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F. Finelli,[1,2] G. Marozzi,[3] A. A. Starobinsky,[4] G. P. Vacca,[5,2] and G. Venturi[5,2]

[1]*INAF/IASF Bologna, Istituto di Astrofisica Spaziale e Fisica Cosmica di Bologna, via Gobetti 101, I-40129 Bologna, Italy*

[2]*INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy*

[3]*GRεCO—Institut d'Astrophysique de Paris, UMR7095, CNRS, Université Pierre and Marie Curie, 98 bis boulevard Arago, 75014 Paris, France*

[4]*Landau Institute for Theoretical Physics, Moscow, 119334, Russia*

[5]*Dipartimento di Fisica, Università degli Studi di Bologna, via Irnerio, 46, I-40126 Bologna, Italy*

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We compute the growth of the mean square of quantum fluctuations of test fields with small effective mass during a slowly changing, nearly de Sitter stage which takes place in different inflationary models. We consider a minimally coupled scalar with a small mass, a modulus with an effective mass $\propto H^2$ (with H the Hubble parameter), and a massless nonminimally coupled scalar in the test field approximation and compare the growth of their relative mean square with the one of gauge-invariant inflaton fluctuations. We find that in most of the single field inflationary models the mean square gauge-invariant inflaton fluctuation grows *faster* than any test field with a non-negative effective mass. Hybrid inflationary models can be an exception: the mean square of a test field can dominate over the gauge-invariant inflaton fluctuation one on suitably chosen parameters. We also compute the stochastic growth of quantum fluctuations of a second field, relaxing the assumption of its zero homogeneous value, in a generic inflationary model; as a main result, we obtain that the equation of motion of a gauge-invariant variable associated, order by order, with a generic quantum scalar fluctuation during inflation can be obtained only if we use the number of e-folds as the time variable in the corresponding Langevin and Fokker-Planck equations for the stochastic approach. We employ this approach to derive some bounds for the case of a model with two massive fields.

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I. INTRODUCTION

The theory of quantum fields in an expanding universe has evolved from its pioneering years [1] into a necessary tool in order to describe the Universe on large scales. The de Sitter background—characterized by the Hubble parameter $H \equiv \dot{a}/a$ being constant in time (for the flat spatial slice), where $a(t)$ is the scale factor of a Friedmann-Robertson-Walker (FRW) cosmological model—has been the main arena to compute quantum effects even before becoming a pillar of our understanding of the early inflationary stage and of the recent acceleration of the Universe.

However, while $|\dot{H}| \ll H^2$ for any inflationary model, $\dot{H}$ may not become zero in a viable model, apart from some isolated moments of time. Indeed, the standard slow-roll expression for the power spectrum of the adiabatic mode of primordial scalar (density) perturbations becomes infinite, i.e., meaningless, if $\dot{H}$ becomes zero during inflation.[1] Outside the slow-roll approximation, $\dot{H}$ may reach zero [3], but for a moment only. Therefore, the study of quantum effects in a nearly de Sitter stage with $\dot{H} \neq 0$, in particular, when the total change in H during inflation is not small compared with its value during the last e-folds of inflation [4–7] (see also the recent papers [8–10]), is not of just pure theoretical interest. Among the main results of previous investigations, it has been shown that the infrared growth of minimally coupled scalar fields with a zero or small mass $m \ll H$ in a background with a practically constant H [11–13] occurs for massive fields when H changes significantly during inflation [4,5], and that the stochastic approach, originally mainly applied to a new inflationary type background with a small change in H during inflation [14], also works in realistic chaotic type inflationary space-times [7] (here we do not discuss its application to eternal inflation [15,16] and to interactions in a de Sitter background [17–19]). A particle production in a realistic inflationary background is so different from the corresponding one in the de Sitter space-time that it has prompted us to reconsider the amplification of nearly massless minimally coupled scalar fields in inflation with a quadratic potential [4,7] and compare it with the dynamics driven by a scalar field condensate.

In this paper we wish to tackle in more detail the moduli problem issue. On one hand we wish to extend our results

[1]This is why the statement sometimes found in literature that, for $H = \text{const}$, a flat (Harrison-Zeldovich) $n_s = 1$ spectrum is generated is also meaningless. Actually, it is a $V(\phi) \propto \phi^{-2}$ inflaton potential that leads to $n_s = 1$ (and $r \neq 0$) in the slow-roll approximation. See [2] for the exact solution for $V(\phi)$ without using this approximation.

in [7] to different inflationary models and to different types of test fields, not only to massive minimally coupled scalar fields, but also to massless nonminimally coupled scalar fields and moduli with an effective mass $\propto H^2$. We investigate effects $\mathcal{O}(\dot{H})$ and we therefore need to consider both these extensions. On the other hand, it is known that the mean square of gauge-invariant inflaton fluctuations grows [5] and the stochastic description for this effect for a general potential has been established [7]; it is therefore interesting to compare the quantum amplification of test fields not only with the background inflaton dynamics, but also with the stochastic growth of gauge-invariant inflaton fluctuations. This comparison aims for a self-consistent understanding of quantum foam during inflation.

We then discuss the diffusion equation for general scalar fluctuations in a generic model of inflation. On using the results obtained by field theory methods, we show that the diffusion equation for the gauge-invariant variable associated with this generic scalar fluctuation should be formulated in terms of the number of e-folds N.

The paper is organized as follows. In Sec. II we choose four representative cases of the inflationary ''zoo'' on which we focus in this paper. In Sec. III we compute the stochastic growth of a minimally coupled scalar with a small mass, a modulus with an effective mass $\propto H^2$, and a massless nonminimally coupled scalar in the test field approximation for the four inflationary models considered. In Sec. IV we review the result obtained in Ref. [7] for gauge-invariant inflaton fluctuations and compare it with the growth of the test fields in the four different inflationary models considered. In Sec. V we examine the stochastic approach for a two field model for two generic self-interacting potentials, and we compare our solution, obtaining some constraints on the parameters, for a particular two quadratic field model. In Sec. VI we illustrate our conclusions.

Our paper does not include a derivation of the stochastic approach. Indeed an introduction of the stochastic method is given in Refs. [14,18] and a comparison between stochastic methods and quantum field theory results is done in Refs. [14,18] and in our previous paper [7].

