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► **To cite this version:**

Nirpeksh Kumar, Subha Chakraborti, Philippe Castagliola. Phase II Exponential Charts for Monitoring Time Between Events Data: Performance Analysis Using Exact Conditional Average Time to Signal Distribution. *Journal of Statistical Computation and Simulation*, 2022, 92 (7), pp.1457-1486. 10.1080/00949655.2021.1998501 . hal-03646713

**HAL Id: hal-03646713**

**<https://hal.science/hal-03646713v1>**

Submitted on 19 Apr 2022

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## Phase II Exponential Charts for Monitoring Time Between Events Data: Performance Analysis Using Exact Conditional Average Time to Signal Distribution

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**Abstract:** In this paper, we study the performance properties of the phase II exponential chart with an unknown in-control (IC) rate parameter, used to monitor high-yield processes. The average time to signal (ATS) is used as the chart performance criterion instead of the usual average run length (ARL). Based on the IC conditional ATS (CATS) distribution, we examine the properties of both the equal-tailed and the ATS-unbiased exponential charts with estimated control limits and adjust the limits so that a nominal IC ATS performance is achieved. Two perspectives are investigated: the unconditional, under which the average of IC CATS distribution is set equal to a nominal ATS and, the conditional so that the IC CATS is set to or exceed a nominal ATS with a high probability. It is shown that the ATS-unbiased exponential chart under the conditional perspective has a better IC performance.

**Keywords:** Conditional and Unconditional perspectives; Equal-tailed and ATS-unbiased exponential chart; Exact distribution; Guaranteed IC performance; Quality control.

### 1. Introduction

Processes with a very low rate of nonconforming items, say, parts per million (ppm) are termed as high-yield processes. High-yield processes can be encountered in the manufacturing of integrated circuits, automated manufacturing processes, in health care surveillance, in the monitoring of earthquake occurrences [1-3]. Monitoring such processes with conventional attribute control charts such as the  $p$ -,  $np$ -charts is known to be not as effective due to several reasons such as increased

FAR (false alarm rate), negative lower control limits, etc. [2]. Instead, for such processes, the time between two consecutive nonconforming items (or events of interest) is recommended as the quality characteristic to be monitored. The control charts which consider the times between nonconforming items (events) are known as TBE (Time Between Events) control charts and they have several advantages over the conventional attribute charts [4]. In such situations, the occurrences of nonconforming items may be modeled by a homogeneous Poisson process with a constant rate of nonconforming, say,  $\lambda$ , and therefore, the TBE, which is also known as the inter-arrival time, follows an exponential distribution with mean  $1/\lambda$ . Several control charts (Shewhart-type or not) have been proposed based on this exponential assumption [1, 3, 5-13]. The readers may also refer to the review papers by Liu et al. [14] and Ali et al. [15]. Recently, Qu et al. [16] proposed a weighted cumulative sum chart based on the exponential distribution using the power parameter of sample mean shifts i.e. using  $(x_i - \frac{1}{\lambda})^w$  in place of  $(x_i - \frac{1}{\lambda})$  in the CUSUM statistic where  $x_i$  is the  $i$ -th observation (TBE data) and  $w > 0$  is the power parameter. In order to improve the sensitivity of the exponential chart against small shifts in the process parameter, Cheng et al. [17] proposed a synthetic exponential chart consisting of an exponential sub-chart and a conforming run length sub-chart. Recently, Shah et al. [18] discussed the TBE chart for the gamma distribution using an exact distribution of the sample TBE mean.

Most often the control charts are evaluated by their run length and associated statistical properties such as the mean, standard deviation and percentiles. The run length may be defined as the total number of charting points to be plotted on the chart to get the first OOC signal. We should mention here that the run length only considers the plotted points on the chart, but, it completely ignores the time needed to obtain these points, which is an important factor in the context of the TBE monitoring. Therefore, in the Statistical Process Control (SPC) literature it is recommended that the times needed to obtain the charting points must be considered in evaluating the chart's performance [4,11]. We denote the time to signal by TS, as the waiting time to get the first OOC signal. Because the TS conveys more information about the process than the run length, its average i.e. the average time to signal (denoted ATS) may be a preferred metric of chart performance over the traditional average of run length i.e. the ARL, which is more meaningful when the plotting points are obtained at equal time intervals. Several authors including Zhang et al. [11], Bourke [19], Yang et al. [20] have used the ATS to evaluate the TBE chart's performance.

It is well accepted that when a chart parameter is unknown, the control limits for monitoring (in phase II) may be constructed by replacing this parameter in the control limits for the known parameter case, by a suitable estimator using an IC retrospective (phase I or a reference) sample. It has been established by several studies that parameter estimation can drastically affect the control chart's properties in a negative way and the resulting chart performs much differently (many more false alarms, for example) from what it is nominally expected [21-26]. Note that most of the studies evaluate the performance of phase II control charts using the unconditional run length distribution (that is averaging over the distribution of the estimator) and some associated characteristics. However, it is now well recognized that the unconditional analysis may not provide a very realistic picture about the control chart's properties for a given estimate from a given phase I sample [21- 22, 27]. The recent papers by Diko et al. [28]; Guo and Wang [29]; Kumar [30] highlighted the importance of the conditional performance analysis and advocated the use of the conditional run length (CRL) distribution and the conditional average run length CARL distribution and its associated properties to evaluate the chart's performance for a given phase I sample. As noted before, for TBE data, the TS is the more suitable random variable of interest and the average time to signal (ATS) has been the chosen performance metric. Thus, following the literature, with estimated parameters from a phase I sample, the phase II control limits of TBE control charts are mainly constructed and examined under the unconditional perspective, that is based on the expected value of conditional ATS (CATS) distribution [11,31].

Note that the CATS is a random variable in the unknown parameter case, whereas in the known parameter case, it is a constant. In this paper, we derive the *exact* distribution of CATS and investigate chart properties under both conditional and unconditional perspectives using various characteristics of the CATS distribution such as the average, standard deviation and percentiles. Note that the conditional approach is advocated to take into account the practitioner-to-practitioner variability in evaluating the chart's performance [27, 32].

When the plotting statistic has a skewed distribution, the common design to construct the control limits of two-sided control charts in the known parameter case is based on the conventional equal-tailed probability limits (henceforth called the equal-tailed chart). However, it is found that the equal-tailed chart based on the ATS leads to an ATS-biased chart [20,31]. A control chart is said to be ATS-biased chart if its ATS function does not achieve the maximum when the process is IC. This implies that the chart takes on average, longer time to raise an OOC signal, than it takes to raise a

false alarm. This phenomenon has been considered as a highly undesirable property of a control chart [11, 31]. To overcome this undesirable situation, ATS-unbiased charts are proposed in the literature [20, 31] which do not have this shortcoming and provide a more balanced guard against both process deterioration and improvement.

Many control charts are designed and evaluated on the basis of their ARL. It has been acknowledged in the literature that when control charts for the known parameters are used with estimated parameters, the variability in the run length distribution increases which shortens the IC ARL, which increases the number of false alarms, and thus reduces the value of process monitoring. In order to rectify the situation, many authors including Aly et al. [33]; Diko et al. [28]; Faraz et al. [34-35]; Gandy and Kvaloy [36]; Goedhart et al. [32, 37]; Hu and Castagliola [38]; Jones and Steiner [39]; Zhao and Driscoll [40] have considered adjusting the control limits. In this paper, we consider adjusting control limits of both equal-tailed and ATS-unbiased exponential charts under both unconditional and conditional perspectives. Under the unconditional perspective, the control limits will be adjusted so that the mean of IC CATS is equal to a nominal  $ATS_0$  value whereas under the conditional perspective, they will be adjusted so that the IC CATS meets or exceeds the nominal  $ATS_0$  value with a high pre-fixed probability.

The rest of this paper is organized as follows. In section 2, the phase II control limits of the exponential chart are constructed and the distribution of the CATS is derived. Section 3 examines the conditional performance of the phase II equal-tailed and ATS-unbiased exponential charts for a given nominal  $ATS_0$  value. The control limits of these charts are then adjusted to guarantee the IC performance in Section 4 and their conditional IC and OOC performances are evaluated. An example is given for illustration in Section 5 and some concluding remarks are summarized in Section 6.

## **2. The Exponential chart and the CATS distribution**

### **2.1. Case of the known parameter**

Let  $X_i, i = 1, 2, \dots$  denote the time between two consecutive nonconforming items in a process which follows an exponential distribution with density function  $f(x) = \lambda \exp(-\lambda x), x > 0, \lambda > 0$ . The process is said to be IC when  $\lambda = \lambda_0$ , otherwise it is OOC, that is when  $\lambda = \lambda_1 (\lambda_1 \neq \lambda_0)$ . Suppose that LCL and UCL denote the lower and upper control limits of the exponential chart for individual data, respectively. For equal-tailed limits with known parameter case, the control limits LCL and

UCL are obtained such that  $P(X < \text{LCL}|IC) = \alpha_0/2$  and  $P(X > \text{UCL}|IC) = \alpha_0/2$ , where  $\alpha_0$  is the nominal false alarm rate (FAR). However, in general setting of unequal-tailed limits that is to have flexibility in the unequal-tailed exponential charts and to attain the desired IC performance, we introduce two design constants such that the probability of an observation  $X$  plotting below the LCL is  $\xi p$  and above the UCL is  $(1 - \xi)p$  that is

$$P(X < \text{LCL}|IC) = \xi p \text{ and } P(X > \text{UCL}|IC) = (1 - \xi)p$$

where and  $0 < p < 1$  and  $0 < \xi < 1$  is a constant (to be determined) that is used to split the FAR towards finding the LCL and the UCL, respectively, in order to maintain the IC performance at a desired level. Thus, for the equal-tailed exponential chart in the known parameter case,  $\xi = 0.5$  and  $p = \alpha_0$  for a fixed FAR,  $\alpha_0$ .

The control limits, the UCL and LCL, can be expressed as

$$\text{LCL} = \frac{-\ln(1-\xi p)}{\lambda_0} = \frac{A_L}{\lambda_0} \text{ and } \text{UCL} = \frac{-\ln((1-\xi)p)}{\lambda_0} = \frac{A_U}{\lambda_0} \quad (1)$$

where

$$A_L = -\ln(1 - \xi p) \text{ and } A_U = -\ln((1 - \xi)p). \quad (2)$$

The center line (CL) of the exponential chart can be considered as the median of the charting statistic  $X$  and is given by

$$\text{CL} = \frac{-\ln(0.5)}{\lambda_0}.$$

Let  $\delta = \frac{\lambda_1}{\lambda_0}$  quantify the standardized shift of the parameter  $\lambda$  from  $\lambda = \lambda_0$  (in-control) to some  $\lambda = \lambda_1$  (out-of-control). Clearly,  $\delta < 1$  i.e.  $\lambda_1 < \lambda_0$  represents an improvement in the process as the mean time between events increases while,  $\delta > 1$  i.e.  $\lambda_1 > \lambda_0$  represents a deterioration in the process as the mean time between events decreases. For the IC process, we have  $\delta = 1$ . Let  $E_i$ ,  $i = 1, 2, \dots$  denote the event  $E_i = (X_i < \text{LCL}) \cup (\text{UCL} < X_i)$  that a charting statistic lies outside the control limits LCL and UCL. This is called a signalling event. When the parameter  $\lambda_0$  is known, the run length RL of the exponential control chart is a geometric random variable with parameter  $\beta(\delta) = P(E_i)$  equal to

$$\beta(\delta) = P[X < \text{LCL} \text{ or } X > \text{UCL}] = 1 - \exp(-\delta A_L) + \exp(-\delta A_U)$$

It is shown in Appendix A that when the parameter  $\lambda_0$  is known, the random variable TS follows an exponential distribution with parameter  $\delta\lambda_0\beta(\delta)$ . Concerning a change in the process parameter it can be assumed to occur either (i) at the beginning of the monitoring period or, more generally, (ii) at some random point in time (this case is known as the random shift model [41]). See also Schuh et al. [42] and Zwetsloot and Woodall [43] for more details. In this paper, in order to simplify the model, we assume that any change in the process parameter occurs at the beginning of the monitoring period and we leave the more general random shift model for future developments.

