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LOCAL EXISTENCE AND UNIQUENESS OF SOLUTIONS TO YANG-MILLS HEAT FLOW PROBLEM

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Abstract: In this paper, we consider the following nonlinear equation

$$\frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r} \partial_r u - 3(d-2)u^2 - (d-2)r^2 u^3$$

where $u : (r, t) \in \mathbb{R}_+^2 \mapsto \mathbb{R}, d \in \mathbb{N}^*$. This equation has been investigated by Grotowski in 2001 in studying the Yang-Mills heat flow connections on Riemann manifolds. In the paper, we prove the local Cauchy problem for above equation that is well-posed in $L_{1+r^2}^\infty(\mathbb{R}_+)$. More precisely, for any initial data $u_0 \in L_{1+r^2}^\infty(\mathbb{R}_+)$, there exists $T(u_0) > 0$ such that the above equation has a unique solution $u(t) \in L_{1+r^2}^\infty(\mathbb{R}_+)$ for all $t \in [0, T(u_0)]$.

Keywords — Local Cauchy problem, Local existence and uniqueness problem, Yang-Mills heat flows, Yang-Mills connections, Geometric flows.

I. INTRODUCTION

In this paper, we are interested in the following nonlinear heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r} \partial_r u - 3(d-2)u^2 - (d-2)r^2 u^3, & (r, t) \in (0, +\infty) \times [0, T), \\ u(0) = u_0 \in L^\infty(\mathbb{R}_+), \end{cases} \quad (1)$$

where $u : (r, t) \in \mathbb{R}_+^2 \mapsto \mathbb{R}, d \in \mathbb{N}^*$. The problem (1) has a strong connection to the problem of Yang Mills connections in $\mathbb{R}^d \times SO(d)$ which is an important part in Yang-Mills theory. We would like to mention that the theory was used to study the weak nuclear forces, governing the nuclear decay of some particles which is considered as a non-commutative version of Maxwell's electromagnetism, see more details in [1]-[6]. In particular, the Yang-Mills heat flow has received a lot of attention from both mathematics and physics communities. Results on existence and uniqueness of weak solutions in other functional spaces were obtained in [7] for $d = 2, 3$, and [8] and [9] for $d = 4$; global existence has proved in [10]-[12]; the regularity was established in [13] in for higher dimensions; stability were proved in [14] and [15]; singularity formations have been studied in [16]-[24], the local well posedness was studied in [25] and [26] in Sobolev spaces. In particular, the study of Yang-Mills connections usually is considered in abstract spaces such as Riemann manifolds which get lot of inconveniences. For that reason, the author in [17] reduced the study of Yang-Mills connections to the problem introduced in (1).

The main goal of this paper is to study the local Cauchy problem to (1) since strong connections to the Yang-Mills problem. Recently, the authors in [24] solved the local Cauchy problem for (1) in $C_0^\infty(\mathbb{R}_+)$ defined as the set of smooth functions with compact supports. However, it remains open if the local Cauchy problem in can be solved $L^\infty(\mathbb{R}_+)$. In this work, we aim to prove the Cauchy problem in $L_{1+r^2}^\infty(\mathbb{R}_+) \subset L^\infty(\mathbb{R}_+)$ which is strictly larger than $C_0^\infty(\mathbb{R}_+)$.

For more convenience, we mention below some important notations which will be used in the proof of the paper. Let Ω be a Lebesgue measurable set in $\mathbb{R}^n, n \in \mathbb{N}^*$, we denote $L^\infty(\Omega)$ as the set of all Lebesgue measurable functions f on Ω satisfying

$$\inf \left\{ B \text{ such that } \mu \left(\left\{ x \in \Omega : |f(x)| > B \right\} \right) = 0 \right\} < +\infty$$

which is a Banach space with the following norm

$$\|f\|_{L^\infty(\Omega)} = \inf \left\{ B \text{ such that } \mu \left(\left\{ x \in \Omega : |f(x)| > B \right\} \right) = 0 \right\}.$$

Similarly, we also define $L_{1+|x|^2}^\infty(\Omega)$

$$L_{1+|x|^2}^\infty(\Omega) = \left\{ f \in L^\infty(\Omega) \text{ such that } \left\| (1+|x|^2) f(\cdot) \right\|_{L^\infty(\Omega)} < +\infty \right\}$$

and it is also a Banach space with the norm

$$\|f\|_{L_{1+|x|^2}^\infty(\Omega)} = \left\| (1+|x|^2) f(\cdot) \right\|_{L^\infty(\Omega)}.$$