II. INFLATIONARY MODELS

The detailed evolution of the expansion during the accelerated stage depends on the inflaton potential and so does the growth of quantum fluctuations. For this reason we consider in the following four different potentials which are representative of the ''inflationary zoo.'' Since we shall study the growth of quantum fluctuations as a function of the number of e-folds

$$N = \log\frac{a(t)}{a(t_i)}, \tag{1}$$

we shall give the evolution of the Hubble parameter as a function of N.

The first obvious model is chaotic quadratic inflation, which we have also used in our previous investigations [4–6]:

$$V(\phi) = \frac{m^2}{2}\phi^2. \tag{2}$$

During the slow-roll trajectory we have

$$H^2 \simeq H_i^2 - \tfrac{2}{3}m^2 N, \tag{3}$$

$$\dot{\phi} \simeq -\sqrt{\tfrac{2}{3}}mM_{\rm pl}, \tag{4}$$

where $M_{\rm pl}^{-2} = 8\pi G$ (these formulas were obtained already in [20] in the context of a closed bouncing FRW universe with two quasi–de Sitter stages during contraction and expansion). Let us note that in the numerical results presented in the figures all dimensional quantities have been rescaled with respect to $m_{\rm pl} = M_{\rm pl}\sqrt{8\pi}$. We then consider the case of a quadratic potential (of arbitrary sign) uplifted with an offset V_0:

$$V(\phi) = V_0 \pm \frac{M^2}{2}\phi^2. \tag{5}$$

With the positive sign, the potential in Eq. (5) is an approximation for the simplest model of hybrid inflation well above the scale of the end of inflation; in this case ϕ decreases during the inflationary expansion. With the negative sign the potential in Eq. (5) is a simple small field inflation model, again far from the end of inflation; in this case ϕ increases during the inflationary expansion. In the following we use the approximate solution for the square of the inflaton as

$$\phi^2(N) \simeq \phi_i^2 e^{\mp 2((M^2M_{\rm pl}^2)/V_0)(N-N_i)}, \tag{6}$$

valid when $H \simeq \sqrt{V_0/(3M_{\rm pl}^2)}$, i.e., $\frac{M^2}{V_0}|\phi^2 - \phi_i^2| \ll 1$.[2] The exact expression can be given in the implicit form $N - N_i = \mp(V_0/(2M^2M_{\rm pl}^2))\log(\phi^2/\phi_i^2) - (\phi^2 - \phi_i^2)/(4M_{\rm pl}^2)$.

As another large field inflationary model we consider an exponential potential

$$V = V_0 e^{-(\lambda/M_{\rm pl})\phi}. \tag{7}$$

This potential leads to a power-law expansion given by

$$a(t) = \left(\frac{t}{t_i}\right)^p, \qquad H(t) = \frac{p}{t} \tag{8}$$

with $p = 2/\lambda^2$ [21] [we consider $a(t_i) = 1$ for the sake of simplicity]. Such a solution is stable for $p > 1$ and the slow-roll conditions are well satisfied for $p \gg 1$. In par-

[2]For a double well spontaneous symmetry breaking potential $V = \lambda(\phi^2 - \bar{\phi}^2)^2$, which may be approximated by Eq. (5) for small field values, the condition becomes $|(\phi^2 - \phi_i^2)/\bar{\phi}^2| \ll 1$ together with $\phi_i^2/\bar{\phi}^2 \ll 1$, which again means that $V(\phi)$ is always considered far from the minima.

ticular one obtains

$$H = H_i e^{-N/p} = \frac{p}{t_i} e^{-N/p}, \tag{9}$$

$$\phi = \sqrt{\frac{2}{p}} M_{\rm pl} N + \phi_i. \tag{10}$$

III. GROWTH OF TEST FIELDS WITH SMALL EFFECTIVE MASS IN THE STOCHASTIC APPROACH

We shall consider a test scalar field with a small effective mass and a zero homogeneous expectation value on an inflationary background driven by an inflaton with potential $V(\phi)$ in the slow-roll approximation. The evolution equation for the renormalized mean square $\langle\chi^2\rangle_{\rm REN}$ (the pedix REN will denote renormalized in the following) in the next three subsections [Eqs. (11), (16), and (22)] follows in a straightforward manner from our previous paper [7]. Note that the right-hand side of Eqs. (11), (16), and (22) representing the contribution of created fluctuations (''particles'') is obtained under the natural assumption of the absence of particles in the in-vacuum state, more exactly that each Fourier mode $\mathbf{k}$ of the quantum field χ was in the adiabatic vacuum state deep inside the Hubble radius and long before the first Hubble radius crossing during inflation, i.e., when its energy $\omega = k/a(t)$ was much larger than $H(t)$.[3] So the explicit time asymmetry of Eq. (11) shows that this in-vacuum is not de Sitter invariant; it is unstable and creation of fluctuations (particles) of light scalar fields, as well as metric perturbations, takes place. In turn, the cause of this instability may be finally traced to the expansion of the Universe.

A. Growth of scalar fields with m_χ^2 in the stochastic approach

The stochastic equation is

$$\frac{d\langle\chi^2\rangle_{\rm REN}}{dN} + \frac{2m_\chi^2}{3H^2(N)}\langle\chi^2\rangle_{\rm REN} = \frac{H^2(N)}{4\pi^2}. \tag{11}$$

Its general solution is

$$\langle\chi^2\rangle_{\rm REN} = \left(\langle\chi^2\rangle_{\rm REN}(N_i) + \int^N dn \frac{H^2(n)}{4\pi^2} \times e^{\int^n ((2m_\chi^2)/(3H^2(\tilde{n})))d\tilde{n}}\right) e^{-\int^N ((2m_\chi^2)/(3H^2(n)))dn}, \tag{12}$$

which is just the integral form of Eq. (13) of Ref. [7] generalized to an arbitrary inflaton potential.

