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**Geometrically consistent approach to stochastic DBI inflation**Larissa Lorenz<sup>\*</sup>*Theoretical and Mathematical Physics Group, Centre for Particle Physics and Phenomenology, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium*Jérôme Martin<sup>†</sup>*Institut d'Astrophysique de Paris, UMR 7095-CNRS, Université Pierre et Marie Curie, 98bis boulevard Arago, 75014 Paris, France*Jun'ichi Yokoyama<sup>‡</sup>*Research Center for the Early Universe, Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan  
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Stochastic effects during inflation can be addressed by averaging the quantum inflaton field over Hubble-patch-sized domains. The averaged field then obeys a Langevin-type equation into which short-scale fluctuations enter as a noise term. We solve the Langevin equation for an inflaton field with a Dirac-Born-Infeld (DBI) kinetic term perturbatively in the noise and use the result to determine the field value's probability density function (PDF). In this calculation, both the shape of the potential and the warp factor are arbitrary functions, and the PDF is obtained with and without volume effects due to the finite size of the averaging domain. DBI kinetic terms typically arise in string-inspired inflationary scenarios in which the scalar field is associated with some distance within the (compact) extra dimensions. The inflaton's accessible range of field values therefore is limited because of the extra dimensions' finite size. We argue that in a consistent stochastic approach the inflaton's PDF must vanish for geometrically forbidden field values. We propose to implement these extra-dimensional spatial restrictions into the PDF by installing absorbing (or reflecting) walls at the respective boundaries in field space. As a toy model, we consider a DBI inflaton between two absorbing walls and use the method of images to determine its most general PDF. The resulting PDF is studied in detail for the example of a quartic warp factor and a chaotic inflaton potential. The presence of the walls is shown to affect the inflaton trajectory for a given set of parameters.

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**I. INTRODUCTION**

Putting the successful inflationary scenario on the firm footing of a fundamental theory is one of the remaining challenges in cosmology. Recent years have seen considerable progress towards this goal with the construction of several concrete string inflation models; for recent reviews, see, e.g., [1–4], and references therein. A top-level distinction among these models is the either closed or open string mode character of the inflaton field. Typical examples of the second class are brane inflation scenarios [5–7]: The inflaton field  $\phi$  corresponds (up to renormalization) to the distance between two branes embedded in a higher-dimensional background. While there is an ongoing debate about the form of the inflationary potential  $V(\phi)$  [7–13], a generic feature of these models is the field's kinetic term, which is of Dirac-Born-Infeld (DBI) type rather than canonic [14,15]. One can understand these DBI dynamics as a geometry-imposed upper limit on the field's velocity  $\dot{\phi}$ , characterized by the so-called warp factor  $T(\phi)$ . This

relativistic speed limit acts like a brake on the inflaton, forcing  $\phi$  to “slow roll” even in regions where the potential is not flat. Hence, open string mode inflaton models provide an additional mechanism to generate quasiexponential expansion.

It is interesting to investigate the DBI analogues of standard inflationary calculations, such as the field perturbations' evolution and spectra [16,17], but also the effects of stochastic inflation [18–27]. Stochastic inflation provides a technique to assess quantum effects on the inflaton  $\phi$ 's trajectory averaged over a scale beyond the Hubble patch. In this way, one can define a coarse-grained field  $\varphi$ , which consists exclusively of the large-scale Fourier components. To zeroth order,  $\varphi$  obeys the slow-roll Klein-Gordon equation, but its full evolution is subject to stochastic noise  $\xi$  from small-scale Fourier modes. For standard inflation with arbitrary potentials, this equation was solved perturbatively up to  $\mathcal{O}(\xi^2)$  in Ref. [20]. By using this perturbative solution, Ref. [20] showed how to obtain the probability density function (PDF) for  $\langle\varphi\rangle$  in the Gaussian approximation, with and without the volume effects (i.e., the size of each averaging domain) taken into account. The reliability of this treatment was further studied in Ref. [21]. In this paper, we generalize both of

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these results to the case of DBI inflation with arbitrary warp factors and potentials.

The Langevin equation for DBI models was given for the first time in Ref. [28], which aimed at studying “eternal inflation” [29,30] in the brane inflation context. Since then, stochastic effects in DBI (or general  $k$ -inflation) models have been studied by several authors [31,32]. In this paper, we use the approach of Refs. [20,21] to calculate the PDFs of a DBI inflaton field (with and without volume effects) for an arbitrary functional form of the potential and warp factor. We illustrate our results by applying them to the example model of “chaotic Klebanov-Strassler (CKS) inflation,” where  $T(\phi)$  and  $V(\phi)$  are known functions. The integrals encountered are, within certain limits, exactly calculable, and we discuss the behavior and reliability of the resulting DBI inflaton probability densities.

However, we argue that all PDFs used so far in the literature suffer from a serious problem: They predict a nonvanishing probability for the moving  $D3$  brane to find itself outside the so-called “Klebanov-Strassler (KS) throat,” i.e., outside the (part of the) extra-dimensional geometry with warp factor  $T(\phi)$ . In fact, this problem is twofold. First, at the bottom end of the throat this means that there is a nonvanishing probability to literally find the brane “outside” the extra dimensions, in other words, “out of space,” which is clearly meaningless. Second, since the metric of the 6D bulk space [and hence the continuation of  $T(\phi)$ ] is typically unknown beyond the KS region, a string-inflationary scenario based on the brane’s motion inside the throat becomes inconsistent beyond its top end. Note that, while the latter question is of a more technical nature and we may hope to resolve it as our understanding of string geometries improves, the former issue is rather severe as the inflaton’s PDF does not respect the fundamentally geometric origin of the scenario.

Put a different way, the compact character of the stringy extra dimensions (for the purposes of concrete model building, this means the well-known KS “corner” of the 6D manifold) can be translated directly into a restricted field range for the DBI inflaton. For consistency, any modifications of the classical trajectory induced by stochastic effects should still respect these geometry-imposed boundaries in field space. Hence, studies of stochastic DBI inflation so far were missing a tool to ensure the consequences of a stringy inflaton’s geometric interpretation at the effective field theory level.

To amend this problem, we propose to install “walls” at the boundaries (i.e., the bottom  $r_0$  and the edge  $r_{UV}$ ) of the inflationary KS throat (whose radial coordinate is denoted by  $r$ ). As a consequence, the stochastically corrected inflaton field value should remain within its allowed range  $\phi_0 < \phi < \phi_{UV}$  at all times. This requires the calculation of a new PDF respecting the boundary conditions imposed by the presence of the reflecting or absorbing walls. For the first time, we then determine this PDF in the presence of

two absorbing walls at  $\phi_0$  and  $\phi_{UV}$ , using the method of images; see, e.g., [33,34]. As a by-product, we obtain the modified stochastic trajectory of the mobile brane within the KS throat and show that, in some cases, the presence of the walls has a significant effect.

This paper is organized as follows. In the next section, we start from the underlying background equations of DBI inflation and discuss how they can be used to formulate their stochastic counterpart, i.e., the DBI Langevin equation. We pay special attention to the normalization factor of the noise term. We then solve the DBI Langevin equation up to second order in the noise and calculate the corresponding PDF along with its volume correction. In Sec. III, we use these expressions to calculate the stochastic effects in the case of chaotic Klebanov-Strassler inflation, for a potential both with and without a constant term. Where applicable, we calculate the domain of validity of the perturbative approach and consider the existence of a regime of eternal inflation [29,30]. In Sec. IV, we use the additional information on the inflaton’s geometric bounds as an argument to implement two absorbing walls into the calculation of the PDF. In this way, it can be assured that even quantum effects do not violate these field space limits. Finally, we summarize our main findings in Sec. V, commenting on eternal inflation as well as on an overall picture for a realistic brane trajectory across the entire compact 6D geometry.

## II. DBI LANGEVIN EQUATION

### A. DBI background equations

As a first step, we quickly recall the basic equations of DBI inflation at the classical level. Since models of this kind descend from (e.g., type IIB) string theory, they are originally represented by a higher-dimensional ( $d = 10$ ) action for the stringy background and the embedded branes. After compactification to four dimensions, the effective field theory typically contains a gravity sector described by general relativity and a four-dimensional inflaton field  $\phi(x, t)$  corresponding to the interbrane distance along one of the compactified dimensions. The model’s effective 4D action therefore reads

$$S = - \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + V(\phi) - T(\phi) + T(\phi) \times \sqrt{1 + \frac{1}{T(\phi)} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} \right], \quad (1)$$

where  $R$  is the four-dimensional scalar curvature and  $\kappa = 8\pi/m_{\text{pl}}^2$ ,  $m_{\text{pl}}$  being the Planck mass. It is useful to introduce (making use of the analogy with special relativity) the so-called “Lorentz factor”  $\gamma(\phi, \partial_\mu \phi)$ , defined as [15]

$$\gamma(\phi, \partial_\mu \phi) = \frac{1}{\sqrt{1 + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / T(\phi)}}. \quad (2)$$

Roughly speaking, this Lorentz factor measures how close the inflaton field's velocity is to the geometry-imposed speed limit  $\sqrt{T(\phi)}$  (see below). In terms of  $\gamma$  (the arguments of which we frequently suppress below), one can rewrite Eq. (1) in the simple form

$$S = - \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + V(\phi) - \frac{\gamma - 1}{\gamma} T(\phi) \right]. \quad (3)$$

Specifying to a Friedmann-Lemaître-Robertson-Walker (FLRW) universe with homogeneous scalar field matter, the Friedmann and Klein-Gordon equations read

$$H^2 = \frac{\kappa}{3} [(\gamma - 1)T(\phi) + V(\phi)], \quad (4)$$

$$- \frac{V'(\phi)}{\gamma^3} = \ddot{\phi} + \frac{3H}{\gamma^2} \dot{\phi} + \frac{3\gamma - \gamma^3 - 2}{2\gamma^3} T'(\phi), \quad (5)$$

where a prime denotes a derivative with respect to the inflaton field  $\phi$ . Note that in the FLRW case,  $\gamma$  from Eq. (2) simplifies to

$$\gamma(\phi, \dot{\phi}) = \frac{1}{\sqrt{1 - \dot{\phi}^2/T(\phi)}}, \quad (6)$$

where the notion of  $\sqrt{T(\phi)}$  as a speed limit is evident. By using the square root's expansion in Eq. (6), it is clear from Eq. (5) that, while  $\gamma \approx 1$ , the inflaton is close to standard dynamics. In this regime,  $\phi$  obeys the usual Klein-Gordon equation and hence can slow roll only if the potential is sufficiently flat. By using the full expression (6), however, one can see that the inflaton velocity can never exceed  $\sqrt{T(\phi)}$  even if the potential is steep. The limit where  $\dot{\phi} \rightarrow \sqrt{T(\phi)}$  (and hence  $\gamma \rightarrow \infty$ ) can therefore be thought of as an additional, "ultrarelativistic" regime of inflation. Let us make this statement more precise by taking the time derivative of the DBI Friedmann equation (4): The DBI condition to maintain accelerated expansion reads

$$\frac{\ddot{a}}{a} = \frac{\kappa}{3} V(\phi) - \frac{\kappa}{6} \frac{(\gamma - 1)(\gamma + 3)}{\gamma} T(\phi) > 0. \quad (7)$$

From this expression it is evident that the potential  $V(\phi)$  still has to dominate the energy density.

Our next step is to notice that from combining Eqs. (4) and (5) one obtains

$$\dot{\phi} = - \frac{2H'}{\kappa\gamma}. \quad (8)$$

This formula has two important consequences. First, by replacing  $\dot{\phi}$  by this expression in Eq. (6), the Lorentz factor  $\gamma$  is easily expressed as a function of  $\phi$  only [28]:

$$\gamma(\phi) = \sqrt{1 + \frac{4H'^2}{\kappa^2 T(\phi)}}. \quad (9)$$

Second, by using the DBI slow-roll condition  $H^2 \approx$

$\kappa V(\phi)/3$ , which is derived and justified in detail in Appendix A, one obtains the following first-order differential equation for  $\phi$ :

$$\dot{\phi} \approx - \frac{V'(\phi)}{3\gamma(\phi)H(\phi)}. \quad (10)$$

Obviously, except for the factor  $\gamma$  appearing in the denominator, this is the standard Klein-Gordon equation in the slow-roll limit. A detailed derivation of Eq. (10) is given in Appendix A.

## B. Stochastic DBI inflation

We now proceed to applying the stochastic approach of Ref. [20] to DBI inflation. In a first step, the "classical" inflaton field  $\phi(t)$  is replaced by the coarse-grained field  $\varphi(t)$ , which is a stochastic process. Based on the previous considerations [see Eq. (8)], we expect  $\varphi$  to obey a Langevin equation of the form

$$\dot{\varphi} = - \frac{2}{\kappa} \frac{H'}{\gamma} + \mathcal{C}\xi(t), \quad (11)$$

where  $\xi(t)$  is a noise term, describing the short wavelength part of the full inflaton field, and obeying the following properties:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'). \quad (12)$$

To proceed from here, the crucial question is how to determine the normalization factor  $\mathcal{C}$ .

In standard (non-DBI) inflation, this question may be resolved in several ways. One possible route to follow is to normalize the prefactor  $\mathcal{C}$  according to the two-point correlation function of a massless test field in a de Sitter background. Indeed, in this case, the above Langevin equation (11) (setting  $\gamma = 1$ ) reduces to  $\dot{\varphi} = \mathcal{C}\xi(t)$ , with its trivial solution  $\varphi(t) = \mathcal{C} \int_{t_{\text{in}}}^t \xi(\tau) d\tau$ . By using Eqs. (12), one finds

$$\langle \varphi^2(t) \rangle = \mathcal{C}^2(t - t_{\text{in}}). \quad (13)$$

Since the exact result is known and reads  $\langle \varphi^2(t) \rangle = H^3 t / (4\pi^2)$  (setting  $t_{\text{in}} = 0$ ), we can read off immediately that  $\mathcal{C} = H^{3/2} / (2\pi)$ . Another way to see this is to interpret Eq. (11) as a Brownian motion [29], the field undergoing quantum kicks of amplitude  $H/(2\pi)$  in every Hubble time interval  $H^{-1}$ . This leads to the expression  $\langle \varphi^2(t) \rangle = [H/(2\pi)]^2 n$ , with the number of steps  $n$  given by  $n = t/H^{-1} = Ht$ .

