Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress tensors

C. Deffayet, S. Deser, G. Esposito-Farèse

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C. Deffayet*

AstroParticule & Cosmologie, UMR 7164-CNRS, Université Denis Diderot-Paris 7, CEA, Observatoire de Paris, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France

S. Deser†

Physics Department, Brandeis University, Waltham, Massachusetts 02454, USA, and Lauritsen Laboratory, California Institute of Technology, Pasadena, California 91125, USA

G. Esposito-Farèse‡

GReCO, Institut d'Astrophysique de Paris, UMR 7095-CNRS, Université Pierre et Marie Curie-Paris 6, 98bis boulevard Arago, F-75014 Paris, France

(Received 10 June 2009; published 10 September 2009)

We extend to curved backgrounds all flat-space scalar field models that obey purely second-order equations, while maintaining their second-order dependence on both field and metric. This extension simultaneously restores to second order the, originally higher derivative, stress tensors as well. The process is transparent and uniform for all dimensions.

DOI: 10.1103/PhysRevD.80.064015 PACS numbers: 04.50.−h, 11.10.−z, 98.80.−k

I. INTRODUCTION

Recently, an interesting scalar field, "Galileon," theory [1], inspired by the decoupling limit of the Dvali-Gabadadze-Porrati (DGP) model [2] and its cosmological consequences [3], was introduced.[1] (This model was previously proposed in [6], also in flat space, with a quite different motivation.) Originally formulated in flat spacetime and dimension $D = 4$, its defining property was that, while the action contains both first and second derivatives, the equations of motion *uniquely* involve the latter. As shown in Ref. [7], the simplest covariantization led to field equations for the scalar π and its stress tensor that contained third derivatives; fortunately, [7] also showed how to eliminate these higher derivatives by introducing suitable nonminimal, curvature, couplings. (This cure's small price was to break an original symmetry of the model, that of shifting the first derivatives of π by a constant vector, which is not meaningful in curved space anyhow.) Although the phenomenological relevance of the nonminimal terms has not been studied, [7] furnished a nontrivial example of "safe," purely second order, class of scalar-tensor couplings. However, it was restricted to $D = 4$ and involved rather complicated algebra.

In the present work, we will provide the transparent and uniform basis in arbitrary D for this, *a priori* surprising, nonminimal completion. To do so, the Galileon model will first be reformulated in Sec. II; in particular, we will exhibit its simplest flat-spacetime properties. Section III will incorporate curved backgrounds, in $D = 4$ for concreteness. This will illustrate how the new formulation leads very directly to the original nonminimal couplings of [7]. The final section completes our results by extending them to arbitrary dimensions and backgrounds. Our results are encapsulated in Eqs. (9) for flat, and (35) for general, background.

To define our framework more precisely, we will exhibit, starting from a transparent "canonical" flat-space action with purely second-derivative field equation (but still unavoidably higher derivative stress tensor), a "minimal" nonminimal gravitational coupling extension that simultaneously guarantees no higher than second derivatives of either field *or* metric in both the field equation *and* stress tensor in any D and background. We do not claim uniqueness for this construction simply because one may add infinitely many (rather trivial because irrelevant) terms, all vanishing in flat space, that also avoid higher derivatives. Examples include Lagrangians such as (any function of) the scalar field times all Gauss-Bonnet-Lovelock or Pontryagin densities, let alone plain scalar curvature. Likewise, starting from a flat "noncanonical" version differing from ours by a total divergence, other nonminimal terms would be generated. Finally, our aim being to avoid higher than second derivatives, we will not discuss, for us trivial, incidental first and zeroth order terms such as $V(\pi)$.