## 2.2. Case of the Unknown parameter

When the parameter  $\lambda_0$  is unknown, the estimated control limits are established by replacing  $\lambda_0$  into Equation (1) by its estimator  $\hat{\lambda}_0$ , obtained from a phase I sample, say,  $Y_1, \dots, Y_m$  of size  $m$ . See Kumar and Chakraborti [44] for a phase I analysis of the exponential chart. Thus, the

estimated (phase II) control limits become

$$\widehat{LCL} = \frac{A_L}{\hat{\lambda}_0} \quad \text{and} \quad \widehat{UCL} = \frac{A_U}{\hat{\lambda}_0}$$

Usually, the UMVUE (uniformly minimum variance unbiased estimator) is used to estimate the IC rate parameter  $\lambda_0$  which is  $\hat{\lambda}_0 = \frac{m-1}{T}$ , where  $T = \sum Y_i$  is a sufficient statistic that follows a gamma distribution with parameters  $m$  and  $\lambda_0$ . Thus, the estimated control limits can be re-written as

$$\widehat{LCL} = \frac{A_L T}{m-1} \quad \text{and} \quad \widehat{UCL} = \frac{A_U T}{m-1} \quad (3)$$

We mention here that when the parameter  $\lambda_0$  is estimated by  $\hat{\lambda}_0$ , the signaling events  $E_i$  are dependent and hence, the TS no longer follows an exponential distribution. However, for a given value of  $\hat{\lambda}_0$ , the events  $E_i$  are independent and consequently, the CTS (conditional time to signal) (given  $\hat{\lambda}_0$ ) follows an exponential distribution with parameter  $\delta\hat{\lambda}_0\hat{\beta}(\delta)$  where

$$\begin{aligned} \hat{\beta}(\delta) &= P[X < \widehat{LCL}] \text{ or } X > \widehat{UCL} ] \\ &= 1 - \exp\left(-\frac{\delta\lambda_0 A_L T}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 A_U T}{m-1}\right) \end{aligned} \quad (4)$$

Thus, the conditional ATS (denoted CATS) which is the mean of the CTS distribution, for the control limits in Equation (3) is given by

$$\text{CATS}(\delta) = \frac{1}{\delta\hat{\lambda}_0\hat{\beta}(\delta)} \quad (5)$$

Using Equation (4), we can re-express the CATS in Equation (5) as follows.

$$\begin{aligned} \text{CATS}(\delta) &= \frac{T}{\delta(m-1)} \left[ 1 - \exp\left(-\frac{\delta\lambda_0 T A_L}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 T A_U}{m-1}\right) \right]^{-1} \\ &= \frac{\varphi(T)}{\delta(m-1)} \end{aligned} \quad (6)$$

where

$$\varphi(T) = T \left[ 1 - \exp\left(-\frac{\delta\lambda_0 T A_L}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 T A_U}{m-1}\right) \right]^{-1} \quad (7)$$

For the known parameter case, the  $\text{ATS} = 1/(\delta\lambda_0\beta(\delta))$  is a constant, but, for the unknown parameter case, the  $\text{CATS}(\delta)$  is a random variable since it's a function of the random variable  $T$ . Because the conditional performance analysis heavily depends on the  $\text{CATS}(\delta)$  and its distributional characteristics, we derive the exact distribution of  $\text{CATS}(\delta)$  in the following theorem.

### 2.1. Distribution of $\text{CATS}(\delta)$

To obtain the exact distribution of  $\text{CATS}(\delta)$  in Equation (6), first we prove the following lemma.

**Lemma 1.** For given  $0 < \xi, p < 1, \lambda_0 > 0, \delta > 0$  and  $m > 1$ , the function  $\varphi(T)$  is an increasing function of  $T$  in  $(0, \infty)$ .

The proof is provided in Appendix B.

**Theorem 1.** For the two-sided exponential chart with control limits defined as in Equation (2), the cumulative distribution function (c.d.f.) of the conditional average time to signal,  $\text{CATS}(\delta)$  is given by

$$G(z) = F_\Gamma(\varphi^{-1}(\delta(m-1)z)), 0 < z < \infty, \delta > 0, m > 1 \quad (8)$$

where the function  $\varphi^{-1}(\cdot)$  is the inverse function of  $\varphi(\cdot)$  shown in Equation (7). The probability density function (p.d.f.) of  $\text{CATS}(\delta)$  is given by

$$g(z) = f_\Gamma(\varphi^{-1}(\delta(m-1)z)) \times \frac{\delta(m-1)}{\varphi'(\varphi^{-1}(\delta(m-1)z))}, 0 < z < \infty \quad (9)$$

where  $F_\Gamma(\cdot)$  and  $f_\Gamma(\cdot)$  are the c.d.f. and p.d.f. of the gamma distribution with shape parameter  $m$  and scale parameter  $\lambda_0$ , respectively. The function  $\varphi'(z)$  is the first derivative of  $\varphi(z)$  with respect to  $z$ .

The proof is provided in Appendix C.



Using this theorem, some properties of the CATS distribution can be obtained as follows. These results play important roles in the developments that follow.

**Theorem 2.** The  $q$ -th quantile of the CATS distribution is given by

$$\text{CATS}_q(\delta) = \frac{1}{\delta(m-1)} \varphi\left(F_{\Gamma}^{-1}(q)\right) \quad (10)$$

where  $F_{\Gamma}^{-1}(\cdot)$  is the  $q$ -th quantile of a gamma distribution with parameters  $m$  and  $\lambda_0$ . The proof is given in Appendix D.

**Theorem 3.** The  $r^{\text{th}}$  moment about origin of the  $\text{CATS}(\delta)$  distribution is given by

$$\mu'_r(\delta) = \left(\frac{1}{\delta\lambda_0(m-1)}\right)^r \frac{\Gamma(m+r)}{\Gamma(m)} \sum_{j=0}^{\infty} \sum_{i=0}^j (-1)^{j+i} \binom{-r}{j} \binom{j}{i} \left[1 + \frac{\delta}{m-1} \{A_L(j-i) + A_U i\}\right]^{-m} \quad (11)$$

The constants  $A_L$  and  $A_U$  are defined in Equation (2). The proof is provided in Appendix E.

Thus, the mean of the  $\text{CATS}(\delta)$ , i.e.  $\text{ACATS}(\delta) = \mu'_1(\delta)$  is given by putting  $r = 1$  in Equation (11) as follows.

$$\mu'_1(\delta) = \left(\frac{m}{\delta\lambda_0(m-1)}\right) \sum_{j=0}^{\infty} \sum_{i=0}^j (-1)^i \binom{j}{i} \left[1 + \frac{\delta}{m-1} \{A_L(j-i) + A_U i\}\right]^{-m}$$

We mention here that in the context of the ARL, Kumar [30] and Kumar et al. [45] suggested two IC performance metrics: the LPB (lower Prediction Bound) which is defined as the smallest IC CARL value that can be attained by the control chart with high probability, say,  $1 - \gamma$ , and the probability that the IC CARL is greater than or equal to a nominal ARL value. The latter is referred to as the exceedance probability criterion (EPC) [46] and has been investigated by several authors including Albers et al. [47]; Gandy and Kvaløy [36], Faraz et al. [34]; Goedhart et al. [32, 37]; Guo and Wang [29] to design the phase II control limits for fixed probability, say, 0.90 to guaranteeing that the IC CARL will meet or exceed the nominal  $\text{ARL}_0$ .

Motivated by these, we define the metrics LPB and the exceedance probability (EP) in terms of IC CATS i.e.  $\text{CATS}(1)$  (thereafter we will denote it by  $\text{CATS}_{\text{IC}}$  by first defining function  $\varphi(T)$  for the IC situation as follows.

$$\varphi_{\text{IC}}(T) = T \left[1 - \exp\left(-\frac{\lambda_0 T A_L}{m-1}\right) + \exp\left(-\frac{\lambda_0 T A_U}{m-1}\right)\right]^{-1}$$

Now we define the  $(1 - \gamma)\%$  LPB for the  $\text{CATS}_{\text{IC}}$ , denoted by  $\text{LPB}_{1-\gamma}$  as

$$P[\text{CATS}_{\text{IC}} \geq \text{LPB}_{1-\gamma}] = 1 - \gamma$$

which gives

$$LPB_{1-\gamma} = \frac{1}{(m-1)} \varphi_{IC}(F_{\Gamma}^{-1}(\gamma))$$

For the control chart with the LPB equal to  $LPB_{1-\gamma}$ , the user has a usually high confidence  $(1-\gamma)\%$  that his  $CATS_{IC}$  will be at least as much as  $LPB_{1-\gamma}$ . Note that the  $LPB_{1-\gamma}$  is in fact the  $\gamma$ -th quantile of the  $CATS_{IC}$  distribution.

On the other hand, the exceedance probability (EP) that the  $CATS_{IC}$  is at least as much as a nominal  $ATS_0$  value is defined as

$$EP = P[CATS_{IC} \geq ATS_0] = 1 - F_{\Gamma}(\varphi_{IC}^{-1}((m-1)ATS_0))$$

Note that it is desirable that both  $LPB_{1-\gamma}$  and EP be high for a good chart.

### 3. Phase II Exponential chart under the unconditional perspective

We should mention here that the exponential charts with known parameter  $\lambda_0$  are often constructed using an equal-tailed probability approach by assigning the half of the FAR,  $\alpha_0$  to the both tails i.e. using  $P[X < LCL|IC] = P[X > UCL|IC] = \alpha_0/2$ . However, when the parameter is unknown, the control limits  $\widehat{LCL}$  and  $\widehat{UCL}$ , being the function of  $\hat{\lambda}$  (or  $T$ ), become random variables. As a result, we set  $E[P[X < \widehat{LCL}|IC]] = E[P[X > \widehat{UCL}|IC]]$  for the equal-tailed exponential chart in the unknown parameter case. The exact expressions for the unconditional probabilities are provided in Appendix F.

In order to construct the equal-tailed exponential chart with estimated parameters under the unconditional perspective for a fixed nominal  $ATS_0$ , we determine the unique pair  $(\xi, p)$  that gives a pair of control limits satisfying the following two equations:

$$\mu'_1(1) = ATS_0 \quad (15)$$

$$1 - \left(1 + \frac{A_L}{m-1}\right)^{-m} = \left(1 + \frac{A_U}{m-1}\right)^{-m} \quad (16)$$

Clearly, Equation (15) ensures that the expected value of  $CATS_{IC}$  i.e. the  $ACATS_{IC}$  (given by  $\mu'(1)$ ) is equal to the nominal  $ATS_0$ , whereas Equation (16) ensures that the unconditional tail probabilities  $E[P[X < \widehat{LCL}|IC]]$  and  $E[P[X > \widehat{UCL}|IC]]$  are equal i.e. a charting statistic has on average an equal chance of lying below the lower control limit and above the upper control limit. As noted above, this

is one analogue of the equal-tailed probability approach used in the known parameter case. Once we find the required pair of  $(\xi, p)$  satisfying Equations (15) and (16), the corresponding  $P[X < \widehat{LCL}|IC]$  and  $P[X < \widehat{UCL}|IC]$  can be obtained from Equation (2) for the phase II equal-tailed exponential chart. Note that the unconditional perspective is defined here in terms of the  $ATS_0$  rather than in terms of the  $ARL_0$  as in [46].