We call f a radially symmetric function on Ω if and only if for all orthogonal matrices A and $x \in \Omega$, then it satisfies that $Ax \in \Omega$ and $f(Ax) = f(x)$. Hence, we introduce

$$L_{rad}^\infty(\Omega) = \left\{ f \in L^\infty(\Omega) \text{ such that } f \text{ radially symmetric} \right\},$$

which is a Banach space with $\|\cdot\|_{L^\infty(\Omega)}$ norm. In particular, we also define the abstract space of Banach-valued functions $X^T, T > 0$ by

$$X^T = L^\infty \left([0, T], L_{rad}^\infty(\Omega) \cap L_{1+|x|^2}^\infty(\Omega) \right)$$

which is also a Banach space with the norm

$$\|z\|_{X^T} = \|Z(t)\|_{L^\infty([0, T])} \text{ where } Z(t) = \|z(t)\|_{L_{1+|x|^2}^\infty(\Omega)}.$$

Next, we denote Δ as Laplace operator in Euclidean space \mathbb{R}^{d+2} by $\Delta = \sum_{j=1}^{d+2} \partial_{x_j}^2$. In particular, taking $\Omega = \mathbb{R}^{d+2}$, we recall the semi-group $\{e^{At}\}_{t>0}$ that

$$e^{At} : L^\infty(\mathbb{R}^{d+2}) \rightarrow L^\infty(\mathbb{R}^{d+2})$$

and

$$e^{At} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} f(y) e^{-\frac{|x-y|^2}{4t}} dy. \quad (2)$$

We would like to mention [27, Proposition 48.4] the following fundamental estimate

$$\|e^{At} f\|_{L^\infty(\mathbb{R}^{d+2})} \leq \|f\|_{L^\infty(\mathbb{R}^{d+2})}. \quad (3)$$

II. MAIN RESULTS

In this section, we aim to prove the local Cauchy problem for equation (1) in $L_{1+|x|^2}^\infty(\mathbb{R}_+)$. However, it is so hard to give a direct proof to equation (1) due to complexity of the linear operator

$$\partial_r^2 + \frac{d+1}{r} \partial_r.$$

To overcome this challenge, we used the idea investigated in [28] where the authors successfully handled the Cauchy problem for harmonic heat flows. Let f be a function defined in \mathbb{R}_+ , then we denote \tilde{f} as f 's extension on \mathbb{R}^{d+2} given by

$$\tilde{f}(x) = f(|x|). \quad (4)$$

We can see that the extension is always a radially symmetric function on \mathbb{R}^{d+2} . In particular, once $f \in C^2(\mathbb{R}_+)$, then we have $\tilde{f} \in C^2(\mathbb{R}^{d+2})$ and the following identity holds

$$\Delta \tilde{f}(x) = \partial_r^2 f + \frac{d+1}{r} \partial_r f(|x|).$$

Let us consider u is a $C^2(\mathbb{R}_+)$ -solution to (1) (so-called classical solution), then the extension \tilde{u} belongs to $C^2(\mathbb{R}^{d+2})$ and \tilde{u} satisfies the following equation

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} - 3(d-2)\tilde{u}^2 - (d-2)|x|^2 \tilde{u}^3, (x, t) \in \mathbb{R}^{d+2} \times (0, T), \\ \tilde{u}(0) = \tilde{u}_0 \in L_{rad}^\infty(\mathbb{R}^{d+2}). \end{cases} \quad (5)$$

From the symmetricity of equation (5), the solution \tilde{u} remains radially symmetric as well as the solution exists. In the following, we aim to prove the local Cauchy problem to equation (5) in $L_{rad}^\infty(\mathbb{R}^{d+2}) \cap L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})$. Firstly, we have the following Lemma

Lemma 2.1. Let us consider α be a positive number and $f \in L^\infty(\mathbb{R}^{d+2})$ satisfying

$$\left\| (1+|x|^\alpha) f(\cdot) \right\|_{L^\infty(\mathbb{R}^{d+2})} \leq A. \quad (6)$$

Then, it holds that

$$\left\| (1+|x|^\alpha) e^{t\Delta} f \right\|_{L^\infty(\mathbb{R}^{d+2})} \leq C(\alpha) A, t > 0,$$

where the semi-group $e^{t\Delta}$ defined as in (2).