For the quadratic inflaton case, we report here the solution given in Ref. [7]:[4]

$$\langle\chi^2\rangle_{\rm REN} = \frac{3H^{2(m_\chi^2/m^2)}}{8\pi^2(2m^2 - m_\chi^2)}(H_i^{4-2(m_\chi^2/m^2)} - H^{4-2(m_\chi^2/m^2)}), \tag{13}$$

where we have assumed $\langle\chi^2\rangle_{\rm REN}(N_i) = 0$ (we shall adopt the same choice afterwards if not otherwise stated). We then consider the potentials in Eq. (5), in the lowest order approximation; that is, for $V \simeq V_0 = 3H_0^2 M_{\rm pl}^2$, we have

$$\langle\chi^2\rangle_{\rm REN} \simeq \frac{3H_0^4}{8\pi^2 m_\chi^2}(1 - e^{-((2m_\chi^2)/(3H_0^2))N}). \tag{14}$$

Let us note that the corrections induced by a nonzero $M^2\phi^2/V_0$ term are typically small both for the case of hybrid inflation as well as for small field inflation (as long as the field does not grow too much due to instability). The corresponding analytic expressions, obtained using Eq. (6), can be written in terms of hypergeometric functions but we do not report them here. For the exponential potential we obtain

$$\langle\chi^2\rangle_{\rm REN} = \frac{p}{8\pi^2} H_i^2 \exp\left(-\frac{p}{3}\frac{m_\chi^2}{H^2}\right)\left[-\exp\left(\frac{p}{3}\frac{m_\chi^2}{H^2}\right)\frac{H^2}{H_i^2} + \frac{p}{3}\frac{m_\chi^2}{H_i^2} Ei\left(\frac{p}{3}\frac{m_\chi^2}{H^2}\right) + \exp\left(\frac{p}{3}\frac{m_\chi^2}{H_i^2}\right) - \frac{p}{3}\frac{m_\chi^2}{H_i^2} Ei\left(\frac{p}{3}\frac{m_\chi^2}{H_i^2}\right)\right], \tag{15}$$

where E_i is the exponential integral function (see, for example, [23]).

B. Growth of moduli fields with $m_\chi^2 = cH^2$ in the stochastic approach

If $|c| \ll 1$, the stochastic equation takes the form

$$\frac{d\langle\chi^2\rangle_{\rm REN}}{dN} + \frac{2c}{3}\langle\chi^2\rangle_{\rm REN} = \frac{H^2(N)}{4\pi^2}. \tag{16}$$

Its general solution is

$$\langle\chi^2\rangle_{\rm REN} = \left(\langle\chi^2\rangle_{\rm REN}(N_i) + \int^N dn \frac{H^2(n)}{4\pi^2} e^{(2/3)cn}\right) e^{-(2/3)cN}. \tag{17}$$

[3] This assumption also means the absence of the so-called trans-Planckian particle creation. See [22] and references therein for a discussion of why trans-Planckian particle creation should be absent in the standard quantum field theory.

[4] For the particular value $m_\chi^2 = 2m^2$ we obtain

$$\langle\chi^2\rangle_{\rm REN} = \frac{3H^4}{4\pi^2 m^2}\log\left(\frac{H_i}{H}\right).$$

For $V(\phi) = m^2\phi^2/2$ we obtain

$$\langle\chi^2\rangle_{\rm REN} = \frac{m^2}{6\pi^2}\left[(1 - e^{-(2/3)cN})\left(\frac{9}{4c^2} + \frac{3}{2c}N_T\right) - \frac{3}{2c}N\right], \tag{18}$$

where $N_T = 3H_i^2/(2m^2)$ is equal to maximal number of possible e-folds in this chaotic model. In the limiting case $c \to 0$ and at the end of inflation ($N = N_T - 3/2$), we recover the result[5]

$$\langle\chi^2\rangle_{\rm REN} \simeq \frac{m^2}{12\pi^2}N_T^2 = \frac{3H_i^4}{16\pi^2m^2}. \tag{19}$$

For the potential in Eq. (5), we consider the lowest approximation as in the previous subsection. Therefore the result can be simply obtained by substituting $m_\chi^2 = cH_0^2$ in Eq. (14):

$$\langle\chi^2\rangle_{\rm REN} = \frac{3H_0^2}{8\pi^2c}(1 - e^{-(2/3)cN}). \tag{20}$$

For the power-law inflation case we obtain

$$\langle\chi^2\rangle_{\rm REN} = \frac{p}{8\pi^2}H_i^2\left(c\frac{p}{3} - 1\right)^{-1}(e^{-2(N/p)} - e^{-(2/3)cN}). \tag{21}$$

Let us note that $m^2 = V_{\phi\phi} = \frac{6}{p}(1 - \frac{1}{3p})H^2$ for this particular model, so the results above are also valid for the case $m_\chi^2 = \tilde{c}m^2$ with $\tilde{c} = \frac{p}{6}(1 - \frac{1}{3p})^{-1}c$.

C. Growth of nonminimally coupled scalar fields in the stochastic approach

The stochastic equation is now

$$\frac{d\langle\chi^2\rangle_{\rm REN}}{dN} + 4\xi(2 - \epsilon)\langle\chi^2\rangle_{\rm REN} = \frac{H^2(N)}{4\pi^2}, \tag{22}$$

where ξ is the nonminimal coupling to the Ricci scalar R and we assume that $|\xi| \ll 1$ (however, ξN may be large). Indeed, the term in the action proportional to $\xi\chi^2R$ gives an effective time dependent mass for χ: $m_\chi^2 = 6\xi H^2(2 - \epsilon)$ where $\epsilon = -\frac{\dot{H}}{H^2}$.

Its general solution is

$$\langle\chi^2\rangle_{\rm REN} = \left[\langle\chi^2\rangle_{\rm REN}(N_i) + \int^N dn\,\frac{H^{2+4\xi}(n)}{4\pi^2H_i^{4\xi}}e^{8\xi n}\right] \times\left(\frac{H_i}{H(N)}\right)^{4\xi}e^{-8\xi N}. \tag{23}$$

If we again consider the chaotic scenario induced by a massive inflaton field as in the previous subsections, the integral can be easily computed in a closed form in terms of the exponential integral function $E_\nu(z)$. Assuming $\langle\chi^2\rangle_{\rm REN} = 0$ initially, one finds

$$\langle\chi^2\rangle_{\rm REN} \simeq \frac{m^2}{6\pi^2}\frac{e^{8\xi(N_T-N)}}{(N_T - N)^{2\xi}}[(N_T - N)^{2+2\xi} \times E_{-1-2\xi}(8\xi(N_T - N)) - (N_T - N_i)^{2+2\xi} \times E_{-1-2\xi}(8\xi(N_T - N_i))]. \tag{24}$$

One can verify that in the limit $\xi \to 0$ at the end of inflation and a fixed large value for N_T, the result of Eq. (19) for a massless modulus is again reobtained.