It is worth mentioning that the equal-tailed exponential chart under the unconditional perspective is ATS-biased as its ATS function does not achieve its maximum at the IC state (i.e.  $\delta = 1$ ), which causes the control chart take more time to give a signal in the OOC state than in the IC state of the process. This is considered undesirable and hence, in this situation, it is worth using the ATS-unbiased exponential charts (see [11]). In order to design the ATS-unbiased chart with the estimated control limits under the unconditional perspective, we first set the  $\mu'_1(1)$  (the mean of the  $CATS_{IC}$ ) is equal to the nominal  $ATS_0$  (see, Equation (17)) and then we set the derivative of  $ACATS(\delta)$ , with respect to  $\delta$ , that is  $\mu'_1(\delta)$  evaluated at  $\delta = 1$  equal to zero (see, for example, [29]) (see, Equation (18)). Thus, the control limits of the ATS-unbiased exponential chart with estimated parameters under the unconditional perspective can be obtained from first solving the following the two equations

$$\mu'_1(1) = ATS_0 \quad (17)$$

$$\left. \frac{d}{d\delta} \mu'_1(\delta) \right|_{\delta=1} = 0 \quad (18)$$

for the  $(\xi, p)$  pair and then using Equations (8) and (9) respectively by plugging in the values of the pair  $(\xi, p)$  of the corresponding chart. Note that Equation (17) ensures that the mean of the  $CATS_{IC}$  is equal to the nominal  $ATS_0$  whereas Equation (18) ensures the unbiasedness that is the maximum of the  $ACATS(\delta)$  is at  $\delta = 1$ . The chart design parameters are calculated and reported in Table 1 for both the phase II equal-tailed and the phase II ATS-unbiased exponential charts corresponding to different values of  $m$ . The control limits of the phase II equal-tailed and the ATS-unbiased exponential charts with unknown parameter can be obtained from Equation (3) by plugging in the appropriate values of the pair  $(\xi, p)$  of the corresponding chart.

### 3.1. In-control Performance of the phase II exponential charts under the unconditional perspective

The c.d.f. and p.d.f. of the phase II equal-tailed and the ATS-unbiased exponential charts with unknown parameter can be obtained from Equations (8) and (9) respectively by plugging in the appropriate values of the pair  $(\xi, p)$  of the corresponding chart. Figures 1 and 2 show the p.d.f.s of the  $CATS_{IC}$  of the phase II equal-tailed and ATS-unbiased exponential charts, respectively, with nominal  $ATS_0=370.4$ ,  $m=20, 50, 100, 500$  and  $\lambda_0 = 1$ . We should mention here that we consider small sample sizes,  $m \leq 30$ , moderate sample sizes,  $30 < m \leq 100$  and large sample sizes,  $m > 100$ . It can be seen from Figure 1 that for smaller values of  $m$ , the p.d.f. of the  $CATS_{IC}$  for the phase II equal-tailed exponential chart is skewed to the left of  $ATS_0$  that is the higher values of the  $CATS_{IC}$  are located to the right of  $ATS_0$  which produces a high EP value. However, as  $m$  increases, the distribution of the  $CATS_{IC}$  becomes more symmetric about  $ATS_0$ . Thus, even when  $m$  is large, the EP values tend to 0.50 which means that there is only about a 50% chance that the  $CATS_{IC}$  will be greater than or equal to  $ATS_0$ . On the other hand, the  $CATS_{IC}$  distribution of the ATS-unbiased exponential chart is more left skewed than the corresponding equal-tailed chart for moderate to large sample sizes (Figure 2). Note that for smaller values of  $m$ , the p.d.f. of the  $CATS_{IC}$  distributions for both control charts reaches its maximum, and then decreases rapidly tending to zero.

Insert Figures 1 and 2

Table 1 presents the values of various performance metrics of the of  $CATS_{IC}$  distribution for both the phase II equal-tailed and ATS-unbiased exponential charts obtained under the unconditional perspective, such as the mean ( $ACATS_{IC}$ ), standard deviation  $SDCATS_{IC}$ , some percentiles, coefficient of variation ( $CV=(SDCATS_{IC}/ACATS_{IC}) \times 100$ ) for different values of  $m$  with  $ATS_0=370.4$ . The following observations can be made from Table 1.

1. Though, the  $ACATS_{IC}$  values are kept fixed at nominal  $ATS_0 = 370.4$ , the corresponding  $SDCATS_{IC}$  are quite high as compared to  $ACATS_{IC}$  especially for small sample sizes for both the charts, for example, when  $m = 20$ , the CV values are 42.24 and 47.18 for the phase II equal-tailed and ATS-unbiased chart, respectively. Thus, looking at the  $ACATS_{IC}$  only conveys half the story and this raises a question about the unconditional perspective. In order to reduce the practitioner-to-practitioner variability, i.e., to reduce  $SDCATS_{IC}$  value to a reasonable level, a huge number of phase I observations are required. Zhang et al. [26] suggested that the variability in IC CARL values should be within 10% of the nominal ARL value. Following them, to keep the  $SDCATS_{IC}$  value within 10% of the desired  $ATS_0$ , it is observed that more than 1000 phase I observations are needed for the equal-tailed exponential chart whereas about 400 phase I

observations are required for the ATS-unbiased exponential chart. Thus, the ATS-unbiased exponential chart needs substantially lower number of observations.

2. It is worth noting that for smaller values of  $m$ , the equal-tailed exponential chart has smaller  $SDCATS_{IC}$  values than the ATS-unbiased exponential chart. For example, when  $m = 20$ , the  $SDCATS_{IC}$  value for the equal-tailed chart is 156.46, whereas it is 174.76 for the ATS-unbiased chart about 12% lower. However, for moderate to large sample sizes  $m$ , the  $SDCATS_{IC}$  values of the ATS-unbiased exponential chart are smaller than that of the equal-tailed exponential chart. For example, for  $m = 150$ , the  $SDCATS_{IC}$  values are 99.80 and 61.86 (about 38% less) for the equal-tailed and the ATS-unbiased exponential charts, respectively.
3. The EP values for both the charts are not particularly appealing, especially, for larger sample sizes. This happens because as  $m$  becomes larger, the  $CATS_{IC}$  distribution for both the charts tends to become symmetric about the  $ATS_0$  which produces EP values close to 0.50. This gives only a 50% probability that the  $CATS_{IC}$  value will be greater than or equal to the nominal  $ATS_0$ . However, the ATS-unbiased chart has higher EP values than the equal-tailed chart except for smaller values of  $m \leq 30$ .

Insert Table 1

Note that in the context of monitoring rare events, it can take a long time to even collect a small number of phase I observations. The study suggests the use of an equal-tailed exponential chart when a small number of phase I observations is available to estimate the parameter. However, under the unconditional perspective neither chart accounts for practitioner-to-practitioner variability which is reflected by their respective low EP values. This provides a motivation for designing the control chart under the conditional perspective using the exceedance probability criterion (EPC), which ensures a guaranteed IC performance in the sense that the  $CATS_{IC}$  meets or exceeds (is at least equal to) a nominal  $ATS_0$  value with a prespecified (high) probability.

### **3.2. Out-of-control Performance of the phase II exponential charts under the unconditional perspective**

In order to examine the conditional OOC performance of the equal-tailed and ATS-unbiased charts under the unconditional perspective, we consider  $\delta = 2.0, 4.0$  to reflect upward shifts and  $\delta = 0.25, 0.5$  for downward shifts and different values of  $m = 20, 30, 50, 100, 200, 500$ . Table 2 provides the mean, standard deviation and percentiles of the conditional OOC CATS, denoted by

$CATS_{OOOC}$ , distributions of the equal-tailed and ATS-unbiased exponential charts under the unconditional perspective when  $ATS_0 = 370.4$ ,  $\lambda_0 = 1$ .

Insert Table 2

It can be observed from Table 2 that when the process improves i.e.  $\delta < 1$  and the phase I sample size  $m$  is small, the equal-tailed exponential chart under the unconditional perspective has larger OOC mean and percentiles of the  $CATS_{OOOC}$  distribution than those for the ATS-unbiased exponential chart under the unconditional perspective whereas for the moderate to large values of  $m$ , the opposite pattern is observed i.e. the OOC mean and percentiles of the equal-tailed exponential chart under the unconditional perspective are smaller than the corresponding values of the ATS-unbiased exponential chart under the unconditional perspective.

However, when the process deteriorates i.e.  $\delta > 1$  and the phase I sample size is small, the OOC mean and percentiles of  $CATS_{OOOC}$  distribution for the equal-tailed chart under the unconditional perspective are smaller than those for the ATS-unbiased exponential chart under the unconditional perspective, however, for moderate to large values of  $m$ , these values for the former chart are larger than the latter.

Overall, for a moderate to large phase I sample size, the ATS-unbiased exponential chart under the unconditional perspective has better OOC performance in terms of lower  $ACATS_{OOOC}$  values than the equal-tailed exponential chart when the process deteriorates. Note that generally speaking, in typical applications, the deterioration case is considered to be more serious than the improvement case in practice. On the other hand, when  $m$  is small the latter performs better than the former. For the improvement case i.e.  $\delta > 1$ , the ATS-unbiased exponential chart under the unconditional perspective has better (worse) performance than the equal-tailed exponential chart under the unconditional perspective for small (moderate to large)  $m$ .

#### **4. Phase II exponential charts under the conditional perspective**

The study in Section 3 shows that both the equal-tailed and ATS-unbiased phase II exponential charts adjusted under the unconditional perspective have poor IC conditional performance since the variation in the  $CATS_{IC}$  distribution is not accounted for. Noting that the  $SDCATS_{IC}$  value decreases as the phase I sample size,  $m$  (See Table 1) increases, the  $SDCATS_{IC}$  can be reduced to reasonable degree, i.e., say within 10% of the desired  $ATS_0$ , by increasing  $m$ . On the other hand, increasing  $m$  reduces the confidence that the  $CATS_{IC}$  will meet or exceed the nominal  $ATS_0$  since the  $CATS_{IC}$

distribution tends to become symmetric around  $ATS_0$ . Thus, when  $m$  is large, there is about 50% chance that the  $CATS_{IC}$  will be less than the  $ATS_0$  which will cause more frequent false alarms and more unwanted process interruptions than desired [29]. We now consider adjusting the control limits of both the phase II equal-tailed and ATS-unbiased exponential charts under the conditional perspective to achieve a fixed EP value, say,  $1 - \gamma$  (usually high, say, .90) i.e.  $CATS_{IC}$  meets or exceeds the desired nominal  $ATS_0$  with a certain probability,  $1 - \gamma$ . In terms of the average run length ARL, this criterion is known as the exceedance probability criterion (see for example, [34, 36]).

First consider the Phase II exponential chart.

Let  $\widehat{LCL}^*$  and  $\widehat{UCL}^*$  be the lower and upper control limits of the phase II exponential chart under conditional perspective such that the resulting  $CATS_{IC}$  is greater than or equal to the nominal  $ATS_0$  with a high probability, say,  $1 - \gamma$ . The adjusted control limits are given by

$$\widehat{LCL}^* = \frac{A_L^*}{\lambda} \text{ and } \widehat{UCL}^* = \frac{A_U^*}{\lambda} \quad (19)$$

where  $A_L^* = -\ln(1 - \xi^* p^*)$  and  $A_U^* = -\ln((1 - \xi^*) p^*)$ . The constants  $\xi^*$  and  $p^*$  for the phase II equal-tailed exponential chart under the conditional perspective can be obtained from the following equations

$$P[CATS_{IC} \geq ATS_0] = 1 - \gamma \quad (20)$$

$$1 - \left(1 + \frac{A_L^*}{m-1}\right)^{-m} = \left(1 + \frac{A_U^*}{m-1}\right)^{-m} \quad (21)$$

and for the ATS-unbiased exponential chart from the following equations

$$P[CATS_{IC} \geq ATS_0] = 1 - \gamma \quad (22)$$

$$\left. \frac{d}{d\delta} \mu_1'(\delta) \right|_{\delta=1} = 0 \quad (23)$$

respectively where  $P[CATS_{IC} \geq ATS_0]$  can be obtained from Equation (14) by replacing  $A_L$  and  $A_U$  by  $A_L^*$  and  $A_U^*$ , respectively in  $\varphi_{IC}(\cdot)$ . Equation (21) is obtained by equating the two unconditional probabilities  $E \left[ P[X < \widehat{LCL}^* | IC] \right]$  and  $E \left[ P[X > \widehat{UCL}^* | IC] \right]$  which can be obtained from Equations (E.1) and (E.2) by replacing  $A_L$  and  $A_U$  by  $A_L^*$  and  $A_U^*$ . The  $\mu_1'(\delta)$  can be obtained from Equation (11) using  $A_1^*$  and  $A_2^*$  instead of  $A_L^*$  and  $A_U^*$ , respectively. The design parameters  $\xi^*$  and  $p^*$  are calculated and reported in Table 3 for both phase II equal-tailed and ATS-unbiased exponential

charts under the conditional perspective corresponding to different values of  $m$ . Using these, the control limits of the equal-tailed and the ATS-unbiased exponential charts under the conditional perspective can be obtained from Equation (3) by replacing  $A_L$  and  $A_U$  by  $A_L^*$  and  $A_U^*$ , respectively. The c.d.f. and p.d.f. of  $CATS(\delta)$  of the respective charts can be obtained from (8) and (9), respectively by plugging in the corresponding design parameters.