II. FLAT-SPACETIME GALILEON

In Ref. [1] it was argued that the most general flat-space action in D dimensions for a scalar field π whose field

*deffayet@iap.fr
†deser@brandeis.edu
‡gef@iap.fr

[1]Aspects of its phenomenology are studied in Refs. [1,4,5].

equations contain *only* second-order (but neither zeroth, first, nor higher) derivatives is obtained by a linear combination of the following Lagrangian densities:[2]

$$\mathcal{L}_{(n+1,0)} = \sum_{\sigma\in S_n} \epsilon(\sigma)[\pi^{\mu_{\sigma(1)}}\pi_{\mu_1}] \times [\pi^{\mu_{\sigma(2)}}{}_{\mu_2}\pi^{\mu_{\sigma(3)}}{}_{\mu_3}\ldots\pi^{\mu_{\sigma(n)}}{}_{\mu_n}], \quad (1)$$

where indices on the scalar field will always denote (ordinary or covariant according to context) derivatives, e.g. $\pi_{\mu\nu}\equiv\pi_{,\mu\nu}$ or $\pi_{;\mu\nu}$, and σ denotes a permutation of signature $\epsilon(\sigma)$ of the permutation group S_n, with $n\le D$. If this last inequality is not satisfied, the above Lagrangian density (1) vanishes identically. Thus in four dimensions, there are only four nontrivial Galileon Lagrangians (1) beyond the nonderivative $\mathcal{L}_{(1,0)}=\pi$ of Ref. [1]; they are $\mathcal{L}_{(2,0)}=\pi_\mu\pi^\mu$, a cubic Lagrangian $\mathcal{L}_{(3,0)}=\pi_\mu\pi^\mu\Box\pi-\pi_\mu\pi^{\mu\nu}\pi_\nu=\frac{3}{2}\pi_\mu\pi^\mu\Box\pi+\text{tot div}$ (the one obtained in the decoupling limit of DGP [2]), and $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(5,0)}$:

$$\mathcal{L}_{(4,0)} = (\Box\pi)^2(\pi_\mu\pi^\mu) - 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu) - (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) + 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho), \quad (2)$$

$$\mathcal{L}_{(5,0)} = (\Box\pi)^3(\pi_\mu\pi^\mu) - 3(\Box\pi)^2(\pi_\mu\pi^{\mu\nu}\pi_\nu) - 3(\Box\pi) \times (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) + 6(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho) + 2(\pi_\mu{}^\nu\pi_\nu{}^\rho\pi_\rho{}^\mu)(\pi_\lambda\pi^\lambda) + 3(\pi_{\mu\nu}\pi^{\mu\nu}) \times (\pi_\rho\pi^{\rho\lambda}\pi_\lambda) - 6(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^{\rho\lambda}\pi_\lambda). \quad (3)$$

The Lagrangian (1) can also be rewritten as

$$\mathcal{L}_{(n+1,0)} = \sum_{\sigma\in S_n}\epsilon(\sigma)g^{\mu_{\sigma(1)}\nu_1}g^{\mu_{\sigma(2)}\nu_2}\ldots g^{\mu_{\sigma(n)}\nu_n}(\pi_{\nu_1}\pi_{\mu_1}) \times (\pi_{\nu_2\mu_2}\pi_{\nu_3\mu_3}\ldots\pi_{\nu_n\mu_n}). \quad (4)$$

As we will see, the key to success will be to rewrite the above Lagrangians in terms of the totally antisymmetric Levi-Civita tensor. We first recall the identity

$$\sum_{\sigma\in S_D}\epsilon(\sigma)g^{\mu_{\sigma(1)}\nu_1}g^{\mu_{\sigma(2)}\nu_2}\ldots g^{\mu_{\sigma(D)}\nu_D} = -\varepsilon^{\mu_1\mu_2\ldots\mu_D}\varepsilon^{\nu_1\nu_2\ldots\nu_D}, \quad (5)$$

valid for any space and dimension, using

$$\varepsilon^{\mu_1\mu_2\ldots\mu_D} = -\frac{1}{\sqrt{-g}}\delta_1^{[\mu_1}\delta_2^{\mu_2}\ldots\delta_D^{\mu_D]}, \quad (6)$$