For the potential in Eq. (5) we have, in the same approximation as in the two previous subsections,

$$\langle\chi^2\rangle_{\rm REN} = \frac{H_0^2}{32\pi^2\xi}(1 - e^{-8\xi N}). \tag{25}$$

As before, let us finish with the case of a power-law model of inflation. We obtain

$$\langle\chi^2\rangle_{\rm REN} = \frac{p}{8\pi^2}H_i^2(-2\xi - 1 + 4p\xi)^{-1} \times (e^{-2(N/p)} - e^{\xi N((4/p)-8)}). \tag{26}$$

IV. COMPARISON WITH THE GROWTH OF INFLATON FLUCTUATIONS

The results of the previous section should be compared with the growth of gauge-invariant inflaton fluctuations $\delta\phi$, the Mukhanov variable [24] which is used to canonically quantize the Einstein-Klein-Gordon Lagrangian. The evolution equation for $\langle\delta\phi^2\rangle_{\rm REN}$ found in [7] can be rewritten as

$$\frac{d\langle\delta\phi^2\rangle_{\rm REN}}{dN} + 2(\eta - 2\epsilon)\langle\delta\phi^2\rangle_{\rm REN} = \frac{H^2(t)}{4\pi^2}, \tag{27}$$

where

$$\epsilon = \frac{M_{\rm pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2, \qquad \eta = M_{\rm pl}^2\frac{V_{\phi\phi}}{V}. \tag{28}$$

In Eq. (27) the positivity of $\eta - 2\epsilon$ is not determined by the convexity of the potential, i.e., $V_{\phi\phi} > 0$, as we would expect in the absence of metric perturbations. The threshold corresponds to the following condition on the potential:

$$\frac{d}{d\phi}\left(\frac{V_\phi}{V}\right) > 0. \tag{29}$$

With the use of the slow-roll expressions for the scalar spectral index n_s and the tensor-to-scalar ratio r, one obtains

$$n_s - 1 = -6\epsilon + 2\eta, \tag{30}$$

$$r = 16\epsilon, \tag{31}$$

and Eq. (27) can be rewritten as

[5]This corresponds to the massless limit of moduli production computed in Eq. (15) of [7] for $\alpha \to 0$.

$$\frac{d\langle\delta\phi^2\rangle_{\rm REN}}{dN} + \left(n_s - 1 + \frac{r}{8}\right)\langle\delta\phi^2\rangle_{\rm REN} = \frac{H^2(t)}{4\pi^2}. \quad (32)$$

Both Eqs. (29) and (32) tell us that power-law inflation, for which $n_s - 1 = -r/8$ holds, lies at the threshold between two opposite behaviors. Power-law inflation with $78 < p < 246$ is allowed at the 95% confidence level [25]. We note that Eq. (32) is the same for a modulus with the mass $m_\chi^2 = cH^2$ and $c = 3(n_s - 1 + r/8)/2$; below the power-law inflation line inflaton fluctuations behave as a modulus with negative c.

The solution of Eq. (27) is

$$\langle\delta\phi^2\rangle_{\rm REN} = \frac{\epsilon(N)}{4\pi^2}\int^N dn\frac{H^2(n)}{\epsilon(n)}. \quad (33)$$

For the quadratic chaotic potential the solution was found in [7]

$$\langle\delta\phi^2\rangle_{\rm REN} = \frac{H_i^6 - H^6}{8\pi^2 m^2 H^2}. \quad (34)$$

For the potential in Eq. (5) we obtain the following in the lowest nontrivial approximation for small M^2-dependent corrections in the potential:

$$\langle\delta\phi^2\rangle_{\rm REN} \simeq \pm\frac{V_0^2}{24\pi^2 M^2 M_{\rm pl}^4}\left[1 - e^{\mp((2M^2M_{\rm pl}^2)/V_0)(N-N_i)}\right], \quad (35)$$

where, as discussed previously, the case with a minus sign in the exponent refers to the hybrid model whereas the other case is associated with the small field inflationary model. Let us note that for a small value of the exponent (an almost constant potential) or for very small $(N - N_i)$, by expanding up to the linear order, one obtains almost the case of a de Sitter background ($M^2 = 0$), with $\langle\delta\phi^2\rangle_{\rm REN}$ linearly growing in N. In this approximation we see that the hybrid model is characterized by $\langle\delta\phi^2\rangle_{\rm REN}$ bounded as $N \to \infty$. This statement remains true even after dropping our approximations (see below).

This leads, by invoking the consistency of the perturbative expansion in field fluctuations through the condition $\langle\delta\phi^2\rangle_{\rm REN} \ll \phi_i^2$, to the following hierarchy which the inflationary model has to satisfy:

$$\frac{V_0}{24\pi^2 M_{\rm pl}^4} \ll \frac{M^2\phi_i^2}{V_0} \ll 1. \quad (36)$$

We have also computed the expression for the fluctuations by solving Eq. (27) using the expression in Eq. (6) with no further approximations. In this more general case we obtain

$$\langle\delta\phi^2\rangle_{\rm REN} \simeq \pm\frac{4V_0^2(1-y) + 3M^4\phi_i^2 y(4M_{\rm pl}^2(N-N_i) + \phi_i^2(1-y)) \pm y(1-y^2)\frac{M^6}{4V_0}}{96\pi^2 M^2 M_{\rm pl}^4(1 \pm y\frac{M^2\phi_i^2}{2V_0})^2}, \quad (37)$$