#### 4.1. Conditional In-control Performance of phase II exponential charts

The p.d.f.s of the proposed phase II equal-tailed and ATS-unbiased exponential charts with fixed  $EP=0.90$  i.e.  $P[CATS_{IC} \geq ATS_0] = 0.90$  and nominal  $ATS_0 = 370.4$  are shown in Figures 3 and 4, respectively for  $m = 20, 50, 100, 500$  and  $\lambda_0 = 1$ . It can be seen from these figures that the distribution of  $CATS_{IC}$  of the ATS-unbiased exponential chart adjusted under the conditional perspective is more skewed to the left than that of the corresponding equal-tailed exponential chart, except for smaller values of  $m$ . Note that the  $CATS_{IC}$  distribution of both exponential charts under the conditional perspective becomes more symmetrical as  $m$  gets larger, however, the distribution of the ATS-unbiased exponential chart under the conditional perspective is more skewed than the equal-tailed chart under the conditional perspective, even for large  $m$ .

Insert Figures 3 and 4

In order to examine the IC performance of the phase II exponential charts under the conditional perspective, various metrics are calculated from the exact distribution of  $CATS_{IC}$ . These metrics are reported in Table 3. The following observations can be made.

1. The mean of  $CATS_{IC}$  i.e.  $ACATS_{IC}$  values of both equal-tailed and ATS-unbiased exponential charts under the conditional perspective are much higher than the nominal  $ATS_0$ . It is the price to be paid to ensure that the  $CATS_{IC}$  will meet or exceed the nominal  $ATS_0$  with a guarantee of 90% probability. Similar observations have been made by Jardim et al. [46] for the Shewhart chart for the mean. It is observed that the  $ACATS_{IC}$  values of the adjusted ATS-unbiased exponential chart under the conditional perspective are closer to  $ATS_0$  than the adjusted equal-tailed exponential chart under the conditional perspective except for smaller values of  $m < 30$ . The difference between the  $ACATS_{IC}$  values of both charts under the conditional perspective becomes smaller with the increase in  $m$ .
2. There is a significant difference in the conditional IC performances of the two types of charts. The study reveals that for smaller values of  $m$ , there is large variability in the  $CATS_{IC}$



values of the ATS-unbiased exponential chart under the conditional perspective relative to the corresponding equal-tailed chart. However, for  $m \geq 30$ , the former has lower  $SDCATS_{IC}$  values than the latter. For example, when  $m = 100$ , the  $SDCATS_{IC}$  is 108.67 for the ATS-unbiased exponential chart under the conditional perspective whereas it is 214.18 for the equal-tailed exponential chart under the conditional perspective.

3. Due to a lower variability in the IC CATS values, the ATS-unbiased exponential chart under the conditional perspective needs less phase I observations than the corresponding adjusted equal-tailed chart to keep  $SDCATS_{IC}$  value under the conditional perspective within 10% of the  $ATS_0$ . It can be observed from the last column of Table 3 that about 500 phase I observations are required for the ATS-unbiased exponential chart under the conditional perspective whereas more than 1000 observations are required for the equal-tailed chart under the conditional perspective to keep  $SDCATS_{IC}$  value below 10% of the  $ATS_0$ . However, as it was stated earlier, collecting such a large number of phase I observations may not always be possible and therefore, one may need a trade off with the IC performance. In this scenario (less phase I observations), the equal-tailed ATS-unbiased exponential chart under the conditional perspective is preferred to the corresponding ATS-unbiased chart.

Insert Table 3

#### 4.2. Conditional Out-of-control performance of the phase II exponential charts

In this section, we examine the conditional performance of the proposed equal-tailed and ATS-unbiased charts under the conditional perspective in an OOC situation. For this, we consider the same  $\delta$  values and the  $m$  values as in the OOC study of the charts under the unconditional perspective. The various performance metrics of the proposed charts are provided in Table 4 when  $ATS_0 = 370.4$ ,  $\lambda_0 = 1$  and  $EP=0.90$ .

It can be observed from Table 4 that the findings are same as found in the OOC study of the equal-tailed and ATS-unbiased exponential charts under the unconditional perspective. For example, when  $\delta < 1$  ( $\delta > 1$ ) and the phase I sample size  $m$  is small, the equal-tailed exponential chart under the conditional perspective has larger (smaller) OOC mean and percentiles of the  $CATS_{OOOC}$  distribution than those for the ATS-unbiased exponential chart under the conditional perspective whereas for the moderate to large values of  $m$ , the opposite pattern is observed i.e. the OOC mean and percentiles of the equal-tailed exponential chart under the conditional perspective are smaller (larger) than the corresponding values of the ATS-unbiased exponential chart under the conditional perspective.

Like the ATS-unbiased exponential chart under the unconditional perspective, for a moderate to large phase I sample size, the ATS-unbiased exponential chart under the conditional perspective has better OOC performance in terms of less  $ACATS_{OOOC}$  values than the equal-tailed exponential chart when the process deteriorates.

Overall, for moderate to large phase I sample size, the ATS-unbiased exponential chart under both perspectives outperforms than the corresponding equal-tailed exponential chart in the deterioration case, however, for the improvement case, the former has worse performance than the latter. We should mention here that though the performance of the phase II (equal-tailed or ATS-unbiased) exponential chart under the conditional perspective seems to be worse than the corresponding exponential chart under the unconditional perspective. However, one should keep in mind that the IC performance of the charts under both perspectives are not comparable in the sense that the  $ACATS_{IC}$  of the exponential charts under the conditional perspective are larger than the corresponding charts under the unconditional perspectives. Note that these observations are similar to the those of [46] that the chart under the conditional perspective neither control the unconditional performance ( $ACATS_{IC}$ ) nor its variability ( $SDCATS_{IC}$ ) and produces larger  $ACATS_{OOOC}$  values than the chart under the unconditional perspective.

Insert Table 4

Finally, in order to help practitioners in implementing the exponential chart with estimated parameters, we provide in Tables 5 and 6 the required charting constants of both the equal-tailed and ATS-unbiased exponential charts for different  $\lambda_0$ . A program to obtain the charting constants, written in MATLAB, is available from the authors on request.

Insert Tables 5,6

## 5. Example

In order to illustrate the application of the proposed phase II equal-tailed and ATS-unbiased exponential charts, adjusted under the conditional perspective, we first use the data given by Jarrett [48]. The data comprises 190 recorded time intervals (in days) between coal-mining disasters in England, from 15 March, 1851 to 22 March, 1962 and have been extensively used in the literature by a number of authors for illustration of various procedures. The data are shown in Table 7 for convenience. It has been tested that the time intervals between successive explosions follow an exponential distribution and we monitor the unknown rate parameter  $\lambda$  of this distribution [48]. We

should mention here that according to Jarrett [48], the first 125 time intervals between the explosions follow an exponential distribution with a mean rate of (an explosion) one every 106 days while for the next 65 explosions, the mean rate is about one every 388 days.

In order to examine the ability of the proposed charts under both perspectives to trace the improvement in the process with small phase I sample, we consider the first  $m = 15$  observations to be from an IC process and construct the proposed exponential charts under both perspective, from which we estimate that  $\hat{\lambda}_0 = 0.0072$  (or, equivalently, the sum of 15 TBEs is 1937 days). The control limits of the charts under the conditional perspective are constructed for the nominal  $ATS_0 = 40,000$  days for which the design parameters  $\xi^*$  and  $p^*$  are obtained solving Equations (20)-(21) and (22)-(23), respectively. Note that the  $ATS_0 = 40,000$  days means a false alarm is expected on the average, every  $40,000/106=377$  explosions. The lower and upper control limits of the equal-tailed exponential chart under the conditional perspective are found to be 0.0839, 1222.4406 respectively whereas these are found to be 0.0331, 1191.3600 respectively for the ATS-unbiased exponential chart under the conditional perspective. For the unconditional perspective, the lower and upper control limits of the equal-tailed exponential chart are found to be 0.2527 and 998.7904, respectively, whereas these limits are 0.2084 and 904.6048, respectively, for the ATS-unbiased exponential chart. The four pairs of the control limits are depicted in Figure 5. It follows from Figure 5 that except the equal-tailed chart under the conditional perspective, all the other charts show an OOC alarm at the 119th time point of the phase II monitoring (corresponds to the 134th observation) whereas the equal-tailed exponential chart under the conditional perspective does not signal at this point. This may be explained by the fact that the conditional perspective produces wider limits which leads to poor OOC performance (see Table 4).

We mention here that the OOC signals indicated by the charts are on the upper side which indicates a decrease in the mean rate of explosions per day. This is consistent with the fact that with more technological advances, coal-mine explosions occur less frequently nowadays. As was stated earlier, Jarrett [48] showed that the last 65 time intervals had an decreased mean rate of explosion per day than the earlier 125 time intervals. The proposed ATS-unbiased exponential chart under the conditional perspective detects the change in the mean rate of explosions earlier, after the 9th time interval since starting the improvement in the process whereas the equal-tailed exponential chart under the conditional perspective detects this change after the 28th time interval of the starting the improvement.

Insert Table 7

Insert Figure 5

## 6. Concluding Remarks

In this paper, we consider the phase II exponential chart, which is one of the most widely used control chart for monitoring TBE data from high-yield processes. To understand the impact of parameter estimation, and to help the user better implement the chart in practice, we study the conditional chart performance in terms of the time to signal and the metric ATS, which is known to be more appropriate than the traditional run length and the ARL in case of TBE data. To study the chart's properties, the exact distribution of the CATS was derived, and the (adjusted) control limits of both the equal-tailed and the ATS-unbiased exponential charts fixed nominal  $ATS_0$  were found under two perspectives for a fixed nominal  $ATS_0$ . Under the unconditional perspective, which sets the average of the IC CATS distribution to a fixed nominal  $ATS_0$  value while finding the adjusted control limits, the study showed that both charts have poor IC conditional performance in terms of very high practitioner-to-practitioner variability ( $SDCATS_{IC}$  values), low LPB and low EP values. Alternatively, under the conditional perspective, the control limits of both charts were calculated so that the  $CATS_{IC}$  is at least equal to a specified value i.e.  $ATS_0$  value with a high probability. For these charts, the performance study revealed good IC conditional performance, even though they have increased mean  $CATS_{IC}$  than the nominal  $ATS_0$ , this may be considered as the price to pay for a guaranteed and improved IC performance. It was observed that for moderate to large phase I sample sizes  $m \geq 30$ , the adjusted ATS-unbiased exponential chart under the conditional perspective has less variability in  $CATS_{IC}$  values than the corresponding equal-tailed chart. Nevertheless, if a practitioner wishes to control both the metrics, for example, to keep  $SDCATS_{IC}$  within 10% of  $ATS_0$  and  $EP=0.90$ , i.e., a guarantee that the  $CATS_{IC}$  will meet or exceed  $ATS_0$  with 90% probability, then about 400 phase I observations are required for the adjusted ATS-unbiased exponential chart whereas more than 1000 observations are needed for the adjusted equal-tailed exponential chart under the conditional perspective. Moreover, the adjusted ATS-unbiased exponential chart under the conditional perspective has a good OOC performance in the deterioration case for moderate to large values of  $m$ . Thus, the phase II ATS-unbiased exponential chart adjusted under the conditional perspective is recommended to monitor TBE individual data from high-yield processes following an exponential distribution with an unknown rate parameter.