In the case at hand here, i.e., stochastic inflation with a DBI field, we need to reproduce the same considerations, but for the modified Langevin equation (11) that now comprises the Lorentz factor  $\gamma$ . For this purpose, let us remark that the fully covariant formulation of the DBI Klein-Gordon equation derived from the action (1) reads

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\gamma^2}{T}g^{\alpha\beta}g^{\mu\nu}(\nabla_\alpha\nabla_\mu\phi)\nabla_\beta\phi\nabla_\nu\phi - \frac{V'}{\gamma} + \frac{T'}{2\gamma}(\gamma^3 - 3\gamma + 2) = 0. \quad (14)$$

To see that this gives back Eq. (5) for spatially homogeneous DBI field  $\phi = \phi(t)$  in a FLRW universe, use the definition (2) [or (6), respectively] of the Lorentz factor  $\gamma$  and note that

$$g^{\alpha\beta}g^{\mu\nu}(\nabla_\alpha\nabla_\mu\phi)\nabla_\beta\phi\nabla_\nu\phi = \frac{\gamma^2 - 1}{\gamma^2}T\ddot{\phi}. \quad (15)$$

At the perturbed level, by setting

$$\begin{aligned} \phi(\mathbf{x}, t) &= \phi(t) + \delta\phi(\mathbf{x}, t), \\ ds^2 &= -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Phi)\delta_{ij}dx^i dx^j, \end{aligned} \quad (16)$$

one can show through repeated use of the Einstein equations for a DBI scalar field (see Appendix B) that

$$\begin{aligned} \delta\ddot{\phi}_k + 3H(1 + \delta_1)\delta\dot{\phi}_k + H^2\left(\frac{k^2}{a^2H^2\gamma^2} + 2\epsilon_1 - \frac{3}{2}\epsilon_2 - 2\epsilon_1^2 - \frac{\epsilon_2^2}{4} + \frac{5}{2}\epsilon_1\epsilon_2 - \frac{\epsilon_2\epsilon_3}{2} + \frac{3}{2}\delta_1 + \frac{3}{2}\delta_1\epsilon_1 - \delta_1\epsilon_2 + \frac{5}{4}\delta_1^2 + \frac{\delta_1\delta_2}{2}\right)\delta\phi_k &= \frac{H^2z}{a\gamma^{3/2}}\Phi_k(2 - 2\epsilon_1 + \epsilon_2 + 2\delta_1) \end{aligned} \quad (17)$$

in Fourier space with comoving wave number  $k$ . Here, we have used the  $(\epsilon_i, \delta_i)$  parameters defined in Appendix A to write Eq. (17) in a compact form. Note that in the limit where  $\gamma \rightarrow 1$ ,  $\delta_i \rightarrow 0$ , this gives back precisely the same equation as in the standard case (see Appendix B).

Ignoring the metric perturbation  $\Phi_k$  amounts to dropping the last term in Eq. (17). Moreover, in the ‘‘DBI slow-roll’’ regime, all the  $(\epsilon_i, \delta_i)$  parameters are small. Hence, if in addition we neglect all terms of at least linear order in these parameters in Eq. (17), we obtain the limit

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2\gamma^2}\delta\phi_k \approx 0, \quad (18)$$

which again precisely corresponds to the standard equation up to the replacement  $k \rightarrow k/\gamma$ .

Usually, however, the perturbed Klein-Gordon equation is not written as in Eq. (17), but in terms of the Mukhanov-Sasaki variable  $v_k$ , which in the DBI case is defined as the combination

$$v_k = a\gamma^{3/2}\delta\phi_k + z\Phi_k, \quad (19)$$

where  $z = a\gamma\sqrt{\epsilon_1}$ . It can be shown (see Appendix B) by inserting this definition into Eq. (17) that  $v_k$  satisfies the following equation of motion [35]:

$$\frac{d^2v_k}{d\eta^2} + \left(\frac{k^2}{\gamma^2} - \frac{1}{z}\frac{d^2z}{d\eta^2}\right)v_k = 0. \quad (20)$$

Here  $\eta$  denotes the conformal time with which the scale factor is expressed as  $a(\eta) = -1/(H\eta)$  during exponential inflation. In this regime we find  $d^2z/d\eta^2 = 2/\eta^2$ , so that the normalized solution of Eq. (20) reads

$$v_k(\eta) = \frac{1}{\sqrt{2kc_s}}\left(1 - \frac{i}{kc_s\eta}\right)e^{-ikc_s\eta}, \quad (21)$$

where we set  $c_s = 1/\gamma$  (which corresponds to the perturbations’ sound speed) and made the usual adiabatic choice of initial conditions. [Note that, as discussed in Ref. [17], this choice corresponds to somewhat more restrictive conditions for scalar DBI perturbations evolving according to Eq. (20), as it would be the case for their standard counterparts.] Thus  $|\delta\phi_k|^2$  behaves as

$$|\delta\phi_k|^2 = \frac{1}{a^2\gamma^3}|v_k|^2 \rightarrow \frac{H^2}{2k^3} \quad (22)$$

in the long wavelength limit. Note that, unlike in the standard case, in the DBI picture the boundary between long and short wavelength regimes is not given by the Hubble radius but by the ‘‘sound horizon’’  $c_sH^{-1} = (\gamma H)^{-1}$ , as can be seen in the solution (21).

Now, the separation of the scalar field into long and short wavelength components,

$$\begin{aligned} \phi(\vec{x}, t) &= \varphi(t) + \int \frac{d^3k}{(2\pi)^{3/2}}\Theta(k - \varepsilon a\gamma H) \\ &\times [a_k\delta\phi_k(t)e^{-ik\cdot x} + a_k^\dagger\delta\phi_k^*(t)e^{ik\cdot x}], \end{aligned} \quad (23)$$

is precisely the essence of the stochastic inflation approach, where  $\varepsilon$  in Eq. (23) is a small parameter (not to be confused with the first slow-roll parameter) and  $a_k$  and  $a_k^\dagger$  are annihilation and creation operators, respectively. By using this expression we find that the noise term in Eq. (11) can be expressed as

$$\begin{aligned} \mathcal{C}\xi(t) &= \varepsilon a(t)\gamma H^2 \int \frac{d^3k}{(2\pi)^{3/2}}\delta(k - \varepsilon a\gamma H) \\ &\times [a_k\delta\phi_k(t)e^{-ik\cdot x} + a_k^\dagger\delta\phi_k^*(t)e^{ik\cdot x}], \end{aligned} \quad (24)$$

with  $\delta\phi_k$  given by the solution of  $v_k$ ; see Eq. (21). Here we have used the condition that time dependence of  $c_s = 1/\gamma$  must be weak:  $|\dot{c}_s/c_s| \ll H$  (which corresponds to  $\delta_1 \ll 1$ ), which is required to justify our analysis of quantum fluctuations based on the mode function (21). We then find the correlation function of the stochastic noise is given by

$$\begin{aligned} \mathcal{C}^2\langle\xi(t)\xi(t')\rangle &= \varepsilon^2 a^2\gamma^2 H^4 \frac{4\pi k^2}{(2\pi)^3}|\delta\phi_k|^2|_{k=\varepsilon a\gamma H} \\ &\times \frac{1}{\varepsilon a\gamma H^2}\delta(t - t') \\ &= \frac{H^3}{4\pi^2}\delta(t - t'). \end{aligned} \quad (25)$$

Thus we conclude that the analogue of the Langevin equa-

tion including a noise term in the DBI case is

$$\dot{\phi} = -\frac{2}{\kappa} \frac{H'}{\gamma} + \frac{H^{3/2}}{2\pi} \xi(t). \quad (26)$$

One notices that the factor  $\gamma$  appears only in the classical term and not in the normalization of the noise term.

By means of this equation, one can estimate in which regime the quantum effects are important. If the field's behavior is dominated by quantum effects, one can neglect the classical drift in Eq. (26). As already mentioned, by using the properties (12) this leads to  $\langle \varphi^2(t) \rangle = H^3 t / (4\pi^2)$ . However, the typical time scale  $\Delta t$  now is  $1/(H\gamma)$ , since the ‘‘horizon’’ felt by the scalar field (and its perturbations) shrinks by a factor of  $1/\gamma$  compared to the standard case. Therefore, the typical quantum kick undergone by the field in the characteristic time scale is

$$\Delta\phi_{\text{qu}} = \sqrt{\langle \varphi^2(t) \rangle} = \sqrt{\frac{H^3}{4\pi^2} \frac{1}{\gamma H}} = \frac{H}{2\pi\gamma^{1/2}}. \quad (27)$$

On the other hand, if the quantum effects are negligible, then the equation determining the behavior of the field is nothing but the slow-roll equation of motion, and this implies that

$$\Delta\phi_{\text{cl}} = -\frac{V'}{3H\gamma} \Delta t = -\frac{V'}{3H^2\gamma^2}. \quad (28)$$

By setting  $\Delta\phi_{\text{qu}} = \Delta\phi_{\text{cl}}$ , one easily concludes that the corresponding vacuum expectation value (vev) of the field  $\phi_*$  obeys the equation

$$H(\phi_*) = \frac{m_{\text{pl}}^2}{4} \frac{V'(\phi_*)}{V(\phi_*)} \frac{1}{[\gamma(\phi_*)]^{3/2}}. \quad (29)$$

(Again, in the standard case  $\gamma = 1$  and one recovers the usual criterion.) This equation allows us to decide for which values of the inflaton quantum effects play an important rôle.

So far, we have used the cosmic time  $t$  as the time variable in the Langevin equation; see Eq. (26). The choice of the time variable, however, is a subtle issue as different choices may lead to physically inequivalent results. It has recently been advocated in Refs. [23,27] that in many cases it would be more appropriate to use the number of  $e$ -folds,  $N = \ln a$ , as the time variable depending on what quantities we wish to calculate. Indeed, in this case, the results obtained from the stochastic formalism coincide with those derived from perturbative quantum field theory [23,27]. Written in terms of the number of  $e$ -folds, the Langevin equation (26) reads

$$\frac{d\phi}{dN} = -\frac{2}{\kappa\gamma} \frac{H'}{H} + \frac{H}{2\pi} \xi(N), \quad (30)$$

where  $\xi(N)$  is a new stochastic process (which, allowing for slightly slippery notation, we still denote by the same symbol) such that  $\langle \xi(N)\xi(N') \rangle = \delta(N - N')$ . It is easy to

check that a free field satisfies  $\langle \varphi^2(N) \rangle = H^2(N - N_{\text{in}})/(4\pi^2)$  as expected. Below, we will carry out our calculations in both time variables, cosmic time  $t$ , and the number of  $e$ -folds  $N$ .

### C. Solving the DBI Langevin equation

We now solve Eq. (26) by using a perturbative expansion in the noise as shown in Ref. [20]. We use the following ansatz for the Hubble-patch-averaged field:

$$\varphi(t) = \varphi_{\text{cl}}(t) + \delta\varphi_1(t) + \delta\varphi_2(t) + \dots, \quad (31)$$

where  $\varphi_{\text{cl}}(t)$  is the classical field,  $\delta\varphi_1(t) \propto \mathcal{O}(\xi)$ , and  $\delta\varphi_2(t) \propto \mathcal{O}(\xi^2)$ . In principle, this expansion can be carried to any order in  $\xi$ . At zeroth order, we get back the classical slow-roll equation (10). At first order, one obtains an equation for  $\delta\varphi_1(t)$ , namely,

$$\begin{aligned} \frac{d\delta\varphi_1(t)}{dt} + \frac{2}{\kappa} \frac{H'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \left[ \frac{H''(\varphi_{\text{cl}})}{H'(\varphi_{\text{cl}})} - \frac{\gamma'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \right] \delta\varphi_1(t) \\ = \frac{H^{3/2}(\varphi_{\text{cl}})}{2\pi} \xi(t). \end{aligned} \quad (32)$$

As expected, this equation differs from the standard one by the presence of the Lorentz factor and its derivative. At second order in the noise, one obtains the equation describing the evolution of  $\delta\varphi_2(t)$ :

$$\begin{aligned} \frac{d\delta\varphi_2(t)}{dt} + \frac{2}{\kappa} \frac{H'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \left[ \frac{H''(\varphi_{\text{cl}})}{H'(\varphi_{\text{cl}})} - \frac{\gamma'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \right] \delta\varphi_2(t) \\ = \frac{3}{4\pi} H'(\varphi_{\text{cl}}) H^{1/2}(\varphi_{\text{cl}}) \xi(t) \delta\varphi_1(t) - \frac{1}{\kappa} \frac{H'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \left[ \frac{H'''(\varphi_{\text{cl}})}{H'(\varphi_{\text{cl}})} \right. \\ \left. - \frac{\gamma''(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} + 2 \frac{\gamma'^2(\varphi_{\text{cl}})}{\gamma^2(\varphi_{\text{cl}})} - 2 \frac{H''(\varphi_{\text{cl}})}{H'(\varphi_{\text{cl}})} \frac{\gamma'(\varphi_{\text{cl}})}{\gamma(\varphi_{\text{cl}})} \right] \delta\varphi_1^2(t). \end{aligned} \quad (33)$$

The same remark as before is valid: The equation for  $\delta\varphi_2(t)$  is modified by the presence of the factor  $\gamma$  and its derivatives. Clearly, the equation contains derivatives of  $\gamma$  up to second order because it is second order in the noise expansion.