where the square bracket denotes unnormalized permutations. From our two ε tensors, it is useful to define the $2n$-contravariant tensor $\mathcal{A}_{(2n)}$ by contracting $D-n$ indices:

$$\mathcal{A}_{(2n)}^{\mu_1\mu_2\ldots\mu_{2n}} \equiv \frac{1}{(D-n)!}\varepsilon^{\mu_1\mu_3\mu_5\ldots\mu_{2n-1}\nu_1\nu_2\ldots\nu_{D-n}} \times \varepsilon^{\mu_2\mu_4\mu_6\ldots\mu_{2n}}{}_{\nu_1\nu_2\ldots\nu_{D-n}}. \quad (7)$$

The numerical factor $1/(D-n)!$ is introduced so that $\mathcal{A}_{(2n)}$ keeps the same expression in terms of products of metric tensors in any dimension $D\ge n$. To further simplify notation, we sometimes replace indices μ_i by their index i whenever $i<10$ (but never larger, reinstating μ_i if needed). For example, (7) now reads

$$\mathcal{A}_{(2n)}^{1234\ldots} = \frac{1}{(D-n)!}\varepsilon^{135\ldots\nu_1\nu_2\ldots\nu_{D-n}}\varepsilon^{246\ldots}{}_{\nu_1\nu_2\ldots\nu_{D-n}}. \quad (8)$$

Note that the tensor $\mathcal{A}_{(2n)}$ is obviously antisymmetric upon permutations of the odd $(1,3,5,\ldots)$, as well as those of even $(2,4,6,\ldots)$, indices. Also, we will only write expressions containing $\mathcal{A}_{(2n)}$ with all indices up, and we will then omit those indices with the convention that lower indices denoted by integers $1,2,\ldots,9$ or by indices μ_i are always contracted with the corresponding upper ones of $\mathcal{A}_{(2n)}$. Hence, we will use a letter different from μ to denote indices that are not contracted with those of $\mathcal{A}_{(2n)}$. It is now easy to see that the Lagrangian (1) can be rewritten as

$$\mathcal{L}_{(n+1,0)} = -\mathcal{A}_{(2n)}(\pi_1\pi_2)(\pi_{34}\pi_{56}\pi_{78}\ldots\pi_{\mu_{2n-1}\mu_{2n}}), \quad (9)$$

while, for example, Lagrangians $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(5,0)}$ given in Eqs. (2) and (3) can be rewritten in the compact form:

$$\mathcal{L}_{(4,0)} = -\varepsilon^{\mu_1\mu_3\mu_5\nu}\varepsilon^{\mu_2\mu_4\mu_6}{}_\nu\pi_{\mu_1}\pi_{\mu_2}\pi_{\mu_3\mu_4}\pi_{\mu_5\mu_6} = -\mathcal{A}_{(6)}\pi_1\pi_2\pi_{34}\pi_{56}, \quad (10)$$

$$\mathcal{L}_{(5,0)} = -\varepsilon^{\mu_1\mu_3\mu_5\mu_7}\varepsilon^{\mu_2\mu_4\mu_6\mu_8}\pi_{\mu_1}\pi_{\mu_2}\pi_{\mu_3\mu_4}\pi_{\mu_5\mu_6}\pi_{\mu_7\mu_8} = -\mathcal{A}_{(8)}\pi_1\pi_2\pi_{34}\pi_{56}\pi_{78}. \quad (11)$$

Clearly, the field equations derived from (9) only contain second derivatives. Indeed, first, upon varying the Lagrangian (9) with respect to π, the twice-differentiated term appearing there gives rise, after integration by parts, to third and fourth order derivatives acting on π, of the form $\pi_{\mu_i\mu_j\mu_k}$ and $\pi_{\mu_i\mu_j\mu_k\mu_l}$. But any expression of the form $\mathcal{A}_{(2n)}\pi_{\mu_i\mu_j\mu_k}$ or $\mathcal{A}_{(2n)}\pi_{\mu_i\mu_j\mu_k\mu_l}$ vanishes identically, because flat-spacetime derivatives commute and such an expression contains at least two indices among $\{i,j,k\}$ having the same parity and, hence, contracted with the same epsilon tensor arising in the definition of $\mathcal{A}_{(2n)}$. So not only does the Lagrangian (9) lead to equations with at most second derivatives, but it also means that when a term with a twice-differentiated π is varied, one must distribute, after integrating by parts, its two derivatives onto the π_1 and π_2 terms. Similarly, when either single