Finally, note that we have only considered the simpler Shewhart-type charts; however, the same principles can be extended to a more general TBE chart such as the  $t_r$ -chart [1, 22], which monitors the time to  $r^{\text{th}}$  event/failure. However, there may be some concern about loss of information with  $t_r$ -type charts since data collection at the end of an aggregation period (say at the  $r^{\text{th}}$  event) may have an adverse effect on the chart's performance [44, 45]. It will be interesting to examine the effect of aggregation on the  $t_r$ -chart's performance with the estimated control limits constructed under the ATS criteria. Note also that when the failures in a process occur due to sudden shock instead of a slow wear and tear, the exponential distribution is a more suitable distribution to model time between events/failures. However, when the failure rate changes over time, then the Weibull distributions is more appropriate and can be considered as a probability model for TBE data. Other non-Shewhart-type charts such as the CUSUM exponential and the EWMA exponential charts can be considered in a future study.

### Acknowledgement

The authors thank the anonymous referees and the editor for their valuable comments.

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## Appendices

### A. Distribution of time to signal (TS)

For the known parameter case, the run length variable,  $R$  follows a geometric distribution with parameter  $\beta(\delta)$ , thus the probability mass function of  $R$  is given by

$$P[R = j] = \beta(\delta)(1 - \beta(\delta))^{j-1}, j = 1, 2, \dots$$

Thus, using the fact that sum of  $j$  independent and identically distributed exponential variables with mean  $\lambda_1 = \delta\lambda_0$  follows a gamma distribution with PDF

$$f_{\Gamma}(t) = \frac{\lambda_1^j}{\Gamma(j)} t^{j-1} e^{-\lambda_1 t}, t > 0, \lambda_1 > 0$$

the probability distribution of time to signal, TS is given by



$$\begin{aligned}
P[\text{TS} \leq s] &= P[\sum_{i=1}^R X_i \leq s] \\
&= \sum_{j=1}^{\infty} P[\sum_{i=1}^j X_i \leq s | R = j] P[R = j] \\
&= \sum_{j=1}^{\infty} \left( \int_0^s \frac{\lambda_1^j}{\Gamma(j)} t^{j-1} e^{-\lambda_1 t} dt \right) \beta(\delta) (1 - \beta(\delta))^{j-1} \\
&= \int_0^s \lambda_1 \beta(\delta) e^{-\lambda_1 t} \left( \sum_{j=1}^{\infty} \frac{(\lambda_1 (1 - \beta(\delta)) t)^{j-1}}{\Gamma(j)} \right) dt \\
&= \int_0^s \lambda_1 \beta e^{-\lambda_1 t} e^{\lambda_1 (1 - \beta(\delta)) t} dt \\
&= 1 - e^{-\lambda_1 \beta(\delta) s}, s > 0
\end{aligned}$$

## B. Proof of Lemma 1

To show  $\varphi(T) = T \left[ 1 - \exp\left(-\frac{\delta\lambda_0 T A_L}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 T A_U}{m-1}\right) \right]^{-1}$  is an increasing function of  $T \in (0, \infty)$ , we must have the first derivative  $\varphi'(T) > 0$  for all  $T \in (0, \infty)$ . Let  $\zeta(T) = \left[ 1 - \exp\left(-\frac{\delta\lambda_0 T A_L}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 T A_U}{m-1}\right) \right]$ , then

$$\varphi'(T) = \frac{\zeta(T) - T\zeta'(T)}{[\zeta(T)]^2}$$

where  $\zeta'(T)$  is the first derivative of  $\zeta(T)$ . Because  $\zeta(T) > 0$  for all  $T$ , then  $\varphi'(T) > 0$  implies  $\zeta(T) - T\zeta'(T) > 0$ .

Let us define  $P(T) = \zeta(T) - T\zeta'(T)$ . It can be easily shown that  $\zeta(T)$  attains its minimum at  $T = M_1 = \frac{\ln(A_U/A_L)}{\kappa(A_U - A_L)}$  so that  $\zeta'(M_1) = 0$  and  $\zeta''(M_1) > 0$ . Further,  $\zeta'(T)$  also attains its maximum at  $T = M_2$  so that  $\zeta''(M_2) = 0$  and  $\zeta'''(M_2) < 0$  where  $M_2 = \frac{2 \ln(A_U/A_L)}{\kappa(A_U - A_L)}$ .

Since,  $P'(T) = -T\zeta''(T)$  and  $P'(T) = 0$  implies either  $T = 0$  or  $\zeta''(T) = 0$  which yields solution  $T = M_2$  in  $(0, \infty)$ . Again,  $P''(T) = -T\zeta'''(T) - \zeta''(T)$  and hence  $P''(0) < 0$  and  $P''(M_2) = -M_2\zeta'''(M_2) > 0$ .

This implies that the function  $P(T)$  attains its minimum at  $T = M_2$  in  $(0, \infty)$ . Thus the minimum value of  $P(T)$  is  $P(M_2) > 0$ . It implies that  $\varphi'(T) > 0$  for all  $T \in (0, \infty)$  and consequently, the function  $\varphi(T)$  is an increasing on  $(0, \infty)$ .

## C. Proof of Theorem 1

By definition, the c.d.f. of the CATS( $\delta$ ) is

$$G(z) = P[\text{CATS}(\delta) \leq z] = P[\varphi(T) \leq \delta(m-1)z]$$

Using Lemma 1, we can write

$$G(z) = P[T \leq \varphi^{-1}(\delta(m-1)z)] = F_{\Gamma}(\varphi^{-1}(\delta(m-1)z))$$

The p.d.f. of the CATS( $\delta$ ) in Equation (9) can be obtained by differentiating the c.d.f. in Equation (8) with respect to  $z$ . Further note that the function  $\varphi^{(-1)}(\cdot)$  has no closed form expression, so must be computed numerically.

#### D. Proof of Theorem 2

The  $q^{\text{th}}$  ( $0 < q < 1$ ) quantile of CATS( $\delta$ ) distribution, denoted by  $\text{CATS}_q(\delta)$ , can be obtained by the equation  $G(\text{CATS}_q(\delta)) = q$ , that is.

$$F_{\Gamma}(\varphi^{-1}(\delta(m-1)\text{CATS}_q(\delta))) = q$$

which gives

$$\varphi^{-1}(\delta(m-1)\text{CATS}_q(\delta)) = F_{\Gamma}^{-1}(q),$$

where  $F_{\Gamma}^{-1}(\cdot)$  is the  $q$ -th quantile of a gamma distribution with parameters  $m$  and  $\lambda_0$ . Using Lemma 1, we have

$$\delta(m-1)\text{CATS}_q(\delta) = \varphi(F_{\Gamma}^{-1}(q))$$

which gives the  $\text{CATS}_q(\delta)$  in Equation (10).

#### E. Proof of Theorem 3

**Proof.** The  $r^{\text{th}}$  moment of CATS( $\delta$ ) is

$$\mu'_r(\delta) = E[\text{CATS}(\delta)]^r = \int_0^{\infty} z^r g(z) dz.$$

Using the transformation  $t \left[ 1 - \exp\left(-\frac{\delta\lambda_0 t A_L}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 t A_U}{m-1}\right) \right]^{-1} = \delta(m-1)z$ , we obtain

$$\mu'_r(\delta) = \frac{1}{(\delta(m-1))^r} \int_0^{\infty} t^r \left[ 1 - \exp\left(-\frac{\delta\lambda_0 A_L t}{m-1}\right) + \exp\left(-\frac{\delta\lambda_0 A_U t}{m-1}\right) \right]^{-r} \frac{\lambda_0^m}{\Gamma(m)} t^{m-1} e^{-\lambda_0 t} dt$$

For  $m > 1, \delta > 0, \lambda_0 > 0$  and  $t > 0$ , we have  $\left| \exp\left(-\frac{\delta\lambda_0 A_L t}{m-1}\right) - \exp\left(-\frac{\delta\lambda_0 A_U t}{m-1}\right) \right| < 1$ . Thus, using the negative binomial series expansion in the above expression, we get

$$\begin{aligned} \mu'_r(\delta) &= \frac{1}{(\delta(m-1))^r} \int_0^\infty \sum_{j=0}^\infty (-1)^j \binom{-r}{j} \left[ \exp\left(-\frac{\delta\lambda_0 A_L t}{m-1}\right) - \exp\left(-\frac{\delta\lambda_0 A_U t}{m-1}\right) \right]^j \\ &\quad \times \frac{\lambda_0^m}{\Gamma(m)} t^{m+r-1} e^{-\lambda_0 t} dt \end{aligned}$$

Again, using the binomial theorem, we can re-express  $\mu'_r(\delta)$  as follows.

$$\begin{aligned} \mu'_r(\delta) &= 1/(\delta(m-1))^r \int_0^\infty \sum_{j=0}^\infty \sum_{i=0}^j (-1)^{j+i} \binom{-r}{j} \binom{j}{i} \exp\left(-\frac{\delta\lambda_0 t}{m-1} [(j-i)A_L + A_U i]\right) \times \\ &\quad \frac{\lambda_0^m}{\Gamma(m)} t^{m+r-1} e^{-\lambda_0 t} dt \end{aligned}$$

Since the summation is absolutely convergent, we can interchange the summation and the integral.

We then have

$$\begin{aligned} \mu'_r(\delta) &= \frac{1}{(\delta(m-1))^r} \sum_{j=0}^\infty \sum_{i=0}^j (-1)^{j+i} \binom{-r}{j} \binom{j}{i} \int_0^\infty \frac{\lambda_0^m}{\Gamma(m) 2^m} t^{m+r-1} \times \exp\left(-\lambda_0 t \left[1 + \frac{\delta}{m-1} \{(j-i)A_L + rA_U\}\right]\right) dy \end{aligned}$$

On simplification, we have

$$\mu'_r(\delta) = \frac{\Gamma(m+r)}{(\delta\lambda_0(m-1))^r \Gamma(m)} \sum_{j=0}^\infty \sum_{i=0}^j (-1)^{j+i} \binom{-r}{j} \binom{j}{i} \left[1 + \frac{\delta}{m-1} \{(j-i)A_L + A_U i\}\right]^{-m}$$

We mention here that the infinite series in Equation (11) is bounded that is

$$\begin{aligned} \frac{\Gamma(m+r)}{(\delta\lambda_0(m-1))^r \Gamma(m)} \left[1 - \left(1 + \frac{\delta A_1}{m-1}\right)^{-m} \left\{1 - \left(1 + \frac{\delta(A_2 - A_1)}{m-1}\right)^{-m}\right\}\right]^{-r} &\leq \mu'_r(\delta) \leq \\ \frac{\Gamma(m+r)}{(\delta\lambda_0(m-1))^r \Gamma(m)} \min\left\{\left(1 - \frac{\delta A_2 r}{m-1}\right)^{-m}, \left(\frac{m}{A_1}\right)^r \frac{\Gamma(m-r)}{\Gamma(m)}\right\} & \end{aligned}$$

and hence, it always converges. The reader may refer to Appendix B of [28] to obtain bounds of series (11).