We are now in a position to solve the above equations. Since they are first-order differential equations, they can be solved by varying the integration constant. One finds for  $\delta\varphi_1(t)$  that

$$\delta\varphi_1(t) = \frac{H'[\varphi_{\text{cl}}(t)]}{2\pi\gamma[\varphi_{\text{cl}}(t)]} \int_{t_{\text{in}}}^t dt' \frac{H^{3/2}[\varphi_{\text{cl}}(t')]}{H'[\varphi_{\text{cl}}(t')]} \gamma[\varphi_{\text{cl}}(t')] \xi(t'), \quad (34)$$

while for  $\delta\varphi_2(t)$  one obtains

$$\begin{aligned}
\delta\varphi_2(t) &= \frac{3}{4\pi} \frac{H'[\varphi_{\text{cl}}(t)]}{\gamma[\varphi_{\text{cl}}(t)]} \int_{t_{\text{in}}}^t dt' H^{1/2}[\varphi_{\text{cl}}(t')] \gamma[\varphi_{\text{cl}}(t')] \\
&\times \xi(t') \delta\varphi_1(t') - \frac{H'[\varphi_{\text{cl}}(t)]}{\kappa\gamma[\varphi_{\text{cl}}(t)]} \int_{t_{\text{in}}}^t dt' \left\{ \frac{H'''[\varphi_{\text{cl}}(t')]}{H'[\varphi_{\text{cl}}(t')]} \right. \\
&- \frac{\gamma''[\varphi_{\text{cl}}(t')]}{\gamma[\varphi_{\text{cl}}(t')]} + 2 \frac{\gamma'^2[\varphi_{\text{cl}}(t')]}{\gamma^2[\varphi_{\text{cl}}(t')]} - 2 \frac{H''[\varphi_{\text{cl}}(t')]}{H'[\varphi_{\text{cl}}(t')]} \\
&\left. \times \frac{\gamma'[\varphi_{\text{cl}}(t')]}{\gamma[\varphi_{\text{cl}}(t')]} \right\} \delta\varphi_1^2(t'). \quad (35)
\end{aligned}$$

From the above expressions and the properties of the noise given by Eq. (12), it is obvious that  $\langle \delta\varphi_1 \rangle = 0$ , and for the second moment we find

$$\langle \delta\varphi_1^2 \rangle = \frac{\kappa}{2} \left( \frac{H'}{2\pi\gamma} \right)^2 \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} d\psi \left[ \frac{H(\psi)\gamma(\psi)}{H'(\psi)} \right]^3. \quad (36)$$

The next step is to calculate  $\langle \delta\varphi_2 \rangle$ . Lengthy but straightforward calculations lead to

$$\begin{aligned}
\langle \delta\varphi_2 \rangle &= \frac{H'}{2\pi m_{\text{Pl}}^2 \gamma} \left\{ \left( \frac{H'}{\gamma} \right)' \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} d\psi \left( \frac{H\gamma}{H'} \right)^3 - \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} d\psi \left[ \left( \frac{H\gamma}{H'} \right)^3 \left( \frac{H'}{\gamma} \right)' - \frac{3}{2} \frac{H^2 \gamma^2}{H'} \right] \right\} \\
&= \frac{(H'/\gamma)'}{2(H'/\gamma)} \langle \delta\varphi_1^2 \rangle + \frac{H'/\gamma}{4\pi m_{\text{Pl}}^2} \left[ \left( \frac{\gamma^2 H^3}{H'^2} \right)_{\varphi_{\text{in}}} - \left( \frac{\gamma^2 H^3}{H'^2} \right)_{\varphi_{\text{cl}}} \right]. \quad (37)
\end{aligned}$$

As in the standard case, everything can be reduced to the calculation of a single quadrature. As expected, in the DBI case this quadrature contains the factor  $\gamma$ .

By using these results, one can now calculate the PDF,  $P_c(\varphi, t)$ , which describes the probability of the stochastic process  $\varphi[\xi]$  to take a given value  $\varphi$  at a given time  $t$  in a single coarse-grained domain (see Ref. [20]),

$$\begin{aligned}
P_c(\varphi, t) &= \langle \delta(\varphi - \varphi[\xi]) \rangle \\
&= \frac{1}{\sqrt{2\pi\langle \delta\varphi_1^2 \rangle}} \exp \left[ - \frac{(\varphi - \varphi_{\text{cl}} - \langle \delta\varphi_2 \rangle)^2}{2\langle \delta\varphi_1^2 \rangle} \right] \\
&\equiv P_g(\varphi - \varphi_{\text{cl}} - \langle \delta\varphi_2 \rangle). \quad (38)
\end{aligned}$$

[In the last line, we have introduced the definition  $P_g$  for later use; see Eq. (89).] If, however, one is interested in spatial averaging over the entire Universe (instead of a single domain), one has to include a weight factor  $a^3(\varphi) = \exp[3 \int d\tau H(\varphi)]$  for the physical volume of each Hubble-sized domain. This leads to

$$\begin{aligned}
P_v(\varphi, t) &= \frac{\langle \delta(\varphi - \varphi[\xi]) e^{3 \int d\tau H(\varphi[\xi])} \rangle}{\langle e^{3 \int d\tau H(\varphi[\xi])} \rangle} \\
&= \frac{1}{\sqrt{2\pi\langle \delta\varphi_1^2 \rangle}} \exp \left[ - \frac{(\varphi - \langle \varphi \rangle - 3I^T J)^2}{2\langle \delta\varphi_1^2 \rangle} \right] \\
&= P_g(\varphi - \langle \varphi \rangle - 3I^T J), \quad (39)
\end{aligned}$$

where again the last line is a definition used in later sections. In Eq. (39),  $\langle \varphi \rangle$  is the single domain averaged mean value  $\langle \varphi \rangle = \varphi_{\text{cl}} + \langle \delta\varphi_2 \rangle$ , to which the volume effects induce the additional correction given by

$$\begin{aligned}
3I^T J &= 3 \int_{t_{\text{in}}}^t d\tau H'(\tau) \langle \delta\varphi_1(t) \delta\varphi_1(\tau) \rangle \\
&= \frac{12H'}{m_{\text{Pl}}^4 \gamma} \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} d\psi \frac{H^4 \gamma^3}{H'^3} - \frac{12\pi\gamma H}{m_{\text{Pl}}^2 H'} \langle \delta\varphi_1^2 \rangle. \quad (40)
\end{aligned}$$

Once this integration is carried out, the volume-weighted distribution  $P_v(\varphi, t)$  as defined in Eq. (39) is also completely determined. Again, by comparing the results of this section with those for the standard case presented in Refs. [20,21], we see that the only changes are the additional powers of  $\gamma$  found in  $\delta\varphi_1(t)$  and  $\delta\varphi_2(t)$ . Below, we calculate  $P_c(\varphi, t)$  and  $P_v(\varphi, t)$  for an example shape of  $V(\varphi)$  and  $T(\varphi)$ .

As discussed above, the Langevin equation can also be written with the number of  $e$ -folds as the time variable. It is straightforward to repeat the above analysis for the corresponding Langevin equation (30). In particular, the first- and second-order corrections obtained in terms of  $e$ -folds read

$$\delta\varphi_1(N) = \frac{1}{2\pi} \frac{H'}{H\gamma} \int_{N_{\text{in}}}^N dN' \frac{H^2(N')\gamma(N')}{H'(N')} \xi(N') \quad (41)$$

and

$$\begin{aligned}
\delta\varphi_2(N) &= \frac{1}{2\pi} \frac{H'}{H\gamma} \int_{N_{\text{in}}}^N dN' H\gamma \delta\varphi_1(N') \xi(N') \\
&- \frac{1}{\kappa} \frac{H'}{H\gamma} \int_{N_{\text{in}}}^N dN' \frac{H\gamma}{H'} \left( \frac{H'}{H\gamma} \right)'' \delta\varphi_1^2(N'). \quad (42)
\end{aligned}$$

In Eq. (42), arguments in the integrands have been partially suppressed where they are evident. As before, it is easy to calculate  $\langle \delta\varphi_1^2 \rangle$  and  $\langle \delta\varphi_2 \rangle$  from these expressions. It is found that

$$\langle \delta\varphi_1^2 \rangle_N = \frac{\kappa}{8\pi^2} \frac{H'^2}{H^2 \gamma^2} \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} \frac{H^5 \gamma^3}{H'^3} d\psi. \quad (43)$$

For the case of a standard kinetic term (hence,  $\gamma = 1$ ), this

equation coincides with Eq. (40) of Ref. [27]. Furthermore, we find from Eq. (42) that

$$\begin{aligned} \langle \delta \varphi_2 \rangle_N &= \frac{H\gamma}{2H'} \left( \frac{H'}{H\gamma} \right)' \langle \delta \varphi_1^2 \rangle \\ &+ \frac{\kappa}{32\pi^2} \frac{H'}{H\gamma} \left[ \left( \frac{H^4 \gamma^2}{H'^2} \right)_{\text{in}} - \left( \frac{H^4 \gamma^2}{H'^2} \right) \right]. \end{aligned} \quad (44)$$

Again, if  $\gamma = 1$ , this equation corresponds to Eq. (48) of Ref. [27]. Let us notice that, as in our previous results of Eqs. (36) and (37), the expression (43) for  $\langle \delta \varphi_1^2 \rangle_N$  is sufficient to obtain  $\langle \delta \varphi_2 \rangle_N$  in Eq. (44), i.e., no additional quadrature is necessary.

By following the same steps, one can also evaluate the PDFs in terms of  $e$ -folds. The corresponding probability  $P_c^{(N)}(\varphi, t)$  is similar to Eq. (38), except that now the formulas (43) and (44) should be used in their respective places. The definition of the volume-weighted distribution also remains the same [see Eq. (39)], but the term  $3I^T J$  now reads

$$\begin{aligned} (3I^T J)_N &= \frac{12}{m_{\text{Pl}}^4} \frac{H'}{H\gamma} \int_{\varphi_{\text{cl}}}^{\varphi_{\text{in}}} d\psi \frac{H^5 \gamma^3}{H'^3} \ln \left( \frac{H}{m_{\text{Pl}}} \right) \\ &- \frac{12\pi}{m_{\text{Pl}}^2} \frac{H\gamma}{H'} \langle \delta \varphi_1^2 \rangle \ln \left( \frac{H}{m_{\text{Pl}}} \right). \end{aligned} \quad (45)$$

This equation should be compared to Eq. (40). Again, assessment of the volume effects requires the calculation of a new quadrature.

### III. APPLICATION TO BRANE INFLATION

In the following, we apply the formalism developed in Sec. II to a popular class of string-inspired inflation models with a DBI kinetic term. We focus on scenarios of the brane inflation type, for which the inflaton field corresponds to the position of a  $D3$  brane embedded in a higher-dimensional background. Successful model building requires that the six extra dimensions be deformed in a way described by a warp factor  $T(\phi)$ ; the resulting geometry is commonly called a Klebanov-Strassler throat. Scenarios of this type have been the subject of a vast body of literature; see, e.g., Refs. [5–13, 28, 36–39]. For our purposes, we denote the warp factor and the inflationary potential by

$$T(\phi) = \frac{\phi^4}{\lambda}, \quad V(\phi) = V_0 \left[ 1 - \left( \frac{\mu}{\phi} \right)^4 \right] + \frac{\varepsilon}{2} m^2 \phi^2, \quad (46)$$

where  $\varepsilon = \pm 1$ . The plus sign identifies so-called ‘‘ultra-violet’’ (UV) models (where the  $D3$  moves from the edge towards the bottom of the throat geometry and hence the field value decreases during inflation), while the minus sign refers to the ‘‘infrared’’ (IR) setting (where the  $D3$  climbs out of the throat and the inflaton’s field value grows with time).

For completeness, the formulation [Eq. (46)] of  $V(\phi)$  includes a Coulomb potential term due to the  $D3$ ’s attraction towards a  $\bar{D}3$  brane sitting at the bottom of the throat. Originally, the small Coulombic attraction resulting from this very flat ( $\propto 1/\phi^4$ ) potential was considered the inflaton’s only driving force, ignoring the (potentially much steeper) second term  $\propto \phi^2$  in Eq. (46) [7]. In fact, it was shown that in this case the DBI dynamics do not affect the inflationary evolution [40], and the stochastic effects were also assessed in the same reference. Therefore, in the following, we ignore the  $\propto 1/\phi^4$  Coulombic contribution in Eq. (46). Note, however, that conceptually the presence of an antibrane at the bottom of the KS throat shall be important for our reasoning in later sections. The quadratic potential term in Eq. (46) has a different status. We shall treat it here as a phenomenological description of the potential that the various background moduli fields produce for the mobile  $D3$  brane. The exact shape of these moduli contributions is still a subject of active research [7–13].

From the cosmological point of view, the inflation model of Eq. (46) has three parameters: the mass  $m$ , the dimensionless constant  $\lambda$ , and  $V_0$  (with dimension  $m_{\text{Pl}}^4$ ). In fact, as we show below, it is rather two dimensionless combinations of  $(m, \lambda, V_0)$ , which we shall call  $(\alpha, \beta)$ , that characterize the evolution. We define these parameters by

$$\alpha \equiv \frac{12\pi m_{\text{Pl}}^2}{\lambda m^2} = \frac{96\pi^2}{\kappa \lambda m^2}, \quad \beta \equiv \frac{V_0}{m^2 m_{\text{Pl}}^2}. \quad (47)$$

Physically,  $\beta$  measures the importance of the constant term relative to the mass term in the potential (recall that we are neglecting the Coulomb term, which involves the parameter  $\mu$ ).

The geometric interpretation (in terms of an extra-dimensional brane position) of the inflaton field enforces some intrinsic consistency conditions. The renormalization relating the inflaton field  $\phi$  to the (radial) throat coordinate  $r$  reads  $\phi = \sqrt{T_3} r$ , where  $T_3 = [(2\pi)^3 g_S \alpha'^2]^{-1}$  is the tension of the  $D3$  brane, calculated from the string coupling  $g_S$  and scale  $\alpha'$ . (We do not consider a possible motion of the brane along angular coordinates of the throat.) Let the edge of the throat correspond to some  $r_{\text{UV}}$ , where the KS corner of the geometry is connected to the compactified six-dimensional bulk. Since the metric outside the throat is unknown, one has to impose  $r < r_{\text{UV}}$  to ensure that the brane stays inside the well-defined KS region. In terms of stringy background parameters, one can express

$$r_{\text{UV}}^4 = 4\pi g_S \alpha'^2 \frac{\mathcal{N}}{v}, \quad (48)$$

where  $\mathcal{N}$  is a positive integer representing the total Ramond-Ramond charge and  $v$  represents the (dimensionless) parameter measuring the volume of the five-dimensional submanifold that forms the basis of the 6D throat in units of the five-sphere volume. Via the inflaton’s



renormalization, this evidently is an upper bound on the inflaton  $\phi$ . Note that, depending on whether we are talking about UV or IR models, this constraint affects the initial or final field value,  $\phi_{\text{in}}^{(\text{UV})} < \phi_{\text{UV}}$  or  $\phi_{\text{end}}^{(\text{IR})} < \phi_{\text{UV}}$ .

