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► **To cite this version:**

Esfandiar Alizadeh, Benjamin Wandelt. Dark matter halo merger and accretion probabilities in the excursion set formalism. *Monthly Notices of the Royal Astronomical Society*, 2010, 409, pp.694-700. <10.1111/j.1365-2966.2010.17332.x>. <hal-03645963>

HAL Id: hal-03645963

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Submitted on 5 Jun 2022

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Dark matter halo merger and accretion probabilities in the excursion set formalism

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Accepted 2010 July 7. Received 2010 May 5; in original form 2009 September 3

ABSTRACT

The merger and accretion probabilities of dark matter haloes have so far only been calculated for an infinitesimal time interval. This means that a Monte Carlo simulation with very small time-steps is necessary to find the merger history of a parent halo. In this paper, we use the random walk formalism to find the merger and accretion probabilities of haloes for a finite time interval. Specifically, we find the number density of haloes at an early redshift that will become part of a halo with a specified final mass at a later redshift, given that they underwent n major mergers, $n = 0, 1, 2, \dots$. We reduce the problem into an integral equation which we then solve numerically. To ensure the consistency of our formalism, we compare the results with Monte Carlo simulations and find very good agreement. Though we have done our calculation assuming a flat barrier, the more general case can easily be handled using our method. This derivation of finite time merger and accretion probabilities can be used to make more efficient merger trees or implemented directly into analytical models of structure formation and evolution.

Key words: galaxies: evolution – galaxies: haloes – cosmology: theory – dark matter.

1 INTRODUCTION

Since the introduction of the spherical collapse model of dark matter haloes by Press and Schechter (Press & Schechter 1974) and its generalizations such as extended Press–Schechter (EPS) (Bond et al. 1991) and ellipsoidal collapse (Sheth & Tormen 2002), it has been used extensively in the cosmological literature as a fast and accurate method to quantify the distribution of collapsed dark matter objects in the Universe. This in turn is the backbone of semi-analytic theories of galaxy formation and evolution (Cole et al. 2000). Modified versions of the excursion set formalism underlying EPS have also found application in different contexts such as ionized bubble growth in the early Universe (Furlanetto, Zaldarriaga & Hernquist 2004). The advantages of this method compared to direct N -body simulation include its superior speed, which allows the exploration of large ranges of parameter space, and redshift range than is currently accessible to N -body simulations.

It must, however, be noted that the excursion set formalism has its own shortcomings. Sheth & Pitman (1997) pointed out the inconsistencies between the excursion set and binary merger trees, and Benson, Kamionkowski & Hassani (2005) showed that the excursion set theory is not self-consistent in the sense that the merger kernel is not symmetric, i.e. $R(m_1, m_2) \neq R(m_2, m_1)$. Also, since

the excursion set theory is based on uncorrelated random walks, the future history of a halo does not depend on its environment. Therefore, it is not possible to explain within this model the recent result from simulations that the formation history of haloes is correlated with their environments (Sheth & Tormen 2004). Nevertheless, simplicity of implementation of the excursion set model together with its surprisingly good description of halo properties has made it an indispensable tool for cosmologists.

In the EPS formalism, one finds the probability $P(M_1, z_1 | M_2, z_2) dM_1$, which gives the probability of a point mass being part of a halo with mass in between M_1 and $M_1 + dM_1$ at redshift z_1 , given that it was (or will be) part of a halo with mass M_2 at redshift z_2 . During this redshift interval, it could merge with any number of haloes whose masses add up to $|M_1 - M_2|$. However, from the viewpoint of galaxy formation, accretion of small haloes into the larger one do not have the ability to change the evolution of the galaxy or galaxies inside it. Indeed, it is assumed that only at *major mergers* in which the mass of the merged halo satisfies a condition to be large enough, have this ability. It is therefore interesting to ask the question: given the criterion for a major merger, is it possible to find an analytical result to describe the progenitor distribution of a parent halo, based on how many times they have undergone a major merger? For example, what is the number density of haloes at redshift, say, $z = 1$ which underwent n major mergers, $n = 0, 1, 2, \dots$, and eventually ended up being a galaxy halo at the present time.

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Here we present an analytical method based exclusively on excursion set assumptions to find these number densities. We reduce the problem to an integral equation which can then be solved numerically. Our derivation is given in Section 2. We show in Section 3 that Monte Carlo (MC) simulations agree with the results of our semi-analytic model.

It is also possible to cast the integral equation into a scale invariant form. This can be a great advantage since we just need to solve the integral equation once and scale the result to find the general formula. This is done in section 4. We discuss how the method presented in this paper can be generalized and conclude in section 5.

A few appendices are added for further clarification. In Appendix A we give a straightforward method to solve the integral equation, and in Appendix B we describe very briefly how we implemented our MC simulation.

2 THE ACCRETION PROBABILITY

The main task of this paper is to find an analytical result for $p_{\text{acc}}(M_2|t_2, M_1, t_1)dM_2$, the probability that a halo of mass M_1 at time t_1 had a progenitor of mass in the range $(M_2, M_2 + dM_2)$ at an earlier time t_2 and that the progenitor halo never merged with any halo of mass larger than a specified mass resolution M_{res} on its journey from t_2 to t_1 . It is, however, more convenient to work with the variables S and ω instead of M and t . Here, S is the rms mass fluctuation inside spheres of mass M :

$$S(M) \equiv \sigma^2(R_M) = \int d \ln k \Delta^2(k) |W(k, R_M)|^2, \quad (1)$$

where $\Delta^2(k) \equiv k^3 P(k)/2\pi^2$ and $P(k)$ is the power spectrum of the mass fluctuations. Also, the barrier height $\omega \equiv \delta_c D(t)/D(t_{\text{now}})$, where $\delta_c \simeq 1.68$ is the linearly extrapolated density to the time of collapse in the spherical collapse model and $D(t)$ is the cosmological linear growth factor.