## F. Unconditional probabilities

The unconditional probability that a charting point lies below the lower control limits  $\widehat{LCL}$  is

$$\int_0^\infty P[X < \widehat{LCL}] f_\Gamma(t) dt = \int_0^\infty \left(1 - e^{-\frac{\delta\lambda_0 A_L t}{m-1}}\right) \frac{\lambda_0^m}{\Gamma(m)} t^{m-1} e^{-\lambda_0 t} dt$$

$$\begin{aligned}
&= 1 - \int_0^{\infty} \frac{\lambda_0^m}{\Gamma(m)} t^{m-1} e^{-\lambda_0 t \left(1 + \frac{\delta A_L}{m-1}\right)} dt \\
&= 1 - \left(1 + \frac{\delta A_L}{m-1}\right)^{-m} \tag{E.1}
\end{aligned}$$

In like manner, we can obtain the unconditional probability that a charting point lies above the upper control limits  $\widehat{UCL}$  which is

$$\int_0^{\infty} P[X > \widehat{UCL}] f_{\Gamma}(t) dt = \left(1 + \frac{\delta A_U}{m-1}\right)^{-m} \tag{E.2}$$

For the IC process, the unconditional probabilities can be obtained from (C.1)-(C.2) respectively by letting  $\delta = 1$ .

**Table 1.** The performance metrics of phase II equal-tailed and ATS-unbiased exponential chart under unconditional perspective with fixed nominal  $ATS_0=370.4$  and  $\lambda_0 = 1$

	Design Parameters			ACATS <sub>IC</sub>	SDCATS <sub>IC</sub>	Percentiles					EP	CV
	$m$	$\xi$	$p$			10%	25%	50%	75%	90%		
Equal-tailed	10	0.797302	0.002491	370.4	152.5	104.2	269.6	443.8	495.5	502.8	0.63	41.17
	15	0.712028	0.002608	370.4	156.8	115.0	251.2	422.8	509.7	532.3	0.59	42.32
	20	0.663459	0.002673	370.4	156.5	126.4	247.7	407.7	510.9	548.0	0.57	42.24
	30	0.611706	0.002732	370.4	151.5	146.1	251.3	391.0	502.7	558.5	0.54	40.91
	50	0.568362	0.002761	370.4	138.9	174.8	263.6	378.6	483.9	552.6	0.52	37.51
	100	0.534765	0.002755	370.4	115.0	215.7	285.5	371.7	456.7	523.4	0.50	31.04
	200	0.517561	0.002736	370.4	89.3	253.2	306.6	370.0	433.8	488.2	0.50	24.11
	500	0.507074	0.002717	370.4	60.2	292.7	328.6	369.9	411.7	448.8	0.50	16.25
	1000	0.503546	0.002709	370.4	43.5	314.5	340.5	370.1	399.9	426.7	0.50	11.75
ATS-unbiased	10	0.365339	0.003731	370.4	248.9	42.5	127.3	361.0	619.0	710.2	0.49	67.19
	15	0.509215	0.003010	370.4	202.4	78.3	186.3	389.8	562.9	628.4	0.52	54.64
	20	0.583302	0.002802	370.4	174.8	109.6	223.0	396.2	530.4	585.6	0.54	47.18
	30	0.653178	0.002682	370.4	142.0	156.0	263.0	395.9	493.9	539.9	0.55	38.34
	50	0.702982	0.002646	370.4	109.1	208.9	296.1	390.1	460.2	498.7	0.56	29.47
	100	0.735050	0.002655	370.4	76.3	262.2	322.0	382.2	429.7	460.5	0.56	20.59
	200	0.748781	0.002672	370.4	53.4	297.1	336.9	376.8	410.3	434.4	0.55	14.40
	500	0.756045	0.002688	370.4	33.4	325.7	349.1	373.1	394.5	411.4	0.53	9.02
	1000	0.758269	0.002694	370.4	23.5	339.3	355.2	371.7	387.0	399.6	0.52	6.35

**Table 2.** OOC performance of the equal-tailed and ATS-unbiased exponential chart under the unconditional perspective with nominal  $ATS_0 = 370.4$ 

		Equal-tailed							ATS-unbiased						
$\delta$	$m$	$ACATS_{IC}$	$SDCATS_{IC}$	10%	25%	50%	75%	90%	$ACATS_{IC}$	$SDCATS_{IC}$	10%	25%	50%	75%	90%
0.25	10	58.5	91.6	10.3	17.5	32.9	64.8	124.3	33.3	41.1	7.9	12.6	21.8	39.0	68.2
	15	38.3	38.1	11.0	16.8	27.5	46.2	75.7	30.9	27.9	9.7	14.5	23.1	37.5	59.5
	20	32.0	24.5	11.7	16.7	25.4	39.2	59.1	29.6	22.0	11.1	15.8	23.7	36.3	54.2
	30	27.2	15.4	12.6	16.9	23.5	33.3	45.9	28.2	16.2	12.9	17.4	24.3	34.5	47.9
	50	24.2	9.9	13.8	17.3	22.3	29.0	36.9	27.1	11.5	15.1	19.0	24.7	32.5	41.8
	100	22.4	6.2	15.3	18.0	21.5	25.8	30.5	26.1	7.6	17.5	20.7	25.0	30.2	36.1
	200	21.6	4.1	16.7	18.6	21.1	24.0	27.0	25.6	5.2	19.4	21.9	25.0	28.6	32.4
	500	21.1	2.5	18.0	19.3	20.9	22.7	24.4	25.2	3.2	21.3	23.0	25.0	27.2	29.5
1000	21.0	1.8	18.8	19.7	20.9	22.1	23.3	25.1	2.3	22.3	23.5	25.0	26.6	28.1	
0.5	10	273.9	369.8	18.9	43.7	119.3	335.5	775.9	136.7	242.6	11.1	22.8	54.3	138.0	330.8
	15	168.4	225.5	20.6	39.8	87.1	197.4	410.2	120.2	173.6	16.1	29.8	62.2	135.3	278.3
	20	126.3	153.3	22.1	38.9	75.5	150.8	284.0	110.5	136.0	20.0	34.8	66.4	130.8	244.9
	30	93.4	90.3	24.6	38.9	66.1	114.7	190.8	99.9	96.8	25.9	41.2	70.5	123.0	205.3
	50	73.6	51.2	28.1	40.0	60.0	91.2	134.2	90.7	64.7	33.4	48.1	73.3	113.0	168.0
	100	62.2	28.4	33.0	42.4	56.3	75.4	98.5	83.4	39.9	42.6	55.5	75.0	101.7	134.4
	200	57.5	17.9	37.5	44.8	54.8	67.2	80.9	79.6	26.1	50.5	61.0	75.4	93.6	113.9
	500	55.0	10.6	42.4	47.5	53.9	61.3	68.9	77.2	15.7	58.5	66.0	75.6	86.6	98.0
1000	54.2	7.3	45.3	49.1	53.7	58.8	63.8	76.4	10.9	63.1	68.7	75.6	83.2	90.8	
2	10	124.5	8.3	124.7	125.9	126.0	126.1	126.1	175.4	24.5	163.2	181.1	183.4	183.6	183.7
	15	133.7	5.9	133.5	134.6	134.8	134.8	134.9	160.5	11.1	158.4	162.5	163.2	163.3	163.4
	20	140.3	4.3	139.9	140.9	141.1	141.1	141.2	151.9	5.8	151.0	152.7	153.1	153.1	153.2
	30	149.2	2.5	148.7	149.5	149.7	149.7	149.8	142.4	2.1	142.1	142.7	142.8	142.9	142.9
	50	159.1	1.2	158.7	159.2	159.4	159.5	159.5	134.4	0.5	134.3	134.5	134.5	134.6	134.6
	100	169.6	0.4	169.3	169.6	169.7	169.8	169.9	128.2	0.1	128.1	128.2	128.2	128.3	128.3
	200	176.5	0.2	176.3	176.5	176.6	176.6	176.7	125.0	0.0	125.0	125.0	125.1	125.1	125.1
	500	181.4	0.1	181.3	181.4	181.5	181.5	181.6	123.2	0.0	123.1	123.1	123.2	123.2	123.2
1000	183.3	0.1	183.2	183.2	183.3	183.3	183.3	122.5	0.0	122.5	122.5	122.5	122.5	122.5	
4	10	31.6	0.2	31.5	31.5	31.6	31.6	31.6	45.9	0.6	45.9	45.9	46.0	46.0	46.0
	15	33.8	0.1	33.7	33.7	33.8	33.8	33.8	40.9	0.1	40.8	40.9	40.9	40.9	40.9
	20	35.3	0.0	35.3	35.3	35.3	35.4	35.4	38.3	0.0	38.3	38.3	38.3	38.4	38.4
	30	37.5	0.0	37.5	37.5	37.5	37.5	37.5	35.8	0.0	35.7	35.8	35.8	35.8	35.8
	50	39.9	0.0	39.9	39.9	39.9	39.9	40.0	33.7	0.0	33.7	33.7	33.7	33.7	33.7
	100	42.5	0.0	42.5	42.5	42.5	42.5	42.5	32.1	0.0	32.1	32.1	32.1	32.1	32.1

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200	44.2	0.0	44.2	44.2	44.2	44.2	44.2	31.3	0.0	31.3	31.3	31.3	31.3	31.3
500	45.5	0.0	45.5	45.5	45.5	45.5	45.5	30.9	0.0	30.8	30.9	30.9	30.9	30.9
1000	45.9	0.0	45.9	45.9	45.9	45.9	45.9	30.7	0.0	30.7	30.7	30.7	30.7	30.7

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**Table 3.** The performance metrics of phase II equal-tailed and ATS-unbiased exponential chart under conditional perspective with fixed  $P[CATS_{IC} \geq ATS_0] = 0.90$ , nominal  $ATS_0 = 370.4$  and  $\lambda_0 = 1$

		Design parameters					Percentiles						
		$m$	$\xi$	$p$	$ACATS_{IC}$	$SDCATS_{IC}$	10%	25%	50%	75%	90%	EP	CV
Equal-tailed	10	0.899340	0.000743		1180.4	440.2	370.4	997.9	1424.2	1491.8	1497.3	0.90	118.85
	15	0.805136	0.000766		1191.1	476.0	370.4	881.5	1407.3	1584.6	1615.2	0.90	128.51
	20	0.737654	0.000835		1132.3	464.8	370.4	791.8	1293.5	1539.8	1604.6	0.90	125.48
	30	0.658787	0.000981		1003.6	412.3	370.4	684.0	1088.9	1369.7	1485.2	0.90	111.33
	50	0.591891	0.001212		835.0	322.9	370.4	585.6	864.4	1106.3	1249.6	0.90	87.19
	100	0.543142	0.001540		661.3	214.2	370.4	502.1	665.6	824.7	945.8	0.90	57.83
	200	0.520437	0.001833		552.9	138.7	370.4	453.5	552.6	652.1	736.1	0.90	37.44
	500	0.507768	0.002131		472.4	79.0	370.4	417.5	471.8	526.7	575.3	0.90	21.32
	1000	0.503784	0.002292		437.9	52.5	370.4	401.8	437.5	473.6	505.9	0.90	14.18
ATS-unbiased	10	0.403709	0.000182		7412.7	5179.5	370.4	1876.0	8029.9	12884.3	13567.8	0.90	1398.46
	15	0.567256	0.000432		2451.2	1377.7	370.4	1144.4	2754.1	3795.9	4035.6	0.90	371.95
	20	0.634341	0.000671		1486.3	716.8	370.4	875.8	1658.7	2153.1	2301.1	0.90	193.54
	30	0.689898	0.001035		938.8	365.3	370.4	666.9	1025.9	1259.5	1351.9	0.90	98.64
	50	0.726065	0.001449		669.8	199.2	370.4	537.4	712.4	834.8	896.5	0.90	53.78
	100	0.747894	0.001865		525.5	108.7	370.4	457.6	544.0	610.4	651.9	0.90	29.34
	200	0.756494	0.002139		462.2	66.7	370.4	420.7	470.7	512.2	541.7	0.90	18.01
	500	0.760345	0.002362		421.2	38.0	370.4	397.2	424.4	448.7	467.8	0.90	10.27
	1000	0.761151	0.002467		404.3	25.7	370.4	387.8	405.8	422.5	436.2	0.90	6.94