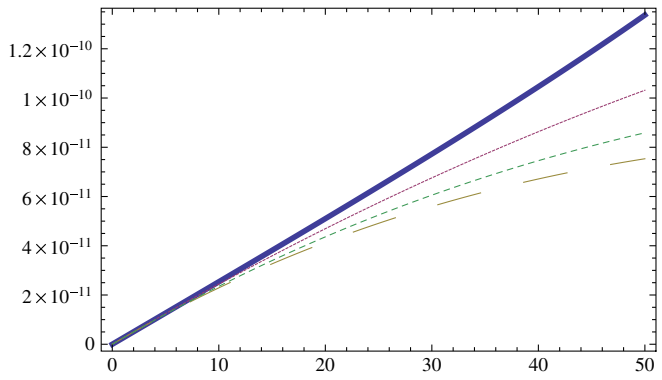

FIG. 1 (color online). Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the quadratic chaotic model. For the inflationary background we have chosen the inflationary trajectory in Eq. (3) with $m = 10^{-6}m_{\rm pl}$ and $H_i = 10m$. The mean square gauge-invariant inflaton fluctuation (thick line) dominates over those of test fields ($m_\chi \simeq 0.3m$ is the solid line, $c = 0.02$ is the dashed line, $\xi = 0.001$ is the dotted line).

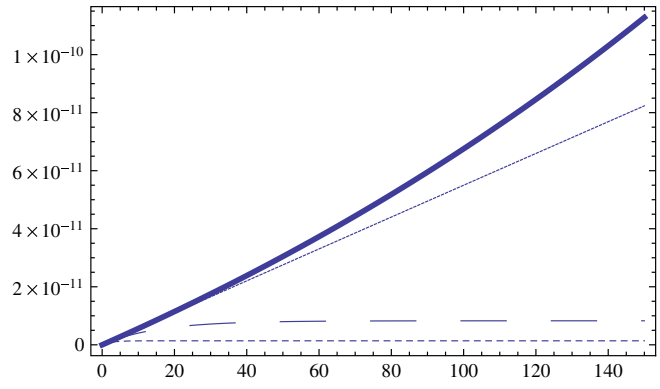

FIG. 2 (color online). Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the small field inflationary model in Eq. (5). For the inflationary background we have chosen $V_0 = 2.6 \times 10^{-12}m_{\rm pl}^4$, $M = 0.85 \times 10^{-6}m_{\rm pl}$, and $\phi_i = 0.3m_{\rm pl}$ as parameters. The mean square gauge-invariant inflaton fluctuation (thick line) dominates over those of test fields ($m_\chi = 10^{-2}H_0$ is the solid line, $c = 0.1$ is the dashed line, $\xi = 0.05$ is the dotted line).

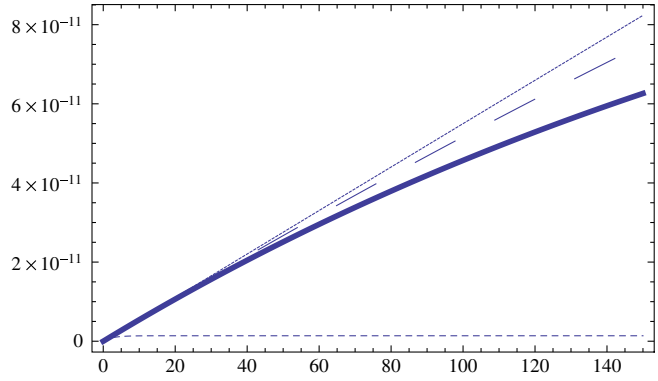

FIG. 3 (color online). Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the hybrid model in Eq. (5). For the inflationary background we have chosen $V_0 = 2.6 \times 10^{-12} m_{\rm pl}^4$, $M = 1.8 \times 10^{-6} m_{\rm pl}$, and $\phi_i = 0.3 m_{\rm pl}$ as parameters. In this case the mean square of moduli can dominate over the mean square of gauge-invariant inflaton fluctuation (thick line): the parameters chosen are $m_\chi = 10^{-2} H_0$ (solid line), $c = 0.002$ (dashed line), $\xi = 0.05$ (dotted line).

where we have set $y = y(N) = e^{\mp((2M^2 M_{\rm pl}^2)/V_0)(N-N_i)}$. From this expression, when analyzing the hybrid inflation case, one can notice that the fluctuations have a maximum for a certain amount of e-folds N and then decay to the asymptotic value for a large number of e-folds. Nevertheless, such a maximum is typically a few percent above the asymptotic value which has been already obtained above using a more crude approximation.

For the exponential potential we find

$$\langle \delta\phi^2 \rangle_{\rm REN} = \frac{p}{8\pi^2}(H_i^2 - H^2), \tag{38}$$

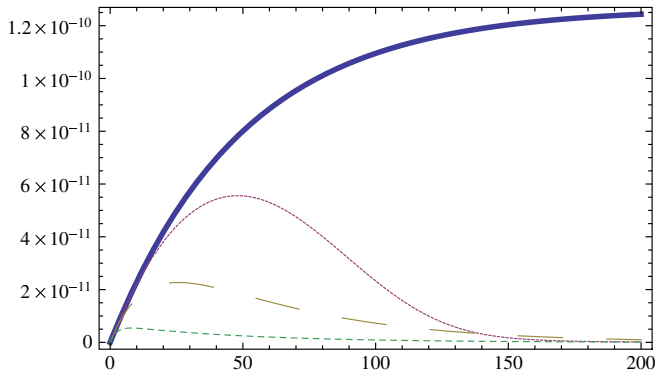

FIG. 4 (color online). Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the exponential potential. For the inflationary background we have chosen the inflationary trajectory in Eq. (9) with $p = 100$ and $t_i = 10^7 m_{\rm pl}^{-1}$. The mean square gauge-invariant inflaton fluctuation (thick line) dominates over those of test fields ($m_\chi = 10^{-6} m_{\rm pl}$ is the solid line, $c = 0.1$ is the dashed line, $\xi = 0.05$ is the dotted line).

which at late times (see also [26]) becomes

$$\langle \delta\phi^2 \rangle_{\rm REN} = \frac{p H_i^2}{8\pi^2}. \tag{39}$$

We show in Figs. 1–4, one for each inflaton potential investigated, the mean square of quantum fluctuations of the three types of test fields together with the gauge-invariant inflaton ones. For the hybrid model, quantum fluctuations of test fields with a small effective mass can dominate the gauge-invariant inflaton one, because of the presence of the leading constant term in the potential.