Finding $p_{\text{acc}}(M_2|t_2, M_1, t_1)dM_2$ in the excursion set formalism is equivalent to finding $f_{\text{acc}}(S_2|\omega_2, S_1, \omega_1)$, the probability that a random walk starting from (S_1, ω_1) has its first upcrossing between S_2 and $S_2 + dS_2$ at the barrier height ω_2 , given that it never had a jump larger than M_{res} between ω_1 and ω_2 (for further explanation of excursion set formalism, see Zentner 2007).

In order to accomplish this, we let a number of random walks start from (S_f, ω_f) (see Fig. 1), where in this paper S_f and ω_f denote $\sigma^2(M_f)$ and $\omega(t_f)$, respectively, and so forth for other subscripts. Setting a barrier at $\omega_i > \omega_f$, that is at earlier times, we know the fraction of random walks that have their first upcrossing in the interval $(S_i, S_i + dS_i)$, regardless of whether they accrete or merge, is

$$f_{\text{tot}}(S_i|\omega_i, S_f, \omega_f) dS_i = \frac{(\omega_i - \omega_f)}{(2\pi)^{1/2}(S_i - S_f)^{3/2}} \exp\left[-\frac{(\omega_i - \omega_f)^2}{2(S_i - S_f)}\right] dS_i. \quad (2)$$

The next step is to note that the fraction of random walks that start from (S_f, ω_f) and have their first upcrossing between S_i and $S_i + dS_i$ at the height ω_i and have one and only one jump from S_1 to S_2 during the interval ω and $\omega + d\omega$ is

$$f_{\text{acc}}(S_1|\omega, S_f, \omega_f) dS_1 f(S_1 \rightarrow S_2; \omega) dS_2 d\omega \times f_{\text{acc}}(S_i|\omega_i, S_2, \omega) dS_i, \quad (3)$$

where $f(S_1 \rightarrow S_2; \omega) dS_2 d\omega$ is the probability that a random walk will have a sudden jump from S_1 to somewhere between S_2 and $S_2 + dS_2$ in an infinitesimal interval $d\omega$. Clearly, we must have

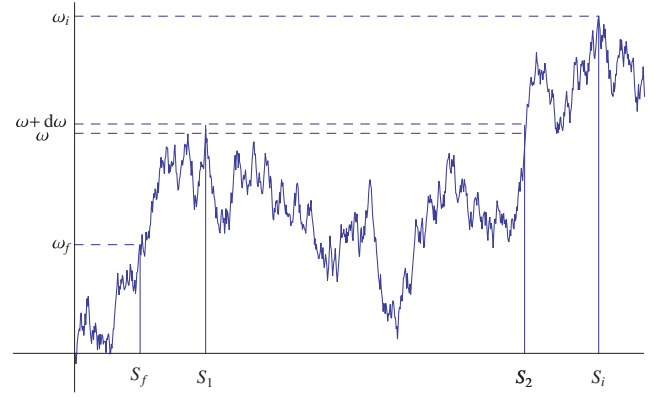


Figure 1. This figure shows a sample trajectory for a point that belongs to a halo of mass S_f at time ω_f in a S versus ω diagram. Increasing S and ω implies decreasing M and t , respectively. From this figure, we can find places where the halo to which this point belongs splits into two smaller haloes of masses larger than M_{res} by finding the places where the first upcrossing of the random walk has a jump larger than $\Delta S = S(M) - S(M - M_{\text{res}})$ when the barrier height is increased from ω to $\omega + d\omega$. In this figure, the mass resolution is such that there is just one jump large enough to be considered a merger, and occurs where the first upcrossing jumps from S_1 to S_2 . The rest of the time the halo will just accrete masses smaller than M_{res} .

$\omega_f \leq \omega \leq \omega_i$. Also from the definitions of merger and accretion, the allowed ranges of the masses are

$$M_i + M_{\text{res}} \leq M_1 \leq M_f \quad (4)$$

$$M_i \leq M_2 \leq M_1 - M_{\text{res}},$$

which translated into S becomes

$$S_f \leq S_1 \leq \sigma^2[M(S_i) + M_{\text{res}}] \quad (5)$$

$$\sigma^2[M(S_1) - M_{\text{res}}] \leq S_2 \leq S_i.$$

For the spherical collapse model, we have (Lacey & Cole 1993)

$$f(S_f \rightarrow S_i; \omega) dS_i d\omega = \frac{1}{(2\pi)^{1/2}(S_i - S_f)^{3/2}} dS_i d\omega. \quad (6)$$

Now, if we integrate over all S_1, S_2 and ω keeping in mind that $M(S_1) - M(S_2)$ must be larger than M_{res} to be classified as a merger, we find the fraction of walks that have undergone one and only one merger to be

$$f_{1\text{merger}}(S_i|\omega_i, S_f, \omega_f) = \int_{\omega_f}^{\omega_i} d\omega \int_{S_f}^{S_i - \Delta'(S_i)} dS_1 \int_{S_1 + \Delta(S_1)}^{S_i} dS_2 \times f_{\text{acc}}(S_1|\omega, S_f, \omega_f) f(S_1 \rightarrow S_2; \omega) f_{\text{acc}}(S_i|\omega_i, S_2, \omega). \quad (7)$$

Here $\Delta(S_1) = \sigma^2[M(S_1) - M_{\text{res}}] - S_1$ is the closest distance that S_2 can be brought to S_1 and still have a jump large enough to be considered as a merger. Analogously define $\Delta'(S_i) = S_i - \sigma^2[M(S_i) + M_{\text{res}}]$.