**Table 4.** OOC performance of the equal-tailed and ATS-unbiased exponential chart under the conditional perspective with nominal  $ATS_0 = 370.4$  and  $P[CATS_{IC} \geq ATS_0] = 0.90$

$\delta$	$m$	Equal-tailed								ATS-unbiased					
		$ACATS_{IC}$	$SDCATS_{IC}$	10%	25%	50%	75%	90%	$ACATS_{IC}$	$SDCATS_{IC}$	10%	25%	50%	75%	90%
0.25	10	124.5	277.8	14.3	26.4	55.0	122.3	265.3	110.5	281.2	13.4	24.4	49.9	108.6	231.5
	15	65.8	84.5	14.9	24.0	42.0	76.6	135.7	61.3	77.3	14.3	22.8	39.7	71.5	125.6
	20	49.5	45.9	15.3	22.9	36.6	60.1	96.0	47.8	43.8	15.0	22.3	35.6	58.1	92.5
	30	37.9	24.5	15.9	21.9	31.6	46.5	66.6	38.3	24.9	16.0	22.1	32.0	47.0	67.5
	50	30.7	13.8	16.6	21.2	27.9	37.0	48.2	32.6	14.9	17.4	22.2	29.4	39.3	51.4
	100	26.1	7.7	17.5	20.7	25.0	30.3	36.2	29.0	8.8	19.1	22.7	27.6	33.7	40.5
	200	23.9	4.8	18.3	20.5	23.4	26.7	30.2	27.3	5.7	20.6	23.3	26.7	30.6	34.8
	500	22.5	2.7	19.1	20.5	22.3	24.2	26.1	26.2	3.4	22.1	23.8	26.0	28.3	30.6
1000	21.9	1.9	19.5	20.6	21.8	23.1	24.3	25.8	2.3	22.9	24.1	25.6	27.3	28.8	
0.5	10	931.1	1316.9	36.7	99.7	334.0	1151.7	2920.9	1776.6	4948.2	32.4	86.3	288.3	1092.8	3945.4
	15	498.1	779.2	37.4	81.2	204.9	543.2	1294.8	538.6	1102.6	34.5	74.2	186.1	500.2	1270.7
	20	318.8	472.2	38.1	73.2	158.3	356.8	752.8	314.7	507.9	36.5	69.8	150.5	339.7	726.8
	30	189.5	222.4	39.2	65.8	120.2	226.0	406.1	192.5	223.1	39.9	67.0	122.5	230.5	413.5
	50	121.5	97.1	40.9	60.3	94.4	150.2	231.1	134.2	106.3	44.6	66.3	104.4	166.9	256.8
	100	85.9	42.6	43.1	56.4	76.5	104.7	139.6	103.8	52.4	51.1	67.4	92.2	127.1	170.3
	200	71.1	23.3	45.3	54.6	67.4	83.5	101.6	91.0	30.8	56.8	69.0	86.0	107.3	131.4
	500	62.4	12.3	47.7	53.6	61.1	69.7	78.7	83.3	17.3	62.9	71.0	81.5	93.6	106.1
1000	59.1	8.1	49.2	53.4	58.5	64.1	69.8	80.4	11.6	66.2	72.2	79.6	87.7	95.7	
2	10	371.8	20.1	373.6	374.4	374.5	374.6	374.7	3307.5	416.1	3301.2	3404.3	3408.5	3408.6	3408.7
	15	403.5	14.2	404.2	405.2	405.3	405.4	405.4	1010.7	60.3	1012.3	1020.8	1021.6	1021.6	1021.7
	20	404.8	9.8	405.0	406.0	406.2	406.2	406.2	584.4	18.6	584.5	586.9	587.3	587.3	587.4
	30	386.3	5.0	386.0	386.8	387.0	387.1	387.1	349.7	4.1	349.5	350.1	350.3	350.3	350.4
	50	348.4	1.9	347.9	348.5	348.7	348.8	348.8	237.7	0.7	237.5	237.8	237.8	237.9	237.9
	100	298.7	0.6	298.4	298.7	298.9	298.9	299.0	179.3	0.1	179.2	179.3	179.4	179.4	179.4
	200	262.0	0.2	261.8	262.0	262.1	262.2	262.2	154.6	0.0	154.5	154.6	154.6	154.6	154.6
	500	231.1	0.1	231.0	231.0	231.1	231.2	231.2	139.3	0.0	139.3	139.3	139.3	139.3	139.3
1000	216.6	0.1	216.5	216.5	216.6	216.6	216.6	133.2	0.0	133.2	133.2	133.2	133.3	133.3	
4	10	93.7	0.4	93.7	93.7	93.7	93.7	93.8	851.9	9.6	852.2	852.2	852.2	852.2	852.3
	15	101.4	0.1	101.4	101.4	101.4	101.4	101.5	255.5	0.5	255.4	255.4	255.5	255.5	255.5
	20	101.6	0.0	101.6	101.6	101.6	101.6	101.6	146.9	0.1	146.9	146.9	146.9	146.9	146.9

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30	96.8	0.0	96.8	96.8	96.8	96.8	96.9	87.6	0.0	87.6	87.6	87.6	87.7	87.7
50	87.3	0.0	87.2	87.2	87.3	87.3	87.3	59.5	0.0	59.5	59.5	59.5	59.5	59.6
100	74.8	0.0	74.8	74.8	74.8	74.8	74.8	44.9	0.0	44.9	44.9	44.9	44.9	44.9
200	65.6	0.0	65.6	65.6	65.6	65.6	65.6	38.7	0.0	38.7	38.7	38.7	38.7	38.7
500	57.9	0.0	57.9	57.9	57.9	57.9	57.9	34.9	0.0	34.9	34.9	34.9	34.9	34.9
1000	54.2	0.0	54.2	54.2	54.2	54.2	54.2	33.4	0.0	33.4	33.4	33.4	33.4	33.4

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**Table 5.** The design constants of the phase II equal-tailed and ATS-unbiased exponential chart under unconditional perspective with fixed nominal  $ATS_0=370.4$  and different  $\lambda_0$

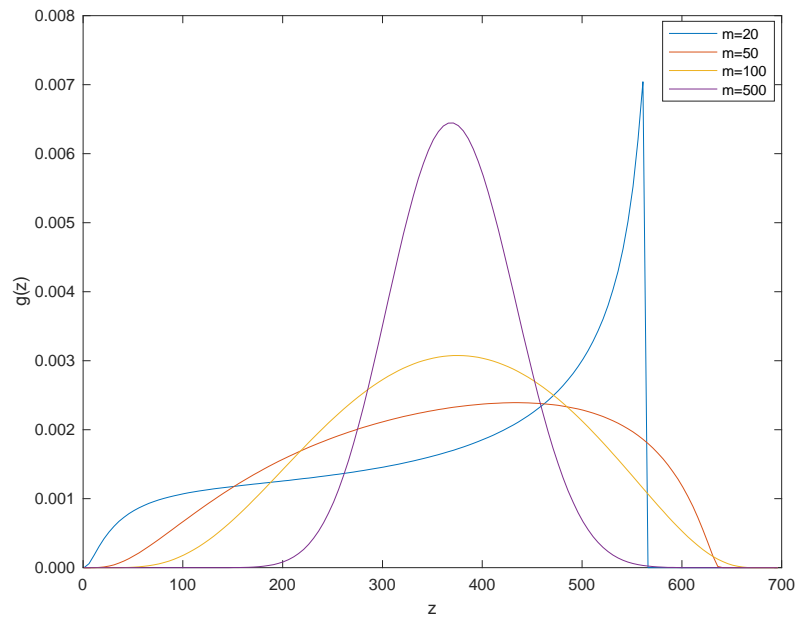
		$\lambda_0$											
		0.01		0.1		0.5		2		5		10	
$m$		$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$
Equal-tailed	10	0.472386	0.306454	0.578325	0.029152	0.726609	0.005220	0.862242	0.001202	0.928880	0.000469	0.961822	0.000234
	15	0.482875	0.294162	0.555894	0.028913	0.658434	0.005407	0.767670	0.001259	0.838260	0.000485	0.884903	0.000237
	20	0.487656	0.288154	0.543449	0.028695	0.621480	0.005492	0.708894	0.001299	0.771148	0.000500	0.817067	0.000244
	30	0.492114	0.282169	0.530104	0.028360	0.582834	0.005551	0.643832	0.001341	0.690283	0.000522	0.727387	0.000255
	50	0.495438	0.277353	0.518689	0.027965	0.550714	0.005557	0.588242	0.001368	0.617726	0.000539	0.642217	0.000266
	100	0.497784	0.273697	0.509617	0.027555	0.525824	0.005516	0.544878	0.001374	0.560014	0.000547	0.572773	0.000272
	200	0.498909	0.271847	0.504885	0.027299	0.513054	0.005470	0.522662	0.001368	0.530310	0.000547	0.536778	0.000273
	500	0.499568	0.270728	0.501974	0.027124	0.505261	0.005431	0.509128	0.001359	0.512208	0.000544	0.514814	0.000272
	1000	0.499784	0.270354	0.500990	0.027062	0.502637	0.005416	0.504575	0.001355	0.506119	0.000542	0.507426	0.000271
ATS-unbiased	10	0.084039	0.372298	0.306853	0.037841	0.352058	0.007502	0.376461	0.001855	0.388755	0.000736	0.396685	0.000366
	15	0.115953	0.334634	0.408015	0.032035	0.483813	0.006137	0.532074	0.001476	0.559415	0.000576	0.578428	0.000282
	20	0.136177	0.316932	0.465428	0.029954	0.553643	0.005718	0.609939	0.001373	0.641637	0.000535	0.663511	0.000263
	30	0.159991	0.300150	0.525791	0.028409	0.621673	0.005456	0.681055	0.001319	0.713642	0.000517	0.735715	0.000254
	50	0.182082	0.287488	0.574953	0.027564	0.672155	0.005355	0.729743	0.001307	0.760393	0.000515	0.780767	0.000255
	100	0.200678	0.278494	0.611560	0.027169	0.706205	0.005345	0.759587	0.001319	0.787124	0.000523	0.805111	0.000260
	200	0.210690	0.274173	0.629574	0.027054	0.721529	0.005364	0.771623	0.001331	0.796868	0.000530	0.813137	0.000264
	500	0.216939	0.271641	0.640240	0.027013	0.730017	0.005383	0.777592	0.001342	0.801073	0.000536	0.816005	0.000267
	1000	0.219063	0.270807	0.643767	0.027004	0.732706	0.005391	0.779318	0.001346	0.802116	0.000538	0.816519	0.000269

**Table 6.** The design constants of the phase II equal-tailed and ATS-unbiased exponential chart under conditional perspective with fixed  $P[CATS_{IC} \geq ATS_0] = 0.90$ , nominal  $ATS_0 = 370.4$  and different  $\lambda_0$

