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Constraints on moduli cosmology from the production of dark matter and baryon isocurvature fluctuations

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We set constraints on moduli cosmology from the production of dark matter—radiation and baryon—radiation isocurvature fluctuations through modulus decay, assuming the modulus remains light during inflation. We find that the moduli problem becomes worse at the perturbative level as a significant part of the parameter space m_σ (modulus mass)— σ_{inf} (modulus vacuum expectation value at the end of inflation) is constrained by the nonobservation of significant isocurvature fluctuations. We discuss in detail the evolution of the modulus vacuum expectation value and perturbations, in particular, the consequences of Hubble scale corrections to the modulus potential, and the stochastic motion of the modulus during inflation. We show, in particular, that a high modulus mass scale $m_\sigma \gtrsim 100$ TeV, which allows the modulus to evade big bang nucleosynthesis constraints is strongly constrained at the perturbative level. We find that generically, solving the moduli problem requires the inflationary scale to be much smaller than 10^{13} GeV.

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I. INTRODUCTION

The existence of scalar fields with gravitational coupling to the visible sector appears to be a generic prediction of particle physics beyond the standard model. This, however, may cause serious cosmological difficulties, as exemplified by the “cosmological moduli problem” [1,2]. Assuming that the mass m_σ of a modulus is of order of the weak scale, as one would expect for soft masses induced by supersymmetry breaking, this field should decay after big bang nucleosynthesis, on a time scale $t_\sigma \sim 10^8 \text{ sec}(m_\sigma/100 \text{ GeV})^{-3/2}$. The ensuing high energy electromagnetic and hadronic cascades would then ruin the success of big bang nucleosynthesis predictions (see [3] for a recent compilation and references therein) unless the modulus energy density were extremely small at that time. By “extremely small,” it is meant about 20 orders of magnitude smaller than what is generically expected for a scalar field oscillating in a quadratic potential with an initial expectation value of order of the Planck scale. Turned around, this cosmological moduli problem reveals the power of big bang nucleosynthesis when used as a probe of high energy physics and early Universe cosmology.

Several classes of solutions have been proposed. The first one argues that the vacuum expectation value (vev) σ_{inf} of the modulus at the end of inflation is much smaller

than the Planck scale [4–8]. This is not a trivial requirement as it demands that the effective minima of the modulus potential at low energy (i.e. well after post-inflationary reheating) and at high energy (i.e. during inflation) coincide with each other [2,8]. Furthermore, quantum fluctuations of the scalar field will generally push the field away from this minimum [9].

An alternative solution to the moduli problem proposes that the modulus mass is so large that the modulus decays before big bang nucleosynthesis, leaving enough time for the high energy cascade to thermalize before the process of nucleosynthesis actually starts [10–14]. This requires $m_\sigma \gtrsim 100$ TeV. Although this lies some 2 orders of magnitude beyond the expected soft scale, such masses can be accommodated in successful models of supersymmetry breaking such as anomaly mediation or no-scale supergravity as argued in Refs. [10–14].

Finally, it has also been proposed to dilute the energy density contained in the moduli through an epoch of low scale inflation [15] or thermal inflation [16,17].

As formulated above, this standard moduli problem is directly expressed as a constraint on the energy density of the modulus field at the time of its decay. Meanwhile, progress in observational cosmology has been such that it is now possible to constrain the nature of density perturbations to a high degree of accuracy. Most notably, the analysis of microwave background temperature fluctuations allows to constrain the fraction of isocurvature modes to a quite low level [18–31]. As we argue in this paper (see also [32]), such constraints on the spectrum and the nature of density perturbations can be translated into constraints

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on moduli cosmology. We will find that the moduli problem becomes worse at the perturbative level. The main reason is that a modulus, being uncoupled to fields of the visible sector, inherits its own fluctuations through inflation. At the time of reheating, there exists an isocurvature fluctuation between the modulus and radiation, which is transformed into a dark matter—radiation or baryon-radiation isocurvature mode when the modulus decays into radiation, dark matter, and baryons. In this way, one may thus picture the modulus as a curvaton field, whose phenomenology has been intensively scrutinized in the past few years (see notably Refs. [33–47]).

The generation of isocurvature fluctuations by modulus decay has been noted before (see [37]), but to our knowledge, a detailed analysis of the ensuing constraints on moduli cosmology has not been given up until now. The present paper furthermore attempts at being general and exhaustive with respect to modulus cosmology. In the course of our discussion, we have thus obtained new results in several places, such as those related to the evolution of the modulus and its perturbations in the presence of supergravity corrections to the modulus potential, or generalized existing discussions, for instance concerning the stochastic behavior of the modulus during inflation. The layout of the paper is as follows. In Sec. II, we describe in general terms how constraints on isocurvature fluctuations can be turned into constraints on moduli cosmology. In Sec. III, we then calculate for various modulus effective potentials (time independent, or accounting for supergravity corrections) the cosmological consequences and present the constraints in the modulus parameter space $(m_\sigma, \sigma_{\text{inf}})$. We summarize our findings and discuss how to evade the modulus problem at the perturbative level in Sec. IV. Finally, the paper ends with three technical appendices, which contain results of importance to the present study but that also possess interest of their own. The first one, Appendix A, is devoted to the calculation of the quantum behavior of the modulus field during inflation, for large field and small field models, accounting for possible supergravity corrections to the modulus potential. The second one, Appendix B, discusses in greater details the evolution of the modulus vev when its potential receives supergravity corrections. The third one, Appendix C, presents some supergravity based, concrete particle physics models for the modulus field. All throughout this paper, $M_{\text{Pl}} = 2.42 \times 10^{18}$ GeV denotes the reduced Planck mass.

II. GENERATION OF ISOCURVATURE PERTURBATIONS

This section sets the stage for the next section, which performs a systematic study of the constraints obtained in moduli parameter space. Here, we introduce the relevant physical parameters and we describe how the amount of isocurvature fluctuations produced through modulus decay can be calculated analytically. The formulas obtained will

be useful to interpret the results of numerical calculations presented in the following section.

A. Background evolution

Let us first sketch the cosmological scenario and the outline of the calculation. We assume that inflation proceeds at the energy scale H_{inf} . At the end of inflation, the slow-roll conditions are violated (or, in the case of hybrid inflation, an instability occurs), the inflaton field ϕ starts oscillating rapidly at the bottom of its potential and the reheating period begins. If the inflaton potential is quadratic, then the universe becomes matter-dominated. At a later stage, the inflaton field decays into radiation, dark matter, and baryons. At this high energy scale, well above the dark matter mass, baryons, radiation, and dark matter are all part of the same “radiation” fluid and the universe is effectively radiation-dominated. The temperature T_{rh} of the radiation fluid at the beginning of this era, the post-inflationary reheating temperature, is a direct function of the decay rate of the inflaton field.

Let us now consider the modulus field σ . In the following, we denote its vev at the end of inflation by σ_{inf} and treat this quantity as a free parameter (however, see below the considerations on the quantum behavior of σ during inflation). In the post-inflationary era, in order to follow the evolution of σ , the shape of the potential is needed. In what follows, one considers two cases: one where the potential is purely quadratic¹ and one where the potential is affected by Hubble scale contributions, meaning that a term of the form $c^2 H^2 (\sigma - \sigma_0)^2$ is added to the quadratic part. If the potential is purely quadratic and if the modulus field is a test field, then its vev remains constant (therefore equal to σ_{inf}) until $H = m_\sigma$. Let us notice that, when $H = m_\sigma$, the energy density of the background is $3M_{\text{Pl}}^2 m_\sigma^2$ while $\rho_\sigma \sim m_\sigma^2 \sigma_{\text{inf}}^2$. Then, the condition that σ is a test field, that is to say that its backreaction on the expansion rate is negligible, i.e. $\rho_\sigma < \rho_\phi$, implies that $\sigma_{\text{inf}} \lesssim M_{\text{Pl}}$. In the opposite limit $\sigma_{\text{inf}} \gtrsim M_{\text{Pl}}$, the modulus would induce a secondary stage of inflation, as envisaged by Linde and Mukhanov [9], leading to a situation similar to that encountered in multiple field inflationary scenarios, with a bunch of possible nontrivial effects both on the spectrum and on the evolution of the adiabatic and entropy perturbations in the postinfla-

¹If the modulus has a more complicated potential, $V(\sigma)$, it will remain constant as long as $H^2 \gg |V''(\sigma)|$, then start to move toward the minimum and oscillate around it as this inequality is violated. If this oscillation is dominated by higher-order terms than the quadratic one, the energy density of the modulus would redshift more rapidly than dust, and the cosmological evolution of the modulus-radiation system would be more complicated. On general grounds, one may expect higher-order terms to provide negligible contribution to the modulus dynamics as soon as $\sigma \ll M_{\text{Pl}}$, in regards of the typical Planck scale suppressions, up to the possible exception of the quartic term $\lambda\sigma^4$. This particular case deserves a dedicated study.

tionary era. In this study, we restrict ourselves to the above test field approximation, i.e. $\sigma_{\text{inf}} \lesssim M_{\text{Pl}}$.

On the other hand, if the Hubble scale corrections are important, σ can never be considered as light (unless $c \ll 1$) since its effective mass is always of the order of the Hubble scale. As a result, the evolution of the modulus between the end of inflation and the time $H = m_\sigma$ can become rather involved since σ has no reason to stay constant anymore as it was the case in the purely quadratic situation. For this reason, we postpone the detailed discussion of the modulus evolution to the following section and we always express our results in terms of the modulus energy density when $H = m_\sigma$.

At late times $H \leq m_\sigma$, Hubble corrections to the mass term indeed become, by definition, negligible. As a consequence, the modulus potential is then given by $m_\sigma^2 \sigma^2/2$. At $H = m_\sigma$, the modulus starts oscillating at the bottom of this potential and $\rho_\sigma \propto a^{-3}$. This occurs at an equivalent temperature scale given by

$$T_{\text{osci}} = \left(\frac{\pi^2 g_{\star, \text{osci}}}{90} \right)^{-1/4} (m_\sigma M_{\text{Pl}})^{1/2} \approx 2.25 \times 10^{11} \text{ GeV} \left(\frac{g_{\star, \text{osci}}}{200} \right)^{-1/4} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{1/2}. \quad (1)$$

Of course, T_{osci} can correspond to the temperature of the radiation bath only if $T_{\text{osci}} \lesssim T_{\text{rh}}$. Otherwise, it should be thought of as the temperature that the radiation bath would have were the energy density contained in radiation. Typically, one has $T_{\text{osci}} > T_{\text{rh}}$ unless the reheating temperature is very high. At the onset of oscillations, the σ field carries a fraction $\Omega_{\sigma, \text{osci}}$ of the total energy density which can be expressed in terms of σ_{inf} , the vacuum expectation value of σ at the end of inflation.

Modulus eventually decays into radiation, dark matter, and baryons when $H = \Gamma_\sigma$, Γ_σ being the gravitational decay width of σ

$$\Gamma_\sigma = \frac{1}{16\pi} \frac{m_\sigma^3}{M_{\text{Pl}}^2} \approx 3.51 \times 10^{-24} \text{ GeV} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^3. \quad (2)$$

As a consequence, the decay of the σ field occurs at a temperature

$$T_d = \left(\frac{\pi^2 g_{\star, \text{dec}}}{90} \right)^{-1/4} (\Gamma_\sigma M_{\text{Pl}})^{1/2} \approx 2.77 \times 10^{-3} \text{ GeV} \left(\frac{g_{\star, \text{dec}}}{10.75} \right)^{-1/4} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{3/2}. \quad (3)$$

Clearly, the decay occurs much after the onset of oscillations and the reheating.

Finally, let us end this short description of the background evolution by mentioning that we assume all throughout this paper that dark matter originates from freeze-out of annihilations. The dark matter freeze-out occurs at a temperature of $\sim m_\chi/x_f$, where m_χ is the dark matter particle mass and $x_f \sim 20\text{--}30$. Since, typically,

$m_\chi \sim \mathcal{O}(100)$ GeV, one obtains a temperature of $\sim 1\text{--}10$ GeV. Therefore, provided $m_\sigma \lesssim 10^7$ GeV, the modulus always decays after dark matter freeze out. On the other hand, the freeze-out of the baryons takes place at a temperature of ~ 20 MeV. Therefore, whether the modulus decay occurs before or after the baryons freeze-out depends on the value of m_σ .

The previous considerations imply that two crucial variables in this study are m_σ and σ_{inf} and, therefore, in the following, we will express our constraints in the plane $(m_\sigma, \sigma_{\text{inf}})$. The mass scale m_σ should in principle be fixed by high energy physics with a preferred range around $10^2\text{--}10^6$ GeV. On the contrary, the vev σ_{inf} is determined by the early cosmological evolution. The mass scale determines, among others, the decay time of the modulus, and together with the vev σ_{inf} , it also determines the magnitude of the modulus energy density at the time of decay, hence the amount of isocurvature perturbations transferred to the dark matter and baryon fluids. The modulus vev σ_{inf} can be expressed as the sum of two parts, one corresponding to the classical trajectory σ_{cl} of the modulus field in its potential during inflation, and the typical spread $\langle \delta\sigma^2 \rangle^{1/2}$ around this trajectory due to quantum effects [48–50]. The standard deviation on scales larger than the Hubble radius $\langle \delta\sigma^2 \rangle^{1/2}$ has been discussed recently by Linde and Mukhanov [9] and Lyth [51]. It is discussed in greater detail in Appendix A, along with the classical trajectory σ_{cl} during inflation.

For both large field and small field models, one may summarize the situation as follows (the case of hybrid inflation is also treated in Appendix A). Consider first the case in which the modulus is effectively massless during inflation, meaning

$$m_{\sigma, \text{eff}} \ll \frac{H_{\text{inf, in}}}{\sqrt{N_T}}, \quad (4)$$

where $H_{\text{inf, in}}$ is the Hubble parameter at the beginning of inflation and N_T the total number of e-folds of inflation. Let us notice that, strictly speaking, this condition is not equivalent to $m_{\sigma, \text{eff}} \ll H_{\text{inf}}$ if the Hubble parameter evolves during inflation and/or if the total number of e-folds is large. The modulus mass is written $m_{\sigma, \text{eff}}$ to encompass two different cases: a fixed mass m_σ or a (supergravity induced) Hubble scale mass $c_i H$, with $0 < c_i < 1$. Here, we have written c_i in order to emphasize (see also Appendix A and see Appendix C for concrete examples) that the effective mass during inflation is not necessarily the same as the effective mass in the postinflationary era (in other words, *a priori*, $c_i \neq c$). For large field $m_\phi^2 \phi^2$ inflation, $H_{\text{inf, in}} N_T^{-1/2} = \sqrt{2/3} m_\phi \approx 1.4 \times 10^{13}$ GeV is fixed by normalization to the cosmic microwave background anisotropies. Given the values $m_\sigma < 10^7$ GeV considered in this paper, this means that, in the pure quadratic case, the modulus field is always massless. Of course, when Hubble scale corrections are present and $c_i \ll 1$,

this is no longer the case. For small field inflation, however, the situation is different. Indeed, the quantity $H_{\text{inf},\text{in}} N_{\text{T}}^{-1/2}$ can be very small or very large depending on the inflationary scale (which can be as low as $\sim \text{TeV}$) and the number of e-folds ($N_{\text{T}} \gtrsim 60$ but is otherwise essentially unbounded, see Appendix A). Therefore, for small field inflation, the pure quadratic case can or cannot correspond to a massless situation. At the classical level, one finds that, in this “massless” field case, $\sigma_{\text{cl}} \sim \sigma_{\text{in}}$, meaning that the classical value has not changed during inflation.

Concerning the contribution of quantum effects, one finds

$$\langle \delta \sigma^2 \rangle^{1/2} \simeq \frac{H_{\text{inf},\text{in}}}{2\pi} N_{\text{T}}^{1/2}. \quad (5)$$

This result holds for small field inflation; for large field $m^2 \phi^2$, it is a factor $\sqrt{2}$ smaller, see Appendix A. This value does not depend on $m_{\sigma,\text{eff}}$ and it diverges in the limit of de Sitter spacetime ($N_{\text{T}} \rightarrow +\infty$), as expected for a massless field. This value can actually be understood simply as follows: every e-fold $H_{\text{inf}} t$, the field performs a random step of length $\pm H_{\text{inf}}/(2\pi)$, which add up randomly, yielding the above random walk behavior. Setting $N_{\text{T}} \gtrsim 60$ yields the following lower bound:

$$\langle \delta \sigma^2 \rangle^{1/2} \gtrsim 5 \times 10^{-6} M_{\text{Pl}} \left(\frac{H_{\text{inf},\text{in}}}{10^{13} \text{ GeV}} \right). \quad (6)$$

For small field inflation, $H_{\text{inf},\text{in}} \sim H_{\text{inf}}$, i.e. the Hubble constant does not change much during inflation. For large field $m_\phi^2 \phi^2$ inflation however, as already noticed before, the numerical prefactor is $1/\sqrt{2}$ times smaller, but $H_{\text{inf},\text{in}} \simeq \sqrt{2N_{\text{T}}} H_{\text{inf}}$, and $H_{\text{inf}} \sim 10^{13} \text{ GeV}$, so that overall the above bound is a factor $\simeq 8$ larger [see Eq. (A33)].

Let us now consider the other limit of a massive field, $m_{\sigma,\text{eff}} \gg H_{\text{inf},\text{in}} N_{\text{T}}^{-1/2}$, yet not too massive in the sense that $m_{\sigma,\text{eff}} \ll H_{\text{inf}}$. This is relevant for the pure quadratic case during small field inflation (depending on the parameters, see above) and when the Hubble scale corrections are present in both cases. Then, at the classical level, one finds that the field evolves and rolls down its potential during inflation. Regarding the magnitude of quantum effects, one obtains

$$\langle \delta \sigma^2 \rangle^{1/2} \simeq \sqrt{\frac{3}{8\pi^2}} \frac{H_{\text{inf}}^2}{m_{\sigma,\text{eff}}}. \quad (7)$$

For large field $m_\phi^2 \phi^2$ inflation and typical moduli masses $\ll 10^{13} \text{ GeV}$, this case only applies if the modulus receives Hubble scale mass corrections $m_{\sigma,\text{eff}} = c_i H$, since $H_{\text{inf},\text{in}} N_{\text{T}}^{-1/2} \sim 10^{13} \text{ GeV}$. Furthermore, the right-hand side in the above equation should be multiplied by an extra factor of $\sqrt{3 + c_i^2}/c_i$ in this case [see Eq. (A44)]. Of course, as $c_i \rightarrow 0$ the field becomes light and one recovers the previous result (see Appendix A). In the case of small

field inflation, the above value can be quite small and all the more so as the scale of inflation is lowered. Note that the above value of $\langle \delta \sigma^2 \rangle^{1/2}$ reproduces the well-known Bunch-Davies expression for a massive field in de Sitter spacetime [52–55]. As discussed in Ref. [9], one can understand this result by considering the same random walk as before, but noting that modes on large wavelengths redshift away in proportion to $\exp[-m_{\sigma,\text{eff}}^2 t/(3H_{\text{inf}})]$, which implies that the maximum contribution to the fluctuations has been generated during the last $\Delta N \sim 3H_{\text{inf}}^2/m_{\sigma,\text{eff}}^2$ e-folds. The product $\Delta N \times H_{\text{inf}}^2/(4\pi^2)$ then reproduces the Bunch-Davies result (squared) to within a factor 2. In this respect, one should note that the previous limit $m_{\sigma,\text{eff}} \ll H_{\text{inf}}/\sqrt{N_{\text{T}}}$ that we considered corresponds to a field so light that modes do not have time to redshift away in N_{T} e-folds. In this limit, ΔN is bounded by N_{T} , hence Eq. (5) is recovered.

Finally, the last case of interest is $m_{\sigma,\text{eff}} \gg H_{\text{inf}}$. In this situation, the field is too massive to be excited, and consequently $\langle \delta \sigma^2 \rangle^{1/2}$ is exponentially suppressed.

B. Evolution of perturbations

Let us now introduce the scenario at the level of perturbations. As we have just done for the background quantities, one can also follow the perturbations of each species throughout the cosmic evolution. To be more precise, we are interested in the curvature perturbation for the species “ α ” defined by [56–59]

$$\zeta_\alpha \equiv -\Phi - H \frac{\Delta \rho_\alpha}{\dot{\rho}_\alpha} \simeq -\Phi + \frac{\Delta_\alpha}{3(1 + \omega_\alpha)}, \quad (8)$$

where Φ is the Bardeen potential, Δ_α the gauge-invariant density contrast, and $\omega_\alpha \equiv p_\alpha/\rho_\alpha$ the equation of state parameter.

After the decay of the inflaton field, the fluctuations in ϕ have been transmitted to radiation, characterized by $\zeta_\gamma^{(i)}$, dark matter, $\zeta_\chi^{(i)}$, and baryons (and antibaryons), $\zeta_b^{(i)}$, $\zeta_b^{(i)}$. Since these fluids share thermal equilibrium, one has [60] $\zeta_\gamma^{(i)} = \zeta_\chi^{(i)} = \zeta_b^{(i)} = \zeta_b^{(i)}$. Let us notice that, at this stage, dark matter and baryons are still relativistic fluids. Indeed, dark matter becomes nonrelativistic at a temperature of $\sim m_\chi = \mathcal{O}(100) \text{ GeV}$. Regarding the baryons, the situation is more complicated since, in principle, they become nonrelativistic at a temperature of $\sim 1 \text{ GeV}$ (about the same as the dark matter freeze-out temperature), that is to say well below the reheating temperature, but in fact at that temperature one still have a quark-gluon plasma.

We define the above initial conditions, indexed with (i) , well into the modulus oscillations era, at $H \ll m_\sigma$ and $T < T_{\text{rh}}$, and before the modulus comes to dominate the energy density. In this era, the modulus can be considered as a pressureless fluid, supergravity contributions to its potential have become negligible, hence previous results on curvaton phenomenology can be applied, as discussed

further below. One needs however to relate the modulus curvature perturbations at this time, $\zeta_\sigma^{(i)}$ to the modulus perturbations acquired through inflation. This obviously depends on the modulus potential at $m_\sigma < H < H_{\text{inf}}$.

For a simple time independent quadratic modulus potential, one can use the results of Ref. [44], which give

$$\zeta_\sigma^{(i)} = -\frac{3}{2}\Phi_{\text{inf}} + \frac{2}{3}\frac{\delta\sigma_{\text{inf}}}{\sigma_{\text{inf}}}. \quad (9)$$

The quantity Φ_{inf} denotes the Bardeen potential at the end of inflation, $\delta\sigma_{\text{inf}}$ denotes the modulus perturbations on large scales and the calculation assumes that the modulus behaves as a test field, so that Φ is approximately constant: $\Phi_{\text{inf}} = \Phi^{(i)}$. It also assumes that radiation dominates at the time at which $\zeta_\sigma^{(i)}$ is defined. It is important to realize that the radiation curvature perturbation is related to the Bardeen potential through $\zeta_\gamma^{(i)} = -3\Phi^{(i)}/2$, so that the initial modulus—radiation isocurvature perturbation can be rewritten as

$$S_{\sigma\gamma}^{(i)} = 2\frac{\delta\sigma_{\text{inf}}}{\sigma_{\text{inf}}}. \quad (10)$$

In Sec. III B, we show that this result holds even when the modulus potential receives a supergravity inspired $+c^2H^2$ quadratic mass term. This result is of importance for the present discussion, since it shows that the modulus—radiation isocurvature fluctuation disappears in the limit $\delta\sigma_{\text{inf}}/\sigma_{\text{inf}} \rightarrow 0$. One way to achieve this is to assume that the modulus is heavy during inflation, either because $H_{\text{inf}} \lesssim m_\sigma$ or because the modulus receives an effective mass term $+c_\sigma^2H^2$ during inflation. Furthermore, Ref. [61] has shown that the isocurvature mode between the inflation and any heavy field actually disappears during inflation because the heavy field is drawn to the minimum of its potential at every point in space, so that in this case, there would not even be an isocurvature fluctuation to start with, at the end of inflation. For this reason, we discard for now this case and assume everywhere that the modulus has remained light during inflation, in which case $\delta\sigma_{\text{inf}}/\sigma_{\text{inf}} \simeq H_{\text{inf}}/(2\pi\sigma_{\text{inf}})$. In Appendix C, we present several concrete models of inflationary model building in a supergravity framework; for both models of D-term inflation, it is found that the modulus remains light during inflation, but acquires a Hubble effective mass after inflation.

In the following, we set $\zeta_\gamma^{(i)} \simeq 10^{-5}$. From the time at which the initial conditions are defined, all the ζ_α remain constant until dark matter freeze-out (in between, the dark matter has become nonrelativistic, see before). During this phase, the radiation, baryons, and modulus fluid perturbations are not affected,

$$\begin{aligned} \zeta_\gamma^{>f} &= \zeta_\gamma^{(i)}, & \zeta_b^{>f} &= \zeta_b^{(i)}, \\ \zeta_b^{>f} &= \zeta_b^{(i)}, & \zeta_\sigma^{>f} &= \zeta_\sigma^{(i)}, \end{aligned} \quad (11)$$

but the dark matter perturbations are modified according to [40,45]

$$\zeta_\chi^{>f} = \zeta_\chi^{(i)} + \frac{(\alpha_f - 3)\Omega_\sigma^{>f}}{2(\alpha_f - 3) + \Omega_\sigma^{>f}}[\zeta_\sigma^{(i)} - \zeta_\gamma^{(i)}], \quad (12)$$

with $\alpha_f \equiv x_f + 3/2$. From the above equation, one sees that the quantity ζ_χ is not modified if $\Omega_\sigma^{>f} \rightarrow 0$ (i.e. the modulus is negligible at dark matter freeze-out) and $\zeta_\sigma^{(i)} = \zeta_\gamma^{(i)}$ in which case the freeze-out surface exactly coincides with the uniform radiation surface.

Then, the ζ_α 's remain constant until the baryons freeze out (assuming it occurs before modulus decay). Through this stage, it is clear that ζ_γ , ζ_χ , and ζ_σ remain unaffected. On the contrary, one expects ζ_b and ζ_σ to evolve. One can consider a ‘‘net baryon number’’ fluid, the energy density of which is given by $\Omega_b - \Omega_{\bar{b}}$. Before baryons freeze out, this fluid is made of baryons and antibaryons in thermal equilibrium (with a small excess of baryons) but after the freeze out of annihilations, it is essentially made of baryons. In the absence of any baryon number violating process, this fluid of ‘‘net baryon number’’ is isolated, hence its curvature perturbation remains constant. Therefore, after baryons freeze-out, one has $\zeta_b = \zeta_b^{>f} = \zeta_b^{(i)}$ and $\zeta_{\bar{b}} = 0$. Notice that the same reasoning would also be valid in the case where the freeze-out occurred after modulus decay. In fact, the above discussion would be modified only if baryon number violation occurred after modulus decay.

Finally, the modulus decays in dark matter, radiation, baryons and antibaryons and it is clear that all the corresponding curvature perturbations are then modified. We obtain (see Refs. [45,47] for details)

$$\begin{aligned} \zeta_\gamma^{>d} &= \zeta_\gamma^{<d} + r^{<d}(\zeta_\sigma^{<d} - \zeta_\gamma^{<d}) \\ &= \zeta_\gamma^{(i)} + r^{<d}[\zeta_\sigma^{(i)} - \zeta_\gamma^{(i)}], \end{aligned} \quad (13)$$

$$\begin{aligned} \zeta_\chi^{>d} &= \zeta_\chi^{<d} + \frac{B_\chi\Omega_\sigma^{>f}}{\Omega_\chi^{>f} + B_\chi\Omega_\sigma^{>f}}(\zeta_\sigma^{<d} - \zeta_\chi^{<d}) \\ &= \zeta_\chi^{>f} + \frac{B_\chi\Omega_\sigma^{>f}}{\Omega_\chi^{>f} + B_\chi\Omega_\sigma^{>f}}[\zeta_\sigma^{(i)} - \zeta_\chi^{>f}], \end{aligned} \quad (14)$$

$$\zeta_b^{>d} = \zeta_b^{<d} = \zeta_b^{(i)}. \quad (15)$$

As explained in Refs. [45,47], in order to obtain these relations, we have assumed that a fraction $B_\chi \equiv \Gamma_{\sigma\chi}/\Gamma_\sigma \ll 1$ of the σ energy density goes into dark matter particles. One can relate this parameter B_χ to the branching ratio of modulus decay into dark matter particles as follows. Assuming that dark matter particles thermalize instantaneously, then if one modulus produces through its decay N_χ particles, one finds that $B_\chi = N_\chi m_\chi/m_\sigma$. In the range of parameters that we are interested in, $m_\chi \ll m_\sigma$ and $N_\chi \lesssim 1$ (see Refs. [13,62] for a detailed discussion).

Strictly speaking, the thermalization is quasi-instantaneous only in the high modulus mass range, $m_\sigma \gtrsim 100$ TeV, while at lower masses some redshifting due to the cosmic expansion occurs. This can be seen as follows. Using the results of Refs. [63,64], one can write down the scattering cross section of Compton-like processes $\chi + \ell \rightarrow \chi + \ell$ (with ℓ an ultrarelativistic lepton of the thermal bath, e.g. a neutrino or an antineutrino at the big bang nucleosynthesis epoch)

$$\sigma_{\chi\ell} \simeq \frac{3C}{128\pi} \frac{(s - m_\chi^2)^2}{m_\chi^2 s^2}, \quad (16)$$

with C a prefactor of order unity defined in Ref. [64], and s the standard center of mass energy squared: $s \simeq 2E_\chi E_\ell + m_\chi^2$ with $E_\ell \simeq 3.15 T$ where T is the temperature of the thermal bath. In the region of interest (at modulus decay), one can check that $2E_\chi E_\ell \ll m_\chi^2$. The ratio of the interaction rate, $\Gamma_{\chi\ell} \equiv n_\ell \sigma_{\chi\ell}$ to the Hubble rate at the time of modulus decay can then be expressed as

$$\left. \frac{\Gamma_{\chi\ell}}{H} \right|_d = 1.4 \times 10^8 C \left(\frac{E_\chi}{m_\sigma} \right)^2 \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{13/2} \left(\frac{m_\chi}{100 \text{ GeV}} \right)^{-6}. \quad (17)$$

As a consequence, if $m_\sigma \gtrsim 30 \text{ TeV} C^{-2/9} (m_\chi/100 \text{ GeV})^{8/9}$, one has $\Gamma_{\chi\ell} \gtrsim H$ for all energies $E_\chi > m_\chi$, which implies that the χ particle becomes nonrelativistic through multiple interactions in less than a Hubble time. In this range, and for our purpose, one can treat the dark matter fluid as a pressureless fluid immediately after modulus decay. Inversely, if $m_\sigma \lesssim 7 \text{ TeV} C^{-2/13} (m_\chi/100 \text{ GeV})^{12/13}$, $\Gamma_{\chi\ell} \lesssim H$ for all energies $E_\chi < m_\sigma/2$: the particle never interacts and simply redshifts to nonrelativistic velocities within $\ln[m_\sigma/(2m_\chi)]$ e-folds. Finally, in the intermediate range, the χ particle loses its energy through interactions to some intermediate value $90 \text{ GeV} C^{-1/2} (m_\sigma/100 \text{ TeV})^{-9/4} (m_\chi/100 \text{ GeV})^3$, then redshifts away down to m_χ through cosmic expansion.

All in all, our above assumption of ‘‘instantaneous thermalization’’ amounts to neglecting this redshifting factor, which in turn slightly overestimates the abundance of σ produced dark matter by a factor which never exceeds $\simeq 35$. This value applies at $m_\sigma \simeq 7$ TeV and it rapidly decreases to unity away from this value. This only affects very marginally the results derived below.

We have also assumed that the fraction of modulus energy density $B_{b+\bar{b}}$ that goes into baryons and antibaryons is very much smaller than unity, as one would expect. With respect to Ref. [47], we have also assumed here that modulus decay preserves baryon number. Finally, the parameter $r^{<d}$ that appears in the first of the above equations has been introduced in Ref. [36]; if dark matter is entirely produced by modulus decay (i.e. $\Omega_\chi^{>f} \approx 0$), then $1 - r^{<d}$ characterizes the amount of initial modulus—radiation

isocurvature mode that is transferred through modulus decay. We use the simple formula

$$r^{<d} \simeq \Omega_\sigma^{<d}, \quad (18)$$

which has been found numerically to be a good approximation [41].

After modulus decay, the ζ_α 's remain constant throughout the subsequent cosmic evolution. The corresponding values can be compared to cosmic microwave background data.

