



**HAL**  
open science

# Proof of a conjecture due to Sun concerning Catalan's constant

John Campbell

► **To cite this version:**

| John Campbell. Proof of a conjecture due to Sun concerning Catalan's constant. 2022. hal-03644515

**HAL Id: hal-03644515**

**<https://hal.science/hal-03644515>**

Preprint submitted on 19 Apr 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Proof of a conjecture due to Sun concerning Catalan's constant

John Maxwell Campbell

York University, Toronto, Ontario, Canada

---

## Abstract

In 2014, Sun had conjectured that the following formula involving Catalan's constant  $G$  holds, letting  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$  denote the  $n^{\text{th}}$  harmonic number:

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} \left( 3H_{2k+1} + \frac{4}{2k+1} \right) = 8G.$$

This had been discovered experimentally, in the context of the study of special values of  $L$ -functions. In this article, we prove this conjecture, using various identities concerning the dilogarithm function, and using an argument relying on a  ${}_3F_2\left(\frac{1}{4}\right)$ -series for Gieseking's constant. It appears that this conjecture due to Sun has not previously been solved, based on the relevant literature.

*Keywords:* Catalan's constant, Harmonic number, Dilogarithm,  $L$ -function

*2010 MSC:* 11Y60, 11B65

---

## 1. Introduction

Sun had conjectured that the equality

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} \left( 3H_{2k+1} + \frac{4}{2k+1} \right) = 8G \quad (1)$$

holds true in the 2015 article [1], and this same conjecture is given in Sun's 2014 preprint [2] listing his conjectures on powers of  $\pi$ , letting  $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$  denote Catalan's constant, and letting  $(H_k : k \in \mathbb{N}_0)$  denote the sequence

---

*Email address:* [jmaxwellcampbell@gmail.com](mailto:jmaxwellcampbell@gmail.com) (John Maxwell Campbell)

of harmonic numbers. Catalan's constant is notable as a special value of the Dirichlet  $L$ -function

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s},$$

with

$$L(s, \chi_4) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s},$$

letting non-principal Dirichlet characters modulo  $n$  be denoted as per usual. We are to make use of values associated with

$$L(2, \chi_6) = \frac{1}{36} \left( \psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right)$$

in our proof introduced in this article that Sun's conjectured formula in (1) holds true.

The formula in (1) is given as Equation (3.18) in [1], which is given as part 5 of part (i) of **Conjecture 3.4** in [1]. The formula in (1) is also reproduced as Equation (1.27) in [2], with **Conjecture 3.4** from [1] being cited. Based on the extant literature citing [1], including [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], along with [22, §18], it appears that the problem of proving (1) had been open, prior to our proof of (1) introduced in this article.

For the sake of brevity, we assume some basic familiarity with the special function

$$\text{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{2}$$

known as the dilogarithm. The Euler–Mascheroni constant is such that  $\gamma = \lim_{n \rightarrow \infty} (H_n - \ln n)$ , and we recall that the *digamma function* refers to the special function such that the following equalities are satisfied [23], §9):

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}.$$

Writing  $\psi = \psi^{(0)}$ , the polygamma function is such that

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi^{(0)}(z).$$

10 **2. Proof of Sun's conjecture**

**Theorem 1.** *Sun's conjectured equality in (1) holds true.*

*Proof.* We are to make use of the moment formula

$$\int_0^1 (-2k-1)x^{2k} \log(1-x) dx = H_{2k+1} \quad (3)$$

together with the dominated convergence theorem, in the following manner. By replacing the summand factor  $H_{2k+1}$  in the series

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} H_{2k+1}$$

with the definite integral in (3), and then rewriting the resultant summand so as to obtain

$$\sum_{k=0}^{\infty} \int_0^1 \frac{(-2k-1)\binom{2k}{k}x^{2k} \log(1-x)}{(2k+1)16^k} dx,$$

we may reverse the order of the operators  $\sum_{k=0}^{\infty} \cdot$  and  $\int_0^1 \cdot dx$  according to the dominated convergence theorem. So, by the generalized binomial theorem, we have that the equality

