cosh(2k)-1)-\beta_{k,+}\xi_-(\sinh(2k)-2k)} =: -Ab_{k,\Xi}.$$

Mimicking the above arguments, we also deduce

$$\begin{aligned}
\mu_c(k,\Xi) &= \max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-|\phi'(-1)|^2+\xi_+|\phi'(1)|^2}{\int_{-1}^1(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2} \\
&= \frac{1}{4k\sinh^2(2k)}\begin{pmatrix}(\sinh(2k)\cosh(2k)-2k)(\xi_++\xi_-) \\ +\begin{pmatrix}(\sinh(2k)-2k\cosh(2k))^2(\xi_++\xi_-)^2 \\ +\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+-\xi_-)^2\end{pmatrix}^{\frac{1}{2}}\end{pmatrix}.
\end{aligned}$$

and that variational problem is attained by functions

$$\phi_k(x) = A(x-b_{k,\Xi})\sinh(kx) + A(b_{k,\Xi}x-1)\tanh k\cosh(kx),$$

where

$$A = \pm\sqrt{\beta_{k,+}}\cosh k\Big(2k(\cosh(2k)-1) - \beta_{k,+}\xi_-(\sinh(2k)-2k)\Big) \times k^{-\frac{1}{2}}\begin{pmatrix}(\sinh(4k)-4k)(\sinh^2(2k)-4k^2)(\beta_{k,+}\xi_-)^2\\ -8k\sinh^2(2k)(\sinh^2(2k)-4k^2)\beta_{k,+}\xi_-\\ +4k^2\sinh^2(2k)(\sinh(4k)-4k)\end{pmatrix}^{-\frac{1}{2}}.$$

Combining all above cases, (3.6) is shown and the proof of the first part of Proposition 3.1 then follows.

Appendix B. Asymptotic behavior of $\mu_c(k,\Xi)$ in low/high regime of wave number

Let us prove Proposition 3.1(2). Clearly, we have that $\mu_c(k,\Xi)$ is a decreasing function in $k>0$. It yields (3.7).

We first consider $k\to 0$. Let us recall the Taylor's expansion of $\sinh(2k)$ and $\cosh(2k)$. We have

$$\sinh(2k) = 2k + \frac{4}{3}k^3 + \frac{4}{15}k^5 + O(k^6), \quad \text{and} \quad \cosh(2k) = 1 + 2k^2 + \frac{2}{3}k^4 + O(k^5).$$

We deduce that

$$\frac{\sinh(2k)\cosh(2k)-2k}{4k\sinh^2(2k)} = \frac{\frac{1}{6}+\frac{2}{15}k^2+O(k^3)}{\frac{1}{2}+\frac{2}{3}k^2+O(k^3)} = \frac{1}{3} - \frac{8}{15}k^2 + O(k^3),$$

that

$$\frac{\sinh(2k)-2k\cosh(2k)}{4k\sinh^2(2k)} = \frac{-\frac{8}{3}-\frac{16}{15}k^2+O(k^3)}{16+\frac{64}{3}k^2+O(k^3)} = -\frac{1}{6} + \frac{7}{45}k^2 + O(k^3)$$

and that

$$\frac{\sinh^2(2k)(\sinh^2(2k)-4k^2)}{16k^2\sinh^4(2k)} = \frac{\frac{16}{3}+\frac{128}{45}k^2+O(k^2)}{64+\frac{256}{3}k^2+O(k^2)} = \frac{1}{12} - \frac{1}{15}k^2 + O(k^3).$$

We deduce that

$$\lim_{k\to 0}\mu_c(k,\Xi) = \frac{1}{3}(\xi_+ + \xi_-) + \sqrt{\frac{1}{36}(\xi_+ + \xi_-)^2 + \frac{1}{12}(\xi_+ - \xi_-)^2}. \tag{B.1}$$

That will imply (3.9), i.e.

$$\mu_c^s(\Xi) = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

Furthermore, we have that

$$\lim_{k\to 0}\frac{\mu_c(k,\Xi)-\mu_c^s(\Xi)}{k^2} = -\frac{2}{15}\Big(4(\xi_+ + \xi_-) + \frac{4\xi_+^2 - \xi_+\xi_- + 4\xi_-^2}{\sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}}\Big). \tag{B.2}$$

(B.1) and (B.2) help us to get (3.8).