We can similarly find the fraction of random walks that have undergone two and only two mergers, $f_{2\text{mergers}}(S_i|\omega_i, S_f, \omega_f)$. We get an expression like equation (7) but with six integrations, and higher terms involve more integrations yet. Adding all of these terms must give

$$f_{\text{tot}}(S_i|\omega_i, S_f, \omega_f) = f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f) + f_{1\text{merger}}(S_i|\omega_i, S_f, \omega_f) + f_{2\text{mergers}}(S_i|\omega_i, S_f, \omega_f) + \dots \quad (8)$$

which says that the fraction of random walks starting from (S_f, ω_f) and passing through a barrier at ω_i for the first time between

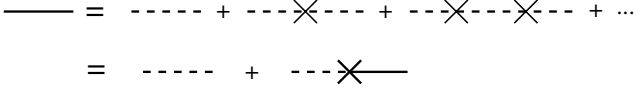


Figure 2. Feynman diagrams illustrating equation (8) for the total probability for a random walk starting from (S_f, ω_f) to have its first upcrossing between S_i and $S_i + dS_i$ at ω_i . A solid line denotes a period of time in which any sequence of accretion and merger events can take place. A dashed line denotes a period of time in which only accretion takes place and a cross indicates a merger. The first equality says that the total probability is equal to the sum of the fractions of walks which underwent no mergers, one merger, two mergers, etc. We can rearrange the sum to give the second equality, which states that the total probability is equal to the sum of the probability to have no mergers and the probability to have at least one merger.

$(S_i, S_i + dS_i)$, no matter what happened during their journey, is equal to the sum of the fractions of walks which underwent no mergers, one merger, two mergers, etc.

Putting the integral formulae for $f_{\text{1merge}}(S_i|\omega_i, S_f, \omega_f)$ and higher order terms into equation (8), we will find an integral equation for the unknown $f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f)$. However, this equation written in this form is not computationally tractable since higher order terms will require a prohibitive number of integrations.

Note that the conditional probability densities in equation (8) are analogous to propagators. It is instructive to visualize each term in equation (8) using a diagrammatic notation, as in Fig. 2. The figure makes it clear that this equation expands the full propagator in terms of bare propagators (the accretion probability) with interactions (mergers). We can use this insight to rearrange the above equation as in the figure, which shows that one can re-sum the terms in equation (8) to write down a tractable integral equation. Writing the second equality of Fig. 2 in the language of equation (8), we find the important result:

$$f_{\text{tot}}(S_i|\omega_i, S_f, \omega_f) = f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f) + \int_{\omega_f}^{\omega_i} d\omega \int_{S_f}^{S_i - \Delta'(S_i)} dS_1 \int_{S_1 + \Delta(S_1)}^{S_i} dS_2 \times f_{\text{acc}}(S_1|\omega, S_f, \omega_f) f(S_1 \rightarrow S_2; \omega) f_{\text{tot}}(S_i|\omega_i, S_2, \omega). \quad (9)$$

This is a Volterra integral equation which we will solve numerically. We refer the interested reader for a discussion of solving this equation numerically to Appendix A. In the next section, we compare these semi-analytic results to MC simulations.

3 COMPARISON WITH MONTE CARLO SIMULATION

There are several methods for generating a merger tree using MC techniques (see Somerville & Kolatt 1997; Cole et al. 2000). Here we choose to use a binary merger tree with accretion method mainly due to its simplicity of implementation. In this scheme the time interval is chosen to be so small that the probability of a merger is very low. This in turn ensures that the probability of more than one merger is negligible in a given time-step. Then a halo one time-step back will have a smaller mass due to accretion or division into two progenitors. The details of the method can be found in Appendix B.

It should be noted that our MC simulation gives number weighted probabilities. However, f_{acc} in equation (9) is a mass-weighted probability. We can easily change f_{acc} to a number-weighted probability p_{acc} using $p_{\text{acc}} = (M_f/M)f_{\text{acc}}$.