[2] In our notation, $\mathcal{L}_{(n,p)}$ is a Lagrangian density that is a sum of monomials, each containing products of n fields π, acted on by first and second derivatives, and p explicit occurrences of the Riemann tensor. Note that Eq. (A4) of Ref. [1] equals n times our Eq. (1).

derivative factor is varied, that derivative must land only on the other, yielding the only contribution, π_{12}. Hence, as announced, the field equations arising from the variation of (9) contain only second derivatives. They read $(n+1)\mathcal{E}_{(n+1,0)} = 0$, where

$$\mathcal{E}_{(n+1,0)} = -\sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{i=1}^{i=n} \pi^{\mu_{\sigma(i)}}{}_{\mu_i} = \mathcal{A}_{(2n)}\pi_{12}\pi_{34}\pi_{56}\ldots\pi_{\mu_{2n-1}\mu_{2n}}. \tag{12}$$

III. GALILEONS IN $D = 4$ CURVED SPACE

In Ref. [7], it was noted that minimal covariantization of (1), just with covariant derivatives (still omitting semicolons),

$$-\int d^D x\sqrt{-g}\,\mathcal{A}_{(2n)}(\pi_1\pi_2)(\pi_{34}\pi_{56}\pi_{78}\ldots\pi_{\mu_{2n-1}\mu_{2n}}), \tag{13}$$

led to third derivatives of the metric, as gradients of curvatures, in the field equation, as well as to third derivatives of π in the stress tensor. This is not very desirable, due to the well-known stability problems caused by higher derivatives in both scalar and gravitational sectors: More initial conditions would have to be specified, and in some backgrounds, new excitations might appear. Note that these problems arise as soon as the Lagrangians (1) contain a product of at least two twice-differentiated π's, as will be seen in detail in Sec. IV. For example, in $D = 4$, this is the case for $\{\mathcal{L}_{(4,0)}, \mathcal{L}_{(5,0)}\}$, but not for $\{\mathcal{L}_{(2,0)}, \mathcal{L}_{(3,0)}\}$. A way out was provided in [7] where it was shown that, in $D = 4$, there exists a unique (in the minimal sense explained in the Introduction) nonminimal term that removes all the third derivatives arising in both variations of the action: the field equations and the stress tensor. Indeed, adding the Lagrangians $\mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,1)}$,

$$\mathcal{L}_{(4,1)} = (\pi_\lambda\pi^\lambda)\pi_\mu[R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R]\pi_\nu, \tag{14}$$

$$\begin{aligned}\mathcal{L}_{(5,1)} = {}& -3(\pi_\lambda\pi^\lambda)(\pi_\mu\pi_\nu\pi_{\rho\sigma}R^{\mu\rho\nu\sigma}) - 18(\pi_\mu\pi^\mu) \\ & \times(\pi_\nu\pi^{\nu\rho}R_{\rho\sigma}\pi^\sigma) + 3(\pi_\mu\pi^\mu)(\Box\pi)(\pi_\nu R^{\nu\rho}\pi_\rho) \\ & + \frac{15}{2}(\pi_\mu\pi^\mu)(\pi_\nu\pi^{\nu\rho}\pi_\rho)R + \text{tot div},\end{aligned} \tag{15}$$