For high wave number, i.e. $k\to+\infty$, we can see that

$$\frac{\sinh(2k)\cosh(2k)-2k}{\sinh^2(2k)} = \frac{1-e^{-8k}-8ke^{-4k}}{1+e^{-8k}-2e^{-4k}} \leqslant 2,$$

that

$$\frac{\sinh(2k)-2k\cosh(2k)}{\sinh^2(2k)} = \frac{1}{2}\frac{1-2k-(1+2k)e^{-4k}}{e^{2k}+e^{-6k}-2e^{-2k}} \leqslant 1.$$

Hence,

$$\mu_c(k,\Xi) = \frac{1}{4k}\left(\begin{array}{l}\dfrac{\sinh(2k)\cosh(2k)-2k}{\sinh^2(2k)}(\xi_+ + \xi_-) \\ + \left(\begin{array}{l}\Big(\dfrac{\sinh(2k)-2k\cosh(2k)}{\sinh^2(2k)}\Big)^2(\xi_+ + \xi_-)^2 \\ + \Big(1 - \dfrac{4k^2}{\sinh^2(2k)}\Big)(\xi_+ - \xi_-)^2\end{array}\right)^{\frac{1}{2}}\end{array}\right)$$
$$\leqslant \frac{1}{4k}\Big(2(\xi_+ + \xi_-) + \sqrt{2(\xi_+^2 + \xi_-^2)}\Big).$$

That implies (3.10). The proof of the second assertion of Proposition 3.1 is complete.

Appendix C. Proof of Proposition 3.1(3)

In this appendix, we prove Proposition 3.1(3). We first show that

$$\mu_c^s(\Xi) = \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} \tag{C.1}$$

Indeed, we write

$$\tilde{\mu}_c(\Xi) = \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2}$$

and then prove that $\mu_c^s(\Xi) = \tilde{\mu}_c(\Xi)$. Clearly, we have $\mu_c(k,\Xi) \leqslant \tilde{\mu}_c(\Xi)$ for all $k \in \mathbf{R}\backslash\{0\}$. It yields $\mu_c^s(\Xi) \leqslant \tilde{\mu}_c(\Xi)$. It suffices to show that $\tilde{\mu}_c(\Xi) \geqslant \mu_c^s(\Xi)$. For any $\varepsilon > 0$, we fix a function $\phi_\varepsilon \in \tilde{H}^2((-1,1))$ such that

$$\frac{\xi_-|\phi_\varepsilon'(-1)|^2 + \xi_+|\phi_\varepsilon'(1)|^2}{\int_{-1}^1 |\phi_\varepsilon''|^2 dx_2} \geqslant \tilde{\mu}_c(\Xi) - \varepsilon.$$

Let $k \neq 0$ be small enough, we then obtain

$$\frac{\xi_-|\phi_\varepsilon'(-1)|^2 + \xi_+|\phi_\varepsilon'(1)|^2}{\int_{-1}^1 (|\phi_\varepsilon''|^2 + 2k^2|\phi_\varepsilon'|^2 + k^4|\phi_\varepsilon|^2)dx_2} > \frac{\xi_-|\phi_\varepsilon'(-1)|^2 + \xi_+|\phi_\varepsilon'(1)|^2}{\int_{-1}^1 |\phi_\varepsilon''|^2 dx_2} - \varepsilon.$$

That implies

$$\mu_c(k,\Xi) > \tilde{\mu}_c(\Xi) - 2\varepsilon.$$

We deduce that $\tilde{\mu}_c(\Xi) = \sup_{k\in\mathbf{R}\backslash\{0\}} \mu_c(k,\Xi)$, i.e. (C.1).