A second consistency condition is the requirement that the volume of the throat be smaller than the total volume of the compactified extra dimensions. This total volume has observational significance since it enters into the four-dimensional Planck mass  $m_{\text{Pl}}$ . From this constraint it follows that

$$\phi < \phi_{\text{UV}} < \frac{m_{\text{Pl}}}{\sqrt{2\pi\mathcal{N}}}. \quad (49)$$

This means that inflation always occurs for sub-Planckian values of  $\phi$ . On the other hand, the bottom of the throat being located at  $r_0$ , one must have  $\phi > \phi_0 \equiv \sqrt{T_3}r_0$ . Moreover, for the model to be valid, the (physical) distance between the branes must be larger than the string length, and one can show that this amounts to

$$\phi > \phi_{\text{strg}} = \phi_0 e^{\sqrt{\alpha'}r_{\text{UV}}}. \quad (50)$$

Note also that the parameters of the warp factor and potential in Eqs. (46) can be calculated in terms of the stringy parameters. Physically,  $T(\phi)$  is the position-dependent brane tension, and it can be written as  $T(\phi) = T_3(\phi/\phi_{\text{UV}})^4$ , which implies [compare Eqs. (46) and (48)] that

$$\lambda = \frac{\mathcal{N}}{2\pi^2\nu}. \quad (51)$$

The constant term  $V_0$  is given by  $V_0 = 4\pi^2\nu\phi_0^4/\mathcal{N}$ , which can also be expressed as

$$V_0 = 2h^4(r_0)T_3, \quad (52)$$

where  $h(\phi) \equiv \phi/\phi_{\text{UV}}$  is the warping function as it appears in the 10D metric [it holds that  $T(\phi) \propto h^4(\phi)$ ].

It turns out that Eq. (49) can be rewritten in a more quantitative way. Since the volume of a Klebanov-Strassler throat is known ( $V_6^{\text{throat}} = 2\pi^4 g_S \mathcal{N} \alpha'^2 r_{\text{UV}}^2$ ) and the total six-dimensional volume is related to the four-dimensional Planck mass [ $V_6^{\text{tot}} = m_{\text{Pl}}^2 (2\pi)^7 g_S^2 \alpha'^4 / (16\pi)$ ], one deduces

for our parameters ( $\alpha, \beta$ ) defined in Eq. (47) that

$$\sqrt{\frac{\beta}{\alpha}} < \frac{1}{\sqrt{24\pi^3}} \frac{h^2(r_0)}{\mathcal{N}} \ll 1. \quad (53)$$

A similar equation [see Eq. (2.10)] was used in Ref. [39].

### A. Chaotic Klebanov-Strassler inflation

In our first example, we set the parameter  $\beta = 0$ ; hence we are considering the case of “pure” CKS inflation without a constant term. The potential and warp factor then are given by

$$V(\phi) = \frac{1}{2}m^2\phi^2, \quad T(\phi) = \frac{\phi^4}{\lambda}. \quad (54)$$

In the slow-roll limit, it follows from Eq. (4) that  $H^2 \simeq 4\pi m^2 \phi^2 / (3m_{\text{Pl}}^2)$  (see Appendix A), and, therefore, the Lorentz factor calculated from Eq. (6) behaves as

$$\gamma(\phi) = \sqrt{1 + \frac{2\lambda m^2}{3\kappa} \frac{1}{\phi^4}} = \frac{m_{\text{Pl}}^2}{\phi^2} \sqrt{\frac{\phi^4}{m_{\text{Pl}}^4} + \frac{1}{\alpha}}. \quad (55)$$

If we plug these expressions into Eq. (36), the calculation of  $\langle \delta\varphi_1^2 \rangle$  can be reduced to an exactly solvable integral, and the final result for  $\langle \delta\varphi_1^2 \rangle$  is

$$\begin{aligned} \langle \delta\varphi_1^2 \rangle &= \frac{4m^2}{3\gamma^2} \left[ \frac{1}{2\alpha} (\gamma - \gamma_{\text{in}}) - \frac{1}{4} \left( \frac{\varphi_{\text{cl}}^4}{m_{\text{Pl}}^4} \gamma - \frac{\varphi_{\text{in}}^4}{m_{\text{Pl}}^4} \gamma_{\text{in}} \right) \right. \\ &\quad \left. + \frac{3}{2\alpha} \ln\left(\frac{\varphi_{\text{in}}}{\varphi_{\text{cl}}}\right) + \frac{3}{4\alpha} \ln\left(\frac{1 + \gamma_{\text{in}}}{1 + \gamma}\right) \right], \end{aligned} \quad (56)$$

where  $\gamma = \gamma(\varphi_{\text{cl}})$  and  $\gamma_{\text{in}} = \gamma(\varphi_{\text{in}})$ . For  $\langle \delta\varphi_2 \rangle$  we have from Eq. (37) that

$$\langle \delta\varphi_2 \rangle = \frac{\gamma^2 - 1}{\gamma^2} \frac{1}{\varphi_{\text{cl}}} \langle \delta\varphi_1^2 \rangle + \frac{m^2}{3\gamma m_{\text{Pl}}^4} (\gamma_{\text{in}}^2 \varphi_{\text{in}}^3 - \gamma^2 \varphi_{\text{cl}}^3). \quad (57)$$

To calculate the volume effects, another integration is necessary. With the potential and warp factor given by Eq. (54), one finds from Eq. (40) that

$$\begin{aligned} 3I^T J &= \frac{16\pi}{\gamma} \frac{m^2}{m_{\text{Pl}}^2} m_{\text{Pl}} \left\{ -\frac{m_{\text{Pl}}}{\varphi_{\text{in}}} \left( \frac{\varphi_{\text{in}}^4}{m_{\text{Pl}}^4} + \frac{1}{\alpha} \right)^{3/2} + \frac{m_{\text{Pl}}}{\varphi_{\text{cl}}} \left( \frac{\varphi_{\text{cl}}^4}{m_{\text{Pl}}^4} + \frac{1}{\alpha} \right)^{3/2} \right. \\ &\quad \left. - \frac{3}{2} (-1)^{-1/4} \alpha^{-5/4} \left[ B\left(-\alpha \frac{\varphi_{\text{in}}^4}{m_{\text{Pl}}^4}, \frac{3}{4}, \frac{3}{2}\right) - B\left(-\alpha \frac{\varphi_{\text{cl}}^4}{m_{\text{Pl}}^4}, \frac{3}{4}, \frac{3}{2}\right) \right] \right\} - 12\pi\gamma \frac{\langle \delta\varphi_1^2 \rangle}{m_{\text{Pl}}^2} \varphi_{\text{cl}}, \end{aligned} \quad (58)$$

where  $B$  is the incomplete Euler's integral of the first kind defined by  $B(z, a, b) \equiv \int_0^z t^{a-1} (1-t)^{b-1} dt$  [41,42]. Notice that, in the above equation, the function  $B$  takes in fact complex values. However, multiplied by the factor  $(-1)^{-1/4}$ , the result is real as it should be.

Let us discuss these results in more detail in the light of the consistency constraint (49). In the limit of small field values compared to the Planck mass, the Lorentz factor is large and can be approximated by  $\gamma(\varphi_{\text{cl}}) \simeq m_{\text{Pl}}^2 / (\sqrt{\alpha} \varphi_{\text{cl}}^2)$ . It is hence easy to show that

$$\langle \delta \varphi_1^2 \rangle \simeq \frac{2m^2}{3\sqrt{\alpha}} \left( \frac{\varphi_{\text{cl}}}{m_{\text{Pl}}} \right)^2, \quad (59)$$

$$\langle \delta \varphi_2 \rangle \simeq \frac{m^2}{3m_{\text{Pl}}^2} \frac{\varphi_{\text{cl}}}{\sqrt{\alpha}}. \quad (60)$$

The expression giving the volume effects can also be simplified. Notice that  $B(z, a, b) = z^a {}_2F_1(a, 1-b, a+1, z)$ , where  ${}_2F_1$  is the hypergeometric function [41,42]. By using the asymptotic behavior of  ${}_2F_1$  [41,42], straightforward calculations show that

$$3I^T J \simeq 8\pi \frac{m^2}{m_{\text{Pl}}^2} \frac{\varphi_{\text{cl}}}{\alpha}. \quad (61)$$

Let us now establish the results in terms of the numbers of  $e$ -folds. By using Eq. (43), one obtains

$$\begin{aligned} \langle \delta \varphi_1^2 \rangle_N &= -\frac{2}{3} \frac{m^2}{\gamma^2} \frac{m_{\text{Pl}}^2}{\phi^2} \left[ \frac{\gamma^3}{3} \frac{\varphi_{\text{cl}}^6}{m_{\text{Pl}}^6} - \frac{\gamma_{\text{in}}^3}{3} \frac{\varphi_{\text{in}}^6}{m_{\text{Pl}}^6} + \frac{\gamma}{\alpha} \frac{\varphi_{\text{cl}}^2}{m_{\text{Pl}}^2} \right. \\ &\quad - \frac{\gamma_{\text{in}}}{\alpha} \frac{\varphi_{\text{in}}^2}{m_{\text{Pl}}^2} - \frac{1}{\alpha^{3/2}} \operatorname{arcsinh} \left( \frac{1}{\alpha^{1/2}} \frac{m_{\text{Pl}}^2}{\varphi_{\text{cl}}^2} \right) \\ &\quad \left. + \frac{1}{\alpha^{3/2}} \operatorname{arcsinh} \left( \frac{1}{\alpha^{1/2}} \frac{m_{\text{Pl}}^2}{\varphi_{\text{in}}^2} \right) \right]. \end{aligned} \quad (62)$$

One can easily check that this expression vanishes at the initial time as expected. The corresponding expression for  $\langle \delta \varphi_2 \rangle$  can be deduced from Eq. (44). The result reads

$$\langle \delta \varphi_2 \rangle_N = \frac{1}{2\varphi_{\text{cl}}} \frac{\gamma^2 - 2}{\gamma^2} \langle \delta \varphi_1^2 \rangle + \frac{1}{3\gamma} \frac{m^2}{\varphi_{\text{cl}}} \left( \gamma_{\text{in}}^2 \frac{\varphi_{\text{in}}^4}{m_{\text{Pl}}^4} - \gamma^2 \frac{\varphi_{\text{cl}}^4}{m_{\text{Pl}}^4} \right). \quad (63)$$

In the limit where the Lorentz factor  $\gamma$  is large, the above expressions can be approximated by

$$\langle \delta \varphi_1^2 \rangle_N \simeq \frac{2m^2}{3\sqrt{\alpha}} \left( \frac{\varphi_{\text{cl}}}{m_{\text{Pl}}} \right)^2 \ln \left( \frac{\varphi_{\text{in}}}{\varphi_{\text{cl}}} \right)^2, \quad (64)$$

$$\langle \delta \varphi_2 \rangle_N \simeq \frac{2m^2}{3m_{\text{Pl}}^2} \frac{\varphi_{\text{cl}}}{\sqrt{\alpha}} \ln \left( \frac{\varphi_{\text{in}}}{\varphi_{\text{cl}}} \right). \quad (65)$$

It is interesting to compare these formulas to Eqs. (59) and (60). We see that working in terms of the number of  $e$ -folds simply introduces (apart from numerical prefactors) a logarithmic correction to the correlation functions: Roughly speaking, the new correlation functions are obtained from the old ones with the replacement  $\varphi_{\text{cl}} \rightarrow \varphi_{\text{cl}} \ln(\varphi_{\text{in}}/\varphi_{\text{cl}})$ . This is confirmed by a calculation of the volume effect. By using Eq. (45), one obtains

$$\begin{aligned} (3I^T J)_N &\simeq \frac{16\pi}{\alpha} \frac{m^2}{m_{\text{Pl}}^2} \varphi_{\text{cl}} \left\{ \ln \left( \frac{\varphi_{\text{cl}}}{\varphi_{\text{in}}} \right) \ln \left[ \left( \frac{4\pi}{3} \right)^{1/2} \frac{m\varphi_{\text{cl}}}{m_{\text{Pl}}^2} \right] \right. \\ &\quad \left. + \frac{1}{2} \ln^2 \left[ \left( \frac{4\pi}{3} \right)^{1/2} \frac{m\varphi_{\text{in}}}{m_{\text{Pl}}^2} \right] - \frac{1}{2} \ln^2 \left[ \left( \frac{4\pi}{3} \right)^{1/2} \frac{m\varphi_{\text{cl}}}{m_{\text{Pl}}^2} \right] \right\}. \end{aligned} \quad (66)$$

This formula should be compared to Eq. (61).

In order to see whether the stochastic effects are important or not, we must normalize the model's parameters to the Cosmic Background Explorer (COBE) observations. This was done in Ref. [17], where it was shown that [see that reference's Eq. (127)]

$$\left( \frac{m}{m_{\text{Pl}}} \right)^2 = \frac{45}{4\pi} \frac{Q^2}{T_{\text{CMB}}^2} \alpha, \quad (67)$$

where the quantity  $Q^2/T_{\text{CMB}}^2$  can be expressed in terms of the cosmic microwave background (CMB) quadrupole according to

$$\frac{Q}{T_{\text{CMB}}} = \sqrt{\frac{5C_2}{4\pi}} \simeq 6 \times 10^{-6}. \quad (68)$$

In Ref. [17] [see Eq. (116)], it was also demonstrated that the first slow-roll parameter  $\epsilon_1$  (see Appendix A for a precise definition of the slow-roll hierarchy), for the model under consideration, can be expressed as  $\epsilon_1 \simeq \sqrt{\alpha}/(4\pi)$ . Therefore, in a realistic inflationary situation we always have  $\alpha \ll 1$ .