C. Transfer of isocurvature modes

As we will see shortly, dark matter—radiation and baryon—radiation isocurvature modes are generated in different regions of the parameter space. The constraints obtained are thus complementary to each other. For this reason, and for the sake of clarity, we discuss the generation of each mode in turn. Let us also recall that the isocurvature perturbations between two fluids ‘‘ α ’’ and ‘‘ β ’’ is defined by $S_{\alpha\beta} \equiv 3(\zeta_\alpha - \zeta_\beta)$.

The dark matter—radiation and baryon isocurvature modes on large scales are given by [45]

$$S_{\chi\gamma}^{>d} \simeq \frac{1}{1 + Y_\chi} \left[\frac{(\alpha_f - 3)\Omega_\sigma^{>f}}{2(\alpha_f - 2) + \Omega_\sigma^{>f}} \frac{\Omega_\chi^{>f}}{\Omega_\chi^{>f} + B_\chi \Omega_\sigma^{>f}} + \frac{B_\chi \Omega_\sigma^{<d}}{\Omega_\chi^{<d} + B_\chi \Omega_\sigma^{<d}} - r^{<d} \right] S_{\sigma\gamma}^{(i)}, \quad (19)$$

$$S_{b\gamma}^{>d} \simeq -\Omega_\sigma^{<d} S_{\sigma\gamma}^{(i)}. \quad (20)$$

These formulas can be straightforwardly deduced from the results quoted in the previous section, except the presence of the parameter Y_χ in $S_{\chi\gamma}^{>d}$. This parameter represents the ratio of the dark matter annihilation rate to the expansion rate immediately after the decay of the modulus field. If this latter produces sufficiently many dark matter particles, these may annihilate with each other. As $Y_\chi \gg 1$, meaning that annihilations are effective, the isocurvature perturbation transfer is partially erased (see Ref. [45] for details). One finds

$$Y_\chi \simeq B_\chi \frac{m_\sigma}{m_\chi} n_\sigma^{<d} \frac{1}{\Gamma_\sigma} \frac{g_{*,f}^{1/2}}{0.076} e^{x_f} \frac{\sqrt{x_f}}{m_\chi M_{\text{Pl}}} \approx 1.68 \times 10^{-8} B_\chi \Omega_\sigma^{<d} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^3 \left(\frac{m_\chi}{200 \text{ GeV}} \right)^{-2} \times \left(\frac{g_{*,f}}{100} \right)^{1/2} \times \sqrt{x_f} e^{x_f}. \quad (21)$$

The above formula neglects the amount of dark matter initially present; this is a good approximation insofar as the annihilation rate of these dark matter particles is very much smaller than the Hubble rate after freeze-out in the absence of modulus decay, which would lead to $Y_\chi \ll 1$

hence to a negligible correction to the equation for the transfer of isocurvature mode. Note that the last factor involving x_f in the above equation may be quite large, being $\sim 2.2 \times 10^9$ for $x_f = 20$ and 5.9×10^{13} for $x_f = 30$. Finally, note that the above formula is only approximate (see Ref. [45] for details). The constraints that we derive further below are obtained through the numerical integration of the full set of equations of motion and are therefore more accurate.

A significant dark matter—radiation isocurvature mode is generated if both following conditions are satisfied:

$$B_\chi \Omega_\sigma^{\leq d} \gg \Omega_\chi^{\leq d}, \quad \Omega_\sigma^{\leq d} \ll 1. \quad (22)$$

The former condition expresses the fact that the amount of moduli produced dark matter particles exceeds that coming from freeze-out of annihilations, while the latter requires that the modulus energy density is not sufficient to affect the radiation content. All in all, this means that the modulus perturbation is transferred to the dark matter fluid but not to radiation. One could also imagine that the modulus perturbations are transferred to the radiation sector but not to the dark matter sector, thereby generating a net isocurvature perturbations. However, this would require either blocking the decay of modulus to dark matter, which is unlikely as the dark matter particle is always much lighter than the modulus in the parameter space we are interested in, or, having the modulus decay after matter—radiation equality, which is forbidden by constraints on cosmic microwave background distortions.

At this stage, we need to compute explicitly the parameters appearing in Eqs. (19) and (20). In particular, from the expression of the dark matter annihilation cross section, one obtains

$$\Omega_\chi^{\gt f} \simeq 1.67 \times 10^{-3} x_f^{3/2} e^{-x_f}. \quad (23)$$

One also needs to evaluate the quantity $\Omega_\sigma^{\gt f}$ and $\Omega_\sigma^{\leq d}$. From the fact that the energy density of the modulus scales as a^{-3} , one obtains that Ω_σ just after the dark matter freeze out can be expressed as

$$\Omega_\sigma^{\gt f} = \Omega_{\sigma, \text{osci}} \frac{x_f}{m_\chi} \min(T_{\text{osci}}, T_{\text{rh}}) \Omega_\gamma^{\gt f}. \quad (24)$$

In this expression, the minimum of T_{osci} and T_{rh} appears because, if the oscillations start before the end of reheating, then the inflaton and modulus energy densities have the same scaling until the reheating stage is completed. Only below T_{osci} or T_{rh} , whichever is smaller, the energy density of the σ oscillations increases relatively to radiation energy density. In order to obtain the above formula, we have also assumed that the modulus can never start a new phase of inflation. In the same manner, immediately prior to decay, the ratio of the energy density contained in σ oscillations to that contained in radiation reads

$$\Omega_\sigma^{\leq d} = \Omega_{\sigma, \text{osci}} \frac{\min(T_{\text{osci}}, T_{\text{rh}})}{T_d} \Omega_\gamma^{\leq d}. \quad (25)$$

We are now in a position where one can calculate the transfer of isocurvature perturbation from modulus—radiation to dark matter—radiation and baryon—radiation. This is done in the section that follows.

III. CONSTRAINTS IN THE MODULUS PARAMETER SPACE

Our present goal is to compute the amount of isocurvature perturbation produced through the differential decay of the modulus into dark matter and radiation, as well as between the baryon and the radiation fluid, which can then be compared to existing bounds obtained through the analysis of cosmic microwave background fluctuations [18–30]. We have chosen to express the amount of isocurvature fluctuation in this matter sector as follows:

$$\delta_{m\gamma} \equiv \frac{\zeta_m - \zeta_\gamma}{(\zeta_m + \zeta_\gamma)/2}, \quad (26)$$

where the subscript “m” comprises all of nonrelativistic matter, i.e. dark matter and baryons (so that, for instance, $\Omega_m \equiv \Omega_\chi + \Omega_b$). The quantity ζ_m can be written in terms of the baryon and dark matter curvature perturbations ζ_b and ζ_χ :

$$\zeta_m \equiv \frac{\Omega_b}{\Omega_m} \zeta_b + \frac{\Omega_\chi}{\Omega_m} \zeta_\chi. \quad (27)$$

The definition (26) can be justified by the fact that the data are in fact sensitive to the quantity defined by $S_{b\gamma}^{\text{eff}} = S_{b\gamma} + \Omega_\chi S_{\chi\gamma}/\Omega_b$ (see Ref. [65]). In this reference, the quantity $B \equiv S_{b\gamma}^{\text{eff}}/\zeta_\gamma$ is constrained using various cosmic microwave background data (including WMAP1). At 95% C.L. it was found that $-0.53 < B < 0.43$ (see Ref. [65]). Our quantity $\delta_{m\gamma}$ is related to B through

$$\delta_{m\gamma} = \frac{2B}{6\Omega_m/\Omega_b + B}, \quad (28)$$

which implies $-0.12 \leq \delta_{m\gamma} \leq 0.089$ where we have taken $\Omega_\chi h^2 \simeq 0.12$ and $\Omega_b h^2 \simeq 0.0225$. The choice (26) is also motivated by the fact that the most recent analysis of Wilkinson Microwave Anisotropy Probe fifth year (WMAP5) data has constrained the same quantity for dark matter only (i.e. $m \rightarrow \chi$) [31]. These results give an upper bound as low as 2.0% (95% C.L.) for fully anticorrelated isocurvature modes and 8.6% (95% C.L.) for uncorrelated modes. In the following, we will present contours on the quantity $\delta_{m\gamma}$ and emphasize the loci of 1% and 10%.

The quantity $\delta_{m\gamma}$ must be calculated at the time of recombination, well after baryon and dark matter freeze-out of annihilations and curvaton decay (and big bang nucleosynthesis). The individual gauge invariant curvature perturbations ζ_b , ζ_χ , and ζ_γ are then constant since the

fluids can be considered as isolated at that time. Given the results of the previous section, in order to compute the constraints in modulus parameter space and to evaluate $\delta_{m\gamma}$, the value of $\Omega_{\sigma,\text{osci}}$, which directly controls the energy density parameter of the modulus at the time of decay, is the only remaining quantity which remains to be specified.

Before embarking on a detailed discussion of these above considerations, one should note that two other cosmological constraints are to be satisfied. One concerns the present day abundance of dark matter and the other the overall amplitude of the total curvature perturbation. Regarding the former, dark matter is produced both thermally (through the freeze-out of annihilations) with present day abundance $\Omega_{\chi,0}^f$, and nonthermally (through modulus decay) with present day abundance $\Omega_{\chi,0}^\sigma$ (immediately after decay $\Omega_\chi^\sigma = B_\chi \Omega_\sigma^{\leq d}$). Hence, the final abundance is controlled by the parameters x_f , m_χ , N_χ , and $\Omega_\sigma^{\leq d}$. In order to minimize the dimensionality of our parameter space, we have chosen to proceed as follows. We use a standard set of values $x_f = 21$ and $m_\chi = 100$ GeV, which implies that $\Omega_{\chi,0}^f = 0.2$; we then tune N_χ for each value of our main parameters m_σ and σ_{inf} such that the total $\Omega_{\chi,0}$ lies within a factor of 3 of its observed value. More precisely, we maintain $N_\chi = 1$ (see above and Refs. [13,62]) whenever $\Omega_{\chi,0}^\sigma < 0.5$ and decrease it in order to saturate this last bound otherwise. This is somewhat arbitrary, but given the remaining freedom in x_f and m_χ , one could always tune slightly the parameters to achieve a better agreement with the known value.

Let us discuss briefly how the results derived below are affected by this choice. If the dark matter annihilation cross section were larger than the standard value (assumed above), then the amount of dark matter that originates from annihilations would be reduced, consequently the contribution from modulus decay would have to be increased, hence the amount of dark matter isocurvature fluctuation would also increase. In an extreme region of parameter space, such that freeze-out of dark matter occurs after modulus decay, the isocurvature constraints would vanish (as discussed in detail in Ref. [45]); this requires a modulus decay temperature $T_d \gg 1$ GeV, hence $m_\sigma \gg 10^7$ GeV.

The above approach is conservative in the sense that we forbid the nonthermal channel to exceed twice the thermal channel which slightly reduces the isocurvature perturbations. In this way, in all of the parameter space scanned in the subsequent figures, the dark matter abundance is correct to within a factor of 2 to 3.

Concerning the overall amplitude of the curvature perturbation, one needs to require that $\zeta_\gamma^{\geq d} \simeq 10^{-5}$. This quantity is determined by Eq. (13) and it can be rewritten as

$$\zeta_\gamma^{\geq d} \simeq \zeta_\gamma^{(i)} + \frac{1}{3} \Omega_\sigma^{\leq d} S_{\sigma\gamma}^{(i)}. \quad (29)$$

Therefore the magnitude of the total curvature perturbation is controlled by several parameters, including σ_{inf} and H_{inf} which determine the scalings of $\Omega_\sigma^{\leq d}$ and $S_{\sigma\gamma}^{(i)}$. In principle, it would be possible to rescale H_{inf} in order to reach the correct magnitude for $\zeta_\gamma^{\geq d}$ at each value of σ_{inf} . However, this would make the interpretation of the figures rather complex. In the following, we have rather chosen to plot the constraints obtained for two values of H_{inf} in each case, in order to gauge the effect of H_{inf} on these constraints. These two values are $H_{\text{inf}} = 10^{13}$ GeV, which provides a natural scale for inflation since it corresponds to the simplest inflaton potential $m_\phi^2 \phi^2$ (see also the discussion about naturalness in Ref. [66]), and $H_{\text{inf}} = 10^9$ GeV. The latter is chosen arbitrarily, but it is such that the total curvature perturbation is of the right order of magnitude at every point of the modulus parameter space. For the former value of H_{inf} , a significant region of modulus parameter space is excluded by the normalization of the total curvature perturbation; however, this region is entirely contained in the region which is excluded by the isocurvature constraints.

A. Quadratic potential

Here, we assume that the potential of the modulus is a simple quadratic potential $V(\sigma) = m_\sigma^2 \sigma^2 / 2$ from the end of inflation onwards. At the onset of oscillations, the σ field carries a fraction $\Omega_{\sigma,\text{osci}}$ of the total energy density:

$$\Omega_{\sigma,\text{osci}} = \frac{1}{6} \left(\frac{\sigma_{\text{inf}}}{M_{\text{Pl}}} \right)^2. \quad (30)$$

Therefore, one can now explicitly evaluate the quantities $\Omega_\sigma^{\geq f}$, $\Omega_\sigma^{\leq d} \sim r^{\leq d}$ [see Eqs. (18) and (23)–(25)]. In particular, using Eqs. (3) and (25), immediately prior to decay, the ratio of the energy density contained in σ oscillations to that contained in radiation now read:

$$\Omega_\sigma^{\leq d} \simeq 6 \times 10^{10} \alpha_{\text{osci/rh}} \left(\frac{\sigma_{\text{inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{-3/2} \left(\frac{g_{\star,\text{dec}}}{10.75} \right)^{1/4} \times \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}} \right) \Omega_\gamma^{\leq d}. \quad (31)$$

The parameter $\alpha_{\text{osci/rh}}$ is defined as follows:

$$\alpha_{\text{osci/rh}} \equiv \min \left(1, \frac{T_{\text{osci}}}{T_{\text{rh}}} \right). \quad (32)$$

This parameter is most likely 1 if one relies on the constraints on the reheating temperature that result from the influence of a moderately massive gravitino on big bang nucleosynthesis. In models in which the gravitino is very massive, $m_{3/2} \gtrsim 100$ TeV, however, such constraints can be evaded. We thus treat $\alpha_{\text{osci/rh}}$ as a free parameter. Obviously, looking at Eq. (31), unless $\sigma_{\text{inf}} \lesssim 10^{-5} M_{\text{Pl}}$, the modulus is bound to dominate the energy density of the Universe at the time of its decay.

Let us now analyze the constraints presented in Fig. 1. This figure shows the contours of the $\delta_{m\gamma}$ quantity calculated numerically, assuming $T_{\text{rh}} = 10^9$ GeV, $H_{\text{inf}} = 10^{13}$ GeV in the left panel and $H_{\text{inf}} = 10^9$ GeV in the right

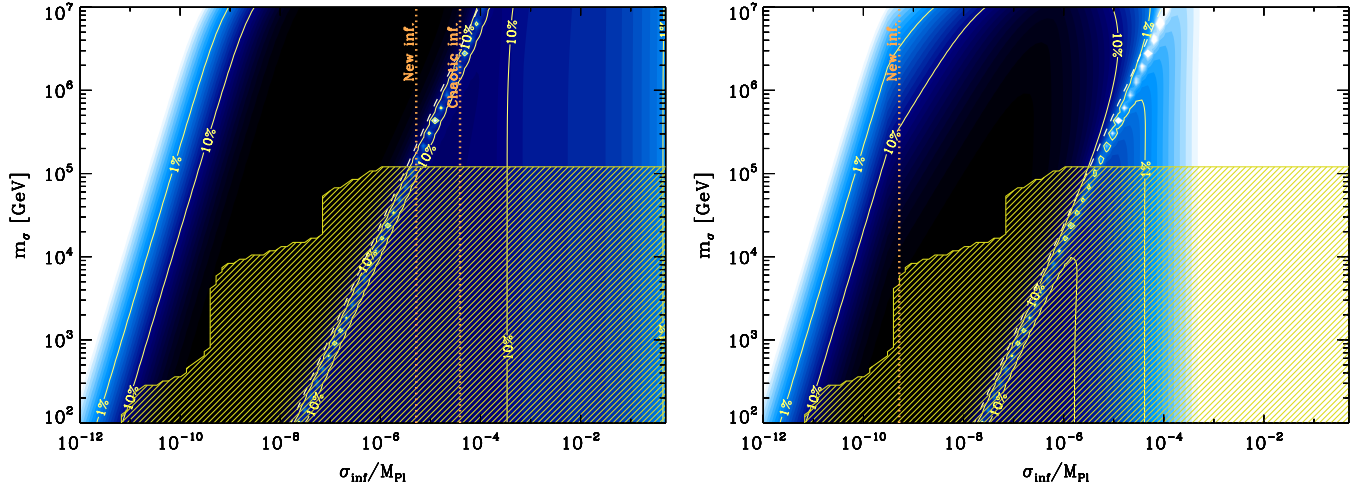


FIG. 1 (color online). Constraints in the $(m_\sigma, \sigma_{\text{inf}})$ plane. The yellow shaded region shows the region excluded by the effect of moduli decay on big bang nucleosynthesis. The blue region give the contours of $\delta_{m\gamma}$ for the matter—radiation isocurvature mode (1% and 10% contours indicated). The white dashed line indicates the place where $\Omega_{\sigma^d} = 0.5$. The orange dotted lines indicate the standard deviation of the modulus expected from stochastic evolution in its potential during inflation, for new and chaotic inflation, as indicated. Left panel: $H_{\text{inf}} = 10^{13}$ GeV and $T_{\text{rh}} = 10^9$ GeV. This value corresponds to the energy scale of inflation during chaotic inflation. The same value is achieved for small field inflation if $\mu \approx M_{\text{Pl}}$. In the case of chaotic inflation, the standard deviation is given by Eqs. (A29) and (A33) since the condition $m_\sigma < H_{\text{inf}} N_T^{-1/2} \approx 1.4 \times 10^{13}$ GeV is always satisfied for $m_\sigma < 10^7$ GeV. In the case of small field inflation, the standard deviation is given by Eq. (A60) assuming the above mentioned condition is also satisfied in this case which amounts requiring that $N_T < 10^{12}$. Right panel: $H_{\text{inf}} = 10^9$ GeV and same reheating temperature. For small field inflation, this corresponds to $\mu \approx 0.22 M_{\text{Pl}}$ assuming $p = 3$. The chaotic inflation standard deviation does not appear in this panel because $H_{\text{inf}} = 10^9$ GeV cannot be realized in this case. For small field inflation the standard deviation is again given by Eq. (A60) which, this time, requires $N_T < 10^4$.

panel, $N_\chi = 1$. This figure assumes initial (gauge invariant) density contrasts $\Delta_\gamma^{(i)} = 2 \times 10^{-5}$ and $\delta\sigma_{\text{inf}}/\sigma_{\text{inf}} = H_{\text{inf}}/(2\pi\sigma_{\text{inf}})$ on large scales. The dashed yellow area is excluded by big bang nucleosynthesis; in order to draw this region, we used the results of Ref. [3] for a hadronic branching ratio of 10^{-3} and initial jet energy 1 TeV. The white dashed line indicates the place where $\Omega_{\sigma^d} = 0.5$ and separates roughly the regions in which either the dark matter or the baryon isocurvature mode dominates (see below). There is actually an accidental cancellation of these two modes close to that line. It has also been assumed that $\zeta_\sigma^{(i)}$ saturates at 0.5 in order for the numerical computations to proceed without errors. In any case, the region in which $\zeta_\sigma^{(i)} \gg 1$, corresponding to $\sigma_{\text{inf}} \ll H_{\text{inf}}$, is an “unlikely” region, in the sense that σ_{inf} is expected to be typically larger than $H_{\text{inf}}/(2\pi)$ due to quantum effects [see Eqs. (5) and (7) for $m_{\sigma,\text{eff}} < H_{\text{inf}}$]. The dotted orange lines in Fig. 1 show the standard deviations $\langle \delta\sigma^2 \rangle^{1/2}$ expected from the stochastic motion of the inflaton, following Appendix A and the formulas Eqs. (5) and (7), for two inflationary scenarios and two inflationary scales. It is important to stress the following: these standard deviations are measured relatively to the “instantaneous” classical value of the modulus field.

Figure 1 provides a clear example of the power of constraints obtained at the perturbative level on moduli

cosmology, as the region excluded by the production of isocurvature fluctuations significantly exceeds that constrained by big bang nucleosynthesis. For instance, at a natural scale $m_\sigma \sim 1$ TeV, the upper bound on σ_{inf} (equivalently, on the modulus energy density) is more stringent than those from big bang nucleosynthesis by some 2 orders of magnitude. This figure also clearly shows that taking $m_\sigma \geq 100$ TeV allows to evade the constraints from big bang nucleosynthesis, but not those from cosmological density perturbations if the inflationary scale is large, $H_{\text{inf}} \sim 10^{13}$ GeV. One may note that this region is also constrained by the possible overproduction of gravitinos through modulus decay [67–69]. It is thus mandatory to require that the modulus potential suffers corrections, in such a way as to reduce considerably the modulus energy density at the time of decay, or that the inflationary scale is much lower. Indeed, if $H_{\text{inf}} \ll 10^{13}$ GeV, a region devoid of constraints opens up at large values of σ_{inf} and large masses $m_\sigma \geq 100$ TeV. The large modulus mass then allows to evade the constraints on entropy injection around big bang nucleosynthesis, while the large value of σ_{inf} ensures that the initial modulus—radiation isocurvature fluctuation has become negligible at such a small inflationary scale. More specifically, using Eq. (10) one finds that $S_{\sigma\gamma}^{(i)} \ll \zeta_\gamma^{(i)}$ when $\sigma \gtrsim M_{\text{Pl}}(H_{\text{inf}}/10^{13} \text{ GeV})$, and the isocurvature fraction constrained by microwave background anisotropies is directly proportional to this ratio.

The constraints obtained in the limit of a high inflationary scale also exclude the possibility of late time entropy production through modulus decay. This result is in itself significant as such entropy production is often invoked to dilute unwanted relics.

In order to understand the relative contributions of dark matter—radiation and baryon—radiation isocurvature fluctuations, it is useful to break Fig. 1 into two subfigures, each showing one of the two contributions. In the left panel of Fig. 2 we plot the same contours as in Fig. 1, assuming arbitrarily (for the sake of demonstration) $\zeta_b^{\gamma d} \sim \zeta_\gamma^{\gamma d}$ (i.e. setting the baryon isocurvature mode to zero). Similarly, in the right panel of Fig. 2, we plot the contours of $\delta_{m\gamma}$ assuming $\zeta_\chi^{\gamma d} \sim \zeta_\gamma^{\gamma d}$ (i.e. no dark matter isocurvature mode). These two plots illustrate the complementarity of the constraints coming from these two isocurvature modes. It is possible to understand each of them as follows.

Assuming for the time being, that $\zeta_b^{\gamma d} \sim \zeta_\gamma^{\gamma d}$ (i.e. no baryon isocurvature mode), the constrained fraction $\delta_{m\gamma}$ can be written in the following way:

$$\delta_{m\gamma} \simeq \frac{\Omega_\chi \zeta_\chi^{\gamma d} / \Omega_m - \Omega_\chi \zeta_\gamma^{\gamma d} / \Omega_m}{[\Omega_\chi \zeta_\chi^{\gamma d} / \Omega_m + (2 - \Omega_\chi / \Omega_m) \zeta_\gamma^{\gamma d}] / 2} = \frac{2x_{\chi\gamma}}{6 + x_{\chi\gamma}}, \quad (33)$$

where

$$x_{\chi\gamma} \equiv \frac{\Omega_\chi}{\Omega_m} \frac{S_{\chi\gamma}^{\gamma d}}{\zeta_\gamma^{\gamma d}}. \quad (34)$$

Therefore, the actual quantity that constrains the magnitude of the final dark matter isocurvature mode is $x_{\chi\gamma}$. This is not surprising since $x_{\chi\gamma}$ is, up to a factor Ω_χ / Ω_m , exactly equal to the quantity B introduced above. Using Eq. (19), one can rewrite $x_{\chi\gamma}$ as:

$$x_{\chi\gamma} \simeq 3 \frac{\Omega_\chi}{\Omega_m} \frac{1}{1 + Y_\chi} \left(\frac{B_\chi \Omega_\sigma^{\leq d}}{\Omega_\chi^{\leq d} + B_\chi \Omega_\sigma^{\leq d}} - r^{\leq d} \right) \frac{\zeta_\sigma^{(i)} - \zeta_\gamma^{(i)}}{\zeta_\gamma^{(i)}}. \quad (35)$$

It is straightforward to verify that the conditions (22) for the generation of isocurvature perturbations lead, if satisfied, to a nonzero value of $x_{\chi\gamma}$. It is however important to note that the magnitude of $x_{\chi\gamma}$ also increases with the ratio $\zeta_\sigma^{(i)} / \zeta_\gamma^{(i)}$. One may rewrite the conditions of existence of a dark matter—radiation isocurvature mode as follows, neglecting the effect of $\zeta_\sigma^{(i)} / \zeta_\gamma^{(i)}$ for clarity:

$$10^{-3} x_f^{1/4} e^{-x_f/2} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{1/2} \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}} \right)^{-1/2} \ll \frac{\sigma_{\text{inf}}}{M_{\text{Pl}}} \ll 4 \times 10^{-6} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{3/4} \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}} \right)^{-1/2}. \quad (36)$$

Out of simplicity, the dependence on the numbers of degrees of freedom has been omitted in these equations. In the above expression, the upper bound comes from the condition $\Omega_\sigma^{\leq d} \ll 1$ using Eq. (31) while the lower bound originates from the condition $B_\chi \Omega_\sigma^{\leq d} \gg \Omega_\chi^{\leq d}$ [see the first formula in Eqs. (22), using the definition of B_χ , the fact that $\Omega_\chi^{\leq d} = \Omega_\chi^{\gamma d} T_f / T_d$ (with $T_f \sim m_\chi / x_f$), and Eqs. (3), (23), and (31)]. These constraints delimit a stripe in the $(\sigma_{\text{inf}}, m_\sigma)$ plane which is consistent with what is observed in Fig. 2. This stripe is actually broader at small values of σ_{inf} due to the large ratio $\zeta_\sigma^{(i)} / \zeta_\gamma^{(i)}$ which enhances the modulus perturbations relative to those of dark matter. The above formula also neglects the effect of annihilations, which is a good approximation as long as $Y_\chi \ll 1$, or

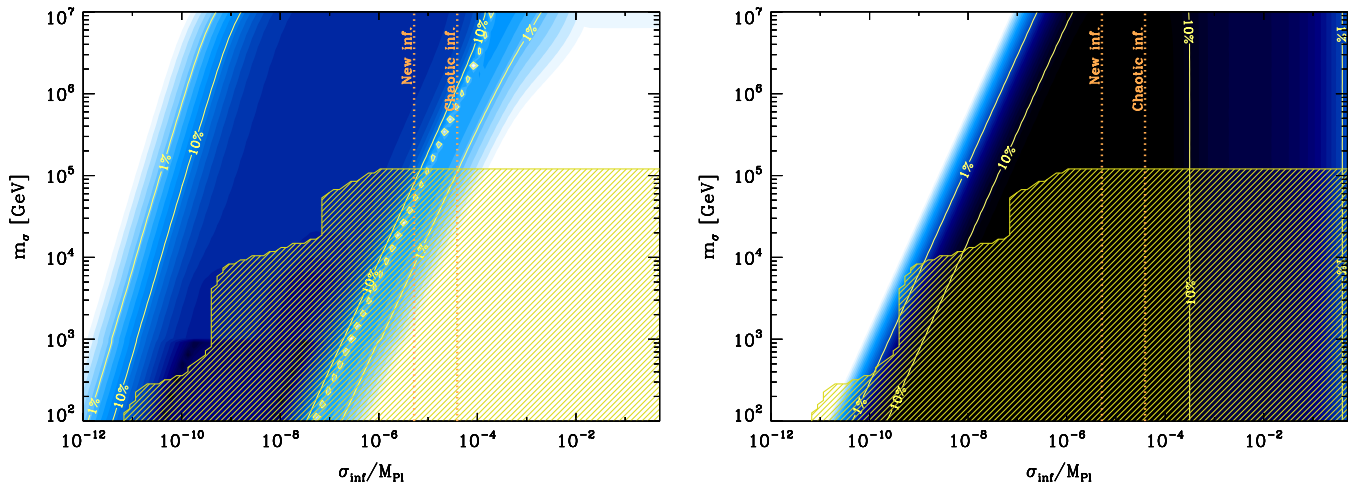


FIG. 2 (color online). Same as the left panel of Fig. 1, but showing only the contribution of dark matter isocurvature modes in the left panel (i.e. setting arbitrarily the baryon isocurvature mode to zero) and the baryon isocurvature mode only in the right panel (setting the dark matter isocurvature mode to zero).

$$\begin{aligned} \left(\frac{\sigma_{\text{inf}}}{M_{\text{Pl}}}\right) &\ll 0.7 \left(\frac{m_\sigma}{100 \text{ TeV}}\right)^{-1/4} \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}}\right)^{-1/2} \\ &\times \left(\frac{m_\chi}{200 \text{ GeV}}\right)^{1/2} x_f^{-1/4} e^{-x_f/2}. \end{aligned} \quad (37)$$

In practice, annihilations will play a role in suppressing the amount of isocurvature mode in the high mass region $m_\sigma \gg 100 \text{ TeV}$ and $\sigma_{\text{inf}} \sim 10^{-6} - 10^{-4} M_{\text{Pl}}$, as can be checked from the above formulas using $x_f \sim 20$.

Turning to the generation of a baryon isocurvature mode, we now assume that $\zeta_\chi^{(f)} \sim \zeta_\gamma^{(f)}$, in which case Eq. (26) can be rewritten as

$$\delta_{m\gamma} \simeq \frac{2x_{b\gamma}}{6 + x_{b\gamma}}, \quad (38)$$

where

$$x_{b\gamma} \simeq -3 \frac{\Omega_b}{\Omega_m} \frac{\Omega_{\sigma^d}^{\leq d} [\zeta_\sigma^{(i)} - \zeta_\gamma^{(i)}]}{\Omega_{\sigma^d}^{\leq d} [\zeta_\sigma^{(i)} - \zeta_\gamma^{(i)}] + \zeta_\gamma^{(i)}}. \quad (39)$$

Therefore the magnitude of the isocurvature mode is proportional to the fraction of energy density of the modulus at the time of decay, times the ratio of initial modulus–radiation isocurvature mode to the initial radiation curvature perturbation. The baryon constraints thus lie at high values of σ_{inf} , since the production of isocurvature fluctuations become dominant when $\Omega_{\sigma^d}^{\leq d}$ is of order unity. However, as $\Omega_{\sigma^d}^{\leq d}$ becomes smaller than unity, its weakness can be compensated to some level by a large value of $\zeta_\sigma^{(i)}/\zeta_\gamma^{(i)}$.

B. Supergravity corrections to the potential

In supergravity, one expects the potential of the modulus to be lifted by an effective term of the form $\pm c^2 H^2 \sigma^2/2$, where the factor c^2 may change between different eras of the thermal history of the Universe (see Appendix C where concrete models are discussed). Including such supergravity corrections, the postinflationary modulus potential may be written as follows:

$$V(\sigma) \simeq \frac{1}{2} m_\sigma^2 \sigma^2 \pm \frac{1}{2} c^2 H^2 (\sigma - \sigma_0)^2. \quad (40)$$

One should recall that we assume the modulus field to be light during inflation, hence the potential (40) refers to the postinflationary epoch only. Moreover, in the following, we denote by “high energy” the regime in which the corrections proportional to $c^2 H^2$ dominate the term proportional to $m_\sigma^2 \sigma^2$ (as mentioned before, although we use the expression “high energy,” these considerations apply to the postinflationary epoch only). Then, the quantity c controls the mass of the field while σ_0 represents its minimum, the so-called “high energy minimum.” One may expect that the vev at the end of inflation (σ_{inf}) be different from σ_0 since the effective potentials during and after inflation *a priori* differ from one another. We also use the terminology “low energy” to characterize the regime

in which the corrections $c^2 H^2$ become subdominant and where the potential reduces to $m_\sigma^2 \sigma^2/2$. Let us recall that we set the “low energy minimum” of the potential at $\sigma = 0$. As a result, the high energy and low energy minima do not coincide if $\sigma_0 \neq 0$ (as one should expect on general grounds).