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}H_{2k+1}}{(2k+1)16^k} = \int_0^1 -\frac{2 \log(1-x)}{\sqrt{4-x^2}} dx \quad (4)$$

holds true. We have determined that an antiderivative for  $-\frac{2 \log(1-x)}{\sqrt{4-x^2}}$  is as below, and this is easily seen by differentiating the following expression and simplifying:

$$\begin{aligned} & -2 \left( i\text{Li}_2 \left( -\frac{2e^{i \sin^{-1}(\frac{x}{2})}}{-i + \sqrt{3}} \right) + i\text{Li}_2 \left( \frac{2e^{i \sin^{-1}(\frac{x}{2})}}{i + \sqrt{3}} \right) + \frac{1}{2} i \sin^{-1} \left( \frac{x}{2} \right)^2 - \right. \\ & \left. \sin^{-1} \left( \frac{x}{2} \right) \log \left( 1 - \frac{e^{i \sin^{-1}(\frac{x}{2})}}{-\frac{\sqrt{3}}{2} + \frac{i}{2}} \right) - \sin^{-1} \left( \frac{x}{2} \right) \log \left( 1 - \frac{e^{i \sin^{-1}(\frac{x}{2})}}{\frac{\sqrt{3}}{2} + \frac{i}{2}} \right) + \right. \\ & \left. \log(1-x) \sin^{-1} \left( \frac{x}{2} \right) \right). \end{aligned}$$

Taking limits as  $x \rightarrow 0$  and  $x \rightarrow 1$ , this gives us that the definite integral in (4) is equal to:

$$2 \left( i \operatorname{Li}_2 \left( -\frac{2}{-i + \sqrt{3}} \right) + i \operatorname{Li}_2 \left( \frac{2}{i + \sqrt{3}} \right) \right) - 2i \operatorname{Li}_2 \left( -\frac{2e^{\frac{i\pi}{6}}}{-i + \sqrt{3}} \right) - \frac{13i\pi^2}{36} + \frac{1}{3}\pi \log \left( \frac{i}{\sqrt{3}} \right) + \frac{1}{3}\pi \log \left( 1 - \frac{e^{\frac{i\pi}{6}}}{-\frac{\sqrt{3}}{2} + \frac{i}{2}} \right).$$

Rewriting the argument of the first dilogarithmic expression shown above as

$$\operatorname{Li}_2 \left( e^{-\frac{5i\pi}{6}} \right)$$

and then applying a series multisection to (2) according to the residue classes of the indices modulo 6, and similarly for the other dilogarithmic expressions in our evaluation of (4), this can be used to show that (4) is also equal to

$$\frac{8G}{3} - \frac{4i\pi^2}{27} + \frac{1}{9}(-\sqrt{3} + i) \psi^{(1)} \left( \frac{1}{3} \right) + \frac{1}{9}(\sqrt{3} + i) \psi^{(1)} \left( \frac{2}{3} \right),$$

and this may be confirmed with Mathematica's `FunctionExpand` applied to our dilogarithmic form for (4). So, we have that Sun's series in (1) is expressible as

$$8G - \frac{4i\pi^2}{9} + \frac{1}{3}i\psi^{(1)} \left( \frac{1}{3} \right) - \frac{\psi^{(1)} \left( \frac{1}{3} \right)}{\sqrt{3}} + \frac{1}{3}i\psi^{(1)} \left( \frac{2}{3} \right) + \frac{\psi^{(1)} \left( \frac{2}{3} \right)}{\sqrt{3}} + 4 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2 16^k}.$$

As it turns out, it is actually known<sup>1</sup> that the series

$$\sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k}{k}}{(1+2k)^2} = 1.0149416064... \quad (5)$$

is equal to a well-known mathematical constant known as *Gieseking's constant* [24], with reference to the OEIS entry [A143298](https://oeis.org/A143298) and the references therein, and it is known that Gieseking's constant is equal to

$$\frac{9 - \psi^{(1)} \left( \frac{2}{3} \right) + \psi^{(1)} \left( \frac{4}{3} \right)}{4\sqrt{3}}.$$

---

<sup>1</sup>See <https://math.stackexchange.com/questions/3162014>, for example.

To show that the series in (5) is equal to Gieseeking's constant, we may start with the Maclaurin series for the inverse sine, so as to write

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k} z^k y^{2k}}{2k+1} = \frac{\sin^{-1}(2y\sqrt{z})}{2y\sqrt{z}},$$

and by indefinitely integrating with respect to  $y$ , we obtain

$$\frac{1}{2\sqrt{z}} \left( \sin^{-1}(2y\sqrt{z}) \log \left( 1 - e^{2i \sin^{-1}(2y\sqrt{z})} \right) - \frac{1}{2} i \left( \sin^{-1}(2y\sqrt{z})^2 + \text{Li}_2 \left( e^{2i \sin^{-1}(2y\sqrt{z})} \right) \right) \right),$$

and this leads us to an expression equivalent to Gieseeking's constant, using previously known dilogarithmic expressions for this constant. So, we have shown that Sun's series in (1) is equal to the following:

$$8G - \frac{4i\pi^2}{9} + \frac{1}{3} i \psi^{(1)} \left( \frac{1}{3} \right) - \frac{\psi^{(1)} \left( \frac{1}{3} \right)}{\sqrt{3}} + \frac{1}{3} i \psi^{(1)} \left( \frac{2}{3} \right) + \frac{\psi^{(1)} \left( \frac{2}{3} \right)}{\sqrt{3}} + \frac{9 - \psi^{(1)} \left( \frac{2}{3} \right) + \psi^{(1)} \left( \frac{4}{3} \right)}{\sqrt{3}}.$$