Then, we show that

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \tag{C.2}$$

Let us consider the Lagrangian functional

$$\mathcal{L}_0(\phi,\beta) = \int_{-1}^1 |\phi''|^2 dx_2 - \beta(\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2 - 1). \tag{C.3}$$

for any $\phi \in \tilde{H}^2((-1,1))$ and $\beta \neq 0$. Owing to Lagrange multiplier theorem, we have to find a stationary point (ϕ_0,β_0) that maximizes $\mathcal{L}_0$, that satisfies

$$\xi_-|\phi_0'(-1)|^2 + \xi_+|\phi_0'(1)|^2 = 1, \tag{C.4}$$

and

$$\int_{-1}^1 \phi_0''\omega'' dx_2 - \beta_0(\xi_-\phi_0'(-1)\omega'(-1) + \xi_+\phi_0'(1)\omega'(1)) = 0 \tag{C.5}$$

for all $\omega \in \tilde{H}^2((-1,1))$. We obtain from (C.5) after restricting $\omega \in C_0^\infty((-1,1))$ and taking integration by parts that

$$\phi_0^{(4)} = 0 \quad \text{on } (-1,1).$$

Hence, ϕ_0 is of the form

$$\phi_0(x) = (x^2-1)(Ax+B).$$

Substituting this form of ϕ_0 into (C.5), we have that for all $\omega \in \tilde{H}^2((-1,1))$,

$$\int_{-1}^{1} (6Ax+2B)\omega''(x)dx = \beta_0\Big(\xi_-(2A-2B)\omega'(-1) + \xi_+(2A+2B)\omega'(1)\Big).$$

Integrating by parts, we observe

$$\begin{cases} 3A+B = \beta_0\xi_+(A+B), \\ 3A-B = \beta_0\xi_-(A-B). \end{cases}$$

Hence,

$$\begin{cases} A(3-\beta_0\xi_+) + B(1-\beta_0\xi_+) = 0, \\ A(3-\beta_0\xi_-) - B(1-\beta_0\xi_-) = 0. \end{cases} \tag{C.6}$$

That system admits a nontrivial solution (A,B) if and only if

$$(3-\beta_0\xi_+)(1-\beta_0\xi_-) + (3-\beta_0\xi_-)(1-\beta_0\xi_+) = 0.$$

It yields

$$\xi_-\xi_+\beta_0^2 - 2(\xi_+ + \xi_-)\beta_0 + 3 = 0. \tag{C.7}$$

We split our arguments into three following cases,

(1) $\xi_+ > 0 = \xi_-$,

(2) $\xi_- > 0 = \xi_+$,

(3) $\xi_\pm > 0$,

The case $\xi_+ > 0 = \xi_-$. We then get $\beta_0 = \frac{3}{2\xi_+}$ and $B = 3A$. So that

$$\phi_0(x) = A(x^2-1)(x+3) \quad (A \neq 0).$$

Plugging into (C.4), we obtain that $64\xi_+A^2 = 1$, i.e. $A = \pm\frac{1}{8\sqrt{\xi_+}}$. It then follows that

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} = \frac{1}{\beta_0} = \frac{2}{3}\xi_+$$

and that variational problem is attained by functions

$$\phi_0(x) = \pm\frac{1}{8\sqrt{\xi_+}}(x^2-1)(x+3).$$

The case $\xi_- > 0 = \xi_+$. We then get $\beta_0 = \frac{3}{2\xi_-}$ and $B = -3A$. By a similar computation to the previous case, we have

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} = \frac{1}{\beta_0} = \frac{2}{3}\xi_-$$

and that variational problem is attained by functions

$$\phi_0(x) = \pm\frac{1}{8\sqrt{\xi_-}}(x^2-1)(x-3)$$

The case $\xi_+\xi_- > 0$. (C.7) rewrites as a quadratic equation of $\frac{1}{\beta_0}$, that is

$$\frac{3}{\beta_0^2} - \frac{2(\xi_+ + \xi_-)}{\beta_0} + \xi_+\xi_- = 0. \tag{C.8}$$