Each panel of Fig. 3 shows p_{acc} versus S_i for a different lookback redshift z_i , with parent mass $M_f = 2.5 \times 10^{12} M_\odot$ and redshift $z_f = 0$. The mass resolution is fixed at $M_{\text{res}} = M_f/10$. The dashed line

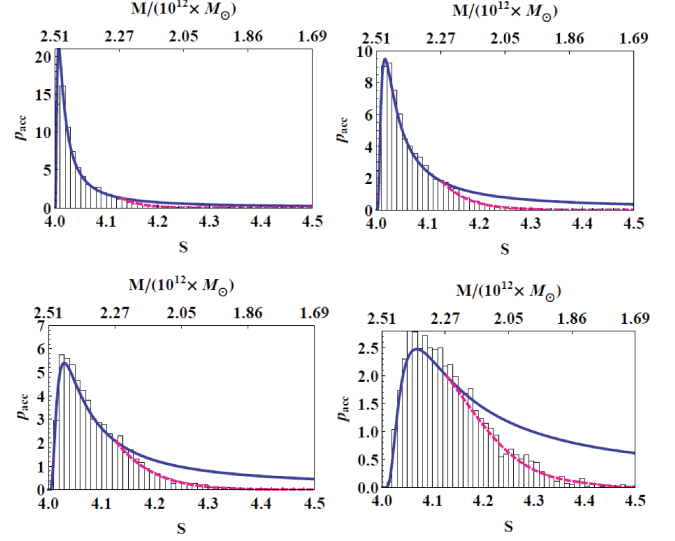


Figure 3. This figure shows p_{acc} versus S_i , with parent mass $M_f = 2.5 \times 10^{12} M_\odot$, mass resolution $M_{\text{res}} = M_f/10$ and redshift $z_f = 0$. The panels show p_{tot} (solid line), our semi-analytic result for p_{acc} (dashed line), and the MC result for p_{acc} (histograms) for increasing redshift from top left to bottom right: $z = 0.184, 0.267, 0.348$ and 0.5 .

is our analytical solution, the histogram shows the result of the MC simulation and the solid line is f_{tot} given by equation (2). We see an excellent agreement between the MC simulation and our analytic result, and note some intuitively sensible trends in the figures that are worth mentioning. First, $p_{\text{acc}} = p_{\text{tot}}$ for $S_i < 4.1$. That is because to have a merger we need $M_i \leq M_f - M_{\text{res}} = 1.8 \times 10^{12}$, which corresponds to $S_i = 4.1$. For S_i smaller than this, the mass jump is always less than mass resolution and only accretion can happen; hence, $p_{\text{acc}} = p_{\text{tot}}$ in this region. For S_i larger than 4.1 mergers are allowed, so the probability of having one or more merger is non-zero and p_{acc} will be less than p_{tot} . Also, the probability of having at least one merger, $p_{\text{tot}} - p_{\text{acc}}$, increases monotonically with increasing redshift as more and more haloes have a chance to undergo a merger. Finally, as we look further back in time, haloes with smaller masses have a chance to reach M_f by just accreting, so p_{acc} spreads to smaller masses with increasing redshift.

Although our method gives the full distribution over the mass of the progenitor haloes, one might only be interested in the fraction of haloes that had no major merger during a given redshift interval. To find this quantity, one can easily integrate the full distribution $p_{\text{acc}}(S|\omega, S_f, \omega_f) dS$ from $S = S_f$ to $S = \infty$. For example, in Fig. 4 this fraction is 0.75, 0.65, 0.56 and 0.43 for the panels from top left to bottom right. As expected, this fraction is a decreasing function of lookback redshift: the longer a halo evolves the more probable it is to have a major merger.

In Fig. 4 we show p_{acc} versus S_i with the same M_f and z_f as above, but here we look at the distribution with different choices of M_{res} at a fixed lookback redshift $z_i = 0.5$. Again the agreement between our analytical result and the MC simulations is very good. Again, there are some intuitively reasonable trends in the figure that should be noted. As we discussed above, p_{acc} must be equal to p_{tot} for $M_i \geq M_{\text{limit}} = M_f - M_{\text{res}}$. As M_{res} gets smaller, this limiting mass becomes larger. Therefore $S_{\text{limit}} = S(M_{\text{limit}})$, the S_i below which $p_{\text{acc}} = p_{\text{tot}}$, becomes smaller, which can easily be seen in the figure. Note that different panels have different scales. Also given a fixed lookback redshift, decreasing the mass resolution increases the number of events we classify as mergers thus raising the probability for a halo

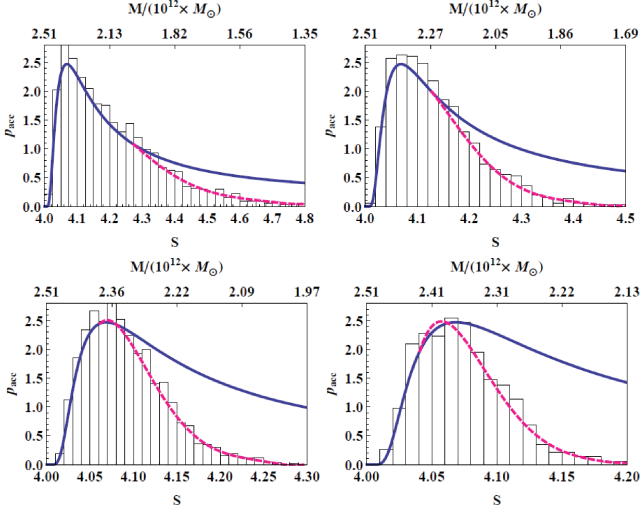


Figure 4. The dashed lines in these figures show our semi-analytic result for p_{acc} versus S_i with the same M_f and z_f as Fig. 3, but here we look at the distribution at a fixed lookback redshift $z_i = 0.5$ for decreasing M_{res} , from top left to bottom right: $M_{\text{res}} = M_f/5, M_f/10, M_f/20$ and $M_f/30$. Solid lines show p_{tot} and histograms the results of MC simulations for p_{acc} .

to have at least one merger. Since p_{tot} is not affected by the choice of M_{res} , p_{acc} accordingly decreases with decreasing M_{res} . The fraction of haloes that have had at least one major merger is 0.38, 0.57, 0.73 and 0.80 from top left to bottom right, respectively.