As a rule of thumb, the inflaton field can be said to behave quantum-mechanically if the correction  $\langle \delta \varphi_2 \rangle$  to the mean value is of the same order as the classical field  $\varphi_{\text{cl}}$ . For a more detailed argument, let us consider the calculation carried out in Ref. [43] (see also the discussion in Ref. [20]). There, the authors compute the number of  $e$ -folds  $\langle N \rangle = \int dt \langle H \rangle$ , which we adapt to the case of DBI inflation as

$$\langle N \rangle = -\frac{\kappa}{2} \int_{\varphi_{\text{in}}}^{\varphi_{\text{cl}}} d\psi \frac{\langle H \rangle}{H_{\text{cl}}} \gamma_{\text{cl}}. \quad (69)$$

In the present case, i.e., for the potential and warp factor of Eqs. (54), this gives

$$\langle N \rangle = -\frac{\kappa}{2} \sqrt{\frac{\kappa}{6}} \int_{\varphi_{\text{in}}}^{\varphi_{\text{cl}}} d\psi \frac{m\varphi_{\text{cl}}\gamma_{\text{cl}}}{H'_{\text{cl}}} \left( 1 + \frac{\langle \delta \varphi_2 \rangle}{\varphi_{\text{cl}}} \right). \quad (70)$$

If  $\langle \delta \varphi_2 \rangle \ll \varphi_{\text{cl}}$ , then  $\langle N \rangle = N_{\text{cl}}$  and the trajectory is indeed classical, confirming our rule of thumb. If we work out the above condition ignoring unimportant numerical factors, we find that

$$\frac{\langle \delta \varphi_2 \rangle}{\varphi_{\text{cl}}} \sim \epsilon_1 \frac{Q^2}{T_{\text{CMB}}^2} \ll 1. \quad (71)$$

We conclude that the stochastic effects do not play any important rôle in the model with  $\beta = 0$  and, therefore, that eternal inflation is not possible. This conclusion, although obtained with a different method, is in agreement with Ref. [28]. In view of this result, we do not discuss the domain of validity of the perturbative approach in the  $\beta = 0$  case but instead turn straight to the case of a CKS potential with constant term.

### B. Chaotic Klebanov-Strassler inflation with a constant term

As a second example, we keep  $T(\phi)$  as in Eq. (54) but take  $\beta \neq 0$ , so that the potential is given by

$$V(\phi) = V_0 + \frac{\varepsilon}{2} m^2 \phi^2. \quad (72)$$

With this potential, the Lorentz factor [compare Eq. (55)] is given by

$$\gamma = \left( \frac{m_{\text{Pl}}}{\phi} \right)^2 \sqrt{\left( \frac{\phi}{m_{\text{Pl}}} \right)^4 + \frac{1}{\alpha} \frac{(\phi/m_{\text{Pl}})^2}{2\beta + \varepsilon(\phi/m_{\text{Pl}})^2}}. \quad (73)$$

Then, in order to determine the corrections to the variance  $\langle \delta\varphi_1^2 \rangle$  and mean value  $\langle \delta\varphi_2 \rangle$ , one has to compute the kernel of the integral in Eq. (36). One obtains

$$\frac{H\gamma}{H'} = \frac{2\varepsilon}{m^2 \phi^2} \left[ \left( V_0 + \frac{\varepsilon}{2} m^2 \phi^2 \right) \times \left[ \frac{V_0 m_{\text{Pl}}^2}{2\alpha\beta} + \phi^2 \left( V_0 + \frac{\varepsilon}{2} m^2 \phi^2 \right) \right]^{1/2} \right]. \quad (74)$$

Unfortunately, the (third power of this) expression is too complicated to perform the integral (36) exactly. However, while the constant term of the potential (72) dominates ( $\beta \gg 1$ ), it is legitimate to approximate  $V_0 + \varepsilon m^2 \phi^2/2$  by  $V_0$ . In this case, the calculation of Eq. (36) can be done and leads to

$$\langle \delta\varphi_1^2 \rangle = -\frac{32\varepsilon}{15} \frac{m^4 \varphi_{\text{cl}}^2}{V_0 \gamma_{\text{cl}}^2} \alpha \beta^4 \left\{ \left[ \frac{1 + 2\alpha\beta(\varphi_{\text{in}}/m_{\text{Pl}})^2}{2\alpha\beta(\varphi_{\text{in}}/m_{\text{Pl}})^2} \right]^{5/2} - \left[ \frac{1 + 2\alpha\beta(\varphi_{\text{cl}}/m_{\text{Pl}})^2}{2\alpha\beta(\varphi_{\text{cl}}/m_{\text{Pl}})^2} \right]^{5/2} \right\}. \quad (75)$$

In the same limit, one can also estimate  $\langle \delta\varphi_2 \rangle$ . The result reads

$$\langle \delta\varphi_2 \rangle = \frac{4\varepsilon\beta^2}{3} \left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\varphi_{\text{cl}}}{\gamma_{\text{cl}}} \left[ \left( \frac{m_{\text{Pl}}}{\phi_{\text{in}}} \right)^2 \gamma_{\text{in}}^2 - \left( \frac{m_{\text{Pl}}}{\varphi_{\text{cl}}} \right)^2 \gamma_{\text{cl}}^2 \right] + \left[ 1 + \frac{1}{2\alpha\beta} \left( \frac{m_{\text{Pl}}}{\varphi_{\text{cl}}} \right)^2 \frac{1}{\gamma_{\text{cl}}^2} \right] \frac{\langle \delta\varphi_1^2 \rangle}{2\varphi_{\text{cl}}}, \quad (76)$$

where the explicit expression of  $\langle \delta\varphi_1^2 \rangle$  is given in Eq. (75). The evolution of  $\langle \delta\varphi_1^2 \rangle$  and  $\langle \delta\varphi_2 \rangle$  in the IR and UV cases is represented in Figs. 1 and 2.

Finally, we have to estimate the volume effects. For this purpose, we have to compute the correction to the mean value of the field given by Eq. (40). In our approximation, one has  $H^4 \gamma^3 / (H')^3 \simeq \sqrt{\kappa V_0/3} [H^3 \gamma^3 / (H')^3]$ . This means that one can express the integral of Eq. (40) in terms of the integral appearing in Eq. (36), i.e., in terms of  $\langle \delta\varphi_1^2 \rangle$ . After carrying out this calculation, one obtains  $3I^T J \simeq 0$ . Therefore, at first order in our approximation, volume effects simply are absent. In fact, one could have guessed this result from the very beginning: At first order, the Hubble parameter is given by  $H \simeq \sqrt{\kappa V_0/3}$ , which is a constant. By looking at Eq. (39), we see that, in this case, the weight related to the volume can be taken outside the integrals in the numerator and denominator. Hence, the corresponding contributions cancel out and, at this order, there is no volume effect.

For the same reason, there is no difference between the correlation functions computed with the time variable  $t$  or with the  $e$ -fold variable  $N$ ; see, for instance, Eqs. (36) and (43). This illustrates the fact that the difference between the two approaches can be important only if the Hubble parameter evolves significantly. This is the case for chaotic inflation [23,27] (although for the DBI version of the chaotic model studied in the last subsection, the corrections were only logarithmic) but not for the model under scrutiny here.

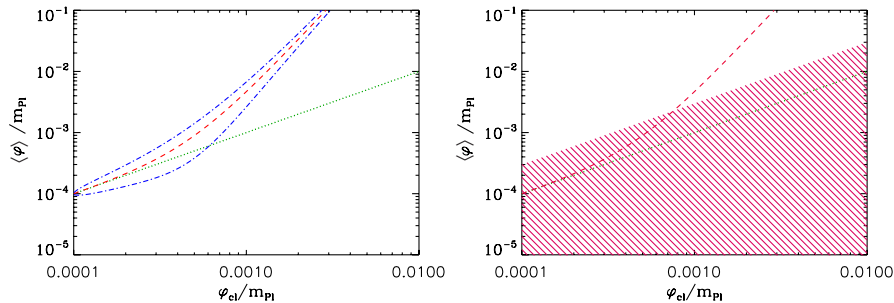


FIG. 1 (color online). Evolution of the (quantum) scalar field in the IR case (hence inflation proceeds from left to right) for  $\alpha = 38$ ,  $\beta = 3.7$ , and  $m \simeq 2.19 \times 10^{-7} m_{\text{Pl}}$ , as implied by the COBE normalization. Note that the choice of the parameters is such that the condition  $\beta/\alpha \ll 1$  is valid. For this example, the spectral index and the running of the model are, respectively, given by  $n_s - 1 \simeq -0.11$  and  $\alpha_s \simeq 0.0023$ , as can be shown by using the results of Ref. [17]. The initial condition is  $\varphi_{\text{in}} = 10^{-4} m_{\text{Pl}}$ . The green dotted line represents the classical evolution without the quantum effects. The red dashed line represents the mean value of the quantum scalar field, namely,  $\varphi_{\text{cl}} + \langle \delta\varphi_2 \rangle$ , while the two blue dashed dotted lines on both side of the mean are  $\varphi_{\text{cl}} + \langle \delta\varphi_2 \rangle \pm \sqrt{\langle \delta\varphi_1^2 \rangle}$ . In the right panel, the hatched region represents the region where the perturbative treatment used in this article is valid. For the parameters chosen here, the perturbative approach breaks down at  $\varphi_{\text{cl}} \simeq 0.001 m_{\text{Pl}}$ .

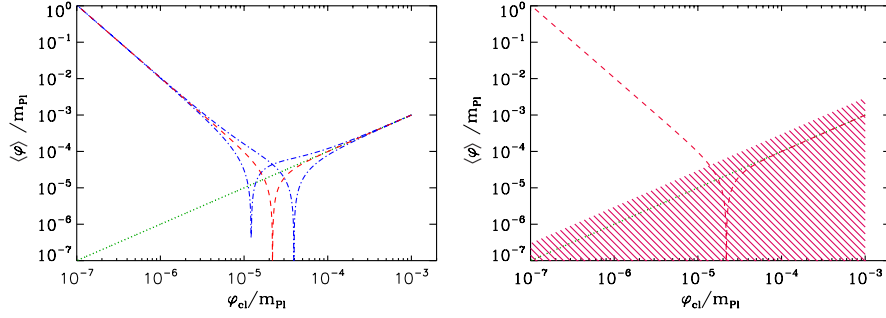


FIG. 2 (color online). Evolution of the (quantum) scalar field in the UV case (hence inflation proceeds from right to left) for  $\alpha = 38$ ,  $\beta = 3.7$ , and  $m \approx 2.19 \times 10^{-7} m_{\text{Pl}}$  as implied by the COBE normalization (this case corresponds to Fig. 9 of Ref. [17]). For this example, the spectral index and the running of the model are, respectively, given by  $n_s - 1 \approx 0.11$  and  $\alpha_s \approx -0.0023$ . The initial condition is  $\varphi_{\text{in}} = 10^{-3} m_{\text{Pl}}$ . The conventions are analogous to those in Fig. 1. The fact that  $\langle \delta \varphi_2 \rangle_{\text{UV}} < 0$  [see Eq. (82)] and, therefore,  $\langle \varphi \rangle < \varphi_{\text{cl}}$  can be clearly observed in the above plot (red dashed line). In the right panel, the hatched region indicates the domain of validity of the perturbative approach.

Before investigating in more detail the previous result, let us establish its domain of validity. It was shown in Ref. [21] that one can trust the perturbative stochastic treatment as long as the mean value of the inflaton field is such that  $\langle \varphi \rangle \in [\varphi_{\text{cl}} - |\Delta \varphi_{\text{min}}(\varphi_{\text{cl}})|, \varphi_{\text{cl}} + \Delta \varphi_{\text{max}}(\varphi_{\text{cl}})]$ , where  $\Delta \varphi_{\text{min}}$  and  $\Delta \varphi_{\text{max}}$  can be found from two conditions. The first of those reads

$$\max_{x \in [\varphi_{\text{cl}}, \varphi_{\text{cl}} + \Delta \varphi(\varphi_{\text{cl}})]} \left| \frac{H^{(4)}(x)}{6} \Delta \varphi^3 \right| \ll \left| \frac{H''_{\text{cl}}}{2} \right| \Delta \varphi^2, \quad (77)$$

where  $H^{(4)}$  denotes the fourth-order derivative. The second condition can be expressed as

$$\max_{x \in [\varphi_{\text{cl}}, \varphi_{\text{cl}} + \Delta \varphi(\varphi_{\text{cl}})]} \left| \frac{[H^{3/2}(x)]''}{2} \right| \Delta \varphi^2 \ll |(H^{3/2})'_{\text{cl}}| \Delta \varphi. \quad (78)$$

These two conditions must be simultaneously satisfied, and, therefore, the tightest bounds on  $\Delta \varphi_{\text{min}}$  and  $\Delta \varphi_{\text{max}}$  that follow from Eqs. (77) and (78), respectively, give the reliability of the perturbative treatment. By replacing again  $V_0 + \varepsilon m^2 \phi^2/2$  by  $V_0$  where applicable, straightforward manipulations show that Eq. (77) leads to  $\Delta \varphi_{\text{max}} = \Delta \varphi_{\text{min}} = 3\varphi_{\text{cl}}$  while Eq. (78) gives  $\Delta \varphi_{\text{max}} = \Delta \varphi_{\text{min}} = 2\varphi_{\text{cl}}$ . Therefore, one concludes that the perturbative approach is correct as long as  $\langle \varphi \rangle \in [\varphi_{\text{cl}} - 2\varphi_{\text{cl}}, \varphi_{\text{cl}} + 2\varphi_{\text{cl}}]$ . The allowed region is represented by the hatched area in Figs. 1 and 2.

We now return to Eqs. (75) and (76). These expressions can be further simplified if we take into account the fact that the vev of the inflaton field (measured in units of the Planck mass) must be small; compare Eq. (49). In this case, the Lorentz factor from Eq. (73) is given by  $\gamma \approx m_{\text{Pl}}/(\sqrt{2\alpha\beta}\varphi_{\text{cl}}) \gg 1$  and

$$\langle \delta \varphi_1^2 \rangle_{\text{IR}} \approx \frac{16}{15\sqrt{2}} \left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\beta^{3/2}}{\alpha^{1/2}} \frac{m_{\text{Pl}}^3 \varphi_{\text{cl}}^4}{\varphi_{\text{in}}^5}, \quad (79)$$

$$\langle \delta \varphi_1^2 \rangle_{\text{UV}} \approx \frac{16}{15\sqrt{2}} \left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\beta^{3/2}}{\alpha^{1/2}} \frac{m_{\text{Pl}}^3}{\varphi_{\text{cl}}}. \quad (80)$$

In the same way, one can also calculate the correction to the vev of the inflaton field. One obtains

$$\langle \delta \varphi_2 \rangle_{\text{IR}} \approx \frac{16}{15\sqrt{2}} \left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\beta^{3/2}}{\alpha^{1/2}} \frac{m_{\text{Pl}}^3 \varphi_{\text{cl}}^3}{\varphi_{\text{in}}^5}, \quad (81)$$

$$\langle \delta \varphi_2 \rangle_{\text{UV}} \approx -\frac{4}{15\sqrt{2}} \left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\beta^{3/2}}{\alpha^{1/2}} \frac{m_{\text{Pl}}^3}{\varphi_{\text{cl}}^2}. \quad (82)$$

Notice, in particular, that, in the UV case, the correction is negative. This is confirmed in Fig. 2.