The discriminant of (C.8) is

$$\Delta_{0,\xi} = (\xi_+ + \xi_-)^2 - 3\xi_-\xi_+ = \xi_+^2 - \xi_+\xi_- + \xi_-^2 > 0.$$

Then, (C.8) has two roots

$$\frac{1}{\beta_{0,\pm}} = \frac{1}{3}\Big(\xi_+ + \xi_- \pm \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

We take the higher value $\frac{1}{\beta_{0,+}}$. As $\beta_0 = \beta_{0,+}$, we have from $(C.6)_2$ that

$$A(3 - \beta_{0,+}\xi_-) = B(1 - \beta_{0,+}\xi_-).$$

It is obvious that

$$\frac{3}{\beta_{0,+}} - \xi_- = \xi_+ + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2} > 0.$$

It yields

$$A = B\frac{1 - \beta_{0,+}\xi_-}{3 - \beta_{0,+}\xi_-}.$$

We continue using (C.4) to find a non-zero B. One deduces

$$4\xi_-|A - B|^2 + 4\xi_+|A + B|^2 = 1.$$

That implies

$$B^2\Big(\xi_- + \xi_+(2 - \beta_{0,+}\xi_-)^2\Big) = \frac{1}{16}(3 - \beta_{0,+}\xi_-)^2. \tag{C.9}$$

Due to $\beta_{0,+}\xi_+(\beta_{0,+}\xi_- - 2) + 3 - 2\beta_{0,+}\xi_- = 0$, (C.9) is equivalent to

$$B^2\Big((\beta_{0,+}\xi_-)^2 - 3\beta_{0,+}\xi_- + 3\Big) = \frac{1}{32}\beta_{0,+}(3 - \beta_{0,+}\xi_-)^2.$$

We then find that

$$B = \pm\frac{\sqrt{\beta_{0,+}}(3 - \beta_{0,+}\xi_-)}{2\sqrt{2}\sqrt{(\beta_{0,+}\xi_-)^2 - 3\beta_{0,+}\xi_- + 3}} \neq 0.$$

That means, we have proved that

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-|\phi'(-1)|^2 + \xi_+|\phi'(1)|^2}{\int_{-1}^1 |\phi''|^2 dx_2} = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

That variational problem is attained by functions

$$\phi_0(x) = \pm\frac{\sqrt{\beta_{0,+}}(3 - \beta_{0,+}\xi_-)}{2\sqrt{2}\sqrt{(\beta_{0,+}\xi_-)^2 - 3\beta_{0,+}\xi_- + 3}}(x^2 - 1)\Big(\frac{1 - \beta_{0,+}\xi_-}{3 - \beta_{0,+}\xi_-}x + 1\Big).$$

We sum up the previous three cases to have (C.2). Combining (C.1) and (C.2), we obtain Proposition 3.1(3).

Appendix D. Comments on paper of Ding, Zi and Li

In [2], the authors Ding, Zi and Li construct an approximate solution generated by the maximal growing mode, $(\sigma^a,\vec{u}^a,q^a)(t,x)=\delta e^{\lambda_1(k)t}\vec{U}_1(\vec{x})$ with k being fixed such that $\frac{2\Lambda}{3}<\lambda_1(k)<\Lambda$. Applying Proposition 5.1, the perturbed problem (2.1)-(2.2) with the initial data

$$(\sigma^\delta,\vec{u}^\delta,q^\delta)(0)=(\sigma^d,\vec{u}^d,q^d)(0).$$

admits a strong solution $(\sigma^\delta,\vec{u}^\delta)\in C^0([0,T^{\max}),H^1\times H^2)$ with an associated pressure $q^\delta\in C^0([0,T^{\max}),L^2)$. Let T^δ such that $\delta e^{\lambda_1 T^\delta}=\epsilon_0\ll 1$. We define

$$\begin{aligned}T^\star&:=\sup\Big\{t\in(0,T^{\max})|\mathcal{E}(\sigma^\delta(t),\vec{u}^\delta(t))\leqslant C\delta_0\}>0,\\ T^{\star\star}&:=\sup\{t\in(0,T^{\max})|\|(\sigma^\delta,\vec{u}^\delta)(t)\|_{L^2(\Omega)}\leqslant C\delta e^{\lambda_1 t}\Big\}>0.\end{aligned}$$