In order to derive the amplitude of the isocurvature perturbations produced in the supergravity case, we need to follow the evolution of the modulus vev and of its perturbations from the end of inflation until the time at which we set the initial conditions. In particular, we need to calculate $S_{\sigma\gamma}^{(i)}$ in terms of the inflaton and modulus perturbations during inflation. In order to do so, we model the introduction of the effective Hubble mass through a potential that accounts for the coupling between the modulus and the inflation, such as $V(\sigma, \phi) = m_\sigma^2 \sigma^2/2 + m_\phi^2 \phi^2 [1 + c^2 \sigma^2/(3M_{\text{Pl}}^2)]/2$, which produces the desired $+c^2 H^2$ effective mass squared. However, the discussion that follows is not restricted to this particular potential. In order to study this two field system, we use the formalism of Gordon, Wands, Bassett, and Maartens [70] (notice that we have changed notations with respect to Ref. [70] since, now, σ now longer represents the adiabatic field but the modulus one). The entropy field is given by the following expression:

$$\delta s \equiv \cos\theta \delta\sigma - \sin\theta \delta\phi \simeq \delta\sigma - \frac{\dot{\sigma}}{\dot{\phi}} \delta\phi, \quad (41)$$

where we have used that

$$\cos\theta \equiv \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + \dot{\sigma}^2}} \simeq 1, \quad \sin\theta \equiv \frac{\dot{\sigma}}{\sqrt{\dot{\phi}^2 + \dot{\sigma}^2}} \simeq \frac{\dot{\sigma}}{\dot{\phi}}, \quad (42)$$

the last equalities following from the test field approximation: $\dot{\sigma} \ll \dot{\phi}$. Then, using the perturbed Einstein equation for the Bardeen potential on large scales

$$\ddot{\Phi} + H\dot{\Phi} = \frac{1}{2M_{\text{Pl}}^2} (\dot{\phi} \delta\phi + \dot{\sigma} \delta\sigma), \quad (43)$$

one can rewrite the entropy perturbation in terms of $\delta\sigma$ and Φ only. One obtains

$$\delta s \simeq \delta\sigma - \frac{HM_{\text{Pl}}^2}{\dot{\phi}^2/2} \dot{\sigma} \Phi = \delta\sigma - \frac{2\dot{\sigma}}{3H} \Phi, \quad (44)$$

where we have used that $\dot{\phi}^2/2 \simeq 3H^2 M_{\text{Pl}}^2/2$ on average. In the above equation, one has neglected $\dot{\sigma}^2/\dot{\phi}^2$ in front of unity. On large scales, the equation of evolution of the entropy perturbation reads [70]

$$\delta\ddot{s} + 3H\delta\dot{s} + (V_{ss} + 3\dot{\theta}^2)\delta s = 0, \quad (45)$$

where the expression of V_{ss} can be found in Ref. [70]. In the test field approximation $\dot{\sigma} \ll \dot{\phi}$, and assuming power-law behaviors of σ and ϕ , one can check that $V_{ss} \simeq V_{\sigma\sigma}$

and $\dot{\theta}^2 \ll V_{\sigma\sigma}$ in Eq. (45), so that δs follows the same quadratic equation than σ , and therefore $\delta s/\sigma$ is conserved. This implies

$$\left[\frac{\delta\sigma}{\sigma} - \frac{2\dot{\sigma}}{3H\sigma} \Phi \right] \Big|_t = \left[\frac{\delta\sigma}{\sigma} - \frac{2\dot{\sigma}}{3H\sigma} \Phi \right] \Big|_{\text{inf}}, \quad (46)$$

hence, with $t = 2/(3H)$, $\Phi \simeq \Phi_{\text{inf}}$ and assuming $\dot{\sigma} = 0$ initially,

$$\frac{\delta\sigma}{\sigma} \Big|_t = \frac{\delta\sigma_{\text{inf}}}{\sigma_{\text{inf}}} + \frac{t\dot{\sigma}(t)}{\sigma(t)} \Phi_{\text{inf}}. \quad (47)$$

This solution happens to match that obtained for a pure time independent modulus potential (see Ref. [44]). However, in the present case, it accounts for the sourcing of the modulus perturbation by the effective Hubble mass. To our knowledge, this result has not been obtained before. In the limiting case $\Phi_{\text{inf}} \rightarrow 0$, one recovers the result obtained in Ref. [42] that $\delta\sigma/\sigma$ is constant. One can now calculate the modulus—radiation isocurvature fluctuation at the initial time, i.e. in the radiation era and after the onset of modulus oscillations, and one recovers the result given in Eq. (10).

Two additional remarks are in order here. First, in the derivation above, we have neglected the preheating effects. This is justified by the following considerations. It turns out that the model investigated here is exactly similar to the two field model $g^2\phi^2\chi^2$ studied in Ref. [71] with a dimensionless coupling constant given by $g^2 = c^2 m_\phi^2 / (6M_{\text{Pl}}^2)$. This means that the quantity $g^2\Phi^2 / (4m_\phi^2) = c^2\Phi^2 / (24M_{\text{Pl}}^2) \ll 1$ and that we are never in the ‘‘broad resonance’’ regime where preheating effects are important [71]. Second, we have also numerically integrated the exact equations of motion for different cases and have checked that the approximations used above are verified. Above all, we have compared the numerical solution for $\delta\sigma$ in the postinflationary epoch to the solution (47) and have found an almost perfect agreement.

1. Case $+c^2H^2/2$

As mentioned previously, the details of the calculations that follow can be found in Appendix B. In order to recompute $\Omega_{\sigma,\text{osci}}$, one must follow the evolution of σ between the end of inflation (where $\sigma = \sigma_{\text{inf}}$) and $H = m_\sigma$ in the situation where the potential is dominated by the Hubble scale corrections. This evolution is characterized by c , σ_0 , and p , the latter defining the evolution of the scale factor: $a \propto t^p$, so that $H = p/t$. These parameters enter in the following combinations:

$$\mu = \frac{3p-3}{2}, \quad \nu^2 = (\mu+1)^2 - p^2c^2, \quad (48)$$

Depending on the magnitude of c , ν may be real or imaginary, which gives rise to different evolutions. We examine each of these cases in turn.

Let us first start with the case $c < (\mu+1)/p$ (real ν). In Appendix B, it is shown that $\Omega_{\sigma,\text{osci}}$ can be expressed as follows [see Eqs. (B19) and (B20)]:

$$\Omega_{\sigma,\text{osci}} \simeq \frac{1}{6} \mathcal{A}_1^2 \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \mathcal{B}_1^2 \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})}, \quad (49)$$

$$T_{\text{osci}} > T_{\text{rh}},$$

$$\Omega_{\sigma,\text{osci}} \simeq \frac{1}{6} \mathcal{A}_2^2 \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \mathcal{B}_2^2 \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \times \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{rh}}} \right)^{2(\mu_{\text{RD}}+1-\nu_{\text{RD}})}, \quad (50)$$

$$T_{\text{osci}} < T_{\text{rh}}.$$

The quantities indexed with ‘‘RD’’ (respectively, ‘‘MD’’) refer to the radiation dominated (respectively, matter dominated) era. These equations are the supergravity counterparts of Eq. (30) in the case where ν is real. In the above expressions, the first term is the contribution originating from the particular solution of the equation of motion while the second term is due to the homogeneous solution. A crucial difference between the scalings of the homogeneous and the particular solution is the redshift factor $(m_\sigma/H_{\text{inf}})^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \ll 1$ for the latter. The prefactors \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{B}_1 , and \mathcal{B}_2 are all of order unity (see Appendix B). The quantity $\Omega_{\sigma,\text{osci}}$ will be dominated by the homogeneous solution contribution whenever

$$|\sigma_{\text{inf}} - \sigma_0| \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^{\mu_{\text{MD}}+1-\nu_{\text{MD}}} \gg |\sigma_0|, \quad (51)$$

assuming for simplicity $T_{\text{rh}} < T_{\text{osci}}$ which is the most generic situation.

Let us now discuss the cosmological consequences for the two cases in which the particular and the homogeneous solution dominates at late times, starting with the former case. Then, the constraints in modulus parameter space are straightforward to derive. Using the results obtained for the purely quadratic case, for which $\Omega_{\sigma,\text{osci}} = (\sigma_{\text{inf}}/M_{\text{Pl}})^2/6$, one can put an upper limit on σ_0 since, in the case considered here, $\Omega_{\sigma,\text{osci}}$ has the same form, σ_{inf} being simply replaced with σ_0 . In Eq. (36), we have established the conditions for the existence of a dark matter isocurvature mode. It is clear that if σ_0 [σ_{inf} in Eq. (36)] is smaller than the lower bound, then there is no isocurvature mode and the scenario is compatible with the cosmic microwave background data. For $x_f \sim 20$, $m_\sigma = 10^5$ GeV, and $T_{\text{rh}} = 10^9$ GeV, this gives

$$\sigma_0 \lesssim 10^{-10} M_{\text{Pl}}. \quad (52)$$

From previous analytical calculations, one expects this bound to scale with the parameters as $H_{\text{inf}}^0 T_{\text{rh}}^{-1/2} m_\sigma^{1/2}$ [see Eq. (36) replacing σ_{inf} by σ_0]. However, such a value

remains well below the typical displacement expected from the quantum jumps of the modulus in its potential during inflation, unless $H_{\text{inf}} \ll 10^9$ GeV [see Eq. (5)].

This brings us to the other extreme case, in which the homogeneous solutions dominate the evolution at late times. As a clear example of this situation, consider $\sigma_0 = 0$, but $\sigma_{\text{inf}} \neq 0$. Physically, this corresponds to the situation in which the minima of the effective potential after inflation coincides at high ($H_{\text{inf}} > H > m_\sigma$) and low energy ($m_\sigma > H$). The vev σ_{inf} is here nonzero, either because the effective minimum during inflation does not coincide with that at latter times, or because σ_{inf} is subject to quantum effects. We thus treat σ_{inf} as a free parameter as before, and one obtains the constraints in the modulus parameter space presented in Fig. 3.

This figure clearly confirms the power of constraints obtained at the perturbative level. With respect to Fig. 1, which corresponds to the purely quadratic case, one can see that the constraints from big bang nucleosynthesis and from the production of isocurvature perturbations have moved towards higher values of σ_{inf} . This is understood easily: in the purely quadratic case, the modulus energy density remains constant between the end of inflation and the onset of oscillations, while it decreases in the present case. The white dashed line, which indicates the locus of $\Omega_{\sigma^d}^{\leq} = 0.5$, serves to delimit the constraints derived from dark matter—radiation and from baryon—radiation isocurvature modes.

Nevertheless, the constraints obtained still preclude the possibility of having a heavy modulus with an arbitrary vev at the end of inflation. In particular, even if the effective minima of the potential during inflation coincides with that lower energy, $|\sigma_{\text{inf}} - \sigma_0|$ should depart from zero by the value expected from quantum motion during inflation. In

the left panel, for $H_{\text{inf}} = 10^{13}$ GeV, it is found that the constraints from the production of isocurvature fluctuations extend significantly below this bound, hence there is no apparent solution to the moduli problem. As H_{inf} becomes much smaller, some region of parameter space opens up at large values of σ_{inf} and large values of m_σ as in the quadratic case, and the typical stochastic displacement also decreases. For $H_{\text{inf}} = 10^9$ GeV, corresponding to the right panel of Fig. 3, there is however little room between this lower limit for σ_{inf} and the region excluded by isocurvature fluctuations.

It is noteworthy to recall that the situation depicted in the above figure is realized by two concrete models of inflation discussed in Appendix C.

Let us now turn to the case $c > (\mu + 1)/p$ (imaginary ν). For $p = 2/3$, this case corresponds to $c > 3/4$. Writing $\nu = i\hat{\nu}$, one has $\hat{\nu} > 0$ growing with c . In this case the energy density stored in the modulus at the onset of oscillations can be expressed as [see Appendix B, especially Eqs. (B34) and (B35)]

$$\Omega_{\sigma, \text{osci}} \simeq \frac{1}{6} \mathcal{A}_3^2 \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \mathcal{B}_3^2 \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}} + 1)}, \quad T_{\text{osci}} > T_{\text{rh}}, \quad (53)$$

$$\Omega_{\sigma, \text{osci}} \simeq \frac{1}{6} \mathcal{A}_4^2 \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \mathcal{B}_4^2 \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right)^{2\mu_{\text{MD}} + 2} \times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{rh}}} \right)^{2\mu_{\text{RD}} + 2}, \quad T_{\text{rh}} > T_{\text{osci}}. \quad (54)$$

These equations are the counterparts of Eqs. (49) and (50) in the case where the quantity ν is complex. The coefficients \mathcal{A}_3 and \mathcal{A}_4 are defined in Eq. (B36). The prefactors

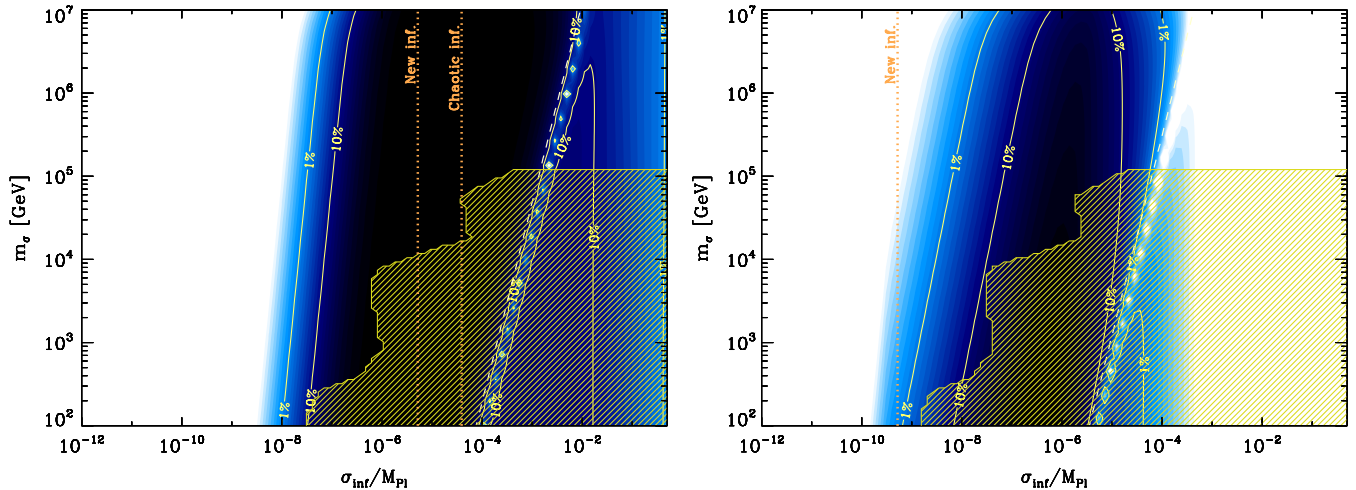


FIG. 3 (color online). Same as Fig. 1, but considering positive squared Hubble scale corrections to the modulus potential, with $c^2 = 0.5$ and $\sigma_0 = 0$. The parameter ν is given by $\nu_{\text{MD}}^2 \simeq 0.027$ and $\nu_{\text{RD}}^2 \simeq -0.063$. For the reheating temperature considered in this figure, one always has $T_{\text{osci}} > T_{\text{rh}}$ and, hence, only the value of ν_{MD}^2 matters [see Eq. (49)]. Therefore, one is indeed in the case $\nu > 0$. Left panel: $H_{\text{inf}} = 10^{13}$ GeV, $T_{\text{rh}} = 10^9$ GeV. Right panel: $H_{\text{inf}} = 10^9$ GeV, $T_{\text{rh}} = 10^9$ GeV. This case is only relevant for small field inflation. The standard deviations due to stochastic motion are calculated as in Fig. 1.

\mathcal{B}_3 and \mathcal{B}_4 are defined in Eqs. (B37) and (B38) and are of order unity. The difference with the case $\nu > 0$ (with prefactors \mathcal{A}_1 and \mathcal{A}_2) comes from the fact that, now, the numerical prefactors \mathcal{A}_3 and \mathcal{A}_4 may become quite small if ν is pure imaginary and its modulus is large. The prefactor \mathcal{A}_3 indeed scales as

$$\mathcal{A}_3 \simeq 2\pi \left(\frac{\hat{\nu}_{\text{MD}}}{2}\right)^{\mu_{\text{MD}}+2} e^{-\hat{\nu}_{\text{MD}}\pi/2 - \mu_{\text{MD}}-3} |\nu_{\text{MD}}| \gg 1. \quad (55)$$

Hence, as c grows beyond 1, the value of $\Omega_{\sigma, \text{part}, \text{osci}}$ decreases exponentially. As argued by Linde [72], this could alleviate the moduli problem, although it is notoriously difficult to construct explicit models in which $c^2 \gtrsim 10$.

Let us now study in more detail the physical consequences of the above expressions. Ignoring the factors of order one for simplicity, and using the explicit expression of the coefficient \mathcal{A}_3 [see Eq. (B36)], the particular solution dominates whenever the following condition is valid:

$$|\sigma_{\text{inf}} - \sigma_0| \left(\frac{m_\sigma}{H_{\text{inf}}}\right)^{\mu_{\text{MD}}+1} \ll |\sigma_0| \Gamma\left(\frac{\mu_{\text{MD}} + 3}{2} + i\frac{\hat{\nu}_{\text{MD}}}{2}\right) \times \Gamma\left(\frac{\mu_{\text{MD}} + 3}{2} - i\frac{\hat{\nu}_{\text{MD}}}{2}\right), \quad (56)$$

assuming for simplicity $T_{\text{osci}} > T_{\text{rh}}$. Then, one finds that the bound on σ_0 obtained previously in the case of a real ν for $H_{\text{inf}} \sim 10^{13}$ GeV is now loosened by

$$\sigma_0 \lesssim 10^{-10} \left[(2\pi)^2 \left(\frac{\hat{\nu}_{\text{MD}}}{2}\right)^{2(\mu_{\text{MD}}+2)} \times e^{-\hat{\nu}_{\text{MD}}\pi - 2(\mu_{\text{MD}}+3)} \right]^{-1} M_{\text{Pl}}, \quad (57)$$

for $m_\sigma = 10^5$ GeV, scaling approximately as $m_\sigma^{1/2}$. To provide concrete estimates, for $c^2 = 10$ ($\nu \simeq 2.04$ assum-

ing matter domination), the right-hand side becomes $2 \times 10^{-7} M_{\text{Pl}}$; for $c^2 = 20$ ($\nu \simeq 2.94$), it is $1.2 \times 10^{-6} M_{\text{Pl}}$. Interestingly, these values always remain smaller or are at most (for $c^2 = 20$) comparable to the standard deviation expected from stochastic motion of the modulus if $H_{\text{inf}} \sim 10^{13}$ GeV. Therefore, solving the moduli problem in this way would require not only a large value of c^2 in order to lessen the modulus energy density, hence the transfer of isocurvature perturbations, but also a small inflationary scale $\ll 10^{13}$ GeV in order to diminish the magnitude of stochastic motion. Note also that, for smaller values of H_{inf} , some parameter space opens up at large values of σ_{inf} and large m_σ , as discussed before.

If, on the contrary, the homogeneous solution dominates, then the situation is slightly different and the constraints in the modulus parameter space are presented in Fig. 4. The comparison with Fig. 3, which presented the constraints for real ν (and $\sigma_0 = 0$) is straightforward. One still obtains an upper bound on $|\sigma_{\text{inf}} - \sigma_0|$ if $H_{\text{inf}} \sim 10^{13}$ GeV, albeit displaced to larger values due to the fact that Ω_σ now scales as $H^{\mu+1}$ instead of $H^{\mu+1-2\nu}$, i.e. the energy density contained in the modulus decreases faster. This upper bound can be written as

$$|\sigma_{\text{inf}} - \sigma_0| \lesssim 3 \times 10^{-6} M_{\text{Pl}}, \quad (58)$$

with the following set of parameters: $H_{\text{inf}} = 10^{13}$ GeV, $T_{\text{rh}} = 10^9$ GeV, $c^2 = 2$ ($\nu \simeq 0.8$ assuming matter domination after inflation), and the prefactor has been evaluated for $m_\sigma = 100$ TeV. As before, one finds that this bound scales as $H_{\text{inf}}^{1/2} m_\sigma^0 T_{\text{rh}}^{-1/2}$. However, even for $c^2 = 2$, this upper bound is smaller than the amplitude of stochastic motion of the modulus in its potential during inflation, so that this cannot be considered as a viable solution. One therefore has to require $H_{\text{inf}} \ll 10^{13}$ GeV, in which case

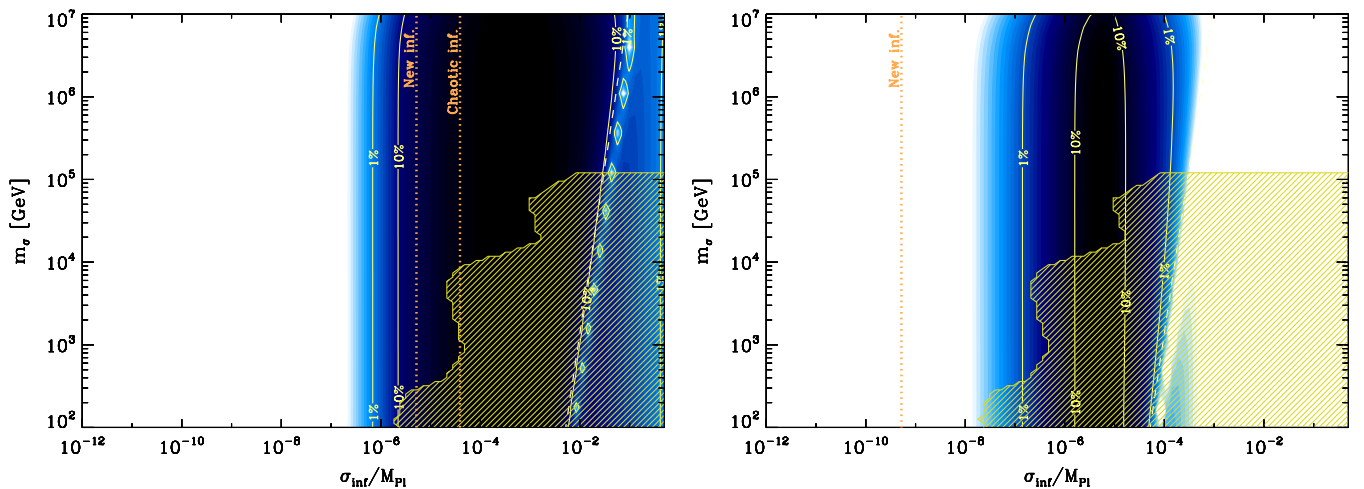


FIG. 4 (color online). Same as Fig. 1, but for a modulus potential receiving a positive Hubble mass squared correction with $c^2 = 2$, and $\sigma_0 = 0$. This implies $\nu_{\text{MD}}^2 \simeq -0.64$ and $\nu_{\text{RD}}^2 \simeq -0.44$. Left panel: $H_{\text{inf}} = 10^{13}$ GeV, $T_{\text{rh}} = 10^9$ GeV. Right panel: $H_{\text{inf}} = 10^9$ GeV, $T_{\text{rh}} = 10^9$ GeV. For this value of the reheating temperature, one always have $T_{\text{osci}} > T_{\text{rh}}$ in the parameter space considered in this figure.

some parameter space opens up at both small values of σ_{inf} (because the amplitude of stochastic motion is smaller at smaller H_{inf}) and at large values of σ_{inf} (where the isocurvature fluctuation becomes much smaller than the curvature perturbation).

Finally, let us end this section with the following remark. Above we have discussed the cases of ν real or pure imaginary in both the radiation dominated and the matter dominated epochs. Of course, there are also two mixed cases corresponding to a real ν during the matter dominated era and a pure imaginary ν in the radiation dominated era and the opposite situation. This would be relevant for the discussion above only if $T_{\text{rh}} > T_{\text{osci}}$, since the matchings would have to be modified. Since this is not the most generic case, we do not consider this situation in this paper.

2. Case $-c^2H^2/2$

If the modulus receives a negative Hubble mass squared contribution after inflation, the minimum of its potential is destabilized and as a result, the modulus will move until this negative contribution is balanced by higher order terms in the potential, $\propto \sigma^4$ or even nonrenormalizable contributions. Let us assume for instance that the next term in the modulus potential takes the form:

$$\frac{\lambda_n}{(n+4)!} \frac{\sigma^{4+n}}{M_{\text{Pl}}^n}. \quad (59)$$

Then, as shown in Ref. [42] the effective potential may be approximated at high energy $H \gg m_\sigma/c$ by

$$V(\sigma) \simeq \frac{1}{2} \tilde{c}^2 H^2 (\sigma - \sigma_n)^2 + \frac{\lambda_n}{(n+4)!} \frac{(\sigma - \sigma_n)^{4+n}}{M_{\text{Pl}}^n}, \quad (60)$$

where the time-dependent quantity σ_n can be expressed as

$$\sigma_n(t) = \left[\frac{(n+3)!}{\lambda_n} c^2 H^2 M_{\text{Pl}}^n \right]^{1/(n+2)}, \quad (61)$$

and $\tilde{c}^2 \equiv (n+2)c^2$. The value σ_n corresponds to the local minimum of the potential. Constant terms in $V(\sigma)$ have been omitted in Eq. (60), as well as subleading terms when compared to the last term on the right-hand side (see Ref. [42]). One crucial difference between this effective potential and that obtained for $+c^2H^2$ is the fact that the local minimum now depends on H and thus evolves in time.

One should distinguish three phases of evolution depending on which term in the potential dominates. If $\sigma_{\text{inf}} \gg \sigma_{n,\text{inf}} \equiv \sigma_n(H = H_{\text{inf}})$, then the field initially evolves in the high order part of the potential given by $V(\sigma) \simeq \lambda_n \sigma^{4+n} / [(n+4)! M_{\text{Pl}}^n]$. As shown in Appendix B, the field is then driven to σ_n within a fraction of e-fold of order $(\sigma_{\text{inf}}/\sigma_{n,\text{inf}})^{-(n+2)/2}$. At this stage, the effective $\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$ of the potential takes over the high order part. In order to model this case $\sigma_{\text{inf}} \gg \sigma_{n,\text{inf}}$, we simply

assume that, starting from the end of inflation, the field evolves in the $\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$ part with an initial value of order $\sigma_{n,\text{inf}}$. In this situation, we also neglect the initial kinetic energy since kinetic energy is strongly damped when the field evolves in the high order part of the potential. Furthermore, this is conservative in the sense that it underestimates the energy density contained in modulus oscillations at late times, and therefore tends to loosen slightly the constraints derived. If $\sigma_{\text{inf}} \lesssim \sigma_{n,\text{inf}}$, then one should directly approximate the potential by $\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$ with σ_{inf} as the initial condition. All in all, it suffices to solve for the evolution of σ in the potential $\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$ with an initial condition $\sigma_{\text{eff,min}} \equiv \min(\sigma_{\text{inf}}, \sigma_{n,\text{inf}})$.

Thus ignoring the high order part of the potential, at high energy $H \gg m_\sigma/c$, the solution for σ comprises a solution to the homogeneous equation σ_{hom} and a particular solution σ_{part} , as before (see Appendix B). The scaling of the homogeneous solution is similar to that obtained in the previous section with a $+c^2H^2$ effective squared mass term, but the particular solution scales differently due to the time dependence of σ_n . As shown in Appendix B, $\sigma_{\text{part}} \propto \sigma_n$ with a prefactor α_n of order unity. The explicit expression of α_n can be found in Appendix B [see Eq. (B43)]. Then $\sigma_{\text{part}} \propto t^{-2/(n+2)}$ so that the energy density of the modulus associated to this particular solution scales as $\rho_{\sigma,\text{part}} \propto H^2 \sigma_{\text{part}}^2 \propto H^{2(n+4)/(n+2)}$. Consequently, the particular solution contribution to the energy density at the onset of oscillations can be written as

$$\Omega_{\sigma,\text{part,osci}} \simeq \frac{1}{6} \left(\frac{\sigma_{\text{part,inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^{4/(n+2)}, \quad (62)$$

where we have ignored all factors of order one. In the above expression, $\sigma_{\text{part,inf}}$ represents the initial value of the particular solution at the beginning of the era driven by $\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$, that is to say

$$\begin{aligned} \sigma_{\text{part,inf}} &= \alpha_n \sigma_{n,\text{inf}} = \alpha_n \left[\frac{(n+3)!}{\lambda_n} c^2 H_{\text{inf}}^2 M_{\text{Pl}}^n \right]^{1/(n+2)} \\ &\simeq H_{\text{inf}}^{2/(n+2)} M_{\text{Pl}}^{n/(n+2)}. \end{aligned} \quad (63)$$

Concerning the homogeneous solution, its evolution is the same as in the previous section (see also Appendix B). This means that one should again distinguish the case where ν is real or imaginary and should treat separately the situation where the onset of oscillations occurs before or after the reheating. Straightforward calculations, similar to the ones already performed in the previous sections, lead to our final expression of $\Omega_{\sigma,\text{osci}}$

$$\begin{aligned} \Omega_{\sigma,\text{osci}} &\simeq \frac{1}{6} \left(\frac{\sigma_{\text{part,inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^{4/(n+2)} \\ &\quad + \frac{1}{6} \left(\frac{\sigma_{\text{eff,inf}} - \sigma_{\text{part,inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^{2[\mu_{\text{MD}} + 1 - \text{Re}(\nu_{\text{MD}})]}, \\ T_{\text{osci}} &> T_{\text{rh}}, \end{aligned} \quad (64)$$

$$\Omega_{\sigma, \text{osci}} \simeq \frac{1}{6} \left(\frac{\sigma_{\text{part, inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^{4/(n+2)} + \frac{1}{6} \left(\frac{\sigma_{\text{eff, inf}} - \sigma_{\text{part, inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{m_\sigma}{H_{\text{rh}}} \right)^{2[\mu_{\text{RD}} + 1 - \text{Re}(\nu_{\text{RD}})]} \times \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right)^{2[\mu_{\text{MD}} + 1 - \text{Re}(\nu_{\text{MD}})]},$$

$$T_{\text{osci}} < T_{\text{rh}}, \quad (65)$$

where we have neglected the factors of order one and where the appearance of the real part of ν accounts for both possibilities (ν real or imaginary).

When discussing the case $+c^2 H^2$, we considered two cases, one in which the particular solution dominates, the other in which the homogeneous solutions dominate. We cannot do so here, because the particular solution is entirely determined by H_{inf} and no longer dependent on the magnitude of σ_0 . Both contributions have to be considered together. Furthermore, we recall that $\sigma_{\text{eff, inf}} = \min(\sigma_{\text{inf}}, \sigma_{n, \text{inf}})$. Therefore, whether the particular or the homogeneous solution dominates in Eqs. (64) and (65) depends on n , μ , ν , and m_σ/H_{inf} (assuming $T_{\text{osci}} > T_{\text{rh}}$).

Regarding the fluctuations of σ , it is not possible to follow analytically the evolution of $\delta\sigma/\sigma$ from the end of inflation until $H = m_\sigma$ due to the nonlinearity of the potential. For the sake of the argument, we thus assume that the initial conditions are the same as in the previous cases, namely, Eqs. (9) and (10).

Let us first discuss the case $n = 0$ and assume for the sake of discussion that $c^2 = 0.5$. Then, ν_{MD} is imaginary since $\tilde{c}^2 = 1$. Furthermore, $2(\mu_{\text{MD}} + 1) = 1$ and $4/(n + 2) = 2$ in this case, so that the particular solution is always negligible in front of the homogeneous solution. The contribution of this latter to the energy density is nevertheless suppressed by m_σ/H_{inf} and $\Omega_{\sigma^d}^{\leq}$ is so small that the constraints from big bang nucleosynthesis are significantly weakened (see Fig. 5); they now allow moduli masses above 300 GeV for all σ_{inf} . This constraint does not depend on σ_{inf} because $\Omega_{\sigma^d}^{\leq}$ hardly depends on σ_{inf} since $(\sigma_{\text{eff, inf}} - \sigma_{\text{part, inf}})^2 \sim \sigma_{\text{part, inf}}^2$ in both limits $\sigma_{\text{inf}} \ll \sigma_{\text{part, inf}}$ and $\sigma_{\text{inf}} \gg \sigma_{\text{part, inf}}$. However, the constraints from the production of isocurvature fluctuations are quite significant in this case because the modulus can perturb significantly the perturbation spectrum of dark matter even though it does not dominate the energy density at the time of decay. Strictly speaking, this is true as long as $H_{\text{inf}} \sim 10^{13}$ GeV, as these isocurvature constraints are less stringent for $H_{\text{inf}} = 10^9$ GeV.