Since Sun's series is real-valued, we may omit the complex terms in the above expansion, giving us that Sun's series equals

$$8G + 3\sqrt{3} - \frac{\psi^{(1)} \left( \frac{1}{3} \right)}{\sqrt{3}} + \frac{\psi^{(1)} \left( \frac{4}{3} \right)}{\sqrt{3}}.$$

Applying an index shift to rewrite  $\psi^{(1)} \left( \frac{4}{3} \right)$  in terms of  $\psi^{(1)} \left( \frac{1}{3} \right)$ , this gives us that Sun's series in (1) is reducible to  $8G$ , as desired.  $\square$

## References

- 15 [1] Z. Sun, New series for some special values of  $L$ -functions, Nanjing Daxue Xuebao Shuxue Bannian Kan 32 (2) (2015) 189–218.
- [2] Z.-W. Sun, List of conjectural series for powers of  $\pi$  and other constants.
- [3] Z.-W. Sun, Supercongruences involving dual sequences, Finite Fields Appl. 46 (2017) 179–216.

- 20 [4] Z. Sun, Open conjectures on congruences, *Nanjing Daxue Xuebao Shuxue Bannian Kan* 36 (1) (2019) 1–99.
- [5] G.-S. Mao, Z.-W. Sun, Two congruences involving harmonic numbers with applications, *Int. J. Number Theory* 12 (2) (2016) 527–539.
- [6] W. Chu, Hypergeometric approach to Apéry-like series, *Integral Transforms Spec. Funct.* 28 (7) (2017) 505–518.
- 25 [7] Z.-W. Sun, New series for powers of  $\pi$  and related congruences, *Electron. Res. Arch.* 28 (3) (2020) 1273–1342.
- [8] W. Wang, C. Xu, Alternating multiple zeta values, and explicit formulas of some Euler-Apéry-type series, *European J. Combin.* 93 (2021) Paper No. 103283, 25.
- 30 [9] Z.-W. Sun, Congruences for two types of apéry-like sequences.
- [10] Z.-H. Sun, Supercongruences involving Apéry-like numbers and binomial coefficients, *AIMS Math.* 7 (2) (2022) 2729–2781.
- [11] C. Lupu, *Analytic Aspects of the Riemann Zeta and Multiple Zeta Values*, University of Pittsburgh, 2018.
- 35 [12] C. Lupu, Another look at zagier’s formula for multiple zeta values involving hoffman elements, *Math. Z.* (2022).
- [13] W. Wang, Y. Chen, Explicit formulas of sums involving harmonic numbers and Stirling numbers, *J. Difference Equ. Appl.* 26 (9-10) (2020) 1369–1397.
- 40 [14] W. Chu, Alternating series of Apéry-type for the Riemann zeta function, *Contrib. Discrete Math.* 15 (3) (2020) 108–116.
- [15] W. Chu, Further Apéry-like series for Riemann zeta function, *Math. Notes* 109 (1-2) (2021) 136–146.
- [16] L. Lai, C. Lupu, D. Orr, Elementary proofs of zagier’s formula for multiple zeta values and its odd variant.
- 45

- [17] C. Xu, J. Zhao, Apéry-type series with summation indices of mixed parities and colored multiple zeta values, ii.
- [18] C. Wang, Z.-W. Sun, Proof of some conjectural hypergeometric supercongruences via curious identities, *J. Math. Anal. Appl.* 505 (1) (2022) Paper No. 125575, 20.
- [19] X. Wang, X. Yuan, Infinite summation formulas involving riemann-zeta function.
- [20] Y. Wang, Y. Li, C. Xu, Evaluations of some Euler-Apéry-type series, *Indian J. Pure Appl. Math.* (202).
- [21] C. Xu, J. Zhao, Sun's three conjectures on apéry-like sums involving harmonic numbers.
- [22] G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part V*, Springer, Cham, 2018.
- [23] E. D. Rainville, *Special functions*, The Macmillan Co., New York, 1960.
- [24] C. C. Adams, The newest inductee in the number hall of fame, *Math. Mag.* 71 (5) (1998) 341–349.