4 SCALED SOLUTION

In general, for any given prescription for M_{res} one can find the solution of integral equation (9) to obtain $f_{\text{acc}}(S_i|\omega_i, S_f, \omega_f)$. Generally, this needs to be solved for each given final halo mass M_f . However, if we impose a special mass resolution for a chosen cosmology, it is possible to cast the integral equation into a scale invariant form. Then we need only solve this equation once. To achieve this, we need to define M_{res} so as to satisfy the following two equations simultaneously:

$$\Delta(S) \equiv \sigma^2[M(S) - M_{\text{res}}] - S = CS \quad (10)$$

and

$$\Delta'(S) \equiv S - \sigma^2[M(S) + M_{\text{res}}] = C'S, \quad (11)$$

where C and C' are constants independent of S . Recall that $\Delta(S)$ and $\Delta'(S)$ appear in the limits of integration of equation (9). For small M_{res} compared to $M(S)$, we can easily see, by Taylor expansion, that to second order in M_{res}/M these equations can be satisfied if we take $C = C'$ and the mass resolution as

$$M_{\text{res}}(S) = -\frac{S}{dS/dM}C. \quad (12)$$

For example, for a scale invariant matter power spectrum with power index n , $P(k) \propto k^n$, the above equation gives

$$M_{\text{res}} = \frac{n+3}{3}CM_{\text{parent}}, \quad (13)$$

i.e. a merger is defined when the mass of any progenitor of the halo is larger than a constant fraction of the parent mass.

Using equations (10) and (11), we now rewrite the integral equation (9) in a scale invariant form. To do so we define the new

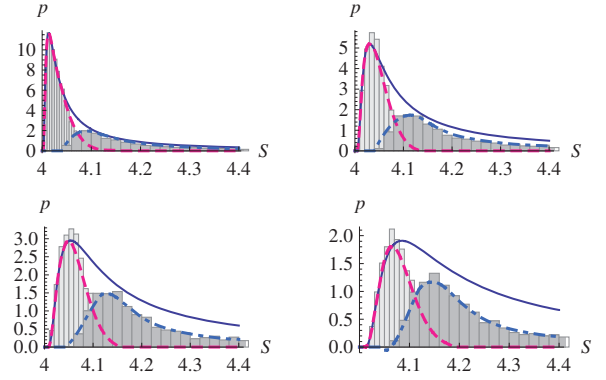


Figure 5. Halo accretion and merger probabilities for a power-law matter power spectrum with $n = -1$ are shown here. We take $C = 0.01$, where C is defined in equation (13). The solid line denotes p_{tot} , the lighter histogram in each panel indicates the result of MC simulation for p_{acc} and the dashed line is our numerical solution of equation (16) scaled according to equation (15). The darker histogram shows the result of our MC simulation for $p_{\text{merger}}(S_i, \omega_i, S_f, \omega_f)$, the number density of haloes in a given range ($S_i, S_i + dS_i$) that have a parent halo of mass corresponding to S_f at time ω_f and have merged once in their journey from ω_i to ω_f . The panels from top left to bottom right are for $\omega - \omega_f = 0.2, 0.3, 0.4$ and 0.5 . Assuming $\omega = 1.69(1+z)$ and $z_f = 0$, these correspond to lookback redshifts 0.118, 0.178, 0.237 and 0.296, respectively.

variables

$$u \equiv S/S_f$$

$$\theta \equiv (\omega - \omega_f)/S_f^{1/2} \quad (14)$$

and the functions \tilde{f}_{acc} and \tilde{f}_{acc} :

$$f_{\text{acc}}(S_1|\omega, S_f, \omega_f) = S_f^{-1} \tilde{f}_{\text{acc}} \left(\frac{S_1}{S_f} \left| \frac{\omega - \omega_f}{S_f^{1/2}} \right. \right)$$

$$\tilde{f}_{\text{acc}}(u_1|\theta) = \frac{\theta}{(2\pi)^{1/2}(u_1 - 1)^{3/2}} \exp \left(-\frac{\theta^2}{2(u_1 - 1)} \right) \tilde{f}_{\text{acc}}(u_1|\theta). \quad (15)$$

With these definitions, equation (9) can be written in the manifestly scale invariant form

$$1 = \tilde{f}_{\text{acc}}(u_i|\theta_i) + \int_0^{\theta_i} d\theta \int_1^{u_i(1-C')} du_1 \int_{u_1(1+C)}^{u_i} du_2 \times \tilde{f}_{\text{acc}}(u_1|\theta) K(\theta_i, u_i, u_1, u_2, \theta), \quad (16)$$

where the kernel for the spherical collapse model is

$$K = \frac{(2\pi)^{1/2}(u_i - 1)^{3/2}}{\theta_i} \exp \frac{\theta_i^2}{2(u_i - 1)}$$

$$\times \frac{\theta}{(2\pi)^{1/2}(u_1 - 1)^{3/2}} \exp \frac{-\theta^2}{2(u_1 - 1)}$$

$$\times \frac{1}{(2\pi)^{1/2}(u_2 - u_1)^{3/2}} \frac{(\theta_i - \theta)}{(2\pi)^{1/2}(u_i - u_2)^{3/2}} \exp \frac{-(\theta_i - \theta)^2}{2(u_i - u_1)}. \quad (17)$$

For a given C and C' this equation can be solved for $\tilde{f}_{\text{acc}}(u|\theta)$. This calculation can be facilitated by noting that the integral over S_2 can be done analytically. Having found $\tilde{f}_{\text{acc}}(u|\theta)$ one can find $f_{\text{acc}}(S_1|\omega_1, S_f, \omega_f)$ for arbitrary S_1, ω_1, S_f and ω_f using equation (15). Note that the spectral index n only enters through equation (13).