respectively to $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(5,0)}$, we obtain covariant Galileon actions whose field equations contain derivatives of order lower or equal to two, both in π and metric variations. We now show how the nonminimal terms $\mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,1)}$ can easily be obtained using our generalized form (9). To match the expressions for $\mathcal{L}_{(n+1,1)}$ derived below, a total derivative must actually be added to Eq. (15), namely 3 times Eq. (18) of Ref. [7], which reads

$$\begin{aligned}\text{tot div} = {}& 3(\pi_\mu\pi^\mu)(\pi_\nu\pi^\nu)(\pi_{\rho\sigma}R^{\rho\sigma}) + 12(\pi_\mu\pi^\mu) \\ & \times(\pi_\nu\pi^{\nu\rho}R_{\rho\sigma}\pi^\sigma) - \tfrac{3}{2}(\pi_\mu\pi^\mu)(\pi_\nu\pi^\nu)(\Box\pi)R \\ & - 6(\pi_\mu\pi^\mu)(\pi_\nu\pi^{\nu\rho}\pi_\rho)R.\end{aligned} \tag{16}$$

Let us first consider $\mathcal{L}_{(5,0)}$, and vary its action with respect to π. Denoting it by $\delta_\pi\mathcal{L}_{(5,0)}$, we have

$$\begin{aligned}\delta_\pi\mathcal{L}_{(5,0)} = {}& -2\mathcal{A}_{(8)}\delta\pi_1\pi_2\pi_{34}\pi_{56}\pi_{78} \\ & - 3\mathcal{A}_{(8)}\pi_1\pi_2\delta\pi_{34}\pi_{56}\pi_{78},\end{aligned} \tag{17}$$

where the coefficients 2 and 3 are easily obtained by a renumbering of the dummy indices μ_i. Upon integration by parts, we see that the first term in the right-hand side above cannot possibly lead to derivatives in the field equations of order higher than two, because such terms could only (after integration by parts) lead to third-order covariant derivatives acting on π. But we know by construction that third derivatives are absent in flat spacetime; hence they can only lead, in curved backgrounds, to terms proportional to (undifferentiated) curvatures times a first derivative of π. The highest order derivatives appearing in such a product are obviously of second order and act on the metric. Hence, the only term which can potentially lead in the equations of motion to derivatives of order higher than 2 (we will call those terms "dangerous" in the following) is

$$\delta_\pi\mathcal{L}_{(5,0)} \sim -3\mathcal{A}_{(8)}\pi_1\pi_2\delta\pi_{34}\pi_{56}\pi_{78}, \tag{18}$$

where a tilde will mean that we only write the dangerous terms and omit the others. Note that no dangerous terms are generated by varying the volume factor $\sqrt{-g}$ in the action, so we may henceforth work at Lagrangian density level and allow integration by parts when writing expressions containing the $\sim$ symbol, with the understanding that such expressions might differ by a total derivative. When integrating the term on the right-hand side of the above equation (18) by parts to obtain the π field equation, we see, for reasons similar to those given above, that the only dangerous terms occur when letting the two derivatives, ∇_{μ_3} and ∇_{μ_4}, act on an already twice-differentiated π. We obtain

$$\delta_\pi\mathcal{L}_{(5,0)} \sim -3\times 2\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi_{5643}\pi_{78}, \tag{19}$$

where the extra factor 2 comes from the possibility that those derivatives act on π_{56} or π_{78}, both of which give the same term, after appropriate renumbering and index permutations. Using similar rearrangements, we can rewrite (19) as

$$\begin{aligned}\delta_\pi\mathcal{L}_{(5,0)} &\sim -3\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2(\pi_{5643} - \pi_{5463})\pi_{78} \\ &\sim -3\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda R_{465\lambda;3}\pi_{78} \\ &\sim -\tfrac{3}{2}\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda(R_{465\lambda;3} + R_{46\lambda3;5})\pi_{78} \\ &\sim \tfrac{3}{2}\delta\pi\mathcal{A}_{(8)}\pi_1\pi_2\pi^\lambda R_{3546;\lambda}\pi_{78},\end{aligned} \tag{20}$$

where the last line uses the Bianchi identity $R_{46[35;\lambda]} = 0$.