With the help of these approximations, let us estimate when the stochastic effects are important. As discussed before, one expects the quantum effects to play a rôle when  $\langle \delta \varphi_2 \rangle / \varphi_{\text{cl}} \approx 1$ . In order to use Eqs. (81) and (82) for this purpose, we must again calculate the COBE normalization; this will allow us to rewrite the parameter combination  $m^2 \beta^{3/2} / (m_{\text{Pl}}^2 \alpha^{1/2})$  appearing in Eqs. (81) and (82). In Ref. [17], it was shown that [see Eq. (153)]

$$\left( \frac{m_{\text{Pl}}}{\varphi_*} \right)^4 = \frac{45}{16\pi} \frac{Q^2}{T_{\text{CMB}}^2} \left( \frac{m_{\text{Pl}}}{m} \right)^2 \frac{\alpha}{\beta^2}, \quad (83)$$

where  $\varphi_*$  is the value of the inflaton field when scales of astrophysical interest crossed out of the DBI sound horizon. Moreover, it was also demonstrated that  $n_s - 1 \sim 4\delta_1$  [see Eq. (155)]. Therefore, we finally arrive at

$$\left( \frac{m}{m_{\text{Pl}}} \right)^2 \frac{\beta^{3/2}}{\alpha^{1/2}} \approx \frac{45\pi^3}{4} \frac{Q^2}{T_{\text{CMB}}^2} (n_s - 1)^4 \left( \frac{\beta}{\alpha} \right)^{3/2}. \quad (84)$$

As a consequence, for the IR case described by Eq. (81), the condition  $\langle \delta \varphi_2 \rangle_{\text{IR}} / \varphi_{\text{cl}} \geq 1$  is equivalent to

$$\frac{\varphi_{\text{cl}}}{m_{\text{Pl}}} \geq \left( \frac{\sqrt{2}}{12\pi^3} \right)^{1/2} \left( \frac{Q}{T_{\text{CMB}}} \right)^{-1} (n_s - 1)^{-2} \left( \frac{\beta}{\alpha} \right)^{-3/4} \left( \frac{\varphi_{\text{in}}}{m_{\text{Pl}}} \right)^{5/2}. \quad (85)$$

Let us discuss this expression in detail. It is interesting to note the dependence on the initial value of the field  $\varphi_{\text{in}}$ .

The smaller  $\varphi_{\text{in}}$  is, the sooner the influence of quantum effects sets in. (Recall that  $\varphi_{\text{in}}$  is bounded from below by the bottom of the throat,  $\varphi_{\text{in}} > \phi_0$ , and that the inflaton field value in this scenario increases as the  $D3$  brane climbs out of the throat.) For  $\varphi_{\text{in}} = 10^{-3.5} m_{\text{Pl}}$ , corresponding to the left panel in Fig. 1, one obtains  $\varphi_{\text{cl}}/m_{\text{Pl}} \gtrsim 0.008$  in good agreement with the plot. For an exemplary initial field value of  $\varphi_{\text{in}} = 10^{-4} m_{\text{Pl}}$ , one has  $\varphi_{\text{cl}}/m_{\text{Pl}} \gtrsim 0.0005$ . Therefore, if the brane starts its evolution deep inside the throat, then the stochastic effects are dominant.

For the UV case of Eq. (82), one can repeat the same discussion. One finds that in this case the limit is given by

$$\frac{\varphi_{\text{cl}}}{m_{\text{Pl}}} \lesssim \left(\frac{3\pi^3}{\sqrt{2}}\right)^{1/3} \left(\frac{Q}{T_{\text{CMB}}}\right)^{2/3} (n_S - 1)^{4/3} \left(\frac{\beta}{\alpha}\right)^{1/2}. \quad (86)$$

Contrary to the IR case, we see that there is no dependence on the initial conditions anymore. For the parameters in Fig. 2, one obtains  $\varphi_{\text{cl}} \lesssim 2.19 \times 10^{-5} m_{\text{Pl}}$  in good agreement with the plot. In addition, one can also check, by comparing the two panels of Fig. 2, that this value does not change much when  $\varphi_{\text{in}}$  is modified. The conclusion is that, in the UV case, the stochastic effects play an important rôle only when the brane is approaching the bottom of the throat, i.e., towards the end of brane inflation in its UV incarnation.

#### IV. STOCHASTIC DBI INFLATION AND THE FINITE SIZE OF EXTRA DIMENSIONS

We have established that, in the model of Eq. (72), the stochastic effects can be dominant in a relevant and realistic inflationary regime. Our next goal is to compute how the (classical) behavior of the inflaton field is modified in the presence of stochastic noise. In principle, the above considerations already answer this question since we have computed in detail the time evolution of  $\langle \delta\varphi_2 \rangle$  and  $\langle \delta\varphi_1^2 \rangle$ . However, it is clear from Figs. 1 and 2 that a first issue is the limited validity of the perturbative regime. Outside the hatched region in the right-hand panels of Figs. 1 and 2, one can no longer follow the evolution of the quantum field. In particular, it seems obvious that the regime of eternal inflation cannot be described in this framework, even if we have established earlier that it is likely to exist. However, even setting aside the issue of validity, there is another, much worse problem that renders our previous treatment highly unsatisfactory. The probability density functions given by Eqs. (38) and (39) can *a priori* extend into the range where  $\varphi < \phi_0$  (possibly even into the range  $\varphi < 0$ ). Clearly, since the inflationary scenario at hand is built on the notion of the inflaton as the KS throat's renormalized radial coordinate  $r_0 < r < r_{\text{UV}}$ , a field value  $\varphi < \phi_0$  is inconsistent. To our knowledge, all works on stochastic inflation in brane inflation published so far are subject to this issue. In fact, we face here a deep conceptual challenge: In brane inflation, the finite size of the extra dimensions plays a fundamental rôle. (In Ref. [40],

for example, it was shown how the finite size condition can be exploited to cut down the allowed parameter space at the effective field theory level.) How can this crucial importance of the geometric restrictions be implemented into the description of stochastic DBI inflation?

We propose to address this issue by introducing a (reflecting or absorbing) wall at  $\phi_0$ , following the reasoning of Ref. [44]. (A similar technique has been used in Ref. [45] in order to study the quantum behavior of the quintessence field. Analogously, the approach we present below may be applied to the case of ‘‘DBI-essence’’ introduced in Ref. [46].) This wall at the bottom of the throat then marks the ‘‘end of the world’’ as imposed by the finite string geometry and prevents the stochastically corrected value of  $\varphi$  from reaching values smaller than  $\phi_0$ . Let us now explore the consequences of this proposal.

Let us recall from Refs. [44,45] that, if we start with a normalized distribution  $P(x)$  for the variable  $x$ , i.e.,  $\int_{-\infty}^{+\infty} P(x) dx = 1$ , then the normalized probability density function in the presence of a reflecting wall at  $x = a$  is given by  $P(x) + P(2a - x)$ , provided that  $P(2a - x)$  is also a solution of the relevant (approximate) Fokker-Planck equation. (This is the case for a Gaussian PDF.) One can check explicitly that this distribution is correctly normalized:  $\int_a^{+\infty} [P(x) + P(2a - x)] dx = 1$ . As a thought experiment, let us install a reflecting wall at the bottom of the KS throat at the position  $\phi_0$ . Then, according to Refs. [44,45], our new distribution would be given by

$$P_{\text{wall}}(\varphi) = \frac{1}{\sqrt{2\pi\langle\delta\varphi_1^2\rangle}} \left\{ \exp\left[-\frac{(\varphi - \varphi_{\text{cl}} - \langle\delta\varphi_2\rangle)^2}{2\langle\delta\varphi_1^2\rangle}\right] + \exp\left[-\frac{(2\phi_0 - \varphi - \varphi_{\text{cl}} - \langle\delta\varphi_2\rangle)^2}{2\langle\delta\varphi_1^2\rangle}\right] \right\}. \quad (87)$$

Notice that the wall distribution is the same with or without the volume effects since we have shown before that  $3I^T J \simeq 0$ .

However, in the present context, when the brane reaches the bottom of the throat, it annihilates with an antibrane fixed at  $\phi_0$ . Note that, while the small Coulombic contribution of the interbrane potential in Eq. (46) was ignored in our calculations above, conceptually the presence of the anti- $D3$  at the bottom of the throat is important for our present argument: Therefore, a correct physical description of this situation involves an absorbing wall rather than a reflecting one. A more difficult question concerns the boundary condition to be chosen at  $\phi_{\text{UV}}$  (where the KS throat joins the 6D bulk with unknown metric).

A more appropriate description of the situation could possibly be given by a ‘‘multithroat’’ scenario: For instance, one can imagine that the mobile  $D3$  brane travels through a 6D compact space made of a (negligibly) small unknown bulk and two well-defined KS throats of different depth. To account for the anti- $D3$ s present at the bottom of each throat, absorbing walls are installed at the correspond-

ing positions (or their field value counterparts, respectively). Clearly, the detailed treatment of such a situation will involve a high degree of technical sophistication because a continuous description of warp factor and inflaton potential is needed across the entire 6D manifold.

In order to render the problem tractable with our present means, here we content ourselves with the assumption that a second absorbing wall is located at  $\phi_{UV}$ . This is a good approximation for a two-throat model in which the second throat is much “shallower” than the primary inflationary throat (which is responsible for most of the exponential expansion). Our hope is that this toy model with absorbing walls at both  $\phi_0$  and  $\phi_{UV}$  will give us an idea about the behavior to be expected in a more realistic case. Of course, placing a reflecting wall at  $\phi_{UV}$  is also technically possible but seems physically less justified at present. Therefore, in the following, we concentrate on the case of two absorbing walls.

### A. Geometrically amended PDF

Our goal is now to derive the PDF in the presence of two absorbing barriers, i.e., the equivalent of Eq. (87) but now with the two boundary conditions

$$P(\varphi = \phi_0) = P(\varphi = \phi_{UV}) = 0. \quad (88)$$

There are two ways to derive the corresponding PDF. The first one is based on the method of images; see, e.g., Refs. [33,34]. We start with the unrestricted PDF  $P_g(\varphi - \phi_{\text{mean}})$ , defined in Eq. (38), centered at  $\phi_{\text{mean}} \equiv \varphi_{\text{cl}} + \langle \delta\varphi_2 \rangle$ . In order to “kill” the contribution of  $P_g$  at  $\phi_{UV}$ , we introduce another “source” located at  $\phi_{UV}^{(1)} \equiv 2\phi_{UV} - \phi_{\text{mean}}$  (this source is the “image” of  $\phi_{\text{mean}}$  with respect to the wall located at  $\phi_{UV}$ ) and weighted with a minus sign. In the same manner, in order to remove the contribution of  $P_g$  at  $\phi_0$ , we introduce a second image located at  $\phi_0^{(1)} \equiv 2\phi_0 - \phi_{\text{mean}}$  still weighted with a minus sign (this source is the image of  $\phi_{\text{mean}}$  with respect to the wall located at  $\phi_0$ ). Therefore, the new PDF is given by

$$\begin{aligned} P_g(\varphi - \phi_{\text{mean}}) - P_g(\varphi - \phi_{UV}^{(1)}) - P_g(\varphi - \phi_0^{(1)}) \\ = P_g(\varphi - \phi_{\text{mean}}) - P_g(\varphi - 2\phi_{UV} + \phi_{\text{mean}}) \\ - P_g(\varphi - 2\phi_0 + \phi_{\text{mean}}). \end{aligned} \quad (89)$$

However, this new distribution function does not yet satisfy the boundary conditions (88) because the source  $\phi_{UV}^{(1)}$  now gives a contribution at  $\phi_0$ . In the same manner, the source  $\phi_0^{(1)}$  gives a contribution at  $\phi_{UV}$ . The cure is obviously to add extra images. Therefore, we introduce the source  $\phi_{UV}^{(2)}$ , which is the image of  $\phi_0^{(1)}$  with respect to the wall located at  $\phi_{UV}$ , i.e.,

$$\phi_{UV}^{(2)} - \phi_{UV} = \phi_{UV} - \phi_0^{(1)} \quad (90)$$

or

$$\phi_{UV}^{(2)} = 2\phi_{UV} + \phi_{\text{mean}} - 2\phi_0. \quad (91)$$

This time the PDF must be weighted with a plus sign since the new contribution is negative. We also introduce the image  $\phi_0^{(2)}$ , the image of  $\phi_{UV}^{(1)}$  with respect to the wall located at  $\phi_0$ . As a consequence, one obtains

$$\phi_0^{(2)} = -2\phi_{UV} + \phi_{\text{mean}} + 2\phi_0. \quad (92)$$

But, as before, the new images will give new contributions on the two walls. Clearly, in order to obtain a distribution that satisfies the boundary conditions (88), this process must be repeated *ad infinitum*; i.e., one must introduce an infinite number of images. The location of the  $n$ th source can be expressed as

$$\phi_0^{(2n)} = \phi_{\text{mean}} + 2n(\phi_0 - \phi_{UV}), \quad (93)$$

$$\phi_{UV}^{(2n)} = \phi_{\text{mean}} + 2n(\phi_{UV} - \phi_0), \quad (94)$$

$$\phi_0^{(2n+1)} = 2\phi_0 - \phi_{\text{mean}} + 2n(\phi_0 - \phi_{UV}), \quad (95)$$

$$\phi_{UV}^{(2n+1)} = 2\phi_{UV} - \phi_{\text{mean}} + 2n(\phi_{UV} - \phi_0). \quad (96)$$

Note the distinction between the positions of the even- and odd-numbered sources ( $n = 0, 1, 2, \dots$ ). Then, the complete distribution is obtained by taking the sum over all  $n$  contributions with their appropriate respective signs, i.e., a minus for the “odd sources” and a plus for the “even sources.” The final expression is given by

$$\begin{aligned} P(\varphi) &= P_g(\varphi - \phi_{\text{mean}}) \\ &- \sum_{n=0}^{\infty} [P_g(\varphi - \phi_0^{(2n+1)}) + P_g(\varphi - \phi_{UV}^{(2n+1)})] \\ &+ \sum_{n=1}^{\infty} [P_g(\varphi - \phi_0^{(2n)}) + P_g(\varphi - \phi_{UV}^{(2n)})] \\ &= \frac{1}{\sqrt{2\pi\langle \delta\varphi_1^2 \rangle}} \\ &\times \sum_{n=-\infty}^{\infty} \left( \exp\left\{ -\frac{[\varphi - \phi_{\text{mean}} + 2n(\phi_{UV} - \phi_0)]^2}{2\langle \delta\varphi_1^2 \rangle} \right\} \right. \\ &\left. - \exp\left\{ -\frac{[\varphi + \phi_{\text{mean}} - 2\phi_0 + 2n(\phi_{UV} - \phi_0)]^2}{2\langle \delta\varphi_1^2 \rangle} \right\} \right). \end{aligned} \quad (97)$$

This formula is very general: It gives the PDF for any Gaussian process (characterized by the variance  $\langle \delta\varphi_1^2 \rangle$  and mean value  $\langle \delta\varphi_2 \rangle$ ) that is restricted between two absorbing barriers.