When $n \geq 1$, $\Omega_{\sigma, \text{osci}}$ becomes large enough to produce substantial isocurvature fluctuations in nearly all of the parameter space (see Fig. 6). For $n = 1$, the homogeneous solution still dominates over the particular solution. The constraints can be derived from the discussion of Sec. III B 1 (i.e. $+c^2 H^2$ potential, in the case in which the homogeneous solution dominates) provided one repla-

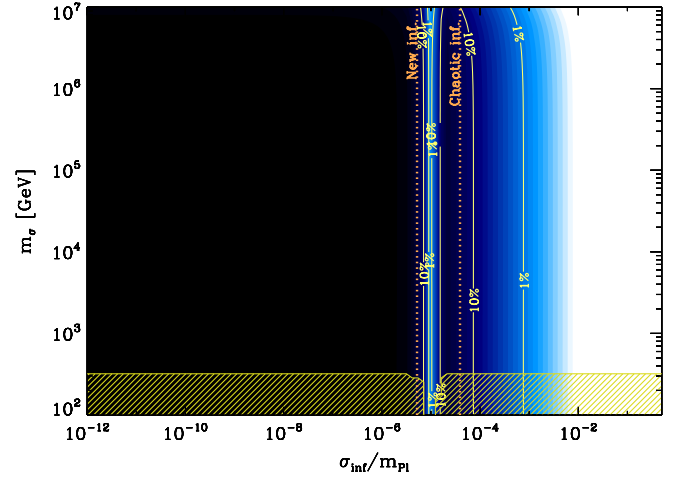


FIG. 5 (color online). Same as Fig. 1, for a potential receiving a negative Hubble mass squared correction with $c^2 = 0.5$, and $n = 0$. $H_{\text{inf}} = 10^{13}$ GeV (for $H_{\text{inf}} = 10^9$ GeV, constraints from isocurvature fluctuations and from big bang nucleosynthesis vanish).

ces $(\sigma_{\text{inf}} - \sigma_0)^2$ with $(\sigma_{\text{inf}} - \sigma_{\text{part, inf}})^2 \simeq \sigma_{\text{part, inf}}^2$, which is a function of n and H_{inf} , but which does not depend on σ_{inf} . Accordingly, $\Omega_{\sigma^d}^{\leq}$ is sufficiently large to produce significant isocurvature perturbations because of the scaling of $\sigma_{\text{part, inf}}$ with n : for $n = 0$, $\sigma_{\text{part, inf}} \sim H_{\text{inf}}$, but for $n \geq 1$, $\sigma_{\text{part, inf}} \gg H_{\text{inf}}$ [see Eq. (63)]. As before, we find that the isocurvature mode becomes small enough to satisfy the constraints from cosmic microwave background data if $H_{\text{inf}} \ll 10^{13}$ GeV and $\sigma_{\text{inf}} \sim M_{\text{Pl}}$.

C. Modulus production by inflaton decay

The above discussion has implicitly assumed that no modulus was produced after inflation. However it seems reasonable to assume that the inflaton can decay into the modulus sector, through possibly Planck suppressed interactions. We take a branching ratio such that each inflaton produces N_σ moduli. Unless the modulus and the inflaton are coupled one to the other, one should expect $N_\sigma \lesssim 1/g_{\star, \text{rh}}$, which means that at most, moduli are produced at the same rate than other light particles. If the inflaton is more strongly coupled to the visible sector than to the modulus sector, one should expect a much lower value of N_σ .

Since the modulus mass is generically much smaller than the inflaton mass, the moduli produced through inflaton decay are ultrarelativistic, with energy $E_{\sigma \rightarrow \phi} \sim m_\phi/2$. These particles do not thermalize but redshift to nonrelativistic velocities. As their momentum redshifts away according to $p_\sigma = m_\phi/2(a_{\text{rh}}/a)$, the particles become nonrelativistic, when $p \sim m_\sigma$ or $a_{\text{rh}}/a_{\text{n-rel}} = 2m_\sigma/m_\phi$. The corresponding temperature $T_{\text{n-rel}}$ is given by

$$g_{\star \text{n-rel}}^{1/3} T_{\text{n-rel}} = g_{\star \text{rh}}^{1/3} T_{\text{rh}} \frac{2m_\sigma}{m_\phi}. \quad (66)$$

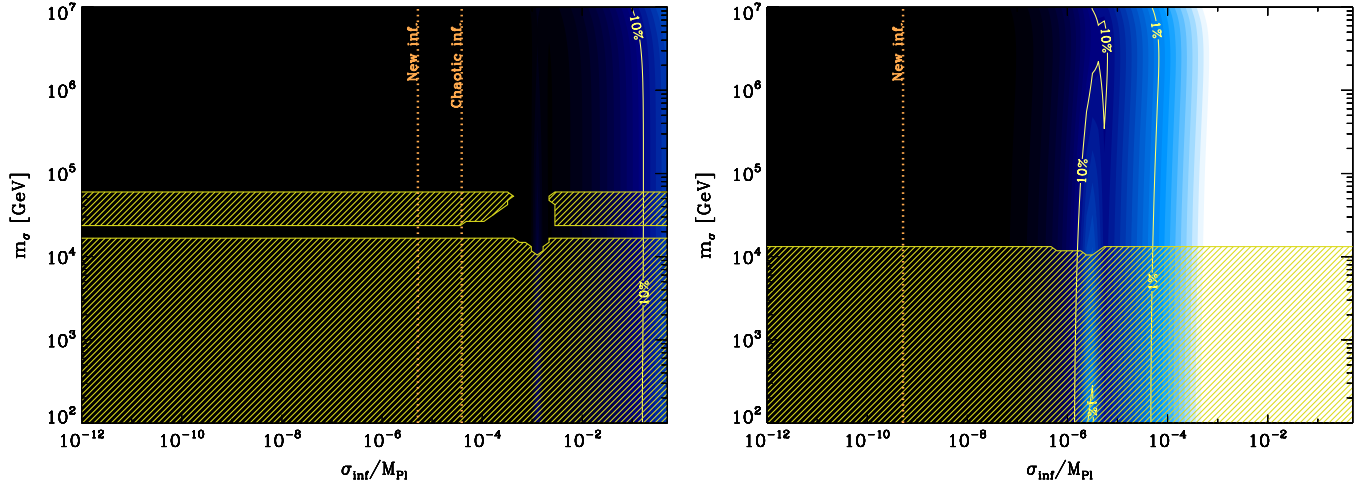


FIG. 6 (color online). Same as Fig. 1, for a potential receiving a negative Hubble mass squared correction with $c^2 = 0.5$, and $n = 1$. Left panel: $H_{\text{inf}} = 10^{13}$ GeV. Right panel: $H_{\text{inf}} = 10^9$ GeV. The standard deviations due to stochastic motion are calculated as in Fig. 1.

For temperatures $T > T_{\text{n-rel}}$, $\rho_{\sigma \leftarrow \phi} \propto 1/a^4$ i.e. it scales as radiation while for $T < T_{\text{n-rel}}$, it scales as a pressureless fluid, $\rho_{\sigma \leftarrow \phi} \propto 1/a^3$. Just after reheating, the energy density of the moduli that were produced by inflaton decay is given by

$$\rho_{\sigma \leftarrow \phi}^{\text{rh}} \simeq N_{\sigma} n_{\phi} \frac{m_{\phi}}{2} = \frac{N_{\sigma}}{2} \rho_{\phi}^{\text{rh}}. \quad (67)$$

If the temperature $T_{\text{n-rel}}$ is smaller than T_{d} , then $\rho_{\sigma \leftarrow \phi}$ will scale as the radiation until the modulus decay. As a consequence

$$\frac{\rho_{\sigma \leftarrow \phi}^{<_{\text{d}}}}{\rho_{\gamma}^{<_{\text{d}}}} \simeq \frac{N_{\sigma}}{2}, \quad T_{\text{n-rel}} < T_{\text{d}}. \quad (68)$$

If, on the contrary, $T_{\text{n-rel}} > T_{\text{d}}$, the energy density $\rho_{\sigma \leftarrow \phi}$ will then increase with respect to that of radiation in the era following $T_{\text{n-rel}}$ and preceding modulus decay. Therefore, the ratio of inflaton produced moduli energy density to radiation energy density immediately before modulus decay can be written as

$$\begin{aligned} \frac{\rho_{\sigma \leftarrow \phi}^{<_{\text{d}}}}{\rho_{\gamma}^{<_{\text{d}}}} &= \frac{N_{\sigma}}{2} \frac{T_{\text{n-rel}}}{T_{\text{d}}} \\ &\simeq 3.61 \left(\frac{N_{\sigma}}{10^{-3}} \right) \left(\frac{g_{\star, \text{rh}}}{g_{\star, \text{n-rel}}} \right)^{1/3} \left(\frac{g_{\star, \text{dec}}}{10.75} \right)^{1/4} \left(\frac{m_{\sigma}}{100 \text{ TeV}} \right)^{-1/2} \\ &\quad \times \left(\frac{m_{\phi}}{10^{13} \text{ GeV}} \right)^{-1} \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}} \right), \quad T_{\text{n-rel}} > T_{\text{d}}, \end{aligned} \quad (69)$$

where one has used the expression (3) of T_{d} . Let us notice that, in most of parameter space, $T_{\text{n-rel}} > T_{\text{d}}$.

The next step is to study how the previous considerations will impact the calculations developed in the previous sections. Obviously the amount of energy density stored

in the modulus oscillations remains unchanged. However the amount of isocurvature fluctuation produced at a same value of $\Omega_{\sigma}^{<_{\text{d}}}$ is reduced [9]. Indeed, inflaton produced moduli inherit the same spectrum of perturbations than radiation, and therefore there is no initial isocurvature perturbation between those moduli and the radiation fluid. Including these inflaton produced moduli, the initial value of the modulus curvature perturbation now reads

$$\zeta_{\sigma}^{(i)'} = \gamma \zeta_{\sigma}^{(i)} + (1 - \gamma) \zeta_{\phi}^{(i)}. \quad (70)$$

The quantities in this equation should be understood as follows: $\zeta_{\sigma}^{(i)}$ and $\zeta_{\phi}^{(i)}$ correspond to the curvature perturbations of the modulus and the inflaton acquired during inflation, as before. The curvature perturbation $\zeta_{\sigma}^{(i)'}$ is the curvature perturbation for the modulus that should be used in Eqs. (35) and (39) for the calculation of the final isocurvature perturbations. Finally, γ denotes the ratio of the energy density of moduli initially present at decay to the total amount of moduli at decay (those initially present together with the inflaton produced moduli). The final effect is to modify the initial modulus—radiation isocurvature perturbation by a factor γ in Eqs. (35) and (39), namely, $x_{\chi\gamma} \rightarrow \gamma x_{\chi\gamma}$. It is also useful to define $\gamma' \equiv \rho_{\sigma \leftarrow \phi}^{<_{\text{d}}} / \rho_{\sigma}^{<_{\text{d}}}$ (with $\rho_{\sigma}^{<_{\text{d}}}$ the amount of energy density stored in the modulus oscillations immediately before decay)

$$\gamma \equiv \frac{1}{\gamma' + 1}, \quad (71)$$

so that, if $T_{\text{n-rel}} > T_{\text{d}}$:

$$\gamma' \simeq 10^{-11} \left(\frac{N_\sigma}{10^{-3}} \right) \frac{1}{\Omega_{\sigma, \text{osci}}} \left(\frac{g_{\star, \text{rh}}}{g_{\star, \text{n-rel}}} \right)^{1/3} \left(\frac{m_\sigma}{100 \text{ TeV}} \right) \times \left(\frac{m_\phi}{10^{13} \text{ GeV}} \right)^{-1}, \quad (72)$$

$$T_{\text{n-rel}} > T_{\text{d}}.$$

In order to obtain this formula, we have used Eq. (69) in order to express $\rho_{\sigma \leftarrow \phi}^{\leftarrow \text{d}}$ and Eq. (25) to express $\rho_{\sigma^{\leftarrow \text{d}}}$ [or, alternatively, Eq. (31) with the term $\sigma_{\text{inf}}^2/M_{\text{Pl}}^2$ replaced by $6\Omega_{\sigma, \text{osci}}$ in order not to be restricted to the quadratic case]. Notice that we have taken $\alpha_{\text{osci/rh}} = 1$.

If, on the contrary, $T_{\text{n-rel}} < T_{\text{d}}$, then the γ' factor can be expressed as

$$\gamma' \simeq 1.38 \times 10^{-15} \left(\frac{N_\sigma}{10^{-3}} \right) \frac{1}{\Omega_{\sigma, \text{osci}}} \left(\frac{g_{\star, \text{dec}}}{10.75} \right)^{-1/4} \times \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{3/2} \left(\frac{T_{\text{rh}}}{10^9 \text{ GeV}} \right)^{-1}, \quad T_{\text{n-rel}} < T_{\text{d}}, \quad (73)$$

where, this time, $\rho_{\sigma \leftarrow \phi}^{\leftarrow \text{d}}$ has been obtained from Eq. (68). In these equations, $\Omega_{\sigma, \text{osci}}$ should be understood as corresponding to the oscillations of the modulus. It does not include, in particular, the moduli produced through inflaton decay.

To study the effect of such modulus production through inflaton decay, we first assume that the potential is purely quadratic, i.e. $c = 0$. The results are presented in Fig. 7. Compared to Fig. 1, one finds that the big bang nucleosynthesis constraints now exclude all moduli masses below 100 TeV. This is expected insofar as the amount of moduli energy density produced through inflaton decay is sufficient to disrupt big bang nucleosynthesis; since this amount does not depend on σ_{inf} , contrary to the amount

of energy density stored in moduli oscillations, the big bang nucleosynthesis constraints also do not depend on σ_{inf} . Note that the inflaton may also decay into gravitinos, with similar consequences for big bang nucleosynthesis (see [73]).

The constraints from isocurvature fluctuations are pushed to larger values of σ_{inf} , since the factor γ becomes small when the energy density produced through inflaton decay far exceeds that stored in modulus oscillations. Conversely, a larger value of σ_{inf} yields a larger value of $\Omega_{\sigma, \text{osci}}$ hence a larger value of γ . Assuming for simplicity $T_{\text{n-rel}} > T_{\text{d}}$, one can check that the suppression of isocurvature fluctuations becomes effective for

$$\sigma_{\text{inf}} \ll 3 \times 10^{-6} M_{\text{Pl}} \left(\frac{m_\sigma}{100 \text{ TeV}} \right)^{1/2} \left(\frac{m_\phi}{10^{13} \text{ GeV}} \right)^{-1/2} \times \left(\frac{N_\sigma}{10^{-3}} \right)^{1/2}. \quad (74)$$

In order to obtain this expression, we have used Eq. (72) and have written the condition $\gamma' \gg 1$ (which is equivalent to $\gamma \ll 1$) in the quadratic case, namely, $\Omega_{\text{osci}} \sim \sigma_{\text{inf}}^2/M_{\text{Pl}}^2$. This allows us to understand, at least qualitatively, the trend shown in Fig. 7.

The effect in the case where the potential receives supergravity corrections is rather straightforward to guess: big bang nucleosynthesis constraints remain unchanged as compared to the above Fig. 7, but the contours depicting the amount of isocurvature fluctuations produced are shifted toward higher values of σ_{inf} , as a result of the redshifting of the energy density stored in modulus oscillations after inflation, yielding a smaller value for γ . For instance, considering the case $c^2 = 2$ as in Fig. 4, one obtains the constraints depicted in Fig. 8.

Quite interestingly, the production of moduli through inflaton decay, while reducing the overall amount of iso-

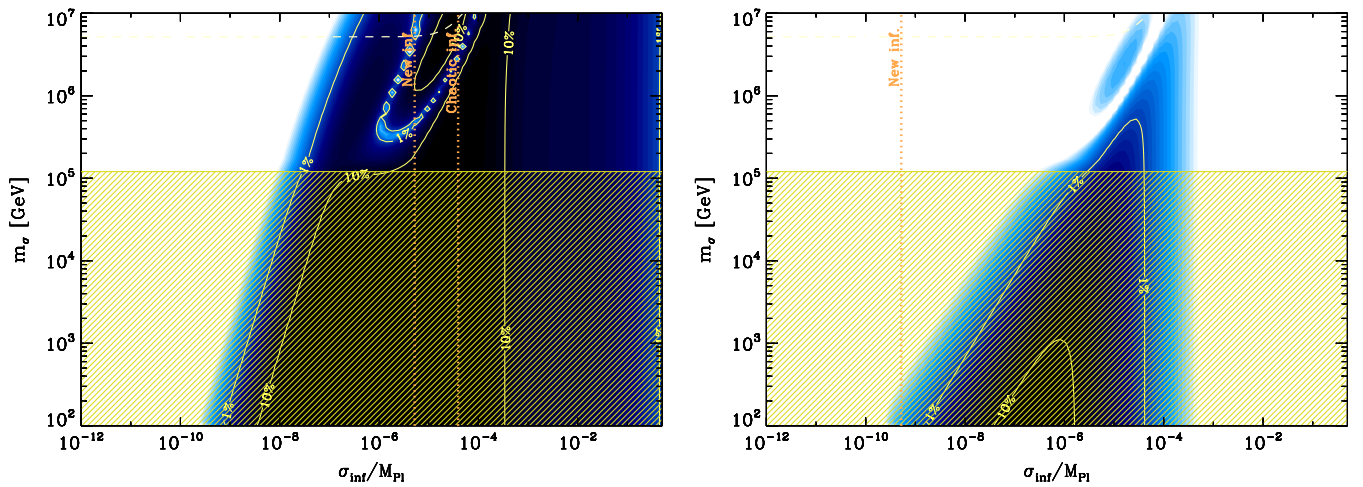


FIG. 7 (color online). Same as Fig. 1, for a quadratic modulus potential, but now accounting for inflaton produced moduli with $N_\sigma = 10^{-3}$ and an inflaton mass $m_\phi = 10^{13}$ GeV. Left panel: $H_{\text{inf}} = 10^{13}$ GeV, $T_{\text{rh}} = 10^9$ GeV. Right panel: $H_{\text{inf}} = 10^9$ GeV, $T_{\text{rh}} = 10^9$ GeV.

curvature fluctuation, does not allow us to find a solution to the moduli problem with arbitrarily high [$\sim \mathcal{O}(M_{\text{Pl}})$ in particular] vev at the end of inflation if $H_{\text{inf}} \approx 10^{13}$ GeV. Strictly speaking, the amount of isocurvature fluctuations produced for this value of the Hubble constant and $\sigma_{\text{inf}} \sim 0.01 - 1M_{\text{Pl}}$ is of the order of a few percent, therefore it is not excluded by present cosmic microwave background data. If present at this level, it could actually be detected by the upcoming generation of instruments. One should also emphasize that we have considered a rather conservative case, in the sense that the modulus is produced at a comparable rate than other light particles; if N_σ is decreased, the amount of isocurvature fluctuations would increase. At smaller values of H_{inf} , the isocurvature constraints have disappeared, due to the combined effect of the partial erasure associated with moduli production in inflaton decay and a smaller initial isocurvature fluctuation. Finally, independently of H_{inf} , the production of moduli through inflaton decay significantly worsens the effect of moduli on big bang nucleosynthesis. The success of big bang nucleosynthesis now requires both $m_\sigma \gtrsim 100$ TeV for all σ_{inf} .

IV. SUMMARY AND CONCLUSIONS

Let us first summarize the results obtained. We have shown that the decay of a generic modulus tends to produce strong isocurvature fluctuations between dark matter and radiation, or between baryons and radiation. The amount of isocurvature fluctuations produced, relatively to the total curvature perturbation, depends on several parameters: the value of the initial modulus—radiation isocurvature perturbation (in units of the total initial curvature perturbation), and the amount of energy density stored in the modulus oscillations at the time of decay, relatively to

that contained in radiation, in particular. We have discussed in some detail the evolution of the modulus energy density and of its perturbations from the end of inflation onwards, for a variety of possible moduli effective potentials, assuming the modulus remains light during inflation. We have then translated the constraints derived from the analysis of cosmic microwave background data into constraints in the modulus parameter space $m_\sigma - \sigma_{\text{inf}}$. We find that the constraints associated with the production of isocurvature fluctuations significantly exceed those from big bang nucleosynthesis in this parameter space. One reason why the constraints obtained cover most of the $m_\sigma - \sigma_{\text{inf}}$ parameter space is that the modulus will produce a large baryon isocurvature mode if $\Omega_\sigma^{\text{d}} \sim 1$, while it will produce a dark matter—radiation isocurvature fluctuation if $\Omega_\sigma^{\text{d}} \ll 1$ (but Ω_σ^{d} large enough to affect the dark matter).

Evading the constraints from big bang nucleosynthesis and from the generation of isocurvature fluctuations requires one of the following conditions to be satisfied.

First, if the modulus potential is time independent (i.e. it does not receive supergravity contributions at any time), it is mandatory that: (i) $H_{\text{inf}} \ll 10^9$ GeV and the modulus initially lies very close to the minimum of its potential, within $\sim 10^{-10}M_{\text{Pl}}$ (depending on the modulus mass, here taken to be ~ 1 TeV [see Fig. 1]); or (ii) $m_\sigma \gtrsim 100$ TeV, $\sigma_{\text{inf}} \gtrsim (H_{\text{inf}}/10^{13} \text{ GeV})M_{\text{Pl}}$ and $H_{\text{inf}} \ll 10^{13}$ GeV. Solution (i) ensures that the moduli energy density at the time of its decay is sufficiently small to affect neither big bang nucleosynthesis nor the dark matter perturbations (which turn out to be more sensitive probes than the former in this region of parameter space). The constraint on the Hubble parameter corresponds to the requirement that the stochastic motion of the modulus during inflation remains small enough as compared to the bound on the final effective displacement of the modulus vev. Such a solution

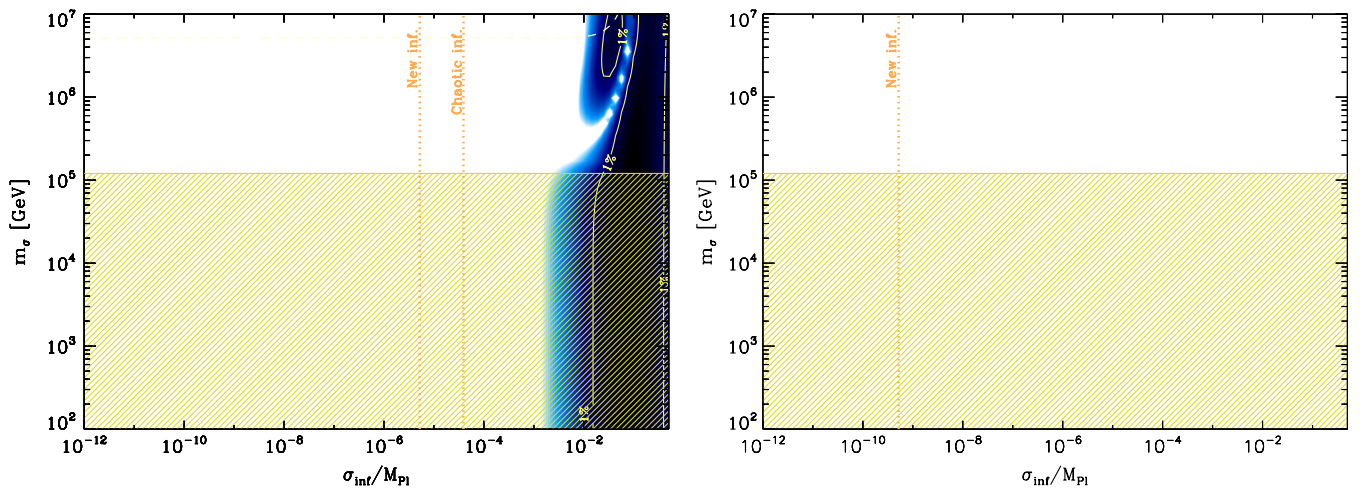


FIG. 8 (color online). Same as Fig. 4, for a receiving a positive squared mass Hubble contribution with $c^2 = 2$, but now accounting for inflaton produced moduli with $N_\sigma = 10^{-3}$ and an inflaton mass $m_\phi = 10^{13}$ GeV. Left panel: $H_{\text{inf}} = 10^{13}$ GeV, $T_{\text{rh}} = 10^9$ GeV. Right panel: $H_{\text{inf}} = 10^9$ GeV, $T_{\text{rh}} = 10^9$ GeV. The stochastic effects are calculated with $c_i^2 = 2$.

might be realized in models in which the modulus is bound to remain close to an enhanced symmetry point in moduli space due to the friction caused by its coupling to other light degrees of freedom [8,74,75]. Solution (ii) is typical of particle physics models which achieve a high modulus mass scale. The constraints on σ_{inf} and most particularly the bound on H_{inf} directly come from the constraints due to isocurvature fluctuations. If $H_{\text{inf}} \ll 10^{13}$ GeV as required by this solution, the tensor modes should be unobservable by upcoming cosmic microwave background missions.

Second, if one assumes that the modulus potential receives supergravity corrections after inflation (but remains light during inflation, as realized in some models of inflation discussed in Appendix C), one has to require $H_{\text{inf}} \ll 10^{13}$ GeV if the minima of the effective modulus potential at high energy ($H \gg m_\sigma$) matches that at low energy ($H \ll m_\sigma$). Then the moduli problem can be solved either for large $\sigma_{\text{inf}} \sim M_{\text{Pl}}$, provided $m_\sigma \gtrsim 100$ TeV, or for vanishingly small σ_{inf} (in units of M_{Pl} [see the main text for details]) and arbitrary m_σ . If the minima do not coincide, the situation is very similar to that discussed for a time independent modulus potential.

We have recalled that if the modulus is heavy during inflation, either because $H_{\text{inf}} \lesssim m_\sigma$, or because the modulus receives an effective Hubble mass during inflation, then the isocurvature fluctuation between the inflaton and the modulus disappears [61]. This provides a natural solution for the moduli problem at the perturbative level, but it does not automatically satisfy the constraints on entropy injection at the time of big bang nucleosynthesis. First of all, either $m_\sigma \gg 100$ TeV, or the minima of the effective modulus potentials during inflation, after inflation, and at low energy ($H \ll m_\sigma$) match one another. In this latter case, one also needs to ensure that the quantum jumps of the modulus in its potential do not result in too large an energy density at the time of big bang nucleosynthesis. This can be done using the discussion of Appendix A, which discusses the stochastic motion of the modulus in the presence of a Hubble effective mass $c_1 H$. Strictly speaking the results are valid in the slow-roll regime, which requires $c_1 < 3/4$, but they should remain valid as long as c_1 is of order unity. For chaotic inflation, the standard deviation $\langle \delta\sigma^2 \rangle^{1/2}$ is given by the Bunch-Davies expression (A40). This leads to the bound of Eq. (A44) which gives $\langle \delta\sigma^2 \rangle^{1/2} \simeq 3.0 \times 10^{-6} M_{\text{Pl}}$ (here for $c_1^2 = 0.5$). For small field inflation, the Bunch-Davies expression is given by Eq. (A74). In the case where $\mu \simeq M_{\text{Pl}}$, which is the case if $H_{\text{inf}} = 10^{13}$ GeV, then one can also use Eq. (A77) and this leads to $\langle \delta\sigma^2 \rangle^{1/2} \simeq 10^{-6} M_{\text{Pl}}$. These values are in conflict with the big bang nucleosynthesis constraints if $m_\sigma \sim 1$ TeV and the modulus potential becomes time independent after inflation, since these constraints require $\sigma_{\text{inf}} < 10^{-10} M_{\text{Pl}}$ (see Fig. 1). When turned into a bound on the Hubble constant during inflation, this becomes $H_{\text{inf}} \ll 10^9$ GeV; a low value indeed. Even if the modulus poten-

tial receives supergravity corrections after inflation, there is a potential conflict, since the big bang nucleosynthesis constraints impose $\sigma_{\text{inf}} < 10^{-6} M_{\text{Pl}}$ for $c^2 = 0.5$, but $\sigma_{\text{inf}} < 10^{-5} M_{\text{Pl}}$ for $c^2 = 2$ (see Figs. 3 and 4).

From the point of view of model building in the modulus sector, the above constraints can also be seen as follows. If the modulus has a vev $\simeq M_{\text{Pl}}$ at the end of inflation, then, in order to escape the constraints associated to the production of isocurvature fluctuations, one needs to achieve $H_{\text{inf}} < 10^{13}$ GeV (or render the modulus heavy during inflation [see above]). Interestingly enough, this suggests that this standard expectation for the modulus behavior could yield detectable isocurvature signals in forthcoming high accuracy cosmic microwave background experiments.

We have also considered the consequences of a possible production of moduli through inflaton decay. Since those moduli inherit the fluctuations of the inflaton, which correspond to those of radiation, this tends to reduce the initial modulus—radiation isocurvature fluctuation, all things being equal. However, it also aggravates the effect of moduli on big bang nucleosynthesis, and one generically finds that all moduli must have masses $m_\sigma \gtrsim 100$ TeV independently of σ_{inf} . Furthermore, one must still require that the effective minima of the modulus potential during inflation, after inflation, and at low energy match one another, or $H_{\text{inf}} \ll 10^{13}$ GeV as above.

Finally, let us recall that the results obtained above rely on some assumptions concerning the high energy underlying theory for the modulus. For instance, we have focused in this study on a quadratic modulus potential, most notably in the low energy regime $H < m_\sigma$. It would be certainly interesting to study the consequences with respect to the above moduli problem of nonquadratic potentials. In particular, it is not obvious whether the result that we have obtained for the evolution of the entropy fluctuations in the post-inflationary era, summarized by Eq. (47), would remain true for a nonquadratic potential. Another interesting avenue of research is to consider the effect of a possible direct coupling between the modulus and the inflaton on the evolution of the modulus energy density and on the entropy perturbations during the reheating phase. In the present work, we have also assumed that baryogenesis occurs well before modulus decay, as it seems most natural from a high energy physics point of view; see Ref. [76] for a detailed discussion of baryogenesis scenarios at low temperatures. One can generalize the present conclusions to the baryon violating late time decaying field, using the techniques developed in Ref. [47].

In short, the moduli problem is worse at the perturbative level. A clear trend emerges from the above calculation, namely, the lower the inflationary scale, the easier it is to solve the moduli problem. A significant amount of late time entropy production could alleviate the moduli problem, provided baryon isocurvature perturbations are not produced at the decay of the entropy producing compo-

ment. As mentioned above, one needs to have a reheating temperature (after entropy production) higher than the baryogenesis scale; alternatively the fluctuations carried by the entropy producing fluid could be similar to those carried by the baryons.

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APPENDIX A: EXPECTED VALUE FOR σ_{inf}

We now analyze the stochastic (quantum) behavior of the modulus field in the case where the total energy density is still dominated by the vacuum energy of the inflaton field. A similar analysis was performed in Ref. [9] but focusing on the chaotic inflationary scenario and for negligible initial values of the modulus field. Moreover, the stochastic nature of the inflaton field was also ignored. Here, we relax these assumptions and generalize the results of Refs. [9,51].

The problem treated here bears close resemblance with the problem tackled in Ref. [77] where the behavior of the quantum quintessence field during inflation was studied (the quantum behavior of the inflaton field being also taken into account). For this reason, we will follow a similar treatment.

According to the formalism of stochastic inflation, the coarse-grained inflaton field ϕ obeys the following Langevin equation [48,49]

$$\frac{d\phi}{dt} + \frac{V'_\phi(\phi)}{3H(\phi)} = \frac{H^{3/2}(\phi)}{2\pi} \xi_\phi(t), \quad (\text{A1})$$

where ξ_ϕ is a white noise field such that $\langle \xi_\phi(t) \xi_\phi(t') \rangle = \delta(t-t')$ and where a prime denotes a derivative with respect to the field. The stochastic evolution of the modulus field σ is also controlled by a Langevin equation which, in the slow-roll approximation, reads

$$\frac{d\sigma}{dt} + \frac{V'_\sigma(\sigma)}{3H(\phi)} = \frac{H^{3/2}(\phi)}{2\pi} \xi_\sigma(t), \quad (\text{A2})$$

where ξ_σ is another white noise field such that

$$\langle \xi_\sigma(t) \xi_\sigma(t') \rangle = \delta(t-t'), \quad \langle \xi_\sigma(t) \xi_\phi(t') \rangle = 0. \quad (\text{A3})$$

The solution of the Langevin Eq. (A2) depends explicitly on ξ_σ but also on the inflaton noise ξ_ϕ through the coarse-grained field ϕ . Since the modulus is considered as a test field, H only depends on ϕ , hence all primes superscript mean differentiation with respect to ϕ ; obviously, all primes superscript on V_ϕ (respectively, V_σ) denote differentiation with respect to ϕ (respectively, σ).