Hence, as already shown in [7], the π field equations contain third derivatives of the metric, as first derivative of the curvature. The above term is the only dangerous one coming from the variation of

$$\int d^4x\sqrt{-g}\mathcal{L}_{(5,0)}. \quad (21)$$

It can be canceled by adding to the above action the following:

$$\frac{3}{4}\int d^4x\sqrt{-g}\mathcal{A}_{(8)}\pi_1\pi_2(\pi_\lambda\pi^\lambda)R_{3546}\pi_{78}, \quad (22)$$

which on the other hand is easily seen not to generate any further dangerous term. In fact, one can check explicitly that this action is identical to the one obtained from $\mathcal{L}_{(5,1)}$, that is

$$\mathcal{L}_{(5,1)} = \tfrac{3}{4}\mathcal{A}_{(8)}\pi_1\pi_2(\pi_\lambda\pi^\lambda)R_{3546}\pi_{78}. \quad (23)$$

It was shown in [7] that the metric variation of the sum (21) plus (23) does not contain derivatives of order higher than two, but as we will see in the next section, this can also easily be checked explicitly using our expressions $\mathcal{L}_{(5,0)}$ and $\mathcal{L}_{(5,1)}$. Before proceeding, let us note that a calculation similar to the one given above leads to a simple expression for the nonminimal term

$$\mathcal{L}_{(4,1)} = \tfrac{1}{4}\mathcal{A}_{(6)}\pi_1\pi_2(\pi_\lambda\pi^\lambda)R_{3546}. \quad (24)$$

IV. ARBITRARY D BACKGROUNDS

We now show how the previous results can be generalized from $D = 4$ to arbitrary D. Namely, we will show that a covariant Galileon model whose field equations have derivatives of order lower or equal to two can be obtained in arbitrary dimensions by a suitable linear combination of Lagrangian densities of the type

$$\mathcal{L}_{(n+1,p)} = -\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}, \quad (25)$$

where $\mathcal{R}_{(p)}$ and $\mathcal{S}_{(q)}$ are defined by

$$\mathcal{R}_{(p)} \equiv (\pi_\lambda\pi^\lambda)^p\prod_{i=1}^{i=p}R_{\mu_{4i-1}\mu_{4i+1}\mu_{4i}\mu_{4i+2}}, \quad (26)$$

$$\mathcal{S}_{(q)} \equiv \prod_{i=0}^{i=q-1}\pi_{\mu_{2n-1-2i}\mu_{2n-2i}}, \quad (27)$$

and one has $q = n - 1 - 2p$. The Lagrangian densities $\mathcal{L}_{(n+1,p)}$ are obtained from $\mathcal{L}_{(n+1,0)}$ by replacing p times a pair of twice-differentiated π, by a product of Riemann tensors by $\pi_\lambda\pi^\lambda$ (with suitable indices). To further streamline the discussion and the notations, we will also use (in the spirit of Petrov notation) an index A_i to denote the four indices $\mu_{4i-1}\mu_{4i+1}\mu_{4i}\mu_{4i+2}$ taken in that order: We will write, e.g.,

$$\mathcal{R}_{(p)} = (\pi_\lambda\pi^\lambda)^p\prod_{i=1}^{i=p}R_{A_i}, \quad (28)$$

and we will also use the convention that $\mathcal{R}_{(p)}$ and $\mathcal{S}_{(q)}$ vanish, respectively, for $p < 0$ and $q < 0$, while $\mathcal{R}_{(0)} = \mathcal{S}_{(0)} \equiv 1$ by consistency of definition (25) with Eq. (9). Let us first look at the variation of $\mathcal{L}_{(n+1,p)}$, denoted by $\delta_\pi\mathcal{L}_{(n+1,p)}$, with respect to π. We find

$$\begin{aligned}\delta_\pi\mathcal{L}_{(n+1,p)} = &-2\mathcal{A}_{(2n)}\delta\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}\\ &- 2p\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p-1)}\delta\pi_\lambda\pi^\lambda R_{A_p}\mathcal{S}_{(q)}\\ &- q\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\delta\pi_{\mu_{4p+3}\mu_{4p+4}}\mathcal{S}_{(q-1)}.\end{aligned} \quad (29)$$