As mentioned above, there exists another way to reach the above result. This second method consists in starting from the first line of Eq. (38), i.e., the definition  $P = \langle \delta(\varphi - \varphi[\xi]) \rangle$ , but with a Dirac  $\delta$  function compatible with the boundary conditions of Eq. (88). The corresponding representation of the Dirac  $\delta$  function is given by

$$\delta(\varphi - \varphi') = \frac{2}{\phi_{UV} - \phi_0} \sum_{n=1}^{\infty} \sin\left(n\pi \frac{\varphi - \phi_0}{\phi_{UV} - \phi_0}\right) \sin\left(n\pi \frac{\varphi' - \phi_0}{\phi_{UV} - \phi_0}\right). \quad (98)$$

This is a  $\delta$  function in the sense that  $\int_{\phi_0}^{\phi_{UV}} d\varphi \delta(\varphi - \varphi') f(\varphi) = f(\varphi')$  for any function  $f(\varphi)$  and any  $\varphi' \in [\phi_0, \phi_{UV}]$ . From this expression, it is straightforward to establish that

$$\begin{aligned} \langle \delta(\varphi - \varphi[\xi]) \rangle &= \frac{1}{\phi_{UV} - \phi_0} \sum_{n=1}^{\infty} \exp\left[-\frac{1}{2} \left(\frac{n\pi}{\phi_{UV} - \phi_0}\right)^2 \langle \delta\varphi_1^2 \rangle\right] \cos\left[\frac{n\pi}{\phi_{UV} - \phi_0} (\varphi - \varphi_{cl} - \langle \delta\varphi_2 \rangle)\right] \\ &\quad - \frac{1}{\phi_{UV} - \phi_0} \sum_{n=1}^{\infty} \exp\left[-\frac{1}{2} \left(\frac{n\pi}{\phi_{UV} - \phi_0}\right)^2 \langle \delta\varphi_1^2 \rangle\right] \cos\left[\frac{n\pi}{\phi_{UV} - \phi_0} (\varphi + \varphi_{cl} + \langle \delta\varphi_2 \rangle - 2\phi_0)\right]. \end{aligned} \quad (99)$$

Then, by using the identity

$$\sum_{n=-\infty}^{\infty} e^{-(\varphi+n\ell)^2} = \frac{\sqrt{\pi}}{\ell} + \sum_{n=1}^{\infty} \frac{2\sqrt{\pi}}{\ell} e^{-(n\pi/\ell)^2} \cos\left(\frac{2n\pi\varphi}{\ell}\right), \quad (100)$$

it is easy to prove that Eq. (99) is in fact exactly Eq. (97). This PDF is the main result of our paper: It is applicable to any model where the range of variation of the stochastic inflaton field is limited by two absorbing walls. Finally, let us notice that the distribution (97) is not normalized as is expected since the two boundary conditions are absorbing walls.

## B. Inflaton between two absorbing walls

In order to study the behavior of the inflaton field in a KS throat with its boundaries marked by absorbing walls, we now calculate the mean field value  $\langle \varphi \rangle$  by using the geometrically consistent PDF (97) obtained in the previous section. It is given by the following expression:

$$\langle \varphi \rangle = \frac{1}{\mathcal{N}} \int_{\phi_0}^{\phi_{UV}} d\varphi P(\varphi) \varphi, \quad (101)$$

where the ‘‘normalization’’  $\mathcal{N}$  can be calculated from

$$\mathcal{N} = \int_{\phi_0}^{\phi_{UV}} P(\varphi) d\varphi. \quad (102)$$

Lengthy but straightforward calculations show that

$$\begin{aligned} \int_{\phi_0}^{\phi_{UV}} P(\varphi) \varphi d\varphi &= \sum_{n=1}^{\infty} \frac{2}{n\pi} \exp\left[-\frac{n^2\pi^2 \langle \delta\varphi_1^2 \rangle}{2(\phi_{UV} - \phi_0)^2}\right] \\ &\quad \times \sin\left[\frac{n\pi(\phi_0 - \phi_{\text{mean}})}{\phi_{UV} - \phi_0}\right] \\ &\quad \times [\phi_{UV} \cos(n\pi) - \phi_0], \end{aligned} \quad (103)$$

while the normalization  $\mathcal{N}$  is given by

$$\begin{aligned} \mathcal{N} &= \sum_{n=1}^{\infty} \frac{4}{n\pi} \exp\left[-\frac{n^2\pi^2 \langle \delta\varphi_1^2 \rangle}{2(\phi_{UV} - \phi_0)^2}\right] \\ &\quad \times \sin\left[\frac{n\pi(\phi_{\text{mean}} - \phi_0)}{\phi_{UV} - \phi_0}\right] \sin^2\left(\frac{n\pi}{2}\right). \end{aligned} \quad (104)$$

Finally, one can push this reasoning further to calculate the

variance of the field from  $\langle \varphi^2 \rangle$  defined as

$$\langle \varphi^2 \rangle = \frac{1}{\mathcal{N}} \int_{\phi_0}^{\phi_{UV}} d\varphi P(\varphi) \varphi^2. \quad (105)$$

By using the same techniques, one obtains

$$\begin{aligned} \langle \varphi^2 \rangle &= \frac{1}{\mathcal{N}} \sum_{n=1}^{\infty} \frac{2}{n\pi} \exp\left[-\frac{n^2\pi^2 \langle \delta\varphi_1^2 \rangle}{2(\phi_{UV} - \phi_0)^2}\right] \\ &\quad \times \sin\left[\frac{n\pi(\phi_{\text{mean}} - \phi_0)}{\phi_{UV} - \phi_0}\right] \left[ \phi_0^2 - 2\left(\frac{\phi_{UV} - \phi_0}{n\pi}\right)^2 \right] \\ &\quad - \cos(n\pi) \left[ \phi_{UV}^2 - 2\left(\frac{\phi_{UV} - \phi_0}{n\pi}\right)^2 \right]. \end{aligned} \quad (106)$$

It is interesting to check the consistency of the above expressions at the initial time (when  $\varphi = \varphi_{\text{in}}$ ). Initially,  $\langle \delta\varphi_2 \rangle = \langle \delta\varphi_1^2 \rangle = 0$  and the series in Eqs. (103) and (104) can be calculated explicitly by using formulas (1.441.1) and (1.441.3) of Ref. [42]. The result reads  $\int_{\phi_0}^{\phi_{UV}} d\varphi P(\varphi) \varphi = \varphi_{\text{in}}$  and  $\mathcal{N} = 1$ . As a consequence, one checks that  $\langle \varphi \rangle_{\text{in}} = \varphi_{\text{in}}$  as expected.

The quantity  $\langle \varphi \rangle$  is represented in Fig. 3 in the case where the two walls are, respectively, located at  $\phi_0 = 10^{-5} m_{\text{Pl}}$  and  $\phi_{UV} = 10^{-3} m_{\text{Pl}}$ . The influence of the walls is clearly visible from a comparison of the two curves. While the unbounded distribution artificially predicts that the brane can be outside the throat, the ‘‘true’’ quantum trajectory accounting for the geometric restrictions shows an interesting behavior: As the brane is approaching the wall, its position starts to oscillate and then becomes stabilized at a value  $> \phi_0$ . Hence, the presence of the wall prevents  $\varphi$  from violating its geometric limit  $\phi_0$ , i.e., from going beyond the bottom of the throat.

Several words of caution are in order here. First, the above conclusion is valid for particular initial conditions. Clearly, if the parameters are such that the stochastic correction  $\langle \delta\varphi_2 \rangle$  remains small, the presence of the wall is not felt. Second, a much more serious problem is that the perturbative approach is not valid in the regime where the brane position oscillates, i.e., on the left part of Fig. 3. Recall, however, that here we are considering the UV scenario in which the inflaton field value decreases during inflation. Therefore, Fig. 3 should be read ‘‘from right to left’’: In the classical and the perturbative stochastic ap-

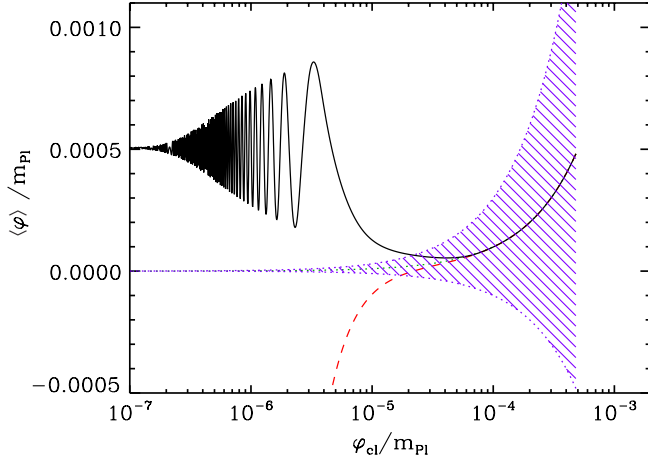


FIG. 3 (color online). Evolution of the (quantum) scalar field in the UV case (hence inflation proceeds from right to left) for  $\alpha = 38$ ,  $\beta = 3.7$ , and  $m \simeq 2.19 \times 10^{-7} m_{\text{Pl}}$  as implied by the COBE normalization (this case corresponds to Fig. 9 of Ref. [17]). The conventions are analogous to those in Fig. 1. The black solid line represents  $\langle \varphi \rangle$  given by Eq. (101) in the case where the two walls are located at  $\phi_0 = 10^{-5} m_{\text{Pl}}$  and  $\phi_{\text{UV}} = 10^{-3}$ . The initial condition is  $\phi_{\text{in}} = 5 \times 10^{-4} m_{\text{Pl}}$ . This figure should be compared to Fig. 2. The influence of the wall on the quantum trajectory is clearly visible.

proach *without the walls* described by Eq. (38) (green dotted and red dashed lines, respectively), the brane travels towards the bottom of the throat, well within the regime of validity of our perturbative treatment. Initially, also the trajectory found *in the presence of the walls* from Eq. (97) (black solid line) overlaps with these, but in the vicinity of  $\phi_0$ , the “geometry-conscious” brane changes direction: It starts climbing upwards in the throat again, as can be seen in Fig. 3. This turnaround occurs within the (hatched) region where the perturbative approach is reliable.

In addition, it seems reasonable to conjecture that the oscillatory behavior mentioned above is real despite the fact that it occurs outside the regime of validity of our approximation. The reason is as follows. In fact, the regime of validity indicates where the calculation of  $\langle \delta \varphi_1^2 \rangle$  and  $\langle \delta \varphi_2^2 \rangle$  is no longer reliable. But it does not limit in any way the validity of Eqs. (97), (103), and (104). Clearly, the oscillatory behavior comes from the peculiar structure of the PDF in Eq. (97), which gives rise to the appearance of trigonometric functions in Eqs. (103) and (104). As long as the stochastic effects grow (and no matter how quantitatively they do, i.e., no matter the detailed behavior of  $\langle \delta \varphi_1^2 \rangle$  and  $\langle \delta \varphi_2^2 \rangle$ ), the brane will feel the wall at some point and, consequently, the trigonometric functions in Eqs. (103) and (104) will start to play a rôle. Hence, it is very likely that the brane position will oscillate even if, with the perturbative method used here, we cannot calculate the fine structure of these oscillations.

Therefore, while the results obtained above probably cannot be considered a fully realistic calculation of the quantum trajectory, we may well take them as an indication that geometric limits are of great importance in stochastic DBI inflation. The finite size of the extra dimensions, modeled by the presence of the two absorbing walls, changes the stochastic corrections to the classical field trajectory considerably.

Finally, we have proven that, for the CKS potential with a constant term, stochastic effects can be dominant and occur near the bottom of the throat. The question of eternal inflation is clearly a more complicated issue. In particular, computing the stationary distribution in this case cannot be done by using the present formalism, even if one can argue (as we have above) that the existence of an eternally inflating regime is likely in brane inflation.

## V. CONCLUSIONS

We now conclude our investigation by revisiting our main results. In this paper, we have generalized the approach of Refs. [20,21] to DBI inflation models, in which the kinetic term of the inflaton is modified through a geometry-imposed upper limit on the field velocity. To this end, we solved the DBI Langevin equation by using a perturbative expansion in the noise. It turns out that the results of Refs. [20,21] essentially only change by additional factors of  $\gamma$ , reflecting the fact that the distinction between long and short wavelength fluctuations, which is at the heart of the stochastic inflationary approach, now is defined with respect to the sound (instead of the Hubble) radius scale.