Then for all $t\leqslant\min\{T^\delta,T^\star,T^{\star\star}\}$, we have

$$\mathcal{E}^2(\sigma^\delta(t),\vec{u}^\delta(t))+\|\vec{u}^\delta_t(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla\vec{u}^\delta_t(\tau)\|^2_{L^2(\Omega)}d\tau\leqslant C\delta^2e^{2\lambda_1 t}.$$

In [2, Proposition 5.2], they claim that the difference functions

$$(\sigma^d,\vec{u}^d,q^d)=(\sigma^\delta,\vec{u}^\delta,q^\delta)-(\sigma^a,\vec{u}^a,q^a)$$

enjoy

$$\|(\sigma^d,\vec{u}^d)\|^2_{L^2(\Omega)}\leqslant C\delta^3e^{3\lambda_1 t}\tag{D.1}$$

for all $\mu>0$. We believe that (D.1) needs to be corrected, not for all $\mu>0$. Precisely, we are in doubt about inequality **(137)** in that paper, that is for all $t\leqslant\min\{T^\delta,T^\star,T^{\star\star}\}$,

$$\begin{aligned}&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d_t(t)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}+\mu\int_0^t\|\nabla\vec{u}^d_t(s)\|^2_{L^2(\Omega)}ds\\&\leqslant\Lambda\Big(\int_0^t\|\sqrt{\rho_0+\sigma^\delta(s)}\vec{u}^d(s)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}\Big)\\&\qquad+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C\delta^3e^{3\lambda_1 t}.\end{aligned}\tag{D.2}$$

Due to (D.2) and the following inequality

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}&=2\int_\Omega(\rho_0+\sigma^\delta(t))\vec{u}^d(t)\cdot\vec{u}^d_t(t)d\vec{x}+\int_\Omega\sigma^\delta_t(t)|\vec{u}^d(t)|^2d\vec{x}\\&\leqslant\frac{1}{\Lambda}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d_t(t)\|^2_{L^2(\Omega)}\\&\qquad+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C\delta^3e^{3\lambda_1 t},\end{aligned}\tag{D.3}$$

it is claimed in [2, **(138)**] that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+\Big(\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d_t(t)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(t)\|^2_{L^2(\Omega)}\Big)\\&\leqslant\Lambda\int_0^t\Big(\|\sqrt{\rho_0+\sigma^\delta(s)}\vec{u}^d_t(s)\|^2_{L^2(\Omega)}+\Lambda\mu\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}\Big)ds\\&\qquad+\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|^2_{L^2(\Omega)}+C\delta^3e^{3\lambda_1 t}.\end{aligned}\tag{D.4}$$

(D.1) is followed by applying Gronwall's inequality to (D.4).

We shall explain the arguments of (D.2) in [2]. Firstly, we still have

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\vec{u}_t^d(s)\|^2_{L^2(\Omega)}ds\\
&\quad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&=\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+\Big(\int_\Omega(\rho_0+\sigma^\delta(t))|u_t^d(t)|^2d\vec{x}\Big)\Big|_{t=0}\\
&\quad+\int_0^t\int_\Omega(2f_t^\delta(s)+2g\vec{u}^\delta(s)\cdot\nabla\sigma^\delta(s)e_3-\sigma_t^\delta(s)\vec{u}_t^d(s))\cdot\vec{u}_t^d(s)ds.
\end{aligned}\tag{D.5}$$

We estimate

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\vec{u}_t^d(s)\|^2_{L^2(\Omega)}ds\\
&\quad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\leqslant\int_\Omega g\rho_0'|u_2^d(t)|^2d\vec{x}+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.6}$$