After integrating by parts, the first term on the right-hand side of the above equation does not lead to dangerous terms (in the terminology of the previous section). Indeed, the only possible dangerous terms it could generate are derivatives of the curvature in the form $R_{A_i;1}$. However, when contracted with $\mathcal{A}_{(2n)}$ those terms vanish by virtue of the Bianchi identity $R_{\mu\nu[\rho\sigma;\kappa]} = 0$. The terms obtained from the second one of Eq. (29) by letting (after integration by parts) the derivative ∇_λ act on $\mathcal{S}_{(q)}$ are *a priori* dangerous, because the index λ is not contracted with one index of $\mathcal{A}_{(2n)}$, and hence our previous argument for discarding third derivatives would fail. However, those terms are exactly compensated (up to nondangerous ones) by those obtained from an integration by parts of the third term of Eq. (29), where the derivatives $\nabla_{\mu_{4p+3}}\nabla_{\mu_{4p+4}}$ act on one of the π_λ of $\mathcal{R}_{(p)}$. We thus find, by a rewriting similar to (20), that the dangerous terms in the variation $\delta_\pi\mathcal{L}_{(n+1,p)}$ read

$$\begin{aligned}\delta_\pi\mathcal{L}_{(n+1,p)} \sim\, &2p^2\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p-1)}\pi^\lambda R_{A_p;\lambda}\mathcal{S}_{(q)}\\ &+ \frac{q(q-1)}{4}\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\pi^\lambda R_{A_{p+1};\lambda}\mathcal{S}_{(q-2)}.\end{aligned} \quad (30)$$

Note that this expression also holds for $p = 0$ and $q = 0$, $q = 1$.

Let us now consider the variation $\delta_g\mathcal{L}_{(n+1,p)}$ of $\mathcal{L}_{(n+1,p)}$ with respect to the metric. Defining the variation of the metric $g_{\mu\nu}$ by $h_{\mu\nu}$, those of $\pi_{\mu_{4p+3}\mu_{4p+4}}$ and of R_{A_p}, denoted by $\delta_g\pi_{\mu_{4p+3}\mu_{4p+4}}$ and $\delta_g R_{A_p}$, respectively, obey

$$\begin{aligned}\delta_g\pi_{\mu_{4p+3}\mu_{4p+4}} = &-\tfrac{1}{2}\pi^\sigma(h_{\sigma\mu_{4p+4};\mu_{4p+3}} + h_{\sigma\mu_{4p+3};\mu_{4p+4}}\\ &- h_{\mu_{4p+3}\mu_{4p+4};\sigma}),\end{aligned} \quad (31)$$

$$\begin{aligned}\mathcal{A}_{(2n)}\delta_g R_{A_p} = &\,2\mathcal{A}_{(2n)}h_{\mu_{4p-1}\mu_{4p+2};\mu_{4p+1}\mu_{4p}}\\ &+ \mathcal{A}_{(2n)}h^\sigma_{\mu_{4p-1}}R_{\sigma\mu_{4p+1}\mu_{4p}\mu_{4p+2}}.\end{aligned} \quad (32)$$