The result of this calculation for a power-law matter power spectrum with $n = -1$ and $C = 0.01$ is shown in Fig. 5. The solid

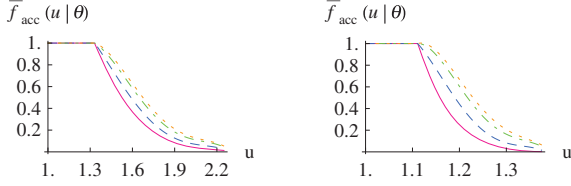


Figure 6. The scaled solutions $\bar{f}_{\text{acc}}(u|\theta)$ (for definition see equation 15) as a function of u and for different choices of θ . The left-hand panel is for a definition of major merger of $r \equiv M_1/M_{\text{parent}} = 1/10$ and from bottom to top, $\theta = 0.2, 0.3, 0.4$ and 0.45 . The right-hand panel is for $r = 1/4$ and from bottom to top: $\theta = 0.4, 0.5, 0.6$ and 0.65 . For a given r , the function \bar{f} needs to be calculated only once and the probabilities of mergers and accretion for any halo mass and redshift interval can be found using equation (15).

line, as usual, denotes p_{tot} , the lighter histogram in each panel indicates the result of MC simulation for p_{acc} and the dashed line on top is our numerical solution of equation (16) scaled according to equation (15). The panels from top left to bottom right are for $\omega - \omega_f = 0.2, 0.3, 0.4$ and 0.5 . Assuming $\omega = 1.69(1+z)$ and $z_f = 0$, these correspond to lookback redshifts 0.118, 0.178, 0.237 and 0.296, respectively. One can see that the agreement is very good for all redshifts considered.

The darker histogram in Fig. 5 shows the result of our MC simulation for $p_{1\text{merger}}(S_i, \omega_i, S_f, \omega_f)$. This is the number density of haloes in a given range $(S_i, S_i + dS_i)$ that have a parent halo of mass corresponding to S_f at time ω_f and have merged once in their journey from ω_i to ω_f . Now that we have found f_{acc} , it is possible to calculate $p_{1\text{merger}}(S_i, \omega_i, S_f, \omega_f)$ using equation (7). $p_{1\text{merger}}$ is nothing but $f_{1\text{merger}}$ multiplied by M_f/M to convert from mass density to number density. This result is shown by the dot-dashed line on top of the histogram. The match is very good. As expected, for small Δz the probability of one merger is much smaller than probability of accretion, which can be seen in the top-left plot. Also in this plot, we can see that the tail of the distribution $p_{1\text{merger}}$ approaches p_{tot} . This says that the probability of having more than one merger is negligible for small Δz , as expected.

On the other hand, when Δz gets larger more and more haloes have a chance to merge, so p_{acc} flattens and $p_{1\text{merger}}$ rises. Also, with a large Δz there is a finite probability of having more than one merger since the tail of $p_{1\text{merger}}$ is considerably below p_{tot} . One can continue this calculation and find $p_{2\text{mergers}}, p_{3\text{mergers}}$ and so on, which we have not shown here. Where this hierarchy should be terminated clearly depends on how far we look back in time; a larger Δz means more chance of a merger and therefore requires higher merger terms.

Though for a general power spectrum, M_{res} must be small compared to M_{parent} for the approximate solution 12 to work, for the special case of a white noise power spectrum, i.e. $n = 0$, one can find an exact analytic solution for equations (10) and (11) for arbitrarily large M_{res} . In this case $M(S) = k/S$, where k is a constant, and if one defines the ratio $r \equiv M_{\text{res}}/M_{\text{parent}}$ then it is easy to show that equations (10) and (11) are satisfied if $C = r/(1-r)$ and $C' = C/(1+2C)$. In Fig. 6 we have shown $\bar{f}(u|\theta)$ as a function of u for a few choices of θ for the case of a white-noise power spectrum. The left-hand panel is calculated for a major merger definition of $M_1/M_2 = 1/3$, which in our notation is equivalent to $r = 1/4$, and the right one for $r = 1/10$, that is $M_1/M_2 = 1/9$. These universal curves can be translated to the physical quantity $f_{\text{acc}}(S_1|\omega, S_f, \omega_f)$ by using equation (15). One can use these curves to get a quick estimate of the fraction of haloes of a given mass that only accreted during a given time interval. To find this quantity one needs to integrate