That implies

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|^2_{L^2(\Omega)}+2\mu\int_0^t\|\nabla\vec{u}_t^d(s)\|^2_{L^2(\Omega)}ds\\
&\quad-2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_{1,t}^d(s,x_1,1)|^2+\xi_-|u_{1,t}^d(s,x_1,-1)|^2)dx_1ds\\
&\leqslant\Lambda^2\int_\Omega(\rho_0+\sigma^\delta(t))|\vec{u}^d(t)|^2d\vec{x}+\Lambda\mu\int_\Omega|\nabla\vec{u}^d(t)|^2d\vec{x}\\
&\quad-\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1+C\delta^3e^{3\lambda_1 t}.
\end{aligned}\tag{D.7}$$

By using the inequality

$$\Lambda\mu\|\nabla\vec{u}^d\|^2_{L^2(\Omega)}\leqslant\Lambda^2\mu\int_0^t\|\nabla\vec{u}^d(s)\|^2_{L^2(\Omega)}ds+\mu\int_0^t\|\nabla\vec{u}_t^d(s)\|^2_{L^2(\Omega)}ds$$

and the identity

$$\begin{aligned}
&\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\\
&=\Lambda^2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds\\
&\quad+\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_{1,t}^d(s,x_1,1)|^2+\xi_-|u_{1,t}^d(s,x_1,-1)|^2)dx_1ds\\
&\quad-\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,-1))dx_1ds,
\end{aligned}$$

it is obtained from (D.7) that (see **(134)** in [2])

$$
\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|_{L^2(\Omega)}^2\\
&\quad+\frac{1}{2}\Lambda\Big(\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\Big)\\
&\quad+\frac{1}{2}\int_0^t\Big(\mu\|\nabla\vec{u}_t^d(s)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|u_{1,t}^d(s,x_1,1)|^2+\xi_-|u_{1,t}^d(s,x_1,-1)|^2)dx_1\Big)ds\\
&\leqslant\Lambda^2\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2(\Omega)}^2+C\delta^3e^{\lambda_1 t}\\
&\quad+\frac{3}{2}\Lambda^2\int_0^t\Big(\mu\|\nabla\vec{u}^d(s)\|_{L^2(\Omega)}^2-\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1\Big)ds\\
&\quad+\frac{3}{2}\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,-1))dx_1ds.
\end{aligned}
\tag{D.8}
$$

Integrating (D.3) in time from 0 to t and using (D.8) and Young's inequality, the authors deduce (D.2) without providing any detailed explanation.

However, we observe by integrating (D.3) in time that

$$
\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}^d(t)\|_{L^2(\Omega)}^2\leqslant\frac{1}{\Lambda}\int_0^t e^{\Lambda(t-s)}\|\vec{u}_t^d(s)\|_{L^2(\Omega)}^2ds+Ce^{3\lambda_1 t}.
$$

Then the l.h.s of (D.2) will be bounded by

$$
\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\vec{u}_t^d(t)\|_{L^2(\Omega)}^2+\Lambda\mu\|\nabla\vec{u}^d(t)\|_{L^2(\Omega)}^2+\mu\int_0^t\|\nabla\vec{u}_t^d(s)\|_{L^2(\Omega)}^2ds\\
&\leqslant\Lambda\int_0^t e^{\Lambda(t-s)}\|\vec{u}_t^d(s)\|_{L^2(\Omega)}^2ds+\frac{1}{2}\Lambda\mu\|\nabla u^d(t)\|_{L^2(\Omega)}^2\\
&\quad+\frac{1}{2}\mu\int_0^t\|\nabla u_t^d(s)\|_{L^2(\Omega)}^2ds+\frac{3}{2}\Lambda^2\mu\int_0^t\|\nabla\vec{u}^d(s)\|_{L^2(\Omega)}^2ds\\
&\quad+\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1\\
&\quad-\Lambda^2\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1\\
&\quad+\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,-1))dx_1+C\delta^3e^{3\lambda_1 t}.
\end{aligned}
\tag{D.9}
$$

We are not clear about the way in [2] to remove all integral terms over $2\pi L\mathbb{T}$ in the r.h.s of (D.9) to get (D.2) for all $\mu>0$, especially the following term

$$
\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-u_{1,t}^d|^2(s,x_1,-1))dx_1.
$$

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