Our calculation yields easy expressions for the PDFs describing the probability for the patch-averaged inflaton to assume a given value in one Hubble domain, or in the entire Universe by taking into account the size of each averaging domain. We calculated and plotted these PDFs for the example of chaotic Klebanov-Strassler inflation, for the cases both with and without a constant term in the potential. In the absence of a constant term  $V_0$ , stochastic effects are found not to alter the classical field trajectory in a significant way. However, in both the UV ( $\varepsilon = +1$ ) and IR ( $\varepsilon = -1$ ) formulation of a potential including a constant term, the quantum behavior of the field plays a dominant rôle for small field values, i.e., at the bottom of the Klebanov-Strassler throat. Hence, in the UV model, where the brane moves downwards in the throat, these effects occur at the end of inflation, whereas in the IR case, with the brane climbing up the throat, they come into play at the very beginning.

The main result of this paper is our demonstration that an additional subtlety arises due to the geometric restriction which confines the field value of the coarse-grained field between  $\phi_0 < \varphi < \phi_{\text{UV}}$ : To prevent the PDFs from extending below  $\phi_0$  (where the throat—and, consequently, the stringy spacetime itself—ends), we introduced two



absorbing walls located at  $\phi_0$  and  $\phi_{UV}$ . This changes the PDF's shape, and within the bounds of validity it is the “absorbing walls probability density functions” that must be used to describe the stochastic behavior of the inflaton field. It should be interesting to put this technique to use for the “DBI version” of quintessence proposed in Ref. [46].

The question of the existence of an eternally self-reproducing regime in the DBI case is more complicated. We have argued that such a regime does not exist if the potential is of the pure  $m^2\phi^2$  type but that, in the case of an additional constant term  $V_0$ , eternal inflation seems possible. Given the intrinsic validity limitations of our present approach, a fully numerical treatment of the DBI Langevin equation would be necessary to answer this question.

Finally, we would like to comment on possible brane trajectories beyond the single throat model. Globally, the string geometric 6D background is given by a compactified bulk Calabi-Yau space (where the metric is unknown) whose “corners” may comprise several (possibly generalized) Klebanov-Strassler throats with different parameters. Some of these throats can contain additional antibranes at their bottom, like in the original scenario of Ref. [7]. One could therefore imagine that a brane starts out at the bottom of an IR throat, moves upwards towards the bulk, and then transverses a short distance within the bulk before dropping down into a UV throat.

Let us consider qualitatively the quantum effects experienced by a brane following such a trajectory: While in the IR throat, the impact of stochastic inflation is strongest at the very bottom; i.e., quantum effects will push the brane upwards even faster than its classical evolution demands. Having made its way across the bulk, the brane will descend into the UV throat first on a purely classical trajectory. However, as it nears the bottom of the second throat, the stochastic influence grows again, pushing the brane away from the bottom and possibly upwards into the bulk again. At the limit, one may therefore imagine a scenario where the brane is propelled out of a UV throat back into the bulk every time it comes sufficiently close to a throat's bottom.

Since we ignore the formulation of the metric in the bulk, it seems difficult to carry out the full calculation for the brane trajectory sketched above. However, we do know how to describe the metric within each individual throat. As a first toy model, we therefore considered the case of a brane trapped between two absorbing walls located at one throat's bottom and edge, respectively. This successfully prevents the brane from meandering “outside of space-time” (below the bottom of the throat at  $\phi_0$ ), and we have therefore solved the severe conceptual issue stated in the introduction of this paper. Near the edge of the throat at  $\phi_{UV}$ , the technical difficulty of describing the stochastic effects remains because to date the 6D bulk metric remains unknown. The next step would be to write down a joint warp factor for two throats of different depth which are

“glued together” at their edges. Given the quantum behavior found in this paper, our expectations might point us towards an increased probability to find the brane near the “throat matching point.”

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## APPENDIX A: SLOW-ROLL LIMIT

In this short appendix, we justify the use of the equation  $H^2 \simeq \kappa V/3$  in the DBI case and present an alternative derivation of the (classical) slow-roll DBI Klein-Gordon equation. In the standard (non-DBI) case, a convenient tool to study the inflationary evolution is the following hierarchy of slow-roll parameters [47–49]:

$$\epsilon_{n+1} = \frac{d \ln |\epsilon_n|}{dN}, \quad \epsilon_0 \equiv \frac{H_{\text{in}}}{H}. \quad (\text{A1})$$

Then, straightforward manipulations of the background Einstein equations lead to the two following equations:

$$H^2 = \frac{\kappa}{3} \frac{V}{1 - \epsilon_1/3}, \quad (\text{A2})$$

$$(3 - \epsilon_1 + \frac{1}{2}\epsilon_2)H\dot{\phi} = -V', \quad (\text{A3})$$

where a prime denotes a derivative with respect to the inflaton field. In the slow-roll limit where  $\epsilon_1, \epsilon_2 \ll 1$ , we therefore obtain the (Friedmann) equation  $H^2 \simeq \kappa V/3$  and the slow-roll version of the Klein-Gordon equation, namely,  $3H\dot{\phi} \simeq -V'$ .

We now derive the same equations, but for DBI inflation. In this case, the hierarchy (A1) is no longer sufficient, and, in order to describe the evolution of the Lorentz factor  $\gamma$  defined in Eq. (6), we now introduce the additional hierarchy of DBI parameters  $\delta_i$ , the so-called “sound flow parameters”; see, e.g., Refs. [16,17]. These are defined as

$$\delta_{n+1} = \frac{d \ln |\delta_n|}{dN}, \quad \delta_0 \equiv \frac{\gamma}{\gamma_{\text{in}}}. \quad (\text{A4})$$

To find the “slow-roll” version of Eqs. (4) and (5) (where by slow-roll we now understand that both of the parameter sets  $\epsilon_i, \delta_i \ll 1$ ), we use the DBI background equations and the definition of  $\gamma$  in Eq. (6) to show that the following expression holds:

$$\dot{\phi}^2 = \frac{2}{\kappa\gamma} H^2 \epsilon_1, \quad (\text{A5})$$

from which we can deduce that

$$\ddot{\phi} = \frac{H}{2} \dot{\phi} (\epsilon_2 - 2\epsilon_1 - \delta_1). \quad (\text{A6})$$

To replace the terms proportional to  $T'$  in Eq. (5), we multiply the expression for  $\delta_1$  (see Ref. [17]) by  $\dot{\phi}$  and use Eq. (A5) to find

$$T' = -\frac{V'}{\gamma-1} - 3H\frac{\gamma}{\gamma-1}\dot{\phi} - H\dot{\phi}\frac{\gamma^3}{(\gamma^2-1)(\gamma-1)}. \quad (\text{A7})$$

When we use these  $\epsilon_i, \delta_i$  to rewrite Eqs. (4) and (5), we therefore find

$$H^2 = \frac{\kappa}{3} \frac{V}{1 - 2\epsilon_1\gamma/(3\gamma+3)}, \quad (\text{A8})$$

$$\frac{\dot{\phi}}{2} \left[ \epsilon_2 - 2\epsilon_1 + \frac{\delta_1}{(\gamma+1)} + \frac{3(\gamma+1)}{\gamma} \right] = -\frac{\gamma+1}{2\gamma^2} \frac{V'}{H}. \quad (\text{A9})$$

As announced, in the limit where both  $\epsilon_i, \delta_i \ll 1$ , one recovers that  $H^2 \simeq \kappa V/3$  and Eq. (10). Note that in Eq. (A9) both the  $\epsilon_i$  and  $\delta_i$  are supposed small to obtain Eq. (10), while it is sufficient to have  $\epsilon_1 \ll 1$  to rederive the slow-roll version of the Friedmann equation.

## APPENDIX B: PERTURBED KLEIN-GORDON EQUATION

In this appendix, we briefly sketch how to obtain the formulation (17) for the DBI Klein-Gordon equation at the linearly perturbed level. For the case of a standard inflaton field, it is well known that the Klein-Gordon equation for linear perturbations of the field  $\delta\phi_k$  and scalar perturbations of the metric  $\Phi_k$  reads [50]

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + V''\right)\delta\phi_k = 4\dot{\phi}\dot{\Phi}_k - 2V'\Phi_k. \quad (\text{B1})$$

One can express  $V'$  and  $V''$  in terms of the slow-roll parameters  $\epsilon_i$  defined in Appendix A,

$$V' = -\frac{z}{a} H^2 \left(3 - \epsilon_1 + \frac{\epsilon_2}{2}\right), \quad (\text{B2})$$

$$V'' = H^2 \left(6\epsilon_1 - \frac{3}{2}\epsilon_2 - 2\epsilon_1^2 - \frac{\epsilon_2^2}{4} + \frac{5}{2}\epsilon_1\epsilon_2 - \frac{1}{2}\epsilon_2\epsilon_3\right), \quad (\text{B3})$$

where the function  $z = a\dot{\phi}/H$ . Moreover, we have [50]

$$\dot{\Phi}_k = H\epsilon_1 \frac{a}{z} \delta\phi_k - H\Phi_k. \quad (\text{B4})$$

Therefore, by using Eqs. (B2)–(B4) we can rewrite Eq. (B1) as

$$\begin{aligned} \delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + H^2 \left( \frac{k^2}{a^2 H^2} + 2\epsilon_1 - \frac{3}{2}\epsilon_2 - 2\epsilon_1^2 - \frac{\epsilon_2^2}{4} \right. \\ \left. + \frac{5}{2}\epsilon_1\epsilon_2 - \frac{\epsilon_2\epsilon_3}{2} \right) \delta\phi_k = \frac{H^2 z}{a} \Phi_k (2 - 2\epsilon_1 + \epsilon_2). \end{aligned} \quad (\text{B5})$$

Note that precisely the same equation is found by working backwards from the well-known evolution of the standard Mukhanov-Sasaki variable  $v_k$  [in conformal time  $\eta$ , where  $dt = a d\eta$ , Eq. (B6) simplifies to  $v_k'' + (k^2 - z''/z)v_k = 0$ ; compare Eq. (20)]:

$$\ddot{v}_k + H\dot{v}_k + \left(\frac{k^2}{a^2} - \frac{\dot{z} + H\dot{z}}{z}\right)v_k = 0, \quad (\text{B6})$$

which is defined as

$$v_k = a\delta\phi_k + z\Phi_k, \quad (\text{B7})$$

To see this, express the time derivatives of  $z$  in terms of the  $\epsilon_i$  parameters and use Eq. (B4) as well as its derivative along with

$$\begin{aligned} \frac{z}{a} \frac{k^2}{a^2} \Phi_k = \frac{\epsilon_1 H^2}{2} (\epsilon_2 - 2\epsilon_1) \delta\phi_k - \epsilon_1 H \delta\dot{\phi}_k \\ + \epsilon_1 H^2 \frac{z}{a} \Phi_k, \end{aligned} \quad (\text{B8})$$

which is a consequence of the Einstein equations [50].

We now proceed to deriving the analogous equations for a scalar field with DBI dynamics. Not surprisingly, the expressions are significantly more involved in this case, and the perturbed DBI Klein-Gordon equation reads

$$\begin{aligned} \delta\ddot{\phi}_k + \left\{ 3H \frac{3 - 2\gamma^2}{\gamma^2} - \frac{3(\gamma^2 - 1)}{\dot{\phi}\gamma^3} [V' + (\gamma - 1)T'] \right\} \delta\dot{\phi}_k \\ + \left\{ \frac{k^2}{a^2\gamma^2} + \frac{V''}{\gamma^3} - \frac{T''}{2\gamma^3} (\gamma^3 - 3\gamma + 2) + \frac{\gamma^2 - 1}{2\gamma} \frac{T'}{T} \right. \\ \left. \times \left[ \frac{6H}{\gamma} \dot{\phi} + \frac{3V'}{\gamma^2} + \frac{3T'}{\gamma^2} (\gamma - 1) \right] \right\} \delta\phi_k \\ = \frac{4}{\gamma^2} \dot{\Phi}_k \dot{\phi} + \frac{1}{\gamma^2} \Phi_k \left[ -\frac{(\gamma^2 - 1)^2}{\gamma^2} 6H\dot{\phi} \right. \\ - \frac{V'}{\gamma^3} (3\gamma^4 - 3\gamma^2 + 2) \\ \left. - \frac{T'}{\gamma^3} (3\gamma^5 - 3\gamma^4 - 4\gamma^3 + 3\gamma^2 + 3\gamma - 2) \right]. \end{aligned} \quad (\text{B9})$$

Despite its complicated appearance, it is easy to see that this equation reduces to Eq. (B1) upon setting  $\gamma = 1$ . Lengthy but straightforward calculations allow one to express each term in Eq. (B9) in terms of the  $(\epsilon_i, \delta_i)$  parameter hierarchy defined in Appendix A, in close analogy to the expressions (B2) and (B3) in the standard case.

In addition, the DBI version of Eq. (B4) reads

$$\dot{\Phi}_k = H\epsilon_1 \frac{a\gamma^{3/2}}{z} \delta\phi_k - H\Phi_k, \quad (\text{B10})$$

now with  $z = a\gamma^{3/2}\dot{\phi}/H$ . Combining all of these ingredients, one arrives at Eq. (17) in Sec. II B, which is equivalent to Eq. (B5) for the case of standard inflation in which  $\gamma = 1$ ,  $\delta_i = 0$ .

Again, Eq. (17) could also have been obtained from

$$\ddot{v}_k + H\dot{v}_k + \left( \frac{k^2}{a^2\gamma^2} - \frac{\ddot{z} + H\dot{z}}{z} \right) v_k = 0, \quad (\text{B11})$$

where in the DBI case the definition of the Mukhanov-

Sasaki variable is

$$v_k = a\gamma^{3/2}\delta\phi_k + z\Phi_k, \quad (\text{B12})$$

but with the new  $z$  defined below Eq. (B10). Here we have used

$$\begin{aligned} \frac{z}{a\gamma^{3/2}} \frac{k^2}{a^2\gamma^2} \Phi_k &= \frac{\epsilon_1 H^2}{2} (\epsilon_2 - 2\epsilon_1 - \delta_1) \delta\phi_k - \epsilon_1 H \delta\dot{\phi}_k \\ &+ \epsilon_1 H^2 \frac{z}{a\gamma^{3/2}} \Phi_k, \end{aligned} \quad (\text{B13})$$

as found from the Einstein equations with DBI scalar matter.

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