$(M_f/M)f_{\text{acc}}(S_1|\omega, S_f, \omega_f)$ over all $S < S_f$. Using equation (15),

$$\begin{aligned} F_{\text{acc}}(\theta) &= \int_{S_f}^{\infty} dS_1 \frac{M_f}{M_1} f_{\text{acc}}(S_1|\omega, S_f, \omega_f) \\ &= \int_1^{\infty} du_1 u_1 \frac{\theta}{(2\pi)^{1/2}(u_1-1)^{3/2}} \exp\left(-\frac{\theta^2}{2(u_1-1)}\right) \bar{f}_{\text{acc}}(u_1|\theta). \end{aligned} \quad (18)$$

As an example, for a definition of major merger of $(M_1:M_2) = (1:3)$, we find that the fraction of the galaxy sized haloes at the present time ($M_f = 10^{12} M_{\odot}$ or $S_f = 5.2$) that have had at least one major merger after redshifts 0.2, 0.4, 0.6, 0.8 are 0.04, 0.13, 0.22 and 0.27, respectively. Though this calculation, strictly speaking, is only valid for the white-noise power spectrum, previous results show that the formulae obtained for the special case of white-noise work surprisingly well for an arbitrary power spectrum (see e.g. Moreno, Giocoli & Sheth 2008). So, it is reasonable to expect this extrapolation to work for our formalism also. In any case, even if this extrapolation turns out to be wrong, one can always solve the more general integral equation (9) for an arbitrary power spectrum.

5 DISCUSSION AND CONCLUSION

We have used the random walk formalism to find the accretion probability, i.e. the probability for a parent halo to have a progenitor in a given mass interval at a given earlier time given that it has not merged with a halo of a mass larger than the mass resolution. As a concrete example, we have worked out the accretion probability in the special case where the barrier is flat, the mass resolution is constant and we look backward in time. However this method can be extended to solve more general problems.

For example, while we have used a constant barrier, it is well known that this barrier shape does not match the results of N -body simulations. However, our formalism can be generalized to the case of a moving barrier, which has proven to give a very good match to N -body simulations. One only needs to find the appropriate formulae for f_{tot} and $f(S_1 \rightarrow S_2; \omega)$ for the moving barrier and solve the integral equation (9). These functions can in general be found numerically, using for example the method of Zhang & Hui (2006). However there are analytical results for simple barriers that reproduce the results of N -body simulations, e.g. the square root barrier (Mahmood & Rajesh 2005; Giocoli et al. 2007). Since the aim of this paper is not to compare with numerical simulations, we will leave this calculation for future work.

Here we have always looked backward in time. However, in certain cases it might be more convenient to find $f_{\text{acc}}(M|M_i)$ in the forward sense. In that case, it gives the probability of a halo of mass M_i at an early time to accrete and become a larger halo in a given mass interval. This problem can be solved with our formalism by using the forward form of equation (9), with the forward form of f_{tot} and f_{merge} (Lacey & Cole 1993). This forward in time probability of accretion is what Somerville & Kolatt (1997) call P_{acc} and is the only missing ingredient in their formalism to find the complete statistics of halo progenitors and therefore to construct a self-consistent merger tree.

Once f_{acc} in the forward sense has been found it can be used for a variety of applications. For example, given a population of objects of mass M at time t , some of these objects will be destroyed in the course of their evolution by merging with other objects. To find the fraction of objects that have survived from the initial time to the observation time (see Verde et al. 2001, for the case of clusters of galaxies) we need to find the fraction of objects whose haloes have

not merged with haloes more massive than a given threshold; in other words, they have only accreted from the initial redshift to the redshift of observation. This is precisely what f_{acc} in the forward sense is.

It is important to note that our method is not equivalent to the MC method, due to the inherent inconsistencies of the excursion set formalism (Sheth & Pitman 1997). Because of these inconsistencies, the halo progenitor probability distribution computed from a large number of small time-steps will not necessarily be equivalent to the prediction of the excursion set formalism for that finite time interval, see e.g. Somerville & Kolatt (1997). For this same reason, there does not exist a unique MC method. In all versions of the MC method, there are some assumptions beyond the excursion set formalism to alleviate the above inconsistency. Therefore, it is not surprising that there is a small discrepancy between our analytic results and our MC method. However, for the special case of a white-noise power spectrum there is a self-consistent algorithm to generate merger trees (Sheth & Pitman 1997). In that case, we expect our semi-analytical result presented in Section 4 to be equivalent to the Sheth and Pitman result. We defer this comparison to future work.

Finally, this method can lead to a major improvement in the speed of merger tree generation in MC simulations. Since there was no formula for the accretion or merger probabilities in a *finite* time interval, past MC codes had to use *infinitesimal* time-steps to be able to use the known formula for merger probabilities (equation 6), making the computation very time consuming. For example, for a given time interval, our unknown coefficients method for solving the integral equation takes an order of magnitude less time to compute than the brute force MC method. It should be noted that since neither of the numerical codes we implemented in this paper are optimally efficient, both computation times are still to be minimized. However, the increase in speed is more likely for the integral equation, since one can use the already existing numerical methods to expedite the code or even find an approximate analytical solution, e.g. perturbationally. It is harder and less straightforward to design a faster MC method.

ACKNOWLEDGMENTS

EA thanks Laura Book for her comments and Akbar Jaefari for his helpful discussion. We acknowledge Andrew Benson for his reading of the manuscript and useful suggestions. BW acknowledges the Friedrich Wilhelm Bessel prize from the Alexander von Humboldt foundation. This work was partially supported by an Arnold O. Beckman award from the University of Illinois.