V. GROWTH OF QUANTUM FLUCTUATION IN TWO FIELD INFLATIONARY MODELS

We now wish to consider a two field model in which an inflaton ϕ and a minimally coupled scalar field χ are present (see [27] for a different approach to the moduli problem). We shall neglect the χ energy density and pressure in the background FRW equations. We expand to second order in the uniform curvature gauge (UCG), in which the inflaton fluctuation φ coincides with the gauge-invariant Mukhanov variable, the Einstein and Klein-Gordon equations.

In the test field expansion $\chi(\vec{x}, t) = \chi_0(t) + \chi^{(1)}(\vec{x}, t) + \chi^{(2)}(\vec{x}, t) + \ldots$, the homogeneous term satisfies

$$\ddot{\chi}_0 + 3H\dot{\chi}_0 + \bar{V}_\chi = 0, \tag{40}$$

while fluctuations satisfy, order by order, for the leading order in the slow-roll approximation and in the long-wavelength limit (neglecting vector and tensor contributions), the following equations:

$$3H\dot{\chi}^{(1)} + \bar{V}_{\chi\chi}\chi^{(1)} = 2\frac{H_\phi}{H}\bar{V}_\chi \varphi^{(1)}, \tag{41}$$

$$\begin{aligned} 3H\dot{\chi}^{(2)} + \bar{V}_{\chi\chi}\chi^{(2)} &= 2\frac{H_\phi}{H}\bar{V}_\chi\varphi^{(2)} \\ &+ \left[\frac{H_{\phi\phi}}{H} - 3\left(\frac{H_\phi}{H}\right)^2\right]\bar{V}_\chi \varphi^{(1)2} \\ &+ 2\bar{V}_{\chi\chi}\frac{H_\phi}{H}\varphi^{(1)}\chi^{(1)} - \frac{\bar{V}_{\chi\chi\chi}}{2}\chi^{(1)2}. \end{aligned} \tag{42}$$

Following the consideration in Sec. VI of Ref. [7], we wish to investigate which time variable in the stochastic equation should be chosen to reobtain, order by order, the equation of motion for the test field χ starting from

$$\frac{d\chi}{dt} = -\frac{\bar{V}_\chi}{3H(\phi)} \tag{43}$$

and expanding order by order. For a general time variable $\tau = \int H(t)^n dt$, the equation becomes

$$\frac{1}{H(t)^n}\frac{d\chi}{dt} = -\frac{\bar{V}_\chi}{3H(\phi)^{n+1}}. \tag{44}$$

As before, expanding to leading order in the slow-roll approximation, we obtain the following equation to the first and second order:

$$\frac{d\chi^{(1)}}{dt} = -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(1)} + \frac{1}{3}(n+1)\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(1)}, \tag{45}$$

$$\begin{aligned}\frac{d\chi^{(2)}}{dt} &= -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(2)} + \frac{1}{3}(n+1)\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(2)} \\ &\quad - \frac{1}{6H}\bar{V}_{\chi\chi\chi}\chi^{(1)2} + \frac{1}{3}(n+1)\frac{H_\phi}{H^2}\bar{V}_{\chi\chi}\varphi^{(1)}\chi^{(1)} \\ &\quad - \frac{1}{6}(n+1)\bar{V}_\chi\Big[-\frac{H_{\phi\phi}}{H^2} + (n+2)\frac{H_\phi^2}{H^3}\Big]\varphi^{(1)2}.\end{aligned} \tag{46}$$

As is easy to verify, we recover the former result only for $n = 1$. So for the case of a test scalar field, evolving in a FRW inflaton driven space-time, in the UCG the right time to consider is the number of e-folds $N = \int H(t)dt$; this recovers the result obtained in [7] for the inflaton fluctuations. As for the case of the standard Mukhanov variable Q, which is defined, order by order, as the value of the inflaton perturbation in the UCG, we can define a generic gauge-invariant Mukhanov variable Q_χ, associated with the perturbation of χ, as the value that this perturbation has in the UCG. In this way $Q_\chi^{(n)} = \chi^{(n)}$ in the UCG and, as seen for variable Q in [7], the equations above can be regarded as the gauge-invariant equations of motion, to first and second order, of this new Mukhanov variable, where one replaces $\chi^{(n)}$ with $Q_\chi^{(n)}$ and $\varphi^{(n)}$ with $Q^{(n)}$.

As for the case of the Mukhanov variable, this result can be considered as a starting point to study the fluctuations of χ in the stochastic approach for an arbitrary potential $\bar{V}$ in the described background. The correct stochastic differential equation is obtained with respect to the number of e-folds [$dN = H(t)dt$] which appears to be the right evolution parameter. One starts from the slow-roll approximation to the Heisenberg equation, which can be interpreted in a general nonperturbative sense, for the large-scale quantum field χ,

$$\frac{d}{dN}\chi = -\frac{1}{3H^2}\bar{V}_\chi + \frac{1}{H}f_\chi,$$

$$\langle f_\chi(N_1, \mathbf{x_1}) f_\chi(N_2, \mathbf{x_2})\rangle = \frac{H^4}{4\pi^2}\delta(N_1 - N_2)\frac{\sin(|\mathbf{x_1} - \mathbf{x_2}|)}{|\mathbf{x_1} - \mathbf{x_2}|},$$

where f_χ is the stochastic noise term given, to the leading order in the slow-roll approximation, by

$$\begin{aligned}f_\chi(t, \mathbf{x}) &= \epsilon a H^2 \int \frac{d^3k}{(2\pi)^{3/2}}\delta(k - \epsilon a H) \\ &\quad \times [\hat{b}_k\chi_k(t)e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_k^\dagger\chi_k^*(t)e^{+i\mathbf{k}\cdot\mathbf{x}}].\end{aligned} \tag{47}$$