In order to find an approximate solution to Eqs. (A1) and (A2), one may try to use the same perturbative technique as the one used in Refs. [78,79]. Therefore, we expand the

inflaton and modulus fields about their classical solution and write

$$\begin{aligned} \phi(t) &= \phi_{\text{cl}}(t) + \delta\phi_1 + \delta\phi_2 + \dots, \\ \sigma(t) &= \sigma_{\text{cl}}(t) + \delta\sigma_1 + \delta\sigma_2 + \dots, \end{aligned} \quad (\text{A4})$$

where the first terms in the expansions are linear in the noise, the second ones are quadratic in the noise, and so on. As mentioned before, the approximation made in Ref. [9] consists in ignoring the stochastic nature of the inflaton field, $\phi = \phi_{\text{cl}}$.

Let us first examine the solution for the inflaton field. Expanding up to second order in the equation of motion, we obtain two linear differential equations for $\delta\phi_1$ and $\delta\phi_2$ (see Refs. [78,79]), namely,

$$\frac{d\delta\phi_1}{dt} + 2M_{\text{Pl}}^2 H''(\phi_{\text{cl}}) \delta\phi_1 = \frac{H^{3/2}(\phi_{\text{cl}})}{2\pi} \xi_\phi(t) \quad (\text{A5})$$

and

$$\begin{aligned} \frac{d\delta\phi_2}{dt} + 2M_{\text{Pl}}^2 H''(\phi_{\text{cl}}) \delta\phi_2 &= -M_{\text{Pl}}^2 H'''(\phi_{\text{cl}}) \delta\phi_1^2 \\ &+ \frac{3}{4\pi} H^{1/2}(\phi_{\text{cl}}) H'(\phi_{\text{cl}}) \\ &\times \delta\phi_1 \xi_\phi(t). \end{aligned} \quad (\text{A6})$$

These equations can be solved by varying the integration constant. Let us first consider the equation for $\delta\phi_1$. If the initial conditions are such that $\delta\phi_1(t = t_{\text{in}}) = 0$, then the solution reads

$$\delta\phi_1(t) = \frac{H'[\phi_{\text{cl}}(t)]}{2\pi} \int_{t_{\text{in}}}^t d\tau \frac{H^{3/2}[\phi_{\text{cl}}(\tau)]}{H'[\phi_{\text{cl}}(\tau)]} \xi_\phi(\tau). \quad (\text{A7})$$

We are now in a position where the various correlation functions can be calculated exactly. Since $\delta\phi_1$ is linear in the noise ξ , the mean value obviously vanishes $\langle \delta\phi_1 \rangle = 0$. The two-point correlation function for $\delta\phi_1$ can be calculated as

$$\langle \delta\phi_1(t_1) \delta\phi_1(t_2) \rangle = \frac{1}{4\pi^2} H'(t_1) H'(t_2) \int_{t_{\text{in}}}^{\min(t_1, t_2)} d\tau \frac{H^3(\tau)}{H^2(\tau)}. \quad (\text{A8})$$

Once the inflaton potential has been specified, it is more convenient to carry out these integrals using the classical value ϕ_{cl} as a variable instead of τ , thanks to the relation derived from the classical trajectory

$$d\phi_{\text{cl}} = -2H' M_{\text{Pl}}^2 dt. \quad (\text{A9})$$

For instance, the two-point correlation function calculated at the same time, i.e. the variance, reads [78,79]

$$\langle \delta\phi_1^2 \rangle = \frac{H^2}{8\pi^2 M_{\text{Pl}}^2} \int_{\phi_{\text{cl}}}^{\phi_{\text{in}}} d\varphi \left(\frac{H}{H'} \right)^3. \quad (\text{A10})$$

Detailed calculations of these integrals will be given below for various prototypical models of inflation.

We now turn to the equation of motion for the second order perturbation $\delta\phi_2$. It can be solved by following exactly the steps that were described before. Then, the solution can be written as [78,79]

$$\begin{aligned} \delta\phi_2(t) = & -H'M_{\text{Pl}}^2 \int_{t_{\text{in}}}^t d\tau \frac{H'''}{H'} \delta\phi_1^2(\tau) + \frac{3H'}{4\pi} \\ & \times \int_{t_{\text{in}}}^t d\tau H^{1/2} \delta\phi_1(\tau) \xi_\phi(\tau). \end{aligned} \quad (\text{A11})$$

As expected the second order perturbation is quadratic in the noise. One can easily evaluate the mean value of $\delta\phi_2(t)$, taking into account a factor 1/2 which originates from the fact that the Dirac δ function appearing in the noise correlation function is centered on an integration limit (see Refs. [78,79]).

Let us now turn to the modulus case when it has a sufficiently flat potential to acquire an independent quantum noise besides the inflaton. As for the inflaton case, it is easy to establish that the equations of motion for the perturbed quantities $\delta\sigma_1$ and $\delta\sigma_2$ are given by the following expressions [77]:

$$\begin{aligned} \frac{d\delta\sigma_1}{dt} + \frac{V'_\sigma(\sigma_{\text{cl}})}{3H(\phi_{\text{cl}})} \delta\sigma_1 = & \frac{V'_\sigma(\sigma_{\text{cl}})H'(\phi_{\text{cl}})}{3H^2(\phi_{\text{cl}})} \delta\phi_1 \\ & + \frac{H^{3/2}(\phi_{\text{cl}})}{2\pi} \xi_\sigma, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} \frac{d\delta\sigma_2}{dt} + \frac{V''_\sigma(\sigma_{\text{cl}})}{3H(\phi_{\text{cl}})} \delta\sigma_2 = & \frac{V'_\sigma(\sigma_{\text{cl}})H'(\phi_{\text{cl}})}{3H^2(\phi_{\text{cl}})} \delta\phi_2 + \frac{V'_\sigma(\sigma_{\text{cl}})H''(\phi_{\text{cl}})}{6H^2(\phi_{\text{cl}})} \delta\phi_1^2 \\ & - \frac{V'_\sigma(\sigma_{\text{cl}})H'^2(\phi_{\text{cl}})}{3H^3(\phi_{\text{cl}})} \delta\phi_1^2 + \frac{V''_\sigma(\sigma_{\text{cl}})H'(\phi_{\text{cl}})}{3H^2(\phi_{\text{cl}})} \delta\phi_1 \delta\sigma_1 \\ & - \frac{V'''_\sigma(\sigma_{\text{cl}})}{6H(\phi_{\text{cl}})} \delta\sigma_1^2 + \frac{3H^{1/2}(\phi_{\text{cl}})H'(\phi_{\text{cl}})}{4\pi} \delta\phi_1 \xi_\sigma. \end{aligned} \quad (\text{A13})$$

Although these equations look quite complicated, they can be solved easily because (by definition) they are linear. Assuming that the modulus potential does not depend explicitly on time (that is, other by its dependence on σ), the solution for $\delta\sigma_1$ reads

$$\begin{aligned} \delta\sigma_1(t) = & V'_\sigma(\sigma_{\text{cl}}) \int_{t_{\text{in}}}^t \left[\frac{H'(\phi_{\text{cl}})}{3H^2(\phi_{\text{cl}})} \delta\phi_1(\tau) \right. \\ & \left. + \frac{H^{3/2}(\phi_{\text{cl}})}{2\pi V'_\sigma(\sigma_{\text{cl}})} \xi_\sigma(\tau) \right] d\tau, \end{aligned} \quad (\text{A14})$$

and, as required, is linear both in the quintessence noise ξ_σ

and (through $\delta\phi_1$) in the inflaton noise ξ . The above formula is not valid anymore if V_σ contains an explicit dependence on time, for instance if the modulus mass receives Hubble term corrections. In this particular case, which will be discussed further below, one has to extract the explicit time dependence out of the potential, then proceed as above. Unless otherwise said, we assume in the following that V_σ does not contain any such explicit dependence on time.

As is obvious, $\delta\sigma_1$ has a vanishing mean value, $\langle\delta\sigma_1\rangle = 0$, but a nonvanishing variance given by the sum of two contributions originating from the inflaton and quintessence noise variances, namely, [77]

$$\begin{aligned} \langle\delta\sigma_1^2\rangle = & \frac{V_\sigma'^2(\sigma_{\text{cl}})}{9} \int_{t_{\text{in}}}^t \int_{t_{\text{in}}}^t \frac{H'(\tau)}{H^2(\tau)} \frac{H'(\eta)}{H^2(\eta)} \\ & \times \langle\delta\phi_1(\tau)\delta\phi_1(\eta)\rangle d\tau d\eta + \frac{V_\sigma'^2(\sigma_{\text{cl}})}{4\pi^2} \\ & \times \int_{t_{\text{in}}}^t \int_{t_{\text{in}}}^t \frac{H^{3/2}(\tau)}{V'_\sigma(\tau)} \frac{H^{3/2}(\eta)}{V'_\sigma(\eta)} \langle\xi_\sigma(\tau)\xi_\sigma(\eta)\rangle d\tau d\eta \end{aligned} \quad (\text{A15})$$

$$\equiv \langle\delta\sigma_1^2\rangle|_{\delta\phi_1\delta\phi_1} + \langle\delta\sigma_1^2\rangle|_{\xi_\sigma\xi_\sigma}. \quad (\text{A16})$$

Let us notice that there is no mixed contribution since the cross correlation $\langle\delta\phi_1\xi_\sigma\rangle = 0$. Using the correlation function of the modulus noise, the term $\langle\delta\sigma_1^2\rangle|_{\xi_\sigma\xi_\sigma}$ can be further simplified, namely, [77]

$$\langle\delta\sigma_1^2\rangle|_{\xi_\sigma\xi_\sigma} = \frac{V_\sigma'^2(\sigma_{\text{cl}})}{4\pi^2} \int_{t_{\text{in}}}^t \frac{H^3(\tau)}{V_\sigma'^2(\tau)} d\tau. \quad (\text{A17})$$

Let us now turn to the second order correction. The solution for $\delta\sigma_2$ can be written as [77]

$$\begin{aligned} \delta\sigma_2(t) = & V'_\sigma(\sigma_{\text{cl}}) \int_{t_{\text{in}}}^t \left\{ \frac{H'(\phi_{\text{cl}})}{3H^2(\phi_{\text{cl}})} \delta\phi_2(\tau) \right. \\ & + \left[\frac{H''(\phi_{\text{cl}})}{6H^2(\phi_{\text{cl}})} - \frac{H'^2(\phi_{\text{cl}})}{3H^3(\phi_{\text{cl}})} \right] \delta\phi_1^2(\tau) \\ & + \frac{V''_\sigma(\sigma_{\text{cl}})H'(\phi_{\text{cl}})}{3V'_\sigma(\sigma_{\text{cl}})H^2(\phi_{\text{cl}})} \delta\phi_1(\tau)\delta\sigma_1(\tau) \\ & - \frac{V'''_\sigma(\sigma_{\text{cl}})}{6V'_\sigma(\sigma_{\text{cl}})H(\phi_{\text{cl}})} \delta\sigma_1^2(\tau) \\ & \left. + \frac{3}{4\pi} \frac{H^{1/2}(\phi_{\text{cl}})H'(\phi_{\text{cl}})}{V'_\sigma(\sigma_{\text{cl}})} \delta\phi_1(\tau)\xi_\sigma \right\} d\tau. \end{aligned} \quad (\text{A18})$$

As expected, one sees that $\delta\sigma_2$ is quadratic in the noises.

From the above expression, one deduces that the mean value of $\delta\sigma_2$ is nonvanishing and is the sum of various terms [77]

$$\begin{aligned} \langle\delta\sigma_2\rangle = & \langle\delta\sigma_2\rangle|_{\delta\phi_2} + \langle\delta\sigma_2\rangle|_{\delta\phi_1^2} + \langle\delta\sigma_2\rangle|_{\delta\phi_1\delta\sigma_1} \\ & + \langle\delta\sigma_2\rangle|_{\delta\sigma_1^2(\xi_\sigma)} + \langle\delta\sigma_2\rangle|_{\delta\sigma_1^2(\xi_\phi)}, \end{aligned} \quad (\text{A19})$$

where the last term in Eq. (A18) does not contribute because $\langle \delta\phi_1 \xi_\sigma \rangle = 0$. Had we not taken into account the stochastic behavior of the inflaton, only the term $\langle \delta\sigma_2 \rangle|_{\delta\sigma_1^2(\xi_\sigma)}$ would have contributed.

At this stage it is interesting to compare the previous considerations to Ref. [9]. In particular the above approach is a perturbative one and a relevant question is its domain of validity [78,79]. In fact, as shown in Ref. [9], it turns out that, for the potential $m_\sigma^2 \sigma^2/2$, the Langevin equation can be integrated exactly. The solution reads

$$\sigma(t) = \sigma_{\text{in}} e^{-\int_{t_{\text{in}}}^t m_\sigma^2/(3H)d\tau} + \frac{1}{2\pi} \times \int_{t_{\text{in}}}^t H^{3/2}(\tau) \xi_\sigma(\tau) e^{\int_{t_{\text{in}}}^\tau m_\sigma^2/(3H)d\eta} d\tau. \quad (\text{A20})$$

From this expression, it is easy to compute the variance. One obtains

$$\begin{aligned} \langle \sigma^2 \rangle &= \frac{1}{4\pi^2} \int_{t_{\text{in}}}^t H^3(\tau) e^{2/3 \int_{t_{\text{in}}}^\tau m_\sigma^2/H d\eta} d\tau \\ &= \frac{m_\sigma^4 \sigma^2(t)}{4\pi^2} \int_{t_{\text{in}}}^t \frac{H^3(\tau)}{m_\sigma^4 \sigma^2(\tau)} d\tau. \end{aligned} \quad (\text{A21})$$

This is exactly the result obtained in Eq. (A17). Therefore, although perturbative in nature, the approach used before has in fact a wider domain of validity and, in the specific case treated above, can be used even if the corrections are not small. The perturbative approach is also interesting for two reasons: first, it allows us to take into account the stochastic behavior of the inflaton field (even if, most of the time, we will show that these corrections are negligible). Second, the method of Eq. (A17) rests on one's ability to solve exactly the Langevin equation which is possible only for a quadratic potential for σ . If the potential is different (that is to say, not quadratic), only the method used here allows us to derive explicit results.

We now turn to the calculation of the various corrections presented above in the following specific inflaton and modulus potentials.

1. Chaotic $m_\phi^2 \phi^2$ inflation

If we assume that $V_\phi = m_\phi^2 \phi^2/2$, then, in the slow-roll approximation, the classical evolution of the inflaton field is given by the following expression:

$$\frac{\phi_{\text{cl}}}{M_{\text{Pl}}} = \sqrt{\left(\frac{\phi_{\text{in}}}{M_{\text{Pl}}}\right)^2 - 4N}, \quad (\text{A22})$$

where N is the number of e-folds defined by $N \equiv \ln(a/a_{\text{in}})$, a_{in} being the initial value of the scale factor at the beginning of inflation, and $\phi_{\text{cl}}(N=0) = \phi_{\text{in}}$. The model remains under control only if the energy density is below the Planck energy density. This amounts to the following constraint on the initial conditions $\phi_{\text{in}}/M_{\text{Pl}} \leq 8\pi\sqrt{2}M_{\text{Pl}}/m_\phi$. Inflation stops when the slow-roll param-

eter $\varepsilon_1 = -\dot{H}/H^2$ is equal to unity corresponding to $\phi_{\text{end}} = \sqrt{2}M_{\text{Pl}}$. As a consequence, one can easily check that the argument of the square root in Eq. (A22) remains always positive. The total number of e-folds during inflation is simply given by $N_{\text{T}} = (\phi_{\text{in}}/M_{\text{Pl}})^2/4 - 1/2$. This number can be huge if the initial energy density of the inflaton field is close to the Planck energy density. Finally, the inflaton mass is fixed by the WMAP normalization

$$\frac{Q_{\text{rms-PS}}^2}{T^2} = \frac{1}{480\pi^2 \epsilon_{1*}} \frac{H_*^2}{M_{\text{Pl}}^2} = \frac{1}{1440\pi^2 \epsilon_{1*}} \frac{V_{\phi_*}}{M_{\text{Pl}}^4}, \quad (\text{A23})$$

where the cosmic microwave background quadrupole is given by $Q_{\text{rms-PS}}/T \simeq 6 \times 10^{-6}$ and a star denotes the time at which the scales of astrophysical interest today crossed the Hubble radius during inflation. Using the fact that $\epsilon_{1*} = 1/(2N_* + 1)$, where N_* is the number of e-folds between the time at which the physical scales left the Hubble radius during inflation and the end of inflation, one obtains

$$\frac{m_\phi}{M_{\text{Pl}}} \simeq \frac{12\pi\sqrt{10}}{2N_* + 1} \frac{Q_{\text{rms-PS}}}{T} \simeq 7 \times 10^{-6}, \quad (\text{A24})$$

where we have used $N_* \simeq 50$. This also implies the Hubble constant at the end of inflation, $H_{\text{inf}} = m_\phi/\sqrt{3} \simeq 10^{13}$ GeV.

Using the perturbative presented before, one can solve Eq. (A1) (let us notice that the Langevin equation can be solved exactly only in the case of a quartic potential) and determine the quantum behavior of the inflaton field. Through straightforward albeit lengthy calculations, one obtains [78,79]

$$\langle \delta\phi_1^2 \rangle = -\frac{m_\phi^2}{192\pi^2 M_{\text{Pl}}^4} (\phi_{\text{cl}}^4 - \phi_{\text{in}}^4). \quad (\text{A25})$$

Note that the classical trajectory obeys $\phi < \phi_{\text{in}}$ during inflation, so that the above is positive as it should be. Similarly, the correction to the mean value reads [78,79]

$$\langle \delta\phi_2 \rangle = -\frac{m_\phi^2}{192\pi^2 M_{\text{Pl}}^4} (\phi_{\text{cl}}^3 - \phi_{\text{in}}^3). \quad (\text{A26})$$

As before, since $\phi < \phi_{\text{in}}$ during inflation, the quantity $\langle \delta\phi_2 \rangle$ is positive.

Concerning the modulus, we will consider two possible potentials, in the spirit of previous sections: one with fixed mass $V_\sigma = m_\sigma^2 \sigma^2/2$, and one with typical supergravity corrections.

a. Modulus potential: $V_\sigma = m_\sigma^2 \sigma^2/2$

Let us first determine the classical trajectory of the modulus during inflation. Solving Eq. (A2) without the noise term leads to the following solution:

$$\sigma_{\text{cl}}(N) = \sigma_{\text{in}} \left[\frac{\phi_{\text{cl}}(N)}{\phi_{\text{in}}} \right]^{m_\sigma^2/m_\phi^2}. \quad (\text{A27})$$

In practice, one has $m_\phi \gg m_\sigma$ and, therefore, the modulus evolves slowly during inflation. Let us also notice that $m_\sigma^2/m_\phi^2 = 2/3(m_\sigma/H_{\text{inf,in}})^2 N_T$ and, therefore, the limit $m_\sigma/m_\phi \rightarrow 0$ corresponds to $m_\sigma \ll H_{\text{inf,in}} N_T^{-1/2}$. Even though the modulus is completely frozen in the limit $m_\sigma/m_\phi \rightarrow 0$, it is necessary to take into account its evolution when computing the integrals in Eqs. (A16) and (A18), since its displacement can be non-negligible if inflation lasts long enough. One then finds

$$\begin{aligned} \langle \delta \sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} &= - \frac{m_\phi^2}{192 \pi^2 M_{\text{Pl}}^4} \frac{1}{1 - m_\sigma^2/(2m_\phi^2)} \\ &\times (\phi_{\text{cl}}^4 - \phi_{\text{in}}^{4-2m_\sigma^2/m_\phi^2} \phi_{\text{cl}}^{2m_\sigma^2/m_\phi^2}), \end{aligned} \quad (\text{A28})$$

which, in the limit $m_\sigma/m_\phi \rightarrow 0$ and $\phi_{\text{cl}} \ll \phi_{\text{in}}$ reduces to

$$\langle \delta \sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} \simeq \frac{m_\phi^2 \phi_{\text{in}}^4}{192 \pi^2 M_{\text{Pl}}^4} = \frac{H_{\text{in}}^2}{8 \pi^2} N_T. \quad (\text{A29})$$

Note that H_{in} refers to the Hubble constant at the onset of inflation, which differs (by $\sqrt{2N_T}$) from the Hubble constant at the end of inflation, noted H_{inf} and used in the rest of our analysis. The two formulas (A28) and (A29) are identical to Eqs. (23) and (24) of Ref. [9]. The interpretation of Eq. (A29) is as follows. If the field is light, then the classical drift in the Langevin equation can be ignored. Then, it is easy to show that the Langevin equation can be integrated exactly and, as a consequence, that $\langle \sigma^2 \rangle = \int H^2 dN / (4\pi^2)$. If the Hubble parameter is approximately constant, then the previous expression reduces to $\langle \sigma^2 \rangle = H^2 N_T / (4\pi^2)$. However, as well known in the case of chaotic inflation, the Hubble parameter can change significantly during inflation, therefore, the integral has to be evaluated exactly. When this is done, this produces the additional factor 1/2 present in Eq. (A29).

Let us now turn to the second contribution originating from the inflaton noise. It reads

$$\begin{aligned} \langle \delta \sigma_1^2 \rangle_{\delta \phi_1 \delta \phi_1} &= - \frac{m_\sigma^2 \sigma_{\text{cl}}^2}{576 \pi^2 M_{\text{Pl}}^4} \frac{m_\sigma^2}{m_\phi^2} \\ &\times \frac{(\phi_{\text{cl}} - \phi_{\text{in}})^3 (\phi_{\text{cl}} + 3\phi_{\text{in}})}{\phi_{\text{cl}}^2} \\ &\simeq \frac{m_\phi^2 \phi_{\text{in}}^4}{192 \pi^2 M_{\text{Pl}}^4} \frac{1}{2} \frac{m_\sigma^4}{m_\phi^4} \frac{\sigma_{\text{cl}}^2}{M_{\text{Pl}}^2}, \end{aligned} \quad (\text{A30})$$

where the last expression is valid at the end of inflation. As a consequence, we note that $\langle \delta \sigma_1^2 \rangle_{\delta \phi_1 \delta \phi_1}$ is negligible in comparison with $\langle \delta \sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}$ since $m_\phi \gg m_\sigma$ and $\sigma_{\text{cl}}/M_{\text{Pl}} \ll 1$.

Let us now calculate the correction to the mean value of the modulus field. The various contributions $\delta \sigma_2$ amount

to

$$\begin{aligned} \langle \delta \sigma_2 \rangle_{\delta \phi_2} &= \frac{m_\sigma^2 \sigma_{\text{cl}}}{192 \pi^2 M_{\text{Pl}}^4} \left(\frac{1}{2} \phi_{\text{cl}}^2 - \frac{3}{2} \phi_{\text{in}}^2 + \frac{\phi_{\text{in}}^3}{\phi_{\text{cl}}} \right), \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1^2} &= - \frac{m_\sigma^2 \sigma_{\text{cl}}}{192 \pi^2 M_{\text{Pl}}^4} \left(\frac{1}{2} \phi_{\text{cl}}^2 + \frac{1}{2} \frac{\phi_{\text{in}}^4}{\phi_{\text{cl}}^2} - \phi_{\text{in}}^2 \right), \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1 \delta \sigma_1} &= - \frac{m_\sigma^2 \sigma_{\text{cl}}}{192 \pi^2 M_{\text{Pl}}^4} \frac{m_\sigma^2}{m_\phi^2} \left(\frac{1}{6} \phi_{\text{cl}}^2 + \frac{4}{3} \frac{\phi_{\text{in}}^3}{\phi_{\text{cl}}} \right. \\ &\quad \left. - \frac{1}{2} \frac{\phi_{\text{in}}^4}{\phi_{\text{cl}}^2} - \phi_{\text{in}}^2 \right), \\ \langle \delta \sigma_2 \rangle_{\delta \sigma_1^2(\xi_\sigma)} &= 0, \end{aligned} \quad (\text{A31})$$

the last result being obtained because $V_\sigma''' = 0$ in our case. Using again the fact that, as inflation proceeds, $\phi_{\text{cl}} \ll \phi_{\text{in}}$, one finally obtains

$$\langle \delta \sigma_2 \rangle \simeq - \frac{m_\sigma^2 \sigma_{\text{cl}} \phi_{\text{in}}^4}{384 \pi^2 M_{\text{Pl}}^4 \phi_{\text{cl}}^2}. \quad (\text{A32})$$

Noticing that $\langle \delta \sigma_2 \rangle$ is maximal at the end of inflation, i.e. for $\phi_{\text{cl}} = \sqrt{2} M_{\text{Pl}}$, one can easily demonstrate that the first order correction is always the dominant one.

Therefore, using the value of the inflaton mass obtained from the WMAP normalization, the expression of the initial value of the inflaton field in terms of the total number of e-folds, and setting $N_T \geq 60$, one obtains the following lower bound:

$$\langle \delta \sigma_1^2 \rangle^{1/2} \geq 3.9 \times 10^{-5} M_{\text{Pl}}. \quad (\text{A33})$$

The above considerations are valid provided one does not enter the regime of eternal inflation where the perturbative treatment of the quantum behavior of the inflaton field breaks down. Eternal inflation starts if $\phi_{\text{in}} \geq (24)^{1/4} \times (m_\phi/M_{\text{Pl}})^{-1/2}$. Therefore, the above calculations are applicable if $N_T \leq 2.8 \times 10^6$ which implies that $\sqrt{\langle \delta \sigma_1^2 \rangle} / M_{\text{Pl}} \leq 1.8$. As was studied in this paper, such values of σ are anyway excluded since the amount of entropy perturbations is too large to be compatible with the cosmic microwave background data.

Finally, it is also interesting to investigate what happens if one considers a more complicated potential for the modulus. As mentioned before, the perturbative approach used here allows us to determine the stochastic behavior of σ even if the potential is not quadratic. Therefore, let us consider the case where

$$V_\sigma(\sigma) = \frac{\lambda_n}{n!} \frac{\sigma^n}{M_{\text{Pl}}^{n-4}}, \quad (\text{A34})$$

where λ_n is a dimensionless constant. The integration of the classical equation of motion leads to the following solution:

$$\frac{\sigma_{\text{cl}}}{M_{\text{Pl}}} = \left[\left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}} \right)^{2-n} + (2-n) \frac{\lambda_n}{(n-1)!} \frac{M_{\text{Pl}}^2}{m_\phi^2} \right. \\ \left. \times \ln \left(\frac{\phi_{\text{cl}}}{\phi_{\text{in}}} \right) \right]^{1/(2-n)}. \quad (\text{A35})$$

In the limit $\lambda_n \rightarrow 0$, the modulus field is almost frozen. Then, one can now compute the variance due to the modulus noise. According to Eq. (A17), it reads

$$\langle \delta \sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} = \frac{H_{\text{in}}^2}{8\pi^2} N_T \left(\frac{\sigma_{\text{cl}}}{M_{\text{Pl}}} \right)^{2-2n} 4^{(3n-4)/(2-n)} \frac{4(n-1)! m_\phi^2}{\lambda_n (2-n) M_{\text{Pl}}^2} \\ \times \exp \left[- \frac{4(n-1)! m_\phi^2}{\lambda_n (2-n) M_{\text{Pl}}^2} \left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}} \right)^{2-n} \right] \\ \times \left\{ \gamma \left[\frac{4-3n}{2-n}, -4 \left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}} \right)^{2-n} \right] \right. \\ \left. - \gamma \left[\frac{4-3n}{2-n}, -4 \left(\frac{\sigma_{\text{cl}}}{M_{\text{Pl}}} \right)^{2-n} \right] \right\}, \quad (\text{A36})$$

where $\gamma(\alpha, x) \equiv \int_0^x t^{\alpha-1} e^{-t} dt$ is the incomplete gamma function.

b. Modulus potential: $V_\sigma = c_i^2 H^2 \sigma^2 / 2$

In this subsection, one considers the case where the modulus mass receives supergravity corrections of the form $c_i^2 H^2$, assuming $c_i < 3/4$, during inflation. A comment is in order at this point. One uses the notation c_i in order to emphasize the fact that the supergravity corrections to the modulus potential are not necessarily the same during inflation and during the postinflationary epoch. As a consequence, one expects $c_i \neq c$. This is of course the same for the minimum of the potential which, during inflation, is not necessarily equal to σ_0 , the minimum of the potential in the postinflationary epoch. In the rest of this appendix, we work in terms of the field displacement (with respect to the inflationary minimum) rather than in terms of the field itself.

Then, if one assumes that the supergravity corrections dominate, the modulus classical motion reads

$$\sigma_{\text{cl}} = \sigma_{\text{in}} \exp \left[- \frac{c_i^2}{3} \int_{t_{\text{in}}}^t d\tau H(\tau) \right] \\ = \exp \left[\frac{c_i^2}{3M_{\text{Pl}}^2} \int_{\phi_{\text{in}}}^{\phi_{\text{cl}}} d\varphi \frac{V(\varphi)}{V'(\varphi)} \right]. \quad (\text{A37})$$

and for $m_\phi^2 \phi^2$ inflation, this gives

$$\sigma_{\text{cl}} = \sigma_{\text{in}} \exp \left[\frac{c_i^2}{12M_{\text{Pl}}^2} (\phi_{\text{cl}}^2 - \phi_{\text{in}}^2) \right]. \quad (\text{A38})$$

Very quickly the argument of the exponential becomes $-c_i^2 \phi_{\text{in}}^2 / (12M_{\text{Pl}}^2) \simeq -c_i^2 N_T / 3 \simeq -(m_\sigma / H_{\text{inf, in}})^2 N_T / 3$ and, therefore, as previously, one recovers that the massless condition is given by $m_\sigma \ll H_{\text{inf, in}} N_T^{-1/2}$. Since $N_T > 60$,

this condition is now always violated (at least provided that $c_i \gtrsim 0.22$).

As mentioned above, Eqs. (A14) and (A18), need to be corrected to account for the explicit time dependence of the potential, which enters through the H prefactor. Making this dependence more explicit in Eq. (A13), one solves these equations as

$$\delta \sigma_1 = \frac{c_i^2}{3} \sigma_{\text{cl}} \int_{t_{\text{in}}}^t d\tau H'(\tau) \delta \phi_1(\tau) + \int_{t_{\text{in}}}^t d\tau \frac{H^{3/2}(\tau)}{2\pi} \\ \times \exp \left[\frac{c_i^2}{3} \int_t^\tau d\eta H(\eta) \right] \xi_\sigma(\tau), \quad (\text{A39})$$

and hence

$$\langle \delta \sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} = \frac{m_\phi^2}{16\pi^2 M_{\text{Pl}}^2 c_i^2} \left[\frac{6M_{\text{Pl}}^2}{c_i^2} + \phi_{\text{cl}}^2 - \left(\frac{6M_{\text{Pl}}^2}{c_i^2} + \phi_{\text{in}}^2 \right) \right. \\ \left. \times e^{-(c_i^2/6M_{\text{Pl}}^2)(\phi_{\text{in}}^2 - \phi_{\text{cl}}^2)} \right] \\ \simeq \frac{3H^4}{8\pi^2 c_i^2 H^2} \left(1 + \frac{6M_{\text{Pl}}^2}{c_i^2 \phi_{\text{cl}}^2} \right). \quad (\text{A40})$$

Therefore, one obtains the Bunch-Davies result (with the mass $c_i^2 H^2$) corrected by a factor the value of which at the end of inflation is $1 + 3/c_i^2$. This result agrees with that obtained in Ref. [9]. In the limit $c_i \rightarrow 0$, one of course recovers the result Eq. (A29).

The contribution to the variance due to the inflaton noise is given by

$$\langle \delta \sigma_1^2 \rangle_{\delta \phi_1 \delta \phi_1} = - \frac{c_i^4 \sigma_{\text{cl}}^2 m_\phi^2}{3456\pi^2 M_{\text{Pl}}^8} \left(\frac{1}{30} \phi_{\text{cl}}^6 - \frac{1}{2} \phi_{\text{cl}}^2 \phi_{\text{in}}^4 \right. \\ \left. + \frac{4}{5} \phi_{\text{cl}} \phi_{\text{in}}^5 - \frac{1}{3} \phi_{\text{in}}^6 \right). \quad (\text{A41})$$

This term is negligible compared to the previous one since the vev of the modulus is small (in Planck units).