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APPENDIX A: NUMERICAL SOLUTION

The goal of this section is to solve equation (9) for $f_{\text{acc}}(S_1|\omega, S_f, \omega_f)$ numerically using the method of unknown coefficients. To begin, it is more convenient to work with the new unknown function $f_d \equiv f_{\text{tot}} - f_{\text{acc}}$ instead of f_{acc} . Suppressing the limits of integration for simplicity of notation, we can then write the integral equation (9) in the form

$$f_d(S_i|\omega_i, S_f, \omega_f) = b(S_i, \omega_j, S_f, \omega_f) - \int d\omega \int dS_1 \int dS_2 f_d(S_1|\omega, S_f, \omega_f) \times f(S_1 \rightarrow S_2; \omega) f_{\text{tot}}(S_i|\omega_i, S_2, \omega), \quad (\text{A1})$$

where

$$b(S_i, \omega_j, S_f, \omega_f) \equiv \int d\omega \int dS_1 \int dS_2 f_{\text{tot}}(S_1|\omega, S_f, \omega_f) \times f(S_1 \rightarrow S_2; \omega) f_{\text{tot}}(S_i|\omega_i, S_2, \omega), \quad (\text{A2})$$

and is a known function. We are seeking a solution of $f_d(S|\omega, S_f, \omega_f)$ for a given (S_f, ω_f) , so we further suppress these two arguments such that $f_d(S, \omega) \equiv f_d(S|\omega, S_f, \omega_f)$. In the method of unknown coefficients one expands the unknown function f_d as a sum of some appropriate basis functions $q_n(S, \omega)$,

$$f_d(S, \omega) = \sum_{n=1}^N a_n q_n(S, \omega) \quad (\text{A3})$$

with the goal of finding the coefficients a_n . The functions $q_n(S, \omega)$ should go to zero when $\omega \rightarrow \omega_f$ or $S \rightarrow S_f + \Delta(S_f)$. The latter limit arises because by definition, for $M - M_f < M_{\text{res}}$ there is no merger, so $f_{\text{acc}} = f_{\text{tot}}$ in this range. The number of terms N we need to have a convergent solution depends on the choice of the basis functions. A simple choice which we used in our calculation is the power series:

$$q_n(S, \omega) \equiv (\omega - \omega_f)^{n_1} \{S - [S_f + \Delta(S_f)]\}^{n_2}, \quad n_1, n_2 \geq 1. \quad (\text{A4})$$

We then plug this into the integral equation and evaluate it for a given choice of $(S, \omega) = (S_j, \omega_j)$:

$$\sum_{n=1}^N a_n q_n(S_j, \omega_j) = b(S_j, \omega_j) - \sum_{n=1}^N a_n w_n(S_j, \omega_j), \quad (\text{A5})$$

where b is defined in equation (A2) and the w_n are

$$w_n(S_j, \omega_j) \equiv \int d\omega \int dS_1 \int dS_2 q_n(S_1, \omega) f(S_1 \rightarrow S_2; \omega) \times f_{\text{tot}}(S_j|\omega_j, S_2, \omega). \quad (\text{A6})$$

Equation (A5) is a simple algebraic equation for the unknown coefficients a_n . If we plug N different choices of (S_j, ω_j) into this equation, we will have N equations for the N unknowns a_n .

APPENDIX B: MONTE CARLO SIMULATION

First, we briefly describe how the binary merger with accretion works. For a small change in $\Delta\omega$, the probability of absorbing a mass ΔS in this time interval is given by

$$P(\Delta S, \Delta\omega) dS = \frac{1}{(2\pi)^{1/2}} \frac{\Delta\omega}{(\Delta S)^{3/2}} \exp\left[-\frac{(\Delta\omega)^2}{2\Delta S}\right]. \quad (\text{B1})$$

Starting from a parent halo with mass S_p at time ω , we go backward in time to $\omega - \Delta\omega$. Then, if $M(\Delta S) < M_{\text{res}}$ we consider that to be accreted mass, stop tracking its history, and take $M_p - M(\Delta S)$ as the new parent halo at redshift $\omega - \Delta\omega$ assuming that this new M_p is larger than M_{res} . If not, we consider that as accreted mass and do not continue to track its history. We choose $\Delta\omega$ small enough to ensure that the probability of having a merger in this time interval becomes small. In other words, we demand

$$\Delta\omega \ll \sqrt{S(M_p - M_{\text{res}}) - S(M_p)}. \quad (\text{B2})$$

Then we generate a random number ΔS consistent with the distribution B1. This is a very easy task to do since equation (B1) can be converted to a Gaussian distribution by a change of variable $x \equiv \Delta\omega/(2\sqrt{\Delta S})$. This procedure is then repeated with the new haloes as the parent haloes until we reach the time ω_i , where we

want to compare our numerical results with the MC simulation. While making the merger tree, we keep track of each halo to know how many times in their history they experienced a merger so we will be able to find the distribution of haloes with no merger, one merger and so forth.

There are several other methods to generate merger trees and it would be interesting to see if the small discrepancy between our results and the MC will disappear using these methods. For example, the method outlined in Moreno et al. (2008), which takes discrete steps in mass rather than time, seems to be using more of the ingredients of the random walk formalism and therefore seems to be a better method to compare to our results. We leave this comparison to future work.

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