Thus, on expanding to the second order, one obtains the following stochastic equations for $\chi^{(1)}$ and $\chi^{(2)}$:

$$\frac{d\chi^{(1)}}{dt} = -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(1)} + \frac{2}{3}\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(1)} + f_\chi, \tag{48}$$

$$\begin{aligned}\frac{d\chi^{(2)}}{dt} &= -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(2)} + \frac{2}{3}\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(2)} - \frac{1}{6H}\bar{V}_{\chi\chi\chi}\chi^{(1)2} \\ &\quad + \frac{2}{3}\frac{H_\phi}{H^2}\bar{V}_{\chi\chi}\varphi^{(1)}\chi^{(1)} \\ &\quad - \frac{1}{3}\bar{V}_\chi\Big[-\frac{H_{\phi\phi}}{H^2} + 3\frac{H_\phi^2}{H^3}\Big]\varphi^{(1)2} + \frac{H_\phi}{H}\varphi^{(1)}f_\chi.\end{aligned} \tag{49}$$

Let us consider the first order stochastic equation. Its general solution with the zero initial condition is given by

$$\chi^{(1)} = \bar{V}_\chi\int_{t_i}^t\Big(\frac{2}{3}\frac{H_\phi}{H^2}\varphi^{(1)} + \frac{f_\chi}{\bar{V}_\chi}\Big)d\tau. \tag{50}$$

Taking into account that $\langle\varphi^{(1)}f_\chi\rangle = 0$, it is easy to derive the expression for the mean square of the first order fluctuation $\langle(\delta\chi^{(1)})^2\rangle$:

$$\begin{aligned}\langle\chi^{(1)2}\rangle &= \bar{V}_\chi^2\int_{t_i}^t d\tau\int_{t_i}^t d\eta\Big[\frac{4}{9}\frac{H_\phi(\tau)}{H(\tau)^2}\frac{H_\phi(\eta)}{H(\eta)^2} \\ &\quad \times\langle\varphi^{(1)}(\tau)\varphi^{(1)}(\eta)\rangle + \frac{1}{\bar{V}_\chi(\tau)}\frac{1}{\bar{V}_\chi(\eta)}\langle f_\chi(\tau)f_\chi(\eta)\rangle\Big] \\ &= \frac{\bar{V}_\chi^2}{4\pi^2}\int_{t_i}^t d\tau\Big[\frac{H(\tau)^3}{\bar{V}_\chi(\tau)^2} - \frac{4}{9M_{\rm pl}^2}\int_{t_i}^\tau d\eta\frac{\dot{H}(\tau)}{H(\tau)^3}\frac{\dot{H}(\eta)}{H(\eta)^3} \\ &\quad \times\int_{t_i}^\eta d\sigma\frac{H(\sigma)^5}{\dot{H}(\sigma)}\Big],\end{aligned} \tag{51}$$

where we have used the stochastic solution

$$\varphi^{(1)} = \frac{V_\phi}{V}\int_{t_i}^t d\tau\Big(\frac{V}{V_\phi}f_\phi\Big), \tag{52}$$

with f_ϕ being the stochastic noise for the inflaton defined analogously to f_χ. Similarly, one can obtain the following solution for a vacuum expectation value of the second order fluctuation:

$$\begin{aligned}\langle\chi^{(2)}\rangle &= \bar{V}_\chi\int_{t_i}^t d\tau\Big[\frac{2}{3}\frac{H_\phi}{H^2}\langle\varphi^{(2)}\rangle - \frac{1}{6H}\frac{\bar{V}_{\chi\chi\chi}}{\bar{V}_\chi}\langle\chi^{(1)2}\rangle \\ &\quad + \frac{2}{3}\frac{H_\phi}{H^2}\frac{\bar{V}_{\chi\chi}}{\bar{V}_\chi}\langle\varphi^{(1)}\chi^{(1)}\rangle \\ &\quad + \frac{1}{3}\Big(\frac{H_{\phi\phi}}{H^2} - 3\frac{H_\phi^2}{H^3}\Big)\langle\varphi^{(1)2}\rangle\Big],\end{aligned} \tag{53}$$

where, to expand further, we should substitute Eq. (51), the

result for $\langle\varphi^{(2)}\rangle$ and $\langle\varphi^{(1)2}\rangle$ obtained in [7] and

$$\langle\varphi^{(1)}\chi^{(1)}\rangle = -\frac{\bar{V}_\chi}{12\pi^2}\frac{\dot{\phi}}{HM_{\rm pl}^2}\int_{t_i}^{t}d\tau\int_{\tau}^{t}d\eta\Big[\frac{H(\tau)^5}{\dot{H}(\tau)}\frac{\dot{H}(\eta)}{H(\eta)^3}\Big]. \tag{54}$$

A. A working example: Two field quadratic model

Let us now consider the particular case $V(\phi) = \frac{m^2\phi^2}{2}$ and $\bar{V}(\chi) = \frac{m_\chi^2\chi^2}{2}$. Classical slow-roll inflation in this model and the evolution of small perturbations in it were calculated in [28], but here we take into account the backreaction of the generated quantum fluctuations of these scalar fields on the evolution of their background values. By solving the background equations, one obtains the following zero order solution for the test field χ:

$$\chi^{(0)}(t) = \chi^{(0)}(t_i)\Big(\frac{H(t)}{H(t_i)}\Big)^{m_\chi^2/m^2}. \tag{55}$$

It remains a test field for the whole duration of the inflation era if

$$\chi^{(0)}(t_i)^2 \ll \Big[1+\frac{\alpha}{9}\frac{m^2}{H^2}\Big]^{-1}\frac{1}{\alpha}\Big(\frac{H}{H_i}\Big)^{2-2\alpha}6\frac{H_i^2}{m^2}M_{\rm pl}^2 \tag{56}$$

for any value of H (where $\alpha = m_\chi^2/m^2$). For the case $\alpha \ll 1$, we obtain the following limiting condition at the end of inflation ($H \simeq m$),

$$\chi^{(0)}(t_i)^2 \ll \frac{6}{\alpha}M_{\rm pl}^2, \tag{57}$$

and for this particular background we can solve Eq. (51) obtaining

$$\langle\chi^{(1)2}\rangle = \frac{3H^{2\alpha}}{8\pi^2m^2(2-\alpha)}(H_0^{4-2\alpha}-H^{4-2\alpha}) - \frac{\alpha^2}{48\pi^2}\frac{\chi^{(0)}(t_i)^2}{M_{\rm pl}^2}\Big(\frac{H}{H_i}\Big)^{2\alpha}\frac{1}{H^4}(H^2-H_i^2)^3. \tag{58}$$