Let us now turn to the corrections $\delta \sigma_2$ to the mean value. As explained before, one needs to modify the results above to take into account the fact that the mass is explicitly time-dependent. Straightforward calculations lead to

$$\langle \delta \sigma_2 \rangle_{\delta \phi_2} = \frac{c_i^2}{3} \sigma_{\text{cl}} \int_{t_{\text{in}}}^t d\tau H'(\tau) \langle \delta \phi_2(\tau) \rangle, \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1^2} = \frac{c_i^2}{3} \sigma_{\text{cl}} \int_{t_{\text{in}}}^t d\tau \left[\frac{H''(\tau)}{2} - \frac{H'^2(\tau)}{H(\tau)} \right] \langle \delta \phi_1^2 \rangle \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1 \delta \sigma_1} = \frac{c_i^2}{3} \int_{t_{\text{in}}}^t H'(\tau) d\tau e^{(c_i^2/3) \int_t^\tau d\eta H(\eta)} \\ \times \langle \delta \phi_1(\tau) \delta \sigma_1(\tau) \rangle, \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1 \xi_\sigma} = 0. \quad (\text{A42})$$

This can be integrated to give

$$\begin{aligned}
\langle \delta\sigma_2 \rangle_{\delta\phi_2} &= \frac{c_i^2 \sigma_{\text{cl}} m_\phi^2}{1152 \pi^2 M_{\text{Pl}}^6} \left(\frac{1}{4} \phi_{\text{cl}}^4 - \phi_{\text{cl}} \phi_{\text{in}}^3 + \frac{3}{4} \phi_{\text{in}}^4 \right), \\
\langle \delta\sigma_2 \rangle_{\delta\phi_1^2} &= -\frac{c_i^2 \sigma_{\text{cl}} m_\phi^2}{1152 \pi^2 M_{\text{Pl}}^6} \left[\frac{1}{4} \phi_{\text{cl}}^4 - \phi_{\text{in}}^4 \log\left(\frac{\phi_{\text{cl}}}{\phi_{\text{in}}}\right) \right. \\
&\quad \left. - \frac{1}{4} \phi_{\text{in}}^4 \right] \\
\langle \delta\sigma_2 \rangle_{\delta\phi_1 \delta\sigma_1} &= -\frac{c_i^4 \sigma_{\text{cl}} m_\phi^2}{6912 \pi^2 M_{\text{Pl}}^8} \left(\frac{1}{30} \phi_{\text{cl}}^6 - \frac{1}{2} \phi_{\text{cl}}^2 \phi_{\text{in}}^4 \right. \\
&\quad \left. + \frac{4}{5} \phi_{\text{cl}} \phi_{\text{in}}^5 - \frac{1}{3} \phi_{\text{in}}^6 \right), \\
\langle \delta\sigma_2 \rangle_{\delta\phi_1 \xi_\sigma} &= 0.
\end{aligned} \tag{A43}$$

Given that $\sigma_{\text{cl}}/M_{\text{Pl}} \ll 1$, it is easy to see that these contributions are subdominant. Therefore, the main contribution is the one given by Eq. (A40). Expressed at the end of inflation ($\phi_{\text{end}} = \sqrt{2}M_{\text{Pl}}$) and normalized to the cosmic microwave background, this expression leads to the following constraint on the value of the modulus:

$$\langle \delta\sigma_1^2 \rangle^{1/2} \gtrsim 0.79 \times 10^{-6} \frac{\sqrt{3 + c_i^2}}{c_i^2} M_{\text{Pl}}. \tag{A44}$$

2. Small field and hybrid inflation

In this section, we turn to another type of inflationary model. We now consider a potential of the form

$$V_\phi = M^4 \left[1 + \epsilon \left(\frac{\phi}{\mu} \right)^p \right], \tag{A45}$$

with $\epsilon = \pm 1$. Such a potential gives rise to small field inflation if $\epsilon = -1$ or hybrid inflation if $\epsilon = +1$. Since all integrals cannot be carried out exactly in this case, we provide the results to leading order in ϕ/μ . Out of simplicity, we use the notation $\Phi \equiv \phi_{\text{cl}}/\mu$ and similarly for Φ_{in} .

Let us now discuss small field inflation in more details. For this model, the slow-roll trajectory is only known implicitly. It can be expressed as

$$N = \frac{1}{2p} \frac{\mu^2}{M_{\text{Pl}}^2} \left(\Phi_{\text{in}}^2 - \Phi^2 + \frac{2}{p-2} \Phi_{\text{in}}^{2-p} - \frac{2}{p-2} \Phi^{2-p} \right). \tag{A46}$$

If $p = 2$ the singular terms must be replaced by a logarithm. From the above formula, one deduces that, given that $\phi_{\text{in}} \ll \phi_{\text{end}}$, the total number of e-folds can be written as

$$N_{\text{T}} \simeq \frac{\mu^2}{M_{\text{Pl}}^2} \frac{1}{p(p-2)} \Phi_{\text{in}}^{2-p}. \tag{A47}$$

In this class of models, the end of inflation occurs by violation of the slow-roll conditions. If $\mu/M_{\text{Pl}} \ll 1$, it

happens at $\Phi_{\text{end}} \simeq [2\mu^2/(p^2 M_{\text{Pl}}^2)]^{1/(2p-2)}$. In this regime, the two first slow-roll parameters are given by

$$\epsilon_{1*} \simeq \exp\left[-4N_* \left(\frac{M_{\text{Pl}}}{\mu}\right)^2\right], \quad \epsilon_{2*} = 4\left(\frac{M_{\text{Pl}}}{\mu}\right)^2, \tag{A48}$$

for $p = 2$, while, for $p \neq 2$, one has

$$\begin{aligned}
\epsilon_{1*} &\simeq \frac{p^2}{2} \left(\frac{M_{\text{Pl}}}{\mu}\right)^2 \left[N_* p (p-2) \left(\frac{M_{\text{Pl}}}{\mu}\right)^2 \right]^{-2(p-1)/(p-2)}, \\
\epsilon_{2*} &= \frac{2}{N_*} \frac{p-1}{p-2}.
\end{aligned} \tag{A49}$$

If $\mu/M_{\text{Pl}} \gtrsim 1$, then the above formulas are no longer valid and the slow-roll parameters must be evaluated numerically. In particular, in the limit $\mu/M_{\text{Pl}} \gg 1$, one has $\phi_{\text{end}} \simeq \mu$.

Our next step is to deduce an expression of the mass scale M . One obtains

$$\begin{aligned}
\left(\frac{M}{M_{\text{Pl}}}\right)^4 &= 720 \pi^2 p^2 \frac{Q_{\text{rms-PS}}^2}{T^2} [p(p-2)N_*]^{-2(p-1)/(p-2)} \\
&\quad \times \left(\frac{\mu}{M_{\text{Pl}}}\right)^{2p/(p-2)},
\end{aligned} \tag{A50}$$

where we have assumed $p \neq 2$. This expression is valid only if $\mu/M_{\text{Pl}} \ll 1$. If $\mu/M_{\text{Pl}} \gtrsim 1$, one can show (see Ref. [80]) that the end of inflation occurs when the vacuum expectation value of the inflaton field is of the order of μ and, as a consequence, that $(M/M_{\text{Pl}})^4 \sim \mathcal{O}(Q_{\text{rms-PS}}^2/T^2)$. Let us also notice that, contrary to the case of chaotic inflation, the Hubble parameter is approximately constant during inflation and given by $H_{\text{inf}} \simeq M^2/(\sqrt{3}M_{\text{Pl}})$. Therefore, if $\mu \gtrsim M_{\text{Pl}}$, $H_{\text{inf}} \simeq 10^{13}$ GeV, but H_{inf} can be much lower if $\mu \ll M_{\text{Pl}}$.

Let us now discuss the constraints on the free parameters p and μ . As shown in Ref. [80], there is no prior independent constraint on the index p . However, if one adopts the theoretical prejudice that the vacuum expectation value of the inflaton field must be smaller than the Planck mass, then μ must be smaller than M_{Pl} and then one can demonstrate that the case $p = 2$ is slightly disfavored by the cosmic microwave background data. In small field inflation, the energy scale of inflation can be very low. But, it can not be smaller than, say, the TeV scale, $V_*/m_{\text{Pl}}^4 \gtrsim 10^{-64}$. If $p = 2$, this implies that $\mu/M_{\text{Pl}} \gtrsim 1.25$. In this case, the formulas (A48) are no longer valid but one can show that the spectral index is still compatible with the data in the regime $\mu/M_{\text{Pl}} \gtrsim 1$ or even $\mu/M_{\text{Pl}} \gg 1$. If $p \neq 2$, the parameter μ is globally unconstrained. The constraint mentioned before implies

$$\begin{aligned}
\frac{\mu}{M_{\text{Pl}}} &\gtrsim \left\{ \frac{4 \times 10^{-64}}{45 p^2} \left(\frac{Q_{\text{rms-PS}}}{T}\right)^{-2} \right. \\
&\quad \left. \times [p(p-2)N_*]^{2(p-1)/(p-2)} \right\}^{(p-2)/(2p)}.
\end{aligned} \tag{A51}$$

This leads to $\mu/M_{\text{Pl}} \geq 0.16$ for $p = 2.1$, $\mu/M_{\text{Pl}} \geq 3.4 \times 10^{-5}$ for $p = 2.5$, $\mu/M_{\text{Pl}} \geq 1.5 \times 10^{-8}$ for $p = 3$, $\mu/M_{\text{Pl}} \geq 9.9 \times 10^{-13}$ for $p = 4$, etc. In this case, one sees that μ/M_{Pl} can be small. For $p \neq 2$, Eqs. (A49) indicate that ϵ_{2*} no longer depends on μ . The spectral index remains compatible with the data in this regime because $\epsilon_{2*} \sim 0.04(p-1)/(p-2)$ still lies in the 2σ contour whatever the value of p . As already mentioned, if, on the contrary, $\mu/M_{\text{Pl}} \gg 1$, Eqs. (A49) are no longer valid but one can also show that the model is still in agreement with the data. The conclusion is that, provided the energy scale of inflation is above the TeV scale, the two regimes $\mu/M_{\text{Pl}} \ll 1$ and $\mu/M_{\text{Pl}} \geq 1$ are still compatible with the data and, hence, the parameter μ remains basically unconstrained [80].

Let us now briefly discuss hybrid inflation. A crucial difference with small inflation is that inflation no longer stops by violation of the slow-roll conditions but by instability. This means that there is one more additional parameter, namely, the value of the inflaton field at which the instability occurs. This makes the analysis more complicated since the parameter space is enlarged. For this reason, although we give all the necessary expressions, we have chosen in this paper to skip a detailed investigation of this case.

Straightforward calculations lead to the following results for the variance and the mean value of the inflaton field

$$\langle \delta \phi_1^2 \rangle \simeq -\frac{M^4 \mu^2}{12\pi^2 M_{\text{Pl}}^4} \frac{1}{\epsilon p(4-3p)} (\Phi^{2-p} - \Phi_{\text{in}}^{4-3p} \Phi^{2p-2}). \quad (\text{A52})$$

Similarly,

$$\langle \delta \phi_2 \rangle = \frac{M^4 \mu}{24\pi^2 M_{\text{Pl}}^4} \frac{1}{\epsilon p(4-3p)} \left[\frac{p-2}{2} \Phi^{1-p} + \frac{4-3p}{2} \Phi^{p-1} \Phi_{\text{in}}^{2-2p} + (p-1) \Phi^{2p-3} \Phi_{\text{in}}^{4-3p} \right]. \quad (\text{A53})$$

These results agree with those of Ref. [78] to leading order.

As regards to the modulus, we again consider two possible potentials. We consider these two possibilities in the following.

a. Modulus potential: $V_\sigma = m_\sigma^2 \sigma^2/2$

The classical trajectory of the modulus reads

$$\sigma_{\text{cl}} = \sigma_{\text{in}} \exp \left[\frac{\mu^2 m_\sigma^2}{\epsilon p(2-p)M^4} (\Phi^{2-p} - \Phi_{\text{in}}^{2-p}) \right]. \quad (\text{A54})$$

Let us notice that the argument of the exponential is always negative. We now discuss the case of small field inflation. The argument of the exponential in Eq. (A54) is always dominated by the following term:

$$\frac{\mu^2 m_\sigma^2}{p(p-2)M^4} \Phi_{\text{in}}^{2-p} = \frac{1}{3} \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^2 N_{\text{T}}. \quad (\text{A55})$$

The argument of the exponential can be small or large depending on the total number of e-folds during inflation and on whether the modulus field is light or heavy. Using the WMAP normalization, one can also express the argument of the exponential in terms of the parameters of the model. In the regime $\mu/M_{\text{Pl}} \ll 1$, one obtains

$$\begin{aligned} \frac{\mu^2 m_\sigma^2}{p(p-2)M^4} \Phi_{\text{in}}^{2-p} &= \frac{1}{3} \left(\frac{m_\sigma}{H_{\text{inf}}} \right)^2 N_{\text{T}} \\ &\simeq 6.6 \times 10^{-21} N_{\text{T}} \left(\frac{m_\sigma}{10^5 \text{ GeV}} \right)^2 \\ &\times \frac{[p(p-2)N_*]^{2(p-1)/(p-2)}}{p^2} \\ &\times \left(\frac{\mu}{M_{\text{Pl}}} \right)^{-2p/(p-2)}. \end{aligned} \quad (\text{A56})$$

Let us give a few examples for different values of p with $m_\sigma = 10^5 \text{ GeV}$ and $N_* = 50$. For $p = 2.1$, this gives $43.78 N_{\text{T}} (\mu/M_{\text{Pl}})^{-42}$; for $p = 2.5$, one has $6.29 \times 10^{-11} N_{\text{T}} (\mu/M_{\text{Pl}})^{-10}$; for $p = 3$, one obtains $3.71 \times 10^{-13} N_{\text{T}} (\mu/M_{\text{Pl}})^{-6}$; and for $p = 4$, this leads to $2.64 \times 10^{-14} N_{\text{T}} (\mu/M_{\text{Pl}})^{-4}$. Let us be more precise and give some numbers. If $p = 2.5$ one can easily check that the argument of the exponential is always large. But this conclusion can be modified if one considers other values of the parameters. For instance, if $p = 3$ and $1.5 \times 10^{-8} \leq \mu/M_{\text{Pl}} \leq 8.4 \times 10^{-3} N_{\text{T}}^{1/6}$, then the argument of the exponential is large but, if $8.4 \times 10^{-3} N_{\text{T}}^{1/6} \leq \mu/M_{\text{Pl}} \ll 1$, it is no longer the case. Let us also notice that the previous considerations are valid only for values of N_{T} such that the number $8.4 \times 10^{-3} N_{\text{T}}^{1/6}$ remains small otherwise the formulas used here would not be valid. This means $N_{\text{T}} \ll 1.84 \times 10^6$ which is not so restrictive. The same analysis is true for $p = 4$, the corresponding intervals being $9.9 \times 10^{-13} \leq \mu/M_{\text{Pl}} \leq 4 \times 10^{-4} N_{\text{T}}^{1/4}$ and $4 \times 10^{-4} N_{\text{T}}^{1/4} \leq \mu/M_{\text{Pl}} \ll 1$. In the present case, the total number of e-folds must satisfy $N_{\text{T}} \ll 3.9 \times 10^9$. On the other hand, in the regime where $\mu/M_{\text{Pl}} \geq 1$, the argument of the exponential can now be written as

$$\frac{\mu^2 m_\sigma^2}{p(p-2)M^4} \Phi_{\text{in}}^{2-p} \simeq 1.87 \times 10^{-18} \left(\frac{m_\sigma}{10^5 \text{ GeV}} \right)^2 N_{\text{T}} \quad (\text{A57})$$

and is small, unless we choose very large values of N_{T} .

We are now in a position where one can compute the variance of the stochastic motion using Eq. (A16). This leads to the following result:

$$\begin{aligned} \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} &= \frac{M^8}{24\pi^2 M_{\text{Pl}}^4 m_\sigma^2} \\ &\times \left[1 - \exp \left[\frac{2m_\sigma^2 \mu^2}{\epsilon p(2-p)M^4} (\Phi^{2-p} - \Phi_{\text{in}}^{2-p}) \right] \right]. \end{aligned} \quad (\text{A58})$$

Let us discuss this result. If the argument of the exponential in Eq. (A54) is large (in absolute value), then the exponential becomes negligible and

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} \simeq \frac{M^8}{24\pi^2 M_{\text{Pl}}^4 m_\sigma^2} = \frac{3H_{\text{inf}}^4}{8\pi^2 m_\sigma^2}, \quad (\text{A59})$$

where we used $H_{\text{inf}}^2 \simeq M^4/(3M_{\text{Pl}}^2)$. It was suggested in Ref. [9] that this formula describes the small field case. But, as was noticed before, the argument of the exponential in Eq. (A54) can also be small and, in this case, one has

$$\begin{aligned} \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} &\simeq \frac{M^4 \mu^2}{12\pi^2 p(2-p)M_{\text{Pl}}^4} (\Phi_{\text{end}}^{2-p} - \Phi_{\text{in}}^{2-p}) \\ &\simeq \frac{H_{\text{inf}}^2}{4\pi^2} N_{\text{T}}. \end{aligned} \quad (\text{A60})$$

We notice that this last expression is similar to Eq. (A29). The only difference is that we do not have the presence of an additional factor 1/2 with respect to the standard case. This is because, in the case of small field inflation, the Hubble parameter is almost constant and, hence, one obtains the de Sitter result. We conclude that if $m_\sigma \leq H_{\text{inf}}/\sqrt{N_{\text{T}}}$, then the variance is given by Eq. (A60) while if $H_{\text{inf}}/\sqrt{N_{\text{T}}} \leq m_\sigma \leq H_{\text{inf}}$, it is given by the Bunch-Davis expression (A59). Finally, if $m_\sigma > H_{\text{inf}}$, then the variance becomes negligible.

As was done for the case of chaotic inflation, one can also express the variance in terms of the parameters of the model. Let us start with the case $\mu \ll M_{\text{Pl}}$. In the regime where the Bunch-Davis term dominates, the variance can be written as

$$\begin{aligned} \frac{\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}}{M_{\text{Pl}}^2} &\simeq 1.63 \times 10^{11} p^4 [p(p-2)N_*]^{-4(p-1)/(p-2)} \\ &\times \left(\frac{m_\sigma}{10^5 \text{ GeV}} \right)^{-2} \left(\frac{\mu}{M_{\text{Pl}}} \right)^{4p/(p-2)}. \end{aligned} \quad (\text{A61})$$

For $p = 2.5$ and $m_\sigma = 10^5 \text{ GeV}$, this gives $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \simeq 1.79 \times 10^{-9} (\mu/M_{\text{Pl}})^{20}$ which implies that $7.6 \times 10^{-99} \lesssim \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \lesssim 1.79 \times 10^{-29}$. On the other hand, if $p = 3$ and $m_\sigma = 10^5 \text{ GeV}$, this leads to $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \simeq 5.15 \times 10^{-5} (\mu/M_{\text{Pl}})^{12}$ which means that $6.7 \times 10^{-99} \lesssim \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \lesssim 6.35 \times 10^{-30} N_{\text{T}}^2$. Finally, if $p = 4$ and $m_\sigma = 10^5 \text{ GeV}$, this means that $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \simeq 0.01 (\mu/M_{\text{Pl}})^8$ which implies that $9.2 \times 10^{-99} \lesssim \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \lesssim 6.5 \times 10^{-30} N_{\text{T}}^2$. We see that

we always find a very small contribution. It is easy to see that it originates from the fact that the energy scale of inflation is very small and that we always have $M^4/M_{\text{Pl}}^4 \ll m_\sigma/M_{\text{Pl}}$. Let us now consider the other regime, given by Eq. (A60) and where the Bunch-Davis term is subdominant. In this case, the variance can be expressed as

$$\begin{aligned} \frac{\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}}{M_{\text{Pl}}^2} &\simeq 2.16 \times 10^{-9} N_{\text{T}} p^2 [p(p-2)N_*]^{-2(p-1)/(p-2)} \\ &\times \left(\frac{\mu}{M_{\text{Pl}}} \right)^{2p/(p-2)}. \end{aligned} \quad (\text{A62})$$

Therefore, if $m_\sigma = 10^5 \text{ GeV}$ and $p = 3$, one obtains $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \simeq 3.84 \times 10^{-17} N_{\text{T}} (\mu/M_{\text{Pl}})^6$. This gives the following range of values: $1.35 \times 10^{-29} N_{\text{T}}^2 \lesssim \langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \lesssim 3.8 \times 10^{-23} N_{\text{T}}$. This leads to a significant constraint only if the number of e-folds is large (remembering that, for $p = 3$, the formula applies only if $N_{\text{T}} \ll 1.84 \times 10^6$, see above). Taking this upper limit, one arrives at $\sqrt{\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}}/M_{\text{Pl}} \simeq 8.3 \times 10^{-9}$ which, besides being an extreme case, remains small in comparison to what was found in the case of large field models. If $p = 4$, one has $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}/M_{\text{Pl}}^2 \simeq 5.4 \times 10^{-16} N_{\text{T}} (\mu/M_{\text{Pl}})^4$ and the same conclusion can be reached. However, the extreme case evoked before becomes more significant as one obtains $\sqrt{\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}}/M_{\text{Pl}} \simeq 1.4 \times 10^{-5}$.

Let us also consider the regime where $\mu/M_{\text{Pl}} \gtrsim 1$ or $\mu/M_{\text{Pl}} \gg 1$. As mentioned above, in this situation, the argument of the exponential in Eq. (A54) is always small and $H_{\text{inf}} \sim 10^{13} \text{ GeV}$. This means that one is always in the case where the Bunch-Davis term is subdominant. Then, given that $N_{\text{T}} > 60$, one obtains the robust lower limit

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}^{1/2} \gtrsim 4.2 \times 10^{-6} M_{\text{Pl}}. \quad (\text{A63})$$

Let us now turn to the other contribution to the variance originating from the inflaton noise. It can be expressed as

$$\begin{aligned} \langle \delta\sigma_1^2 \rangle_{\delta\phi_1 \delta\phi_1} &= - \frac{\mu^4 m_\sigma^4 \sigma_{\text{cl}}^2}{24\pi^2 M_{\text{Pl}}^4 M^4} \frac{1}{\epsilon p(4-3p)} \\ &\times \left[\frac{1}{(4-2p)(4-p)} \Phi^{4-p} \right. \\ &+ \frac{4-3p}{p^2(4-2p)} \Phi^p \Phi_{\text{in}}^{4-2p} - \frac{1}{2p^2} \Phi^{2p} \Phi_{\text{in}}^{4-3p} \\ &\left. - \frac{4-3p}{2p^2(4-p)} \Phi_{\text{in}}^{4-p} \right]. \end{aligned} \quad (\text{A64})$$

The analysis is complicated by the fact that the dominant term depends on whether $p < 4$ or $p > 4$. Since, previously, we have mainly considered situations where $p < 4$, we will restrict ourselves to this case. In addition, very

large values of p appear rather unnatural from a high energy physics point of view. Therefore, in the case of small field inflation, the dominant term can be written as

$$\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1} \simeq \frac{\mu^4 m_\sigma^4 \sigma_{\text{cl}}^2}{24\pi^2 M_{\text{Pl}}^4 M^4} \times \frac{1}{p(4-3p)(4-2p)(4-p)} \Phi_{\text{end}}^{4-p}. \quad (\text{A65})$$

In the regime where $\mu/M_{\text{Pl}} \ll 1$, the term can be re-expressed as

$$\frac{\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}}{M_{\text{Pl}}^2} \simeq 4.7 \times 10^{-50} \frac{[p(p-2)N_*]^{2(p-1)/(p-2)}}{p^3(4-3p)(4-2p)(4-p)} \times \left(\frac{2}{p^2}\right)^{(4-p)/(2p-2)} \left(\frac{m_\sigma}{10^5 \text{ GeV}}\right)^4 \left(\frac{\sigma_{\text{cl}}}{M_{\text{Pl}}}\right)^2 \times \left(\frac{\mu}{M_{\text{Pl}}}\right)^{p(p-4)/[(p-2)(p-1)]}. \quad (\text{A66})$$

If $p = 2.5$ and $m_\sigma = 10^5 \text{ GeV}$, then one obtains $\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}/M_{\text{Pl}}^2 \simeq 1.9 \times 10^{-41} (\sigma_{\text{cl}}/M_{\text{Pl}})^2 (\mu/M_{\text{Pl}})^{-5}$. But we have seen before that, if $p = 2.5$, the argument of the exponential in Eq. (A54) is always large (and negative). This means that the modulus is exponentially killed during inflation and, hence, negligible at the end of inflation. In this case, the contribution $\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}/M_{\text{Pl}}^2$ is also negligible. If $p = 3$, the story is slightly different. Indeed, for $m_\sigma = 10^5 \text{ GeV}$, one obtains $\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}/M_{\text{Pl}}^2 \simeq 6 \times 10^{-44} (\sigma_{\text{cl}}/M_{\text{Pl}})^2 (\mu/M_{\text{Pl}})^{-3/2}$. If $1.5 \times 10^{-8} < \mu/M_{\text{Pl}} < 8.4 \times 10^{-8} N_{\text{T}}^{1/6}$, the argument of the exponential in Eq. (A54) is large and the same conclusion as before applies. But when $8.4 \times 10^{-8} N_{\text{T}}^{1/6} < \mu/M_{\text{Pl}} \ll 1$, the argument is small and the modulus is almost frozen, $\sigma_{\text{cl}} \simeq \sigma_{\text{in}}$. In this case, this implies that $1.9 \times 10^{-42} (\sigma_{\text{in}}/M_{\text{Pl}})^2 \lesssim \langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}/M_{\text{Pl}}^2 \lesssim 7.8 \times 10^{-41} N_{\text{T}}^{1/4} (\sigma_{\text{in}}/M_{\text{Pl}})^2$. We see that this contribution remains very small.

Let us now consider the case where $\mu/M_{\text{Pl}} > 1$. As already mentioned, this means that $\phi_{\text{end}} \simeq \mu$. In addition, we have shown before that, in this regime, the modulus is almost frozen during inflation. As a consequence, the variance due to the inflaton noise can be expressed as

$$\frac{\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}}{M_{\text{Pl}}^2} \simeq 3.3 \times 10^{-66} \frac{1}{p(4-3p)(4-2p)(4-p)} \times \left(\frac{m_\sigma}{10^5 \text{ GeV}}\right)^4 \left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}}\right)^2 \left(\frac{\mu}{M_{\text{Pl}}}\right)^4. \quad (\text{A67})$$

We conclude that this contribution is negligible unless one takes very large and unrealistic values of μ .

After having estimated the variance, one can now calculate the correction to the mean value. The nonzero contributions to $\langle \delta\sigma_2 \rangle$ read

$$\langle \delta\sigma_2 \rangle|_{\delta\phi_2} = \frac{\mu^2 m_\sigma^2 \sigma_{\text{cl}}}{96\pi^2 M_{\text{Pl}}^4} \frac{1}{\epsilon p(4-3p)} \times \left[\Phi^{2-p} - \frac{4-3p}{p} \Phi^p \Phi_{\text{in}}^{2-2p} - \Phi^{2p-2} \Phi_{\text{in}}^{4-3p} + \frac{4-3p}{p} \Phi_{\text{in}}^{2-p} \right], \quad (\text{A68})$$

$$\langle \delta\sigma_2 \rangle|_{\delta\phi_1^2} = \frac{\mu^2 m_\sigma^2 \sigma_{\text{cl}}}{48\pi^2 M_{\text{Pl}}^4} \frac{1}{\epsilon p(4-3p)} \left[\frac{p-1}{2-p} \Phi^{2-p} - \frac{1}{2} \Phi^{2p-2} \Phi_{\text{in}}^{4-3p} + \frac{4-3p}{2(2-p)} \Phi_{\text{in}}^{2-p} \right], \quad (\text{A69})$$

$$\langle \delta\sigma_2 \rangle|_{\delta\phi_1 \delta\sigma_1} = -\frac{\mu^4 m_\sigma^4 \sigma_{\text{cl}}}{48\pi^2 M_{\text{Pl}}^4 M^4} \frac{1}{\epsilon p(4-3p)} \times \left[\frac{1}{(4-2p)(4-p)} \Phi^{4-p} + \frac{4-3p}{p^2(4-2p)} \Phi^p \Phi_{\text{in}}^{4-2p} - \frac{1}{2p^2} \Phi^{2p} \Phi_{\text{in}}^{4-3p} - \frac{4-3p}{2p^2(4-p)} \Phi_{\text{in}}^{4-p} \right]. \quad (\text{A70})$$

We notice that the corrections to the mean value are of the same order of magnitude as $\langle \delta\sigma_1^2 \rangle_{\delta\phi_1, \delta\phi_1}^{1/2}$. Therefore, following the above analysis, one can safely conclude that these contributions are negligible.

b. Modulus potential: $V_\sigma = c_i^2 H^2 \sigma^2 / 2$

If the modulus mass is dominated by a Hubble term contribution from supergravity effects, one obtains the following results for the classical trajectory and the quantum corrections:

$$\sigma_{\text{cl}} = \sigma_{\text{in}} \exp\left[\frac{c_i^2 \mu^2}{3M_{\text{Pl}}^2} \frac{1}{\epsilon p(2-p)} (\Phi^{2-p} - \Phi_{\text{in}}^{2-p}) \right]. \quad (\text{A71})$$

At this stage, one can reproduce the discussion of the previous section and study the argument of the exponential. It is easy to show that it can be expressed as

$$-\frac{c_i^2 \mu^2}{3M_{\text{Pl}}^2 p(p-2)} \Phi_{\text{in}}^{2-p} \simeq -\frac{c_i^2}{3} N_{\text{T}}. \quad (\text{A72})$$

We see that the above formula is nothing but Eq. (A54) with the time-dependent mass $c_i H$. The argument of the exponential can be large or small depending on the parameter c_i and the total number of e-folds. But, as already mentioned in the section devoted to chaotic inflation, if $c_i \gtrsim 0.22$, then it is always greater than one given the fact that $N_{\text{T}} > 60$.

The variance of the first order correction is made of the combination of the following two terms, as above. The term due to the modulus noise reads

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} = \frac{M^4}{8\pi^2 M_{\text{Pl}}^2 c_i^2} \left\{ 1 - \exp \left[\frac{2c_i^2 \mu^2}{3M_{\text{Pl}}^2} \frac{1}{\epsilon p(2-p)} \right] \times (\Phi^{2-p} - \Phi_{\text{in}}^{2-p}) \right\}. \quad (\text{A73})$$

If the argument of the exponential in Eq. (A71) is large, then one has

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} \simeq \frac{M^4}{8\pi^2 M_{\text{Pl}}^2 c_i^2} = \frac{3H_{\text{inf}}^4}{8\pi^2 (c_i H_{\text{inf}})^2}, \quad (\text{A74})$$

and one recovers the Bunch-Davis term with a mass cH_{inf} . On the contrary, if the argument of the exponential in Eq. (A71) is small, then the expression of the variance reads

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} \simeq \frac{M^4 N_{\text{T}}}{12\pi^2 M_{\text{Pl}}^2} = \frac{H_{\text{inf}}^2}{4\pi^2} N_{\text{T}}, \quad (\text{A75})$$

and one recovers the de Sitter result. In particular, the term c_i^2 has cancelled out, as, in the corresponding situation, the modulus mass m_σ cancelled out in the previous section.

As before, one can also express the above results directly in terms of the relevant parameters. Let us start with the situation where the argument of the exponential is large. In the regime $\mu/M_{\text{Pl}} \ll 1$, this gives

$$\frac{\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}}{M_{\text{Pl}}^2} \simeq 3.24 \times 10^{-9} \frac{p^2}{c_i^2} [p(p-2)N_*]^{-2(p-1)/(p-2)} \times \left(\frac{\mu}{M_{\text{Pl}}} \right)^{2p/(p-2)}. \quad (\text{A76})$$

If $p = 2.5$, this gives $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} / M_{\text{Pl}}^2 \simeq 3.39 \times 10^{-19} c_i^{-2} (\mu/M_{\text{Pl}})^{10}$ and if $p = 3$, one has $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} / M_{\text{Pl}}^2 \simeq 5.76 \times 10^{-17} c_i^{-2} (\mu/M_{\text{Pl}})^6$. Given that the above estimates are valid in the regime where μ/M_{Pl} is small, we conclude that the variance is always very small unless c_i takes tiny and unrealistic values.