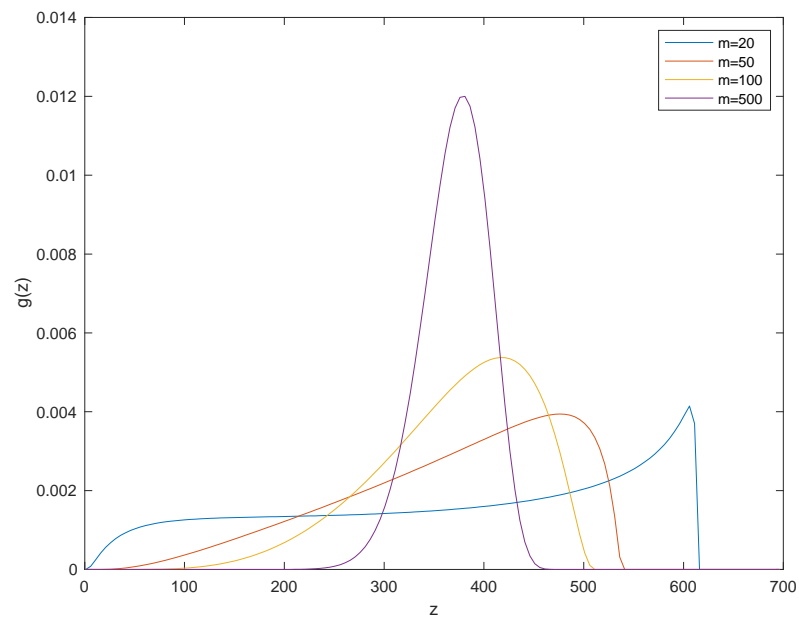
		$\lambda_0$												
		0.01		0.1		0.5		2		5		10		
		$m$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$	$\xi$	$p$
Equal-tailed	10	0.498121	0.121296	0.692139	0.007513	0.848639	0.001412	0.936908	0.000406	0.968826	0.000192	0.982887	0.000111	
	15	0.495740	0.144321	0.621741	0.009267	0.750921	0.001567	0.853746	0.000387	0.906058	0.000164	0.935864	0.000088	
	20	0.495535	0.159252	0.586650	0.010689	0.689553	0.001757	0.784809	0.000406	0.842046	0.000162	0.879445	0.000083	
	30	0.496085	0.177953	0.553362	0.012781	0.623250	0.002100	0.696294	0.000464	0.746954	0.000176	0.784560	0.000086	
	50	0.497134	0.197614	0.529137	0.015337	0.570079	0.002590	0.615899	0.000569	0.650451	0.000212	0.678217	0.000101	
	100	0.498331	0.218144	0.513109	0.018354	0.532552	0.003245	0.555025	0.000732	0.572637	0.000274	0.587329	0.000131	
	200	0.499089	0.233029	0.506057	0.020724	0.515343	0.003804	0.526185	0.000883	0.534773	0.000337	0.542011	0.000162	
	500	0.499610	0.246454	0.502254	0.022959	0.505811	0.004359	0.509981	0.001041	0.513297	0.000404	0.516100	0.000197	
	1000	0.499799	0.253291	0.501086	0.024122	0.502826	0.004655	0.504868	0.001128	0.506493	0.000442	0.507868	0.000217	
ATS-unbiased	10	0.257128	0.102276	0.361959	0.004515	0.393694	0.000481	0.412317	0.000068	0.422052	0.000019	0.428441	0.000007	
	15	0.298149	0.128457	0.475469	0.007616	0.542908	0.001028	0.589779	0.000181	0.617339	0.000057	0.636808	0.000024	
	20	0.314384	0.145115	0.527788	0.009960	0.606435	0.001514	0.659854	0.000297	0.690623	0.000101	0.712012	0.000045	
	30	0.324657	0.165612	0.575998	0.013076	0.660908	0.002220	0.715863	0.000482	0.746477	0.000176	0.767312	0.000082	
	50	0.324059	0.186875	0.611127	0.016375	0.697773	0.003003	0.750866	0.000699	0.779486	0.000267	0.798604	0.000129	
	100	0.311044	0.209124	0.634736	0.019676	0.721022	0.003788	0.770932	0.000919	0.796958	0.000361	0.814044	0.000178	
	200	0.293558	0.225610	0.644853	0.021925	0.730661	0.004309	0.778275	0.001062	0.802472	0.000421	0.818132	0.000209	
	500	0.271949	0.241007	0.649391	0.023843	0.735213	0.004737	0.781227	0.001178	0.804063	0.000470	0.818626	0.000234	
	1000	0.258848	0.249123	0.650090	0.024783	0.736220	0.004941	0.781735	0.001232	0.804082	0.000492	0.818230	0.000246	

**Table 7.** Time intervals in days between explosions in mines, from March 15, 1851 to March 22, 1962 (to be read down columns), reproduced from Jarrett [43]

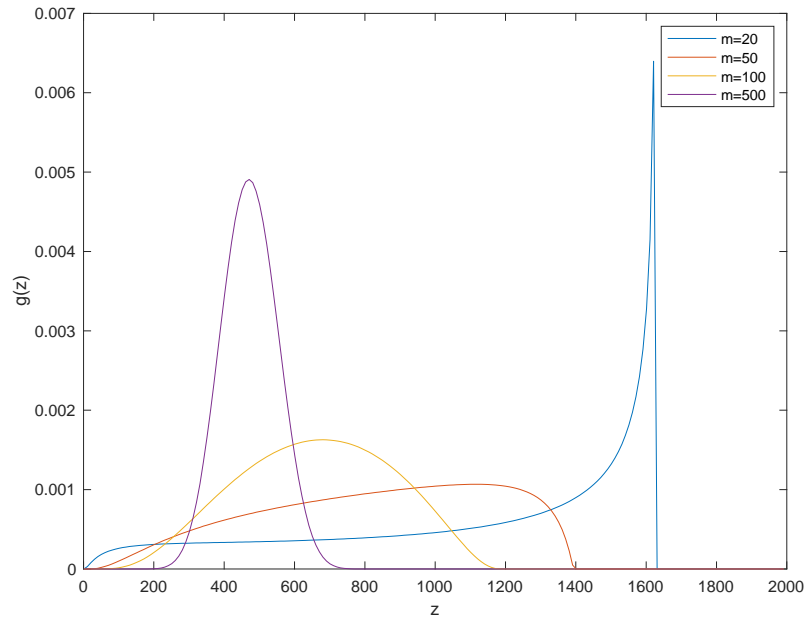
157	65	53	93	127	176	22	1205	1643	312
123	186	17	24	218	55	61	644	54	536
2	23	538	91	2	93	78	467	326	145
124	92	187	143	0	59	99	871	1312	75
12	197	34	16	378	315	326	48	348	364
4	431	101	27	36	59	275	123	745	37
10	16	41	144	15	61	54	456	217	19
216	154	139	45	31	1	217	498	120	156
80	95	42	6	215	13	113	49	275	47
12	25	1	208	11	189	32	131	20	129
33	19	250	29	137	345	388	182	66	1630
66	78	80	112	4	20	151	255	292	29
232	202	3	43	15	81	361	194	4	217
826	36	324	193	72	286	312	224	368	7
40	110	56	134	96	114	354	566	307	18
12	276	31	420	124	108	307	462	336	1358
29	16	96	95	50	188	275	228	19	2366
190	88	70	125	120	233	78	806	329	952
97	225	41	34	203	28	17	517	330	632



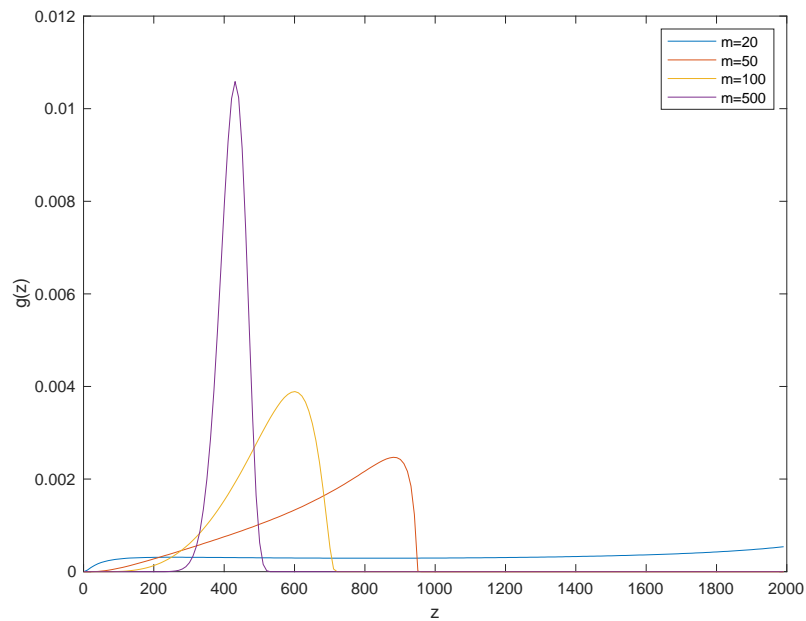
**Figure 1.** The p.d.f. of the  $CATS_{IC}$  distribution for the equal-tailed exponential chart under the unconditional perspective for  $m = 20, 50, 100$  and  $500$  for nominal  $ATS_0 = 370.4, \lambda_0 = 1$ .



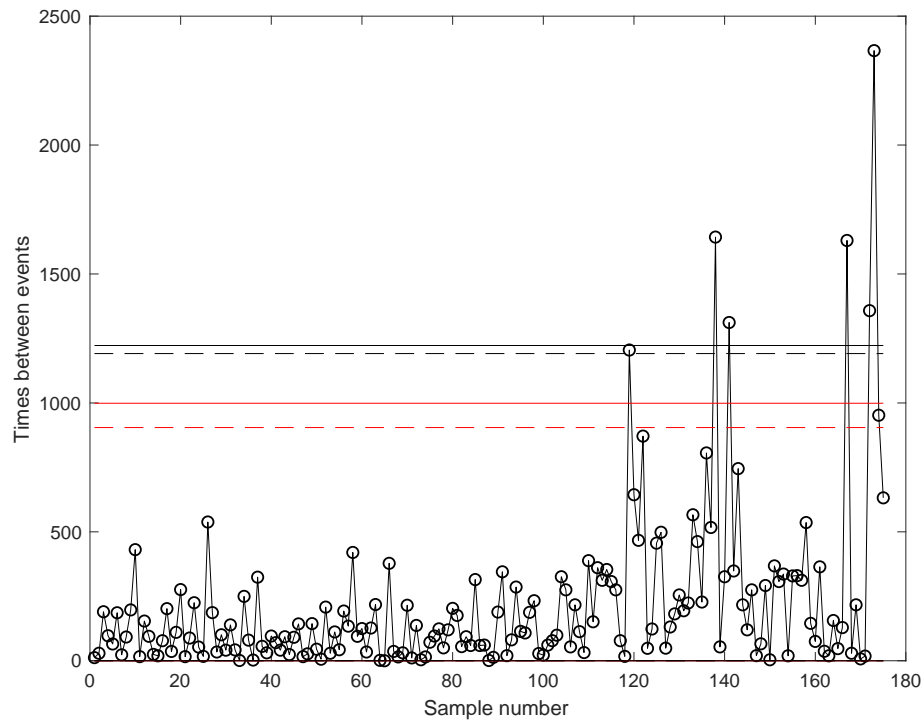
**Figure 2.** The p.d.f.s of the  $CATS_{IC}$  distribution of the ATS-unbiased exponential chart under the unconditional perspective for  $m = 20, 50, 100$  and  $500$  for nominal  $ATS_0 = 370.4, \lambda_0 = 1$ .



**Figure 3.** The p.d.f.s of the  $CATS_{IC}$  distribution of the equal-tailed exponential chart under the conditional perspective for  $m = 20, 50, 100$  and  $500$  for  $EP=0.90$  with nominal  $ATS_0 = 370.4, \lambda_0 = 1$ .



**Figure 4.** The p.d.f.s of the  $CATS_{IC}$  distribution of the ATS-unbiased exponential chart under conditional perspective for  $m = 20, 50, 100$  and  $500$  for  $EP=0.90$  with nominal  $ATS_0 = 370.4, \lambda_0 = 1$ .



**Figure 5.** Control limits of equal-tailed exponential chart under the unconditional perspective (red and solid lines), ATS-unbiased exponential chart under the unconditional perspective (red and dashed lines), equal-tailed exponential chart under the conditional perspective (black and solid lines) and ATS-unbiased exponential chart under the conditional perspective (black and dashed lines).