Thus, one obtains the term already considered in [7] plus a new term which depends on the background value of χ. At the end of inflation, the leading value of this second term is negligible with respect to the leading value of the first one, for $\alpha < 2$, if

$$\chi^{(0)}(t_i)^2 \ll \frac{18}{2-\alpha}\frac{1}{\alpha^2}\frac{M_{\rm pl}^2m^2}{H_i^2}. \tag{59}$$

This condition is different from, and can be stronger than, the condition (57). If we consider the particular case $\alpha \ll 1$ and require that (57) implies (59), we obtain the following condition on α:

$$\alpha \ll \frac{3}{2}\frac{m^2}{H_i^2}. \tag{60}$$

Analogously we can evaluate Eq. (53),

$$\langle\chi^{(2)}\rangle = \frac{\alpha}{8\pi^2}\frac{\chi^{(0)}(t_i)}{M_{\rm pl}^2}\Big(\frac{H}{H_i}\Big)^{\alpha}\Big[-\frac{H_i^6}{H^4}\frac{1-\alpha/2}{6}+\frac{H_i^4}{H^2}\frac{1-\alpha}{4} + H_i^2\frac{\alpha}{4}-H^2\frac{1+\alpha}{12}\Big], \tag{61}$$

which to leading order gives

$$\langle\chi^{(2)}\rangle = -\frac{\alpha}{48\pi^2}\frac{\chi^{(0)}(t_i)}{M_{pl}^2}\Big(\frac{H_i}{H}\Big)^{4-\alpha}H_i^2\Big(1-\frac{\alpha}{2}\Big). \tag{62}$$

VI. CONCLUSIONS

Motivated by previously found differences for gravitational particle production in the de Sitter background and in realistic inflationary models with $\dot{H} \neq 0$, we have studied in detail the growth of quantum fluctuations for the latter case. We have selected four different potentials as representative examples of the inflationary zoo and different types of nearly massless fluctuations, including inflaton ones. We have rewritten in Eq. (32) the diffusion equation for the gauge-invariant inflaton fluctuations found in [7], emphasizing the role of the slope of the spectrum of curvature perturbations n_s and of the tensor-to-scalar ratio r, i.e., the relevant observable quantities.

We have found that for most of the inflationary models, the mean square of the gauge-invariant inflaton fluctuations dominates over those of moduli with a non-negative effective mass. Hybrid inflationary models can be an exception: the mean square of a test field can dominate over that of the gauge-invariant inflaton fluctuations on choosing parameters appropriately. Our findings show that the understanding of inflaton dynamics including its quantum fluctuations is more important than the moduli problem in most of the inflationary models.

We have then discussed the stochastic approach for general scalar fluctuations, which may have a nonzero homogeneous mode, in a generic model of inflation. We show, by using the field theory results as a guideline, that the stochastic equations for the gauge-invariant variable associated with such scalar fluctuations are naturally formulated as a flow in terms of the number of e-folds N. Finally we have studied the particular case of a massive inflaton and a second massive scalar field χ, for which we show how to extract some bounds for the homogeneous mode of χ and for its mass.

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[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[2] A. A. Starobinsky, JETP Lett. **82**, 169 (2005).
[3] O. Seto, J. Yokoyama, and H. Kodama, Phys. Rev. D **61**, 103504 (2000).
[4] F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, Phys. Rev. D **65**, 103521 (2002).
[5] F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, Phys. Rev. D **69**, 123508 (2004).
[6] F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, Phys. Rev. D **74**, 083522 (2006).
[7] F. Finelli, G. Marozzi, A. A. Starobinsky, G. P. Vacca, and G. Venturi, Phys. Rev. D **79**, 044007 (2009).
[8] Y. Urakawa and K.-I. Maeda, Phys. Rev. D **77**, 024013 (2008); **78**, 064004 (2008).
[9] T. M. Janssen, S. P. Miao, T. Prokopec, and R. P. Woodard, Classical Quantum Gravity **25**, 245 013 (2008).
[10] T. M. Janssen and T. Prokopec, arXiv:0906.0666.
[11] A. D. Linde, Phys. Lett. **116B**, 335 (1982).
[12] A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[13] A. Vilenkin and L. Ford, Phys. Rev. D **26**, 1231 (1982).
[14] A. A. Starobinsky, in *Field Theory, Quantum Gravity and Strings*, Lect. Notes in Physics Vol. 246, edited by H. J. De Vega and N. Sanchez (Springer, New York, 1986), p. 107.
[15] A. D. Linde, Phys. Lett. B **175**, 395 (1986).
[16] A. D. Linde, D. A. Linde, and A. Mezhlumian, Phys. Rev. D **49**, 1783 (1994).
[17] N. C. Tsamis and R. P. Woodard, Nucl. Phys. **B724**, 295 (2005).
[18] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).
[19] E. O. Kahya and V. K. Onemli, Phys. Rev. D **76**, 043512 (2007).
[20] A. A. Starobinsky, Sov. Astron. Lett. **4**, 82 (1978).
[21] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).
[22] E. W. Kolb, A. A. Starobinsky, and I. I. Tkachev, J. Cosmol. Astropart. Phys. 07 (2007) 005.
[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).
[24] V. F. Mukhanov, JETP Lett. **41**, 493 (1985); Sov. Phys. JETP **68**, 1297 (1988).
[25] F. Finelli, J. Hamann, S. M. Leach, and J. Lesgourgues, J. Cosmol. Astropart. Phys. 04 (2010) 011.
[26] G. Marozzi, Phys. Rev. D **76**, 043504 (2007).
[27] M. Lemoine, J. Martin, and J. Yokoyama, Phys. Rev. D **80**, 123514 (2009); Europhys. Lett. **89**, 29 001 (2010).
[28] D. Polarski and A. A. Starobinsky, Nucl. Phys. **B385**, 623 (1992).