In the regime $\mu/M_{\text{Pl}} \gtrsim 1$, one obtains

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}^{1/2} \simeq 6.8 \times 10^{-7} \frac{M_{\text{Pl}}}{c_i^2}. \quad (\text{A77})$$

Contrary to the previous case, the above formula indicates that the quantum effects can now be significant, especially when the parameter c_i is small ($c_i < 0.22$).

If the argument of the exponential in Eq. (A71) is small, then it is easy to see that the variance is now equal to $2N_{\text{T}}c_i^2/3$ times the variance given in Eq. (A74). In the regime $\mu/M_{\text{Pl}} \ll 1$, the factor $N_{\text{T}}c_i^2$ is unlikely to com-

pensate the smallness of the variance obtained in Eq. (A74) (unless the total number of e-folds is huge), and we conclude that the quantum effects can become arbitrarily small in this regime.

On the other hand, if $\mu/M_{\text{Pl}} \gtrsim 1$, then one obtains $\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma} / M_{\text{Pl}}^2 \simeq 3 \times 10^{-13} N_{\text{T}}$ and, given the fact that the total number of e-folds must be larger than 60, one obtains the following lower bound:

$$\langle \delta\sigma_1^2 \rangle_{\xi_\sigma \xi_\sigma}^{1/2} \gtrsim 4.2 \times 10^{-6} M_{\text{Pl}}. \quad (\text{A78})$$

Let us now consider the second contribution to the variance due to the inflaton noise. Its expression can be written as

$$\begin{aligned} \langle \delta\sigma_1^2 \rangle_{\delta\phi_1 \delta\phi_1} &= -\frac{c_i^4 \sigma_{\text{cl}}^2 M^4 \mu^4}{216\pi^2 M_{\text{Pl}}^8} \frac{1}{\epsilon p(4-3p)} \\ &\times \left[\frac{1}{(4-2p)(4-p)} \Phi^{4-p} \right. \\ &+ \frac{4-3p}{2p^2(2-p)} \Phi^p \Phi_{\text{in}}^{4-2p} - \frac{1}{2p^2} \Phi^{2p} \Phi_{\text{in}}^{4-3p} \\ &\left. - \frac{4-3p}{2p^2(4-p)} \Phi_{\text{in}}^{4-p} \right]. \quad (\text{A79}) \end{aligned}$$

As discussed in the previous section in a similar context, the amplitude of this term depends on whether $p < 4$ or $p > 4$. Here we restrict ourselves to the case $p < 4$. Moreover, if the modulus significantly evolves during inflation, then the above contribution becomes negligible. So we assume that $c_i^2 N_{\text{T}}/3 \ll 1$ which implies that c_i is small. In this situation, if $\mu/M_{\text{Pl}} \ll 1$, one has

$$\begin{aligned} \langle \delta\sigma_1^2 \rangle_{\delta\phi_1 \delta\phi_1} &= \frac{c_i^4 \sigma_{\text{cl}}^2 M^4 \mu^4}{216\pi^2 M_{\text{Pl}}^8} \frac{1}{p(4-3p)(4-2p)(4-p)} \\ &\times \frac{1}{(4-2p)(4-p)} \Phi_{\text{end}}^{4-p} \\ &\simeq 1.2 \times 10^{-10} \frac{p c_i^4}{(4-3p)(4-2p)(4-p)} \\ &\times \left(\frac{2}{p^2} \right)^{(4-p)/(2p-2)} [p(p-2)N_*]^{-2(p-1)/(p-2)} \\ &\times \left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}} \right)^2 \left(\frac{\mu}{M_{\text{Pl}}} \right)^{(5p^2-8)/[(p-2)(p-1)]} M_{\text{Pl}}^2. \quad (\text{A80}) \end{aligned}$$

This term is obviously tiny, in particular, because the power of the term μ/M_{Pl} is positive. Hence, the contribution due to the inflaton noise can be neglected.

The case where $\mu/M_{\text{Pl}} \gtrsim 1$ remains to be studied. Straightforward considerations lead to

$$\frac{\langle \delta \sigma_1^2 \rangle_{\delta \phi_1 \delta \phi_1}}{M_{\text{Pl}}^2} \simeq 1.68 \times 10^{-14} \frac{c_i^4}{p(4-3p)(4-2p)(4-p)} \times \left(\frac{\sigma_{\text{in}}}{M_{\text{Pl}}} \right)^2 \left(\frac{\mu}{M_{\text{Pl}}} \right)^4. \quad (\text{A81})$$

$$\begin{aligned} \langle \delta \sigma_2 \rangle_{\delta \phi_2} &= \frac{c_i^2 \sigma_{\text{cl}} \mu^2 M^4}{288 \pi^2 M_{\text{Pl}}^6} \frac{1}{\epsilon p(4-3p)} \left[\Phi^{2-p} - \frac{4-3p}{p} \Phi^p \Phi_{\text{in}}^{2-2p} - \Phi^{2p-2} \Phi_{\text{in}}^{4-3p} + \frac{4-3p}{p} \Phi_{\text{in}}^{2-p} \right], \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1^2} &= \frac{c_i^2 \sigma_{\text{cl}} \mu^2 M^4}{288 \pi^2 M_{\text{Pl}}^6} \frac{1}{\epsilon p(4-3p)} \left[\frac{2(p-1)}{2-p} \Phi^{2-p} - \Phi^{2p-2} \Phi_{\text{in}}^{4-3p} + \frac{4-3p}{2-p} \Phi_{\text{in}}^{2-p} \right], \\ \langle \delta \sigma_2 \rangle_{\delta \phi_1 \delta \sigma_1} &= -\frac{c_i^4 \sigma_{\text{cl}} \mu^4 M^4}{432 \pi^2 M_{\text{Pl}}^8} \frac{1}{\epsilon p(4-3p)} \left[\frac{1}{(4-2p)(4-p)} \Phi^{4-p} + \frac{4-3p}{2p^2(2-p)} \Phi^p \Phi_{\text{in}}^{4-2p} - \frac{1}{2p^2} \Phi^{2p} \Phi_{\text{in}}^{4-3p} \right. \\ &\quad \left. - \frac{4-3p}{2p^2(4-p)} \Phi_{\text{in}}^{4-p} \right]. \end{aligned} \quad (\text{A82})$$

It is clear that the order of magnitude of those terms is similar to $\langle \delta \sigma_1^2 \rangle_{\delta \phi_1 \delta \phi_1}^{1/2}$. Hence one concludes that the corrections to the mean value can be safely neglected, at least when μ/M_{Pl} is not too large.

APPENDIX B: DETAILS OF THE SOLUTIONS TO THE EQUATIONS OF MOTION IN PRESENCE OF HUBBLE SCALE MASS CORRECTIONS

In this appendix, we study in detail the solutions of the modulus equation of motion in the postinflationary epoch in the case where the potential receives supergravity corrections.

1. Case $+c^2 H^2/2$

The solution to the equation of motion has been obtained by Lyth & Stewart [17]. The field equation can indeed be put in the following form:

$$\begin{aligned} (m_\sigma t)^2 \tilde{\sigma}'' + (m_\sigma t) \tilde{\sigma}' + \left[(m_\sigma t)^2 + p^2 c^2 - \frac{(3p-1)^2}{4} \right] \tilde{\sigma} \\ = p^2 c^2 \sigma_0 (m_\sigma t)^{(3p-1)/2}, \end{aligned} \quad (\text{B1})$$

with the following definition:

$$\tilde{\sigma} \equiv (m_\sigma t)^{(3p-1)/2} \sigma. \quad (\text{B2})$$

A prime denotes derivative with respect to $m_\sigma t$, while p characterizes the global equation of state of the Universe: $H = p/t$, with $p = 2/3$ for matter domination or $p = 1/2$ for radiation domination. The above field equations takes the form of a Lommel equation. Out of convenience, we write the general solution as the sum of a homogeneous solution with constants α_1 and α_2 , and a particular solution given in terms of a hypergeometric function:

This contribution is small unless $\mu/M_{\text{Pl}} \gg 1$.

Finally, the (nonzero) second order corrections to the mean value of the modulus vacuum expectation value read:

$$\begin{aligned} \sigma(t) &= \alpha_1 (m_\sigma t)^{-(\mu+1)} [J_\nu(m_\sigma t) + J_{-\nu}(m_\sigma t)] \\ &\quad + \alpha_2 (m_\sigma t)^{-(\mu+1)} [J_\nu(m_\sigma t) - J_{-\nu}(m_\sigma t)] \\ &\quad + \sigma_{01} F_2 \left[1; \frac{\mu-\nu+3}{2}; \frac{\mu+\nu+3}{2}; -\frac{(m_\sigma t)^2}{4} \right], \end{aligned} \quad (\text{B3})$$

where we have defined

$$\mu = \frac{3p-3}{2}, \quad \nu^2 = (\mu+1)^2 - p^2 c^2. \quad (\text{B4})$$

The initial conditions are defined at $t = p/H_{\text{inf}}$ as follows: $\sigma = \sigma_{\text{inf}}$ and $\sigma' \simeq 0$. As a consequence, the coefficients α_1 and α_2 can then be expressed as

$$\begin{aligned} \alpha_1 &= (\sigma_{\text{inf}} - \sigma_0) \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^{\mu+1} \left[\frac{\Gamma(1+\nu)}{2^{2-\nu}} \left(1 + \frac{\mu+1}{\nu} \right) \right. \\ &\quad \left. \times \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^{-\nu} + \frac{\Gamma(1-\nu)}{2^{2+\nu}} \left(1 - \frac{\mu+1}{\nu} \right) \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^\nu \right] \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \alpha_2 &= (\sigma_{\text{inf}} - \sigma_0) \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^{\mu+1} \left[\frac{\Gamma(1+\nu)}{2^{2-\nu}} \left(1 + \frac{\mu+1}{\nu} \right) \right. \\ &\quad \left. \times \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^{-\nu} - \frac{\Gamma(1-\nu)}{2^{2+\nu}} \left(1 - \frac{\mu+1}{\nu} \right) \left(\frac{pm_\sigma}{H_{\text{inf}}} \right)^\nu \right]. \end{aligned} \quad (\text{B6})$$

Obviously, if $\sigma_{\text{inf}} = \sigma_0$, meaning that the vev of σ at the end of inflation corresponds with the high energy minimum, then $\alpha_1 = \alpha_2 = 0$. The solution then reduces to the particular solution.

We now study the behavior of the solutions according to the values of the parameters. Let us start with the situation where ν is real, namely, $c < (\mu+1)/p$. At early times, $m_\sigma t \ll 1$ (hence $H \gg m_\sigma$) and at late times, $m_\sigma t \gg 1$ (hence $H \ll m_\sigma$), the particular and homogeneous solu-

tions evolve differently. At early times, the particular solution remains constant,

$$\sigma_{\text{part}}(t) \simeq \sigma_0, \quad m_{\sigma} t \ll 1. \quad (\text{B7})$$

Therefore, before the onset of oscillations, the particular solution does not undergo any redshift. On the contrary, at late times, one obtains

$$\begin{aligned} \sigma_{\text{part}}(t) &\simeq \sigma_0 (m_{\sigma} t)^{-\mu-3/2} \frac{2^{\mu+3/2}}{\sqrt{\pi}} \Gamma\left(\frac{\nu}{2} + \frac{\mu+3}{2}\right) \\ &\times \Gamma\left(-\frac{\nu}{2} + \frac{\mu+3}{2}\right) \sin\left(m_{\sigma} t + \frac{\pi}{4} - \frac{\mu+1}{2}\pi\right), \\ m_{\sigma} t &\gg 1, \end{aligned} \quad (\text{B8})$$

and, in this regime, one has that $\sigma_{\text{part}}(t) \propto a^{-3/2}$ regardless of p . Since, in the limit $m_{\sigma} t \gg 1$, the supergravity corrections are by definition negligible, one can write $\rho_{\sigma} \simeq m_{\sigma}^2 \sigma_{\text{part,osci}}^2 (a/a_{\text{osci}})^{-3}/2$. From the above formula, one deduces that the quantity $\sigma_{\text{part,osci}}$ can be expressed as

$$\sigma_{\text{part,osci}} \simeq \frac{2^{\mu+3/2}}{\sqrt{\pi}} \Gamma\left(\frac{\nu}{2} + \frac{\mu+3}{2}\right) \Gamma\left(-\frac{\nu}{2} + \frac{\mu+3}{2}\right) \sigma_0. \quad (\text{B9})$$

Let us notice that the modulus energy density at $m_{\sigma} t \sim 1$ (or $H \simeq m_{\sigma}$) is not given by $m_{\sigma}^2 \sigma_{\text{part,osci}}^2/2$ only. Indeed, at $m_{\sigma} t \sim 1$, the term $c^2 H^2$ can not be neglected and participate to the energy density at the onset of oscillations. Therefore, strictly speaking, $m_{\sigma}^2 \sigma_{\text{part,osci}}^2/2$ only represents the contribution of the term $m_{\sigma}^2 \sigma^2/2$ at $H = m_{\sigma}$. However what really matters is the modulus energy density at the time of dark matter freeze-out. In this regime, the supergravity corrections are negligible. As a consequence, one only needs to take into account the contribution due to the term $m_{\sigma}^2 \sigma^2/2$ and to express it in terms of its value at the onset of oscillations. Then the contribution of the particular solution to the energy density contained in the modulus is given by

$$\begin{aligned} \Omega_{\sigma,\text{part,osci}} &\simeq \frac{2^{2\mu+2}}{3\pi} \Gamma^2\left(\frac{\nu}{2} + \frac{\mu+3}{2}\right) \\ &\times \Gamma^2\left(-\frac{\nu}{2} + \frac{\mu+3}{2}\right) \left(\frac{\sigma_0}{M_{\text{Pl}}}\right)^2. \end{aligned} \quad (\text{B10})$$

Note that this result differs from that obtained for the purely quadratic potential: in the latter case, $\Omega_{\sigma,\text{osci}}$ is controlled by the value of σ at the end of inflation, while here, it is controlled by the value of the local minimum σ_0 generated by supergravity corrections.

Another issue concerns the values of the parameters μ and ν that should be inserted in the above equation. If $T_{\text{th}} > T_{\text{osci}}$, then one should use μ_{RD} and ν_{RD} , where quantities indexed with ‘‘RD’’ (respectively, ‘‘MD’’) are to be evaluated for a radiation dominated (respectively, matter dominated) era, that is to say with $p = 1/2$ (respectively, $p = 2/3$). The justification is the following one. Just after inflation, the universe is matter dominated

and the solution (B3) with $p = 2/3$ should be used in order to describe the evolution of σ . Since, in the present case, we study a situation where the onset of oscillations happens after reheating, one is in fact in the regime $m_{\sigma} t \ll 1$ during the whole matter dominated era and, as a result, the particular solution remains constant and equal to σ_0 during this phase. Then, after the reheating is completed, the universe becomes radiation dominated and, in principle, one should solve again the equation of motion but, this time, with $p = 1/2$ rather than $p = 2/3$ and with ‘‘new initial conditions’’ inherited from the reheating era. However, at the beginning of the radiation era, we are still before the onset of oscillations (i.e. still in the regime $m_{\sigma} t \ll 1$) and the solution with $p = 1/2$ is still constant since this asymptotic behavior does not depend on p . Consequently, the matching of the two solutions is in fact trivial. This means that, in the radiation dominated era, the relevant solution is nothing but the solution (B3) with $p = 1/2$. Therefore, at late times, it is sufficient to use Eq. (B8) with $p = 1/2$, hence the above claim.

If $T_{\text{osci}} > T_{\text{th}}$, the situation is slightly more complicated. This time, the onset of oscillations occurs during the matter dominated phase and the matching between the solution with $p = 2/3$ and $p = 1/2$ should be done in the $m_{\sigma} t \gg 1$ regime. However, as before, the two solutions are the same because σ scales as $a^{-3/2}$ independently of p (see above). As a result, one can ignore the matching and still work with the solution with $p = 2/3$ at late times in the radiation dominated era. Therefore, if $T_{\text{osci}} > T_{\text{th}}$, one should use μ_{MD} and ν_{MD} .

Let us now consider the homogeneous solution. At early times, it can be approximated as

$$\begin{aligned} \sigma_{\text{hom}}(t) &\simeq \frac{1}{2} (\sigma_{\text{inf}} - \sigma_0) \left(1 + \frac{1+\mu}{\nu}\right) \left(\frac{tH_{\text{inf}}}{p}\right)^{-\mu-1+\nu} m_{\sigma} t \\ &\ll 1, \end{aligned} \quad (\text{B11})$$

where one has used the expressions (B5) and (B6). In particular, the last terms in Eqs. (B5) and (B6) become negligible which implies $\alpha_1 \simeq \alpha_2$ and, therefore, the terms proportional to $J_{-\nu}$ cancel out. One notices that the homogeneous solution does not remain constant prior to the onset of oscillations as it was the case for the particular solution. One finds that $\rho_{\sigma,\text{kin}}$ and $\rho_{\sigma,\text{pot}}$ scale as $\propto t^{2(-\mu-2+\nu)}$. For a matter dominated era, one therefore finds $\rho_{\sigma} \propto a^{-9/2+3\nu}$, while for a radiation dominated era one has $\rho_{\sigma} \propto a^{-5+4\nu}$. On the other hand, the late time evolution of σ reads as follows:

$$\begin{aligned} \sigma_{\text{hom}}(t) &\simeq (m_{\sigma} t)^{-\mu-3/2} \left[\sqrt{\frac{8}{\pi}} \alpha_1 \cos\left(m_{\sigma} t - \frac{\pi}{4}\right) \cos\left(\nu \frac{\pi}{2}\right) \right. \\ &\quad \left. + \sqrt{\frac{8}{\pi}} \alpha_2 \sin\left(m_{\sigma} t - \frac{\pi}{4}\right) \sin\left(\nu \frac{\pi}{2}\right) \right], \\ m_{\sigma} t &\gg 1. \end{aligned} \quad (\text{B12})$$

As expected, one finds the standard evolution at late times, with $\rho_\sigma \propto a^{-3}$ since $\mu + 3/2 = 3p/2$. If one uses the expressions of α_1 and α_2 given in Eqs. (B5) and (B6), this can be rewritten as

$$\begin{aligned} \sigma_{\text{hom}}(t) &\simeq (m_\sigma t)^{-\mu-3/2} (\sigma_{\text{inf}} - \sigma_0) \left(\frac{p m_\sigma}{H_{\text{inf}}} \right)^{\mu+1-\nu} \\ &\times \sqrt{\frac{2}{\pi}} \frac{\Gamma(1+\nu)}{2^{1-\nu}} \left(1 + \frac{\mu+1}{\nu} \right) \\ &\times \cos\left(m_\sigma t - \frac{\pi}{4} - \frac{\pi\nu}{2}\right), \\ m_\sigma t &\gg 1. \end{aligned} \quad (\text{B13})$$

As before, one should distinguish two different cases. If $T_{\text{osci}} > T_{\text{rh}}$, then the onset of oscillations occurs in the matter dominated era. As a consequence, at the end of the reheating stage (in the regime $m_\sigma t \gg 1$), the homogeneous solution is given by

$$\begin{aligned} \sigma_{\text{hom}}(t) &\simeq (m_\sigma t)^{-\mu_{\text{MD}}-3/2} \sqrt{\frac{2}{\pi}} (\sigma_{\text{inf}} - \sigma_0) \\ &\times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{\mu_{\text{MD}}+1-\nu_{\text{MD}}} \frac{\Gamma(1+\nu_{\text{MD}})}{2^{1-\nu_{\text{MD}}}} \\ &\times \left(1 + \frac{\mu_{\text{MD}}+1}{\nu_{\text{MD}}} \right) \cos\left(m_\sigma t - \frac{\pi}{4} - \frac{\pi\nu_{\text{MD}}}{2}\right). \end{aligned} \quad (\text{B14})$$

Then, one should match this solution to the solution valid in the radiation dominated era. However, as in the case of the particular solution, since one is in the regime $m_\sigma t \gg 1$, the two solutions scale as $a^{-3/2}$ and the matching becomes trivial. As a consequence, one finds

$$\begin{aligned} \Omega_{\sigma, \text{hom,osci}} &\simeq \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \\ &\times \frac{2^{2\nu_{\text{MD}}-1}}{\pi} \Gamma^2(1+\nu_{\text{MD}}) \left(1 + \frac{\mu_{\text{MD}}+1}{\nu_{\text{MD}}} \right)^2. \end{aligned} \quad (\text{B15})$$

Finally, the case $T_{\text{rh}} > T_{\text{osci}}$ remains to be treated. In this situation, one needs to take into account the change in the values of μ and ν and there is no other choice than performing the matching explicitly at the time of reheating. This time, the matching turns out to be nontrivial. The whole matter dominated era occurs in the regime $m_\sigma t \ll 1$ and, to leading order [using again the expressions (B5) and (B6)] one obtains that

$$\sigma|_{\text{rh}} \simeq \frac{\sigma_{\text{inf}} - \sigma_0}{2} \left(1 + \frac{\mu_{\text{MD}}+1}{\nu_{\text{MD}}} \right) \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{\nu_{\text{MD}} - \mu_{\text{MD}} - 1}, \quad (\text{B16})$$

$$\begin{aligned} t \frac{d\sigma}{dt} \Big|_{\text{rh}} &\simeq \frac{\sigma_{\text{inf}} - \sigma_0}{2} \left(1 + \frac{\mu_{\text{MD}}+1}{\nu_{\text{MD}}} \right) (-\mu_{\text{MD}} - 1 + \nu_{\text{MD}}) \\ &\times \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{\nu_{\text{MD}} - \mu_{\text{MD}} - 2}. \end{aligned} \quad (\text{B17})$$

In the radiation dominated era, the solution is given by $\sigma = \beta_1 (m_\sigma t)^{-\mu_{\text{RD}}-1} J_{\nu_{\text{RD}}}(m_\sigma t) + \beta_2 (m_\sigma t)^{-\mu_{\text{RD}}-1} J_{-\nu_{\text{RD}}}(m_\sigma t)$ and one should determine the coefficients β_1 and β_2 by matching, at reheating, the above solution to the values (B16) and (B17). Straightforward calculations show that $\beta_1 \gg \beta_2$. Then, using the asymptotic behavior of the Bessel function $J_{\nu_{\text{RD}}}(m_\sigma t)$ that appears in the dominant branch of the solution in the regime $m_\sigma t \gg 1$, i.e. after the onset of oscillations, one can check that this solution scales as $a^{-3/2}$, the proportionality coefficient being, as before, directly related to $\Omega_{\sigma, \text{hom,osci}}$. One finds

$$\begin{aligned} \Omega_{\sigma, \text{hom,osci}} &\simeq \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{rh}}} \right)^{2(\mu_{\text{RD}}+1-\nu_{\text{RD}})} \\ &\times \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \frac{2^{2\nu_{\text{RD}}-3}}{\pi} \frac{\Gamma^2(1+\nu_{\text{RD}})}{\nu_{\text{RD}}^2} \\ &\times \left(1 + \frac{\mu_{\text{RD}}+1}{\nu_{\text{RD}}} \right)^2 \\ &\times (\mu_{\text{MD}} - \mu_{\text{RD}} + \nu_{\text{MD}} + \nu_{\text{RD}})^2. \end{aligned} \quad (\text{B18})$$

As expected, one can check that, in the limit where all the quantities labeled ‘‘RD’’ becomes equal to their counterparts labeled ‘‘MD’’, the above expression exactly reduces to Eq. (15).

Summarizing our result, one has

$$\begin{aligned} \Omega_{\sigma, \text{osci}} &\simeq \frac{2^{2\mu_{\text{MD}}+2}}{3\pi} \Gamma^2\left(\frac{\mu_{\text{MD}}+3+\nu_{\text{MD}}}{2}\right) \\ &\times \Gamma^2\left(\frac{\mu_{\text{MD}}+3-\nu_{\text{MD}}}{2}\right) \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \\ &\times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \frac{2^{2\nu_{\text{MD}}-1}}{\pi} \\ &\times \Gamma^2(1+\nu_{\text{MD}}) \left(1 + \frac{\mu_{\text{MD}}+1}{\nu_{\text{MD}}} \right)^2, \\ T_{\text{osci}} &> T_{\text{rh}}, \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \Omega_{\sigma, \text{osci}} &\simeq \frac{2^{2\mu_{\text{RD}}+2}}{3\pi} \Gamma^2\left(\frac{\mu_{\text{RD}}+3+\nu_{\text{RD}}}{2}\right) \\ &\times \Gamma^2\left(\frac{\mu_{\text{RD}}+3-\nu_{\text{RD}}}{2}\right) \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \\ &\times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{rh}}} \right)^{2(\mu_{\text{RD}}+1-\nu_{\text{RD}})} \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1-\nu_{\text{MD}})} \\ &\times \frac{2^{2\nu_{\text{RD}}-3}}{\pi} \frac{\Gamma^2(1+\nu_{\text{RD}})}{\nu_{\text{RD}}^2} \left(1 + \frac{\mu_{\text{RD}}+1}{\nu_{\text{RD}}} \right)^2 \\ &\times (\mu_{\text{MD}} - \mu_{\text{RD}} + \nu_{\text{MD}} + \nu_{\text{RD}})^2, \\ T_{\text{osci}} &< T_{\text{rh}}. \end{aligned} \quad (\text{B20})$$

Let us emphasize that, in order to obtain the above expressions, we have assumed that the total energy density at the onset of oscillations is only made of two pieces, one originating from the particular solution and the other from the homogeneous solution. This means that we have neglected the cross terms that, in principle, should contribute. This is justified by the fact that, in practice, one solution always dominates the other. Which one is dominating depends on the region explored in the parameter space. It is also convenient to define

$$\mathcal{A}_1 \equiv \frac{2^{\mu_{\text{MD}}+3/2}}{\sqrt{\pi}} \Gamma\left(\frac{\mu_{\text{MD}} + \nu_{\text{MD}} + 3}{2}\right) \Gamma\left(\frac{\mu_{\text{MD}} - \nu_{\text{MD}} + 3}{2}\right), \quad (\text{B21})$$

and \mathcal{A}_2 expressed in the same way as \mathcal{A}_1 above, but with μ_{MD} and ν_{MD} replaced by μ_{RD} and ν_{RD} ; The coefficients \mathcal{B}_1 and \mathcal{B}_2 are given by

$$\mathcal{B}_1 \equiv \sqrt{\frac{2}{\pi}} \frac{\Gamma(1 + \nu_{\text{MD}})}{2^{1-\nu_{\text{MD}}}} \left(1 + \frac{\mu_{\text{MD}} + 1}{\nu_{\text{MD}}}\right), \quad (\text{B22})$$

$$\mathcal{B}_2 \equiv \frac{2^{\nu_{\text{RD}}-3/2}}{\sqrt{\pi}} \frac{\Gamma(1 + \nu_{\text{RD}})}{\nu_{\text{RD}}} \left(1 + \frac{\mu_{\text{RD}} + 1}{\nu_{\text{RD}}}\right) \times (\mu_{\text{MD}} - \mu_{\text{RD}} + \nu_{\text{MD}} + \nu_{\text{RD}}). \quad (\text{B23})$$

With these definitions, the above expressions (B19) and (B20) exactly match the ones used in the main text [see Eqs. (49) and (50)].

The other possibility is when ν is imaginary, namely, $c > (\mu + 1)/p$. For $p = 2/3$, this case corresponds to $c > 3/4$. Writing $\nu = i\hat{\nu}$, one has $\hat{\nu} > 0$ growing with c . This brings in nontrivial modifications for the scaling of the modulus energy density. Following the same reasoning as before, one finds that the contribution due to the particular solution can be expressed as

$$\Omega_{\sigma,\text{part,osci}} \simeq \frac{2^{2\mu+2}}{3\pi} \Gamma^2\left(\frac{\mu+3}{2} + i\frac{\hat{\nu}}{2}\right) \times \Gamma^2\left(\frac{\mu+3}{2} - i\frac{\hat{\nu}}{2}\right) \left(\frac{\sigma_0}{m_{\text{Pl}}}\right)^2. \quad (\text{B24})$$

As before, in the above equation, one should use quantities labeled ‘‘MD’’ or ‘‘RD’’ according to $T_{\text{osci}} > T_{\text{rh}}$ or $T_{\text{osci}} < T_{\text{rh}}$. The difference with Eq. (B10) comes from the fact that the numerical prefactors, which are of order unity if ν is real, may become quite small if ν is pure imaginary and its modulus is large. This point is elaborated further in more detail in the main text.

Turning to the homogeneous solution, we find that it scales as follows. At early times, one no longer finds $\alpha_1 \simeq \alpha_2$ and this implies that

$$\sigma(t) \simeq t^{-\mu-1}, \quad m_{\sigma}t \ll 1. \quad (\text{B25})$$

Keeping in mind that ν is imaginary, this gives the following scaling of the kinetic energy density: $\rho_{\text{kin,hom}} \propto t^{2(-\mu-2)}$, and it is easy to see that at early times, the

potential energy density scales similarly. At late times, the homogeneous solution evolves as

$$\sigma(t) \simeq (m_{\sigma}t)^{-\mu-3/2} \left[\alpha_1 \sqrt{\frac{8}{\pi}} \cos\left(m_{\sigma}t - \frac{\pi}{4}\right) \cosh\left(\frac{\hat{\nu}\pi}{2}\right) + i\alpha_2 \sqrt{\frac{8}{\pi}} \sin\left(m_{\sigma}t - \frac{\pi}{4}\right) \sinh\left(\frac{\hat{\nu}\pi}{2}\right) \right], \quad (\text{B26})$$

$m_{\sigma}t \gg 1.$

As expected and as it was the case before for the case of ν real, one can check that σ scales as $a^{-3/2}$ regardless of p . At this point, as was done before in the case where ν was real, one has to distinguish the case where the onset of oscillations occurs before or after the reheating. Let us start with the case $T_{\text{osci}} > T_{\text{rh}}$. Since the oscillations start in the matter dominated era, one is interested in the regime $m_{\sigma}t \gg 1$ with $p = 2/3$. The corresponding solution is given by Eq. (B26). Expressing the coefficients α_1 and α_2 using Eqs. (B5) and (B6), one obtains the following result:

$$\sigma \simeq (m_{\sigma}t)^{-\mu_{\text{MD}}-3/2} \frac{\sigma_{\text{inf}} - \sigma_0}{2} \sqrt{\frac{8}{\pi}} \left(\frac{p_{\text{MD}}m_{\sigma}}{H_{\text{inf}}}\right)^{\mu_{\text{MD}}+1} \times \sqrt{\frac{\pi\hat{\nu}_{\text{MD}}}{\sinh(\pi\hat{\nu}_{\text{MD}})}} \sqrt{1 + \frac{(\mu_{\text{MD}} + 1)^2}{\hat{\nu}_{\text{MD}}^2}} \times \left[\cos Y_{\text{MD}} \cos\left(m_{\sigma}t - \frac{\pi}{4}\right) \cosh\left(\frac{\pi\hat{\nu}_{\text{MD}}}{2}\right) + \sin Y_{\text{MD}} \sin\left(m_{\sigma}t - \frac{\pi}{4}\right) \sinh\left(\frac{\pi\hat{\nu}_{\text{MD}}}{2}\right) \right], \quad (\text{B27})$$

where $Y_{\text{MD}} \equiv \hat{\nu}_{\text{MD}} \ln[p m_{\sigma}/(2H_{\text{inf}})] - \Theta_{\text{MD}} - \Psi_{\text{MD}}$, Θ_{MD} being the phase of the complex number $\Gamma(1 + i\hat{\nu}_{\text{MD}})$ and Ψ_{MD} the phase of $1 - i(\mu_{\text{MD}} + 1)/\hat{\nu}_{\text{MD}}$. As usual, this function scales as $a^{-3/2}$. Then one should match this solution with the solution in the radiation dominated era (still in the regime $m_{\sigma}t \gg 1$). However, the solution in the radiation dominated era also scales as $a^{-3/2}$. As a consequence, the matching is trivial. Therefore, using the above equation in the limit where $|\nu| \gg 1$ (which simply amounts to replace, in the above expression, $\sinh x$ and $\cosh x$ by $e^x/2$), one obtains

$$\Omega_{\sigma,\text{hom,osci}} \simeq \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}}\right)^2 \left(\frac{p_{\text{MD}}m_{\sigma}}{H_{\text{inf}}}\right)^{2(\mu_{\text{MD}}+1)} \times \hat{\nu}_{\text{MD}} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}}\right)^2\right], \quad (\text{B28})$$

$T_{\text{osci}} > T_{\text{rh}},$

where we have ignored the remaining trigonometric functions since they give contributions of order one. It is remarkable that all the exponential dependence in $\hat{\nu}$ has cancelled out in this expression.