From those equations, it follows that $\delta_g\mathcal{L}_{(n+1,p)}$ contains the dangerous terms,

$$\begin{aligned}\delta_g \mathcal{L}_{(n+1,p)} \sim \frac{q(q-1)}{2} &\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\pi^\sigma\pi_{\mu_{4p+5}\mu_{4p+6}\sigma}\\ &\times \mathcal{S}_{(q-2)}h_{\mu_{4p+3}\mu_{4p+4}} + \frac{pq}{2}\mathcal{A}_{(2n)}\pi_1\pi_2\\ &\times \mathcal{R}_{(p-1)}\pi^\sigma R_{A_p;\sigma}\pi_\lambda\pi^\lambda\mathcal{S}_{(q-1)}h_{\mu_{4p+3}\mu_{4p+4}}\\ &- 2pq\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p-1)}\pi_\lambda\pi^\lambda\\ &\times \pi_{\mu_{4p+3}\mu_{4p+4}\mu_{4p}\mu_{4p+1}}\mathcal{S}_{(q-1)}h_{\mu_{4p-1}\mu_{4p+2}}\\ &- 4p^2\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p-1)}\pi^\lambda\pi_{\lambda\mu_{4p}\mu_{4p+1}}\\ &\times \mathcal{S}_{(q)}h_{\mu_{4p-1}\mu_{4p+2}}.\end{aligned} \tag{33}$$

From a rewriting again similar to (20), it is easily seen that the second and third terms on the right-hand side of the above equation cancel each other. Then, after some relabeling and permutation of dummy indices, one is left with

$$\begin{aligned}\delta_g \mathcal{L}_{(n+1,p)} \sim \frac{q(q-1)}{2} &\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\pi^\lambda\pi_{\mu_{4p+5}\mu_{4p+6}\lambda}\\ &\times \mathcal{S}_{(q-2)}h_{\mu_{4p+3}\mu_{4p+4}} + 4p^2\mathcal{A}_{(2n)}\pi_1\pi_2\\ &\times \mathcal{R}_{(p-1)}\pi^\lambda\pi_{\lambda\mu_{4p+2}\mu_{4p+1}}\mathcal{S}_{(q)}h_{\mu_{4p-1}\mu_{4p}}.\end{aligned} \tag{34}$$

Using the above expressions (30) and (34), it is then easy to see that the action given by

$$I = \int d^D x\sqrt{-g}\sum_{p=0}^{p_{\max}}\mathcal{C}_{(n+1,p)}\mathcal{L}_{(n+1,p)}, \tag{35}$$

with $p_{\max}$ the integer part of $(n-1)/2$ (i.e., the number of pairs of twice-differentiated π in $\mathcal{S}_{(n-1)}$), leads to field equations (both for π and the stress tensor) with no more than second derivatives, provided the coefficients $\mathcal{C}_{(n+1,p)}$ satisfy the recurrence relation

$$\mathcal{C}_{(n+1,p)} = -\frac{(n+1-2p)(n-2p)}{8p^2}\mathcal{C}_{(n+1,p-1)}. \tag{36}$$

The latter is easily solved by (setting $\mathcal{C}_{(n+1,0)}$ to one)

$$\begin{aligned}\mathcal{C}_{(n+1,p)} &= \left(-\frac{1}{8}\right)^p\frac{(n-1)!}{(n-1-2p)!(p!)^2}\\ &= \left(-\frac{1}{8}\right)^p\binom{n-1}{2p}\binom{2p}{p}.\end{aligned} \tag{37}$$

These coefficients correspond to those of $(xy)^p$ in the expansion of $(1+x-y/8)^{n-1}$. Remarkably, they suffice to ensure the disappearance of dangerous terms in both the metric and π field equations.

V. CONCLUSIONS

We have presented, in arbitrary D and gravitational backgrounds, the ‘‘minimally’’ most general scalar models whose field equations and stress tensors depend on second field derivatives and (undifferentiated) curvatures. Whatever their ultimate physical usefulness, it is remarkable that these models exist at all and even more that they can be systematized in so uniformly simple a manner. Their construction is tantalizingly reminiscent of gravitational Gauss-Bonnet-Lovelock models.

ACKNOWLEDGMENTS

We thank the authors of Ref. [1] and J. Mourad for discussions, and D. Fairlie for bringing Refs. [6] to our attention. The work of S. D. was supported by NSF Grant No. PHY 07-57190 and DOE Grant No. DE-FG02-92ER40701.

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