When $T_{\text{rh}} > T_{\text{osci}}$, the onset of oscillations occurs in the radiation dominated era and one additional matching is required as explained previously. At reheating, we have

$$\begin{aligned} \sigma|_{\text{rh}} &\simeq (\sigma_{\text{inf}} - \sigma_0) \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{-\mu_{\text{MD}}-1} \\ &\times \sqrt{1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2} \cos \left[\hat{\nu}_{\text{MD}} \ln \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right) + \Psi_{\text{MD}} \right], \end{aligned} \quad (\text{B29})$$

$$\begin{aligned} t \frac{d\sigma}{dt} \Big|_{\text{rh}} &\simeq -(\sigma_{\text{inf}} - \sigma_0) \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{-\mu_{\text{MD}}-2} \\ &\times \hat{\nu}_{\text{MD}} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2 \right] \sin \left[\hat{\nu}_{\text{MD}} \ln \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right) \right]. \end{aligned} \quad (\text{B30})$$

$$\begin{aligned} \sigma(t) &\simeq (m_\sigma t)^{-\mu_{\text{RD}}-3/2} (\sigma_{\text{inf}} - \sigma_0) \sqrt{\frac{2}{\pi}} \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{-\mu_{\text{MD}}-1} \left(\frac{m_\sigma p}{H_{\text{rh}}} \right)^{\mu_{\text{RD}}+1} \frac{e^{\pi \hat{\nu}_{\text{RD}}/2}}{2 \hat{\nu}_{\text{RD}}} \sqrt{1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2} \sqrt{\frac{\pi \hat{\nu}_{\text{RD}}}{\sinh(\pi \hat{\nu}_{\text{RD}})}} \\ &\times \left\{ -\hat{\nu}_{\text{MD}} \sqrt{1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2} \sin \left[\hat{\nu}_{\text{MD}} \ln \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right) \right] \sin \left[\hat{\nu}_{\text{RD}} \ln \left(\frac{2H_{\text{rh}}}{m_\sigma p} \right) + \Theta_{\text{RD}} + m_\sigma t - \frac{\pi}{4} \right] \right. \\ &\left. + \hat{\nu}_{\text{RD}} \sqrt{1 + \left(\frac{\mu_{\text{RD}} + 1}{\hat{\nu}_{\text{RD}}} \right)^2} \cos \left[\hat{\nu}_{\text{MD}} \ln \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right) + \Psi_{\text{MD}} \right] \cos \left[\hat{\nu}_{\text{RD}} \ln \left(\frac{2H_{\text{rh}}}{m_\sigma p} \right) + \Theta_{\text{RD}} + \Psi_{\text{RD}} + m_\sigma t - \frac{\pi}{4} \right] \right\}. \end{aligned} \quad (\text{B31})$$

As expected, this solution scales as $a^{-3/2}$. Again, it is interesting to observe how any exponential dependence in $\hat{\nu}_{\text{RD}}$ is in fact exactly cancels out. It is also interesting to study the case where all the quantities labeled RD becomes equal to their counterparts labeled MD. In this case, lengthy calculations using trigonometric identities show that the term in the curled bracket reduces to

$$\begin{aligned} \{-\} &= \hat{\nu}_{\text{MD}} \sqrt{1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2} \cos \Psi_{\text{MD}} \\ &\times \left[\cos Y_{\text{MD}} \cos \left(m_\sigma t - \frac{\pi}{4} \right) \right. \\ &\left. + \sin Y_{\text{MD}} \sin \left(m_\sigma t - \frac{\pi}{4} \right) \right], \end{aligned} \quad (\text{B32})$$

and recalling the definition of Ψ_{MD} (which implies that $\cos \Psi_{\text{MD}}$ exactly cancels the square root in the above formula), it is easy to demonstrate that Eq. (B31) exactly reduces to Eq. (B27). Finally, one obtains

$$\begin{aligned} \Omega_{\sigma, \text{osci}} &\simeq \frac{1}{6} \frac{(\sigma_{\text{inf}} - \sigma_0)^2}{M_{\text{Pl}}^2} \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{-2\mu_{\text{MD}}-2} \left(\frac{m_\sigma p_{\text{MD}}}{H_{\text{rh}}} \right)^{2\mu_{\text{RD}}+2} \\ &\times \frac{1}{\hat{\nu}_{\text{RD}}} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2 \right] \{-\}^2, \\ T_{\text{rh}} &> T_{\text{osci}}. \end{aligned} \quad (\text{B33})$$

This completes our analysis of the case where ν is complex.

As before, the total $\Omega_{\sigma, \text{osci}}$ is roughly given by the sum of $\Omega_{\sigma, \text{hom,osci}}$ and $\Omega_{\sigma, \text{part,osci}}$. Summarizing our results one

These equations should be compare with Eqs. (B16) and (B17). As before, in the radiation dominated era, the solution is given by $\sigma = \beta_1 (m_\sigma t)^{-\mu_{\text{RD}}-1} J_{\nu_{\text{RD}}}(m_\sigma t) + \beta_2 (m_\sigma t)^{-\mu_{\text{RD}}-1} J_{-\nu_{\text{RD}}}(m_\sigma t)$, where ν_{RD} is now a complex number. The coefficients β_1 and β_2 are determined by matching the previous solution to the values (B29) and (B30). Then, the solution is completely specified. In the regime $m_\sigma t \gg 1$, one obtains

obtains

$$\begin{aligned} \Omega_{\sigma, \text{osci}} &\simeq \frac{2^{2\mu_{\text{MD}}+2}}{3\pi} \Gamma^2 \left(\frac{\mu_{\text{MD}} + 3}{2} + i \frac{\hat{\nu}_{\text{MD}}}{2} \right) \\ &\times \Gamma^2 \left(\frac{\mu_{\text{MD}} + 3}{2} - i \frac{\hat{\nu}_{\text{MD}}}{2} \right) \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \left(\frac{\sigma_{\text{inf}} - \sigma_0}{M_{\text{Pl}}} \right)^2 \\ &\times \left(\frac{p_{\text{MD}} m_\sigma}{H_{\text{inf}}} \right)^{2(\mu_{\text{MD}}+1)} \hat{\nu}_{\text{MD}} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\nu_{\text{MD}}} \right)^2 \right], \\ T_{\text{osci}} &> T_{\text{rh}}, \end{aligned} \quad (\text{B34})$$

$$\begin{aligned} \Omega_{\sigma, \text{osci}} &\simeq \frac{2^{2\mu_{\text{RD}}+2}}{3\pi} \Gamma^2 \left(\frac{\mu_{\text{RD}} + 3}{2} + i \frac{\hat{\nu}_{\text{RD}}}{2} \right) \\ &\times \Gamma^2 \left(\frac{\mu_{\text{RD}} + 3}{2} - i \frac{\hat{\nu}_{\text{RD}}}{2} \right) \left(\frac{\sigma_0}{M_{\text{Pl}}} \right)^2 + \frac{1}{6} \frac{(\sigma_{\text{inf}} - \sigma_0)^2}{M_{\text{Pl}}^2} \\ &\times \left(\frac{H_{\text{inf}}}{H_{\text{rh}}} \right)^{-2\mu_{\text{MD}}-2} \left(\frac{m_\sigma p_{\text{MD}}}{H_{\text{rh}}} \right)^{2\mu_{\text{RD}}+2} \frac{1}{\hat{\nu}_{\text{RD}}} \\ &\times \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2 \right] \{-\}^2, \\ T_{\text{rh}} &> T_{\text{osci}}. \end{aligned} \quad (\text{B35})$$

These equations are the counterparts of Eqs. (49) and (50) in the case where the quantity ν is complex. As was done before, it is convenient to define a prefactor \mathcal{A}_3 (the equivalent of \mathcal{A}_1 , see before) by

$$\mathcal{A}_3 \equiv \frac{2^{\mu_{\text{MD}}+1}}{\sqrt{\pi}} \Gamma \left(\frac{\mu_{\text{MD}} + 3}{2} + i \frac{\hat{\nu}_{\text{MD}}}{2} \right) \Gamma \left(\frac{\mu_{\text{MD}} + 3}{2} - i \frac{\hat{\nu}_{\text{MD}}}{2} \right). \quad (\text{B36})$$

As before, if $T_{\text{osci}} < T_{\text{rh}}$, one should rather use a prefactor \mathcal{A}_4 defined as \mathcal{A}_3 above, but with all μ_{MD} and ν_{MD} replaced by their values for the radiation dominated era, μ_{RD} and ν_{RD} . Finally, for the homogeneous solution, one introduces the coefficients

$$\mathcal{B}_3 \equiv \hat{\nu}_{\text{MD}}^{1/2} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2 \right]^{1/2}, \quad (\text{B37})$$

$$\mathcal{B}_4 \equiv \frac{1}{\sqrt{\hat{\nu}_{\text{RD}}}} \left[1 + \left(\frac{\mu_{\text{MD}} + 1}{\hat{\nu}_{\text{MD}}} \right)^2 \right]^{1/2} \{-\}. \quad (\text{B38})$$

The above expressions (B34) and (B35) reduce exactly to the expressions used in the main text, namely, Eqs. (53) and (54).

2. Case $-c^2 H^2/2$

As explained in the main text, the potential can be written under the approximate form:

$$V(\sigma) \simeq \frac{1}{2} \tilde{c}^2 H^2 (\sigma - \sigma_n)^2 + \frac{\lambda_n}{(n+4)!} \frac{(\sigma - \sigma_n)^{4+n}}{M_{\text{Pl}}^n}, \quad (\text{B39})$$

with

$$\sigma_n = \left[\frac{(n+3)!}{\lambda_n} c^2 H^2 M_{\text{Pl}}^n \right]^{1/(n+2)}, \quad \tilde{c}^2 = (n+2)c^2. \quad (\text{B40})$$

The vev σ_n indicates the (time-dependent) local minimum of the effective potential.

As discussed in the main text, the potential can be further approximated by $V(\sigma) \simeq \lambda_n \sigma^{n+4} / [(n+4)! M_{\text{Pl}}^n]$ in the limit $\sigma \gg \sigma_n$. Then one can show, as follows, that the field will evolve to values of order σ_n in a fraction of an e-fold of order $(\sigma/\sigma_n)^{-(n+2)/2}$. In the following, we write $\sigma_{n,\text{inf}}$ the value $\sigma_n(H = H_{\text{inf}})$. Consider the field to be initially at rest, $\dot{\sigma}_{\text{inf}} = 0$. Then, neglecting the damping term $H\dot{\sigma}$ in the equation of motion, one can rewrite this latter equation as

$$H^{-2} \ddot{\sigma} \Big|_{H=H_{\text{inf}}} \simeq -c^2 \left(\frac{\sigma_{\text{inf}}}{\sigma_{n,\text{inf}}} \right)^{n+2}, \quad (\text{B41})$$

which shows that the field initially evolves on the e-folding scale announced above. As soon as the field starts to evolve, it converts its potential energy into kinetic energy; since this occurs in a fraction of an e-fold, expansion damping can be ignored and the conversion is nearly fully efficient: $\rho_{\sigma,\text{kin}} \simeq V(\sigma_{\text{inf}}) - V(\sigma)$. Once the kinetic energy dominates over the potential energy, $\rho_{\sigma,\text{kin}} \propto a^{-6}$, and one finds the approximate solution:

$$\sigma \simeq \sigma_{\text{inf}} - \frac{2\sqrt{2V(\sigma_{\text{inf}})}}{3H_{\text{inf}}} \left(1 - \frac{H}{H_{\text{inf}}} \right), \quad (\text{B42})$$

assuming matter domination after inflation. The factor

$\sqrt{2V(\sigma_{\text{inf}})}$ is introduced to approximate the value $\dot{\sigma}$ at the time at which the kinetic energy of the modulus dominates over the potential energy. The ratio $\sqrt{V(\sigma_{\text{inf}})}/H_{\text{inf}} \approx \sigma_{\text{inf}}(\sigma_{\text{inf}}/\sigma_{n,\text{inf}})^{(n+2)/2}$, therefore the above solution shows that σ evolves down to σ_n on the aforementioned e-folding scale.

When $\sigma \ll \sigma_n$ (and $H \gg m_\sigma/c$), one can approximate the potential (B39) with the low order term $+\tilde{c}^2 H^2 (\sigma - \sigma_n)^2$. In this case, the solution of the equation of motion for σ is the sum of a particular and an homogeneous solution, namely,

$$\sigma(t) = \alpha_n \sigma_n(t) + \sigma_{\text{hom}}(t), \quad (\text{B43})$$

with the definition

$$\alpha_n \equiv \left[\frac{4}{(n+2)^2} + \frac{2}{n+2} - \frac{6p}{n+2} + p^2 \tilde{c}^2 \right]^{-1} p^2 \tilde{c}^2. \quad (\text{B44})$$

The homogeneous solutions can be expressed as follows. Reintroducing the notation: $\nu^2 = (\mu+1)^2 - p^2 \tilde{c}^2$, the solutions read differently according to whether ν is real or imaginary. Let us start with the case ν real. In this case, the solution normalized to negligible initial kinetic energy and initial value of the modulus σ_1 reads [42]

$$\sigma_{\text{hom}}(t) = \sigma_1 \left(\frac{H}{H_{\text{inf}}} \right)^{\mu+1} \left[\frac{1}{2} \left(1 + \frac{\mu+1}{\nu} \right) \left(\frac{H}{H_{\text{inf}}} \right)^{-\nu} + \frac{1}{2} \left(1 - \frac{\mu+1}{\nu} \right) \left(\frac{H}{H_{\text{inf}}} \right)^{\nu} \right]. \quad (\text{B45})$$

The time dependence is contained in the dependence of $H(t)$. Of course σ_1 must be chosen so that the sum of $\sigma_{\text{hom}} + \sigma_{\text{part}}$ takes the right value at the end of inflation. The question of the initial conditions in this case is discussed in more detail in the main text (see Sec. III B 2). Depending on the value of n , either the particular solution or the homogeneous solutions may dominate at late times. In general, however, one should expect the homogeneous solution to decay less rapidly, since this corresponds to: $n < 2(-\mu + \nu)/(\mu + 1 - \nu)$, and during matter domination, $\mu = -1/2$.

Let us now study the case where ν is imaginary. As usual, we write $\nu = i\hat{\nu}$ and, then, the solution reads [42]

$$\sigma_{\text{hom}}(t) = \sigma_1 \left(\frac{H}{H_{\text{inf}}} \right)^{\mu+1} \left\{ \cos \left[\hat{\nu} \log \left(\frac{H}{H_{\text{inf}}} \right) \right] - \frac{\mu+1}{\hat{\nu}} \sin \left[\hat{\nu} \log \left(\frac{H}{H_{\text{inf}}} \right) \right] \right\}. \quad (\text{B46})$$

The homogeneous solution scales less rapidly with time than the particular solution if $n < -2\mu/(\mu+1)$.

APPENDIX C: SOME CONCRETE MODELS OF MODULI EVOLUTION

In this appendix, we present specific supergravity examples of inflationary scenarios which give different predictions of modulus evolution. Since the moduli fields are scalar fields with a flat potential which eventually get a supersymmetry-breaking mass of order of the gravitino mass $m_{3/2}$, the simplest example is the supersymmetry-breaking field or the Polonyi field itself (denoted S in what follows). Therefore, we consider a system consisting of an inflaton sector and a separate Polonyi sector. In the rest of this appendix, we assume that the Polonyi sector is characterized by a minimal Kähler potential, $K_S = |S|^2$, and a superpotential given by $W_S \equiv M_S^2(\beta + S)$. Here β is a constant chosen as $\beta = (2 - \sqrt{3})M_{\text{Pl}}$ so that the potential energy vanishes at its minimum $S_{\text{min}} = (\sqrt{3} - 1)M_{\text{Pl}}$ when the inflaton sector is absent. At the potential minimum the gravitino mass reads $m_{3/2}M_{\text{Pl}} = \langle e^{K_{S_{\text{min}}}/2} W_{S_{\text{min}}} \rangle = M_S^2 e^{2-\sqrt{3}}$. Below we mostly take $M_{\text{Pl}} = 1$ but recover it when appropriate.

1. Small-field inflation models induced by the F-term

First we consider the case inflation is driven by an F-term according to a small-field scenario. We denote the fields in the inflaton sector by I collectively although this sector usually contains two or more fields. We assume a minimal Kähler potential K_I and a superpotential W_I that we do not specify in details. In the following, we define the F-term scalar potential of each sector by

$$V_j^{(F)} = e^{K_j} [|D_j W_j|^2 - 3|W_j|^2], \quad (\text{C1})$$

where $D_j W_j \equiv \partial_j W_j + \partial_j K W_j$. Our fundamental assumption is that the inflaton and modulus sectors are separated. Technically, this means $K = K_I + K_S$ and $W = W_I + W_S$. Then, the total potential reads

$$\begin{aligned} V_{\text{tot}}^{(F)} &= e^K [|D_I W|^2 + |D_S W|^2 - 3|W|^2] \\ &= e^{|S|^2} V_I^{(F)}(I) + e^{K_I} V_S^{(F)}(S) + V_{\text{int}}^{(F)}(I, S), \end{aligned} \quad (\text{C2})$$

where the interaction potential is given by

$$\begin{aligned} V_{\text{int}}^{(F)} &= e^{|S|^2 + K_I} [D_I W_I (\partial_I K_I W_S)^\dagger + (D_I W_I)^\dagger \partial_I K_I W_S \\ &\quad + |\partial_I K_I W_S|^2 + D_S W_S S W_I^\dagger + (D_S W_S)^\dagger S^\dagger W_I \\ &\quad + |S|^2 |W_I|^2 - 3W_I^\dagger W_S - 3W_I W_S^\dagger]. \end{aligned} \quad (\text{C3})$$

Note that the energy scales of V_I and V_S are given by $V_I(I) \approx 3H_I^2 M_{\text{Pl}}^2$ and $V_S(S) = M_S^4 \approx m_{3/2}^2 M_{\text{Pl}}^2$ where H_I is the Hubble parameter during inflation. Let us also remark that the structure of Eq. (C2) is in fact typical of supergravity [81]: although the two sectors are separated, there is an interaction between them because they ‘‘communicate’’ through gravity. In the minimal supersymmetric standard model, this is a possible way to transmit the

breaking of supersymmetry to the observable sector [81]. The same mechanism has also been used to argue that the quintessence and inflaton fields must interact [82] or to compute the interaction of dark energy with the observable sector (see Ref. [83]).

Since we are assuming a small field inflation here one can easily see that $|D_I W_I| > |W_I|$, and $V_I \approx |D_I W_I|^2 \approx 3H_I^2 M_{\text{Pl}}^2$. In this case the modulus acquires a mass $\sim V_I/M_{\text{Pl}}^2 \approx 3H_I^2$ during inflation. After inflation, $|I|$ takes a value $\sim M_{\text{Pl}}$ and oscillates around the potential minimum $I \equiv I_{\text{min}}$ with $V_I(I_{\text{min}}) = 0$, so that W_I and $D_I W_I$ are of the same order of magnitude on average. If the field oscillation is dominated by a quadratic potential, we find the time average of the potential energy $\overline{V_I(I)} = \rho_{\text{tot}}(t)/2 = 3H^2(t)M_{\text{Pl}}^2/2$, so that we can parametrize $\overline{D_I W_I} \equiv c_1 H M_{\text{Pl}}$, $\overline{W_I} \equiv c_2 H M_{\text{Pl}}^2$ with c_1 and c_2 parameters of order of unity. If we take $K_I = |I - I_{\text{min}}|^2$, the effective potential for S averaged over an inflaton oscillation period reads

$$\begin{aligned} \overline{V}_{\text{tot}}(S, I_{\text{min}}) &= \left[\left(\frac{3}{2} + c_2^2 \right) H^2 - f_1 m_{3/2} H \right] |S \\ &\quad - \frac{f_2 m_{3/2} M_{\text{Pl}} H}{(3/2 + c_2^2) H^2 - f_1 m_{3/2} H} \Big|^2 \\ &\quad - \frac{|f_2|^2 m_{3/2}^2 M_{\text{Pl}}^2 H^2}{(3/2 + c_2^2) H^2 - f_1 m_{3/2} H} + V_S(S) + \dots \end{aligned} \quad (\text{C4})$$

Here f_1 and f_2 are numerical coefficients defined by $f_1 = 2\beta(c_2 + c_2^*)e^{\sqrt{3}-2}$ and $f_2 = 2c_2 e^{\sqrt{3}-2}$, respectively. Thus in the early field oscillation stage when H is larger than $m_{3/2}$, the modulus has a mass larger than H and its minimum is located at

$$S_{\text{min}} \approx \frac{f_2 m_{3/2} M_{\text{Pl}} H}{(3/2 + c_2^2) H^2 - f_1 m_{3/2} H}. \quad (\text{C5})$$

As H decreases, it eventually settles down to the absolute minimum $S_{\text{min}} = (\sqrt{3} - 1)M_{\text{Pl}}$ with a mass $\sqrt{2}m_{3/2}$ as determined by $V_S(S)$. Hence this model can be regarded as an example of models where the modulus acquires a mass larger than the Hubble parameter during both inflation and the subsequent field oscillation regimes with a shift of the minimum of order of M_{Pl} typically.

2. D-term hybrid inflation

Next we consider the case in which inflation is induced by the D-term potential. As the simplest realization we consider a hybrid inflation model originally proposed in Refs. [84,85]. Although this model is not observationally viable any more due to the largeness of the cosmic string tension produced after inflation, a number of remedies have been proposed e.g. in Ref. [86]. In order to avoid inessential complexity, we stick to the original model here to adopt the superpotential with three chiral superfields

$W_I = \lambda Q \phi_+ \phi_-$ together with $W_S = M_S^2(\beta + S)$ as in the previous subsection. ϕ_{\pm} has a U(1) gauge charge ± 1 , while Q is neutral. Assuming the minimal Kähler potential, $K = |Q|^2 + |\phi_+|^2 + |\phi_-|^2 + |S|^2$, the scalar potential is given by $V_{\text{tot}} = V_{\text{tot}}^{(F)} + V^{(D)}$, with

$$\begin{aligned} V_{\text{tot}}^{(F)} &= e^K [\lambda^2 (|Q\phi_+|^2 + |Q\phi_-|^2 + |\phi_+\phi_-|^2 \\ &\quad + 3|Q\phi_+\phi_-|^2) + (|\phi_+|^2 + |\phi_-|^2 + |S|^2 + |Q|^2) \\ &\quad \times |\lambda Q\phi_+\phi_- + M_S^2(\beta + S)|^2 - 3M_S^4\beta^2 + M_S^4 \\ &\quad - 2M_S^4\beta(S + S^\dagger) - M_S^4|S|^2 + \lambda M_S^2 S Q^\dagger \phi_+^\dagger \phi_-^\dagger \\ &\quad + \lambda M_S^2 S^\dagger Q \phi_+ \phi_-], \\ V^{(D)} &= \frac{g^2}{2} (|\phi_-|^2 - |\phi_+|^2 - \xi)^2, \end{aligned} \quad (\text{C6})$$

where the last term is the D-term contribution with $\xi > 0$ being the Fayet-Illiopoulos term. For $|Q| > g\sqrt{\xi}/\lambda \equiv Q_{\text{cri}}$, the potential minimum with respect to ϕ_{\pm} is found at $\phi_{\pm} = 0$ and inflation is induced by the D-term energy density $g^2\xi^2/2$. Without W_S the F-term potential vanishes in this regime and the motion of the inflaton is governed by a potential generated by quantum corrections. Then terms of the potential relevant to S are given by

$$\begin{aligned} V_{\text{tot}}^{(F)}(S, Q, \phi_{\pm} = 0) &= M_S^4 e^{|\phi_-|^2 + |Q|^2} [(|S|^2 + |Q|^2)|\beta + S|^2 \\ &\quad - 3\beta^2 - 2\beta(S + S^\dagger) - |S|^2 + 1], \end{aligned} \quad (\text{C7})$$

so that the modulus acquires an extra mass squared,

$$\delta m_{\text{Seff}}^2 \sim \frac{M_S^4}{M_{\text{Pl}}^4} |Q|^2 \sim \frac{|Q|^2}{M_{\text{Pl}}^2} m_{3/2}^2, \quad (\text{C8})$$

which is smaller than the original one as long as $|Q|$ is smaller than M_{Pl} . The shift of the potential minimum of S is of order of $|Q|^2/M_{\text{Pl}}$. Thus in this model the modulus mass remains much smaller than the Hubble parameter during inflation, and the shift of the minimum may also be much smaller than M_{Pl} depending on the value of Q . Note, however, that the field configuration of S is determined by long-wave quantum fluctuations in this case.

As Q becomes smaller than Q_{cri} the instability occurs with respect to ϕ_- and inflation is terminated. In this regime the potential for S acquires an additional term, $\delta V_S(S) = e^{|\phi_-|^2 + \dots} \lambda^2 |Q\phi_-|^2$. This term induces a correction to the modulus mass term

$$\delta m_{\text{Seff}}^2 = \frac{\lambda^2 |Q|^2 |\phi_-|^2}{M_{\text{Pl}}^2} = \frac{\lambda^2 |Q|^2}{M_{\text{Pl}}^2} \left(\xi - \frac{\lambda^2}{g^2} |Q|^2 \right), \quad (\text{C9})$$

where we have inserted the minimum of $|\phi_-|^2$ for $Q < Q_{\text{cri}}$. The last expression takes its maximum at $|Q|^2 = |Q_{\text{cri}}|^2/2$ and is given by $\delta m_{\text{Seff}}^2 = g^2\xi^2/4$. This means that after inflation the effective mass of the Polonyi field rises significantly to the level comparable to the Hubble parameter at that time. When this induced mass term is

operative, the potential minimum for $|S|$ takes a value $\mathcal{O}(m_{3/2}^2/H^2)M_{\text{Pl}}$ which departs from the eventual minimum by $\sim M_{\text{Pl}}$.

3. Hybrid inflation with both F- and D-term contributions

Next we consider a variant of D-term hybrid inflation model with a nonvanishing F-term potential as an example of models where the modulus mass remains much smaller than the Hubble parameter during inflation but it acquires a large correction of order H just after inflation. Specifically we consider sneutrino hybrid inflation model proposed in Ref. [87] with the minimal Kähler potential. The effects of nonminimal Kähler potential are discussed, for example, in Ref. [88] for the sneutrino inflation and in Ref. [89] for D-term inflation.

The inflaton sector of the model consists of three species of chiral superfields, N_i^c containing (s)neutrinos ($i = 1, 2, 3$), and ϕ_{\pm} with U(1) gauge charge ± 1 as in the above model. The relevant part of the superpotential reads

$$W_I = \frac{\lambda}{M_{\text{Pl}}} N_i^c N_i^c \phi_- \phi_+ + \frac{1}{2} M_i N_i^c N_i^c, \quad (\text{C10})$$

besides the interaction between N_i^c and lepton and Higgs fields. One can always choose a basis for N_i^c so that their mass matrix is diagonal, and we take N_i^c to be Majorana mass eigenstate fields with real mass M_i . We assume, without loss of generality, the inflaton sneutrino is the lightest heavy sneutrino $M_1 \ll M_2, M_3$ and we are interested in the lower range of the preferred values of heavy neutrino masses $M_i = 10^{10} \sim 10^{15}$ GeV. D-term contribution to the scalar potential with nonvanishing Fayet-Illiopoulos term $\xi > 0$ is given by $V_D(\phi_+, \phi_-)$ in Eq. (C6) as before.

The large amplitude of inflaton sneutrino gives a large effective mass to the slepton and the Higgs field H_u and they stay at the origin and do not affect the inflation dynamics. ϕ_+ has a positive mass during and after inflation and it stays at the origin all the time. For the discussion of inflationary dynamics, therefore, we discuss the evolution of \tilde{N}_1 and ϕ_- . The tree-level potential for the inflaton sector is then given by

$$\begin{aligned} V_I^{(F)} + V_I^{(D)} &= e^{K_I} \left[M^2 |\tilde{N}_1|^2 + \frac{1}{4} M^2 |\tilde{N}_1|^4 + \frac{1}{4} M^2 |\tilde{N}_1|^6 \right. \\ &\quad \left. + \left(\frac{\lambda}{M_{\text{Pl}}} \right)^2 |\tilde{N}_1|^4 |\phi_-|^2 + \frac{1}{4} M^2 |\tilde{N}_1|^4 |\phi_-|^2 \right] \\ &\quad + \frac{g^2}{2} (\xi - |\phi_-|^2)^2. \end{aligned} \quad (\text{C11})$$

We expect $\lambda \gg M/M_{\text{Pl}}$, and therefore find $\phi_- = 0$ for $|\tilde{N}_1|^4 > g^2\xi M_{\text{Pl}}^4/\lambda^2 \equiv \tilde{N}_{\text{cri}}^4$ during inflation. ϕ_- destabilizes for $\tilde{N}_1 < \tilde{N}_{\text{cri}}$ to reach its minimum at $\phi_- = \sqrt{\xi}$ after inflation. We find $\tilde{N}_{\text{cri}} \lesssim 2M_{\text{Pl}}$ for cosmologically relevant parameter values [87].

The correction to the Polonyi mass during inflation can be found from the F-term potential with $\phi_{\pm} = 0$. The interaction term reads

$$\begin{aligned}
 V_{\text{int}}(I, S) = & M_S^2 e^{K_I + |S|^2} \left\{ \left(M \tilde{N}_1 + \frac{1}{2} M |\tilde{N}_1|^2 \tilde{N}_1 \right) \tilde{N}_1 (\beta + S^\dagger) \right. \\
 & + \left(M \tilde{N}_1^\dagger + \frac{1}{2} M |\tilde{N}_1|^2 \tilde{N}_1^\dagger \right) \tilde{N}_1^\dagger (\beta + S) \\
 & + M_S^2 |\tilde{N}_1^\dagger (\beta + S)|^2 + [1 + S^\dagger (\beta + S)] \\
 & \times S \frac{1}{2} M (\tilde{N}_1^\dagger)^2 + [1 + S(\beta + S^\dagger)] S^\dagger \frac{1}{2} M \tilde{N}_1^2 \\
 & \left. - \frac{3}{2} M (\tilde{N}_1^\dagger)^2 (\beta + S) - \frac{3}{2} M \tilde{N}_1^2 (\beta + S^\dagger) \right\} \\
 & + e^{K_I + |S|^2} |S|^2 \frac{1}{2} M \tilde{N}_1^2 |^2. \quad (\text{C12})
 \end{aligned}$$

For $\tilde{N}_1 \sim M_{\text{Pl}}$ we find the last term has the largest contribution to the effective modulus mass, $\mathcal{O}(M)$. This correction, however, is still much smaller than the Hubble parameter during inflation thanks to the assumption that inflation is driven by the D-term potential energy which ensures that

$$V_I^{(D)} = \frac{g^2}{2} \xi^2 \gg V_I^{(F)} \supset M^2 |\tilde{N}_1|^2. \quad (\text{C13})$$

As \tilde{N}_1 gets smaller than \tilde{N}_{cri} , a phase transition occurs to make ϕ_- nonvanishing and inflation is terminated. Its minimum is located at

$$|\phi_-|^2 \approx \xi - \left(\frac{\lambda}{g M_{\text{Pl}}} \right)^2 |\tilde{N}_1|^4. \quad (\text{C14})$$

We therefore find

$$\begin{aligned}
 e^{|S|^2} V_I^{(F)} \supset & e^{|S|^2 + K_I} \left(\frac{\lambda}{M_{\text{Pl}}} \right)^2 |\tilde{N}_1|^4 |\phi_-|^2 \\
 = & e^{|S|^2 + K_I} \left(\frac{\lambda}{M_{\text{Pl}}} \right)^2 |\tilde{N}_1|^4 \left[\xi - \left(\frac{\lambda}{g M_{\text{Pl}}} \right)^2 |\tilde{N}_1|^4 \right] \\
 \leq & e^{|S|^2 + K_I} \frac{1}{4} g^2 \xi^2, \quad (\text{C15})
 \end{aligned}$$

where the equality holds at $|\tilde{N}_1|^4 = |\tilde{N}_{\text{cri}}|^4/2$ in the last inequality. Thus one can see that in the early field oscillation regime after inflation, the modulus acquires an effective mass of order of the Hubble parameter during inflation.

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