Proof: By the explicit formula in (2), we write as follows

$$e^{At} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} e^{-\frac{|x-y|^2}{4t}} f(y) dy.$$

Let us consider $R > 0$ large enough and fixed later. For all $|x| \leq R$, we use (6) to derive

$$\left| (1+|x|^\alpha) e^{At} f(x) \right| \leq \frac{A(1+R^\alpha)}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \frac{e^{-\frac{|x-y|^2}{4t}}}{1+|y|^\alpha} dy \leq C(R, \alpha) A.$$

It remains to the case $|x| \geq R$. Let us define $K_1 = \left\{ y : |y| \leq \frac{|x|}{2} \right\}$ and $K_2 = \left\{ y : |y| > \frac{|x|}{2} \right\}$. It is obvious that $K_1 \cup K_2 = \mathbb{R}^{d+2}$. Then,

we have the decomposition

$$(1+|x|^\alpha) e^{At} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} (1+|y|^\alpha) f(y) dy = I_1 + I_2,$$

where

$$I_1 = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_1} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} (1+|y|^\alpha) f(y) dy,$$

and

$$I_2 = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_2} \frac{1+|x|^\alpha}{1+|y|^\alpha} e^{-\frac{|x-y|^2}{4t}} (1+|y|^\alpha) f(y) dy.$$

+ For I_1 : We using the facts that $t \in (0, 1)$ and $y \in K_1, |x-y| \geq |x| - |y| \geq \frac{|x|}{2}$. So we have the estimate

$$|I_1| \leq \frac{A}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_1} (1+|x|^\alpha) e^{-\frac{|x-y|^2}{4t}} dy.$$

Using the changing variable $z = \frac{x-y}{2\sqrt{t}}$, we get the following

$$\begin{aligned}
& \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_1} \left(1 + |x|^\alpha\right) e^{-\frac{|x-y|^2}{4t}} dy \\
&= \frac{\left(1 + |x|^\alpha\right)}{(\pi)^{\frac{d+2}{2}}} \int_{y \in K_1, z = \sqrt{t}y-x} e^{-z^2} dz \\
&\leq \frac{\left(1 + |x|^\alpha\right)}{(\pi)^{\frac{d+2}{2}}} \int_{\frac{|x|}{2} \leq |z| \leq \frac{3|x|}{2}} e^{-z^2} dz \leq C(1 + |x|)^{d+2+\alpha} e^{-\frac{|x|^2}{4}} \\
&\leq C(R, \alpha),
\end{aligned}$$

which implies $|I_1| \leq C(R, \alpha) A$.

+ For I_2 : Using the fact that $y \in K_2, |y| \geq \frac{|x|}{2}$, then it follows $\frac{1 + |x|^\alpha}{1 + |y|^\alpha} \leq C(\alpha)$. Hence, we estimate I_2 as follows

$$|I_2| \leq \frac{C(\alpha)A}{(4\pi t)^{\frac{d+2}{2}}} \int_{K_2} e^{-\frac{|x-y|^2}{4t}} dy \leq C(\alpha)A.$$

Thus, we conclude that for all $|x| \geq R$, we have

$$\left| (1 + |x|^\alpha) e^{\Delta t} f(x) \right| \leq C(R, \alpha) A.$$

Finally, we get the conclusion of the proof.

Now, we have the following result:

Proposition 2.2. Let $\tilde{u}_0 \in L_{rad}^\infty(\mathbb{R}^{d+2}) \cap L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})$ arbitrarily, there exists $T = T(\tilde{u}_0) > 0$ such that problem (5) has a unique solution on $[0, T]$ and $\tilde{u}(t) \in L_{rad}^\infty(\mathbb{R}^{d+2}) \cap L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})$ for all $t \in [0, T]$. In particular, the following estimate holds

$$\|\tilde{u}\|_{X^T} \leq \|\tilde{u}_0\|_{L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})} + 1.$$

Proof: The result follows from the Banach fixed point theorem (the unique fixed point of a constructive mapping). Let us consider initial data $\tilde{u}_0 \in L_{rad}^\infty(\mathbb{R}^{d+2}) \cap L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})$ and $T > 0$ and we define

$$B_0 = \left\{ \tilde{u} \in X^T \text{ such that } \|\tilde{u} - e^{t\Delta}(\tilde{u}_0)\|_{X^T} \leq 1 \right\}.$$

In addition, we define Γ a mapping on X^T

$$\Gamma(\tilde{u})(t) = e^{t\Delta}\tilde{u}_0 + \int_0^t e^{(t-s)\Delta} F(s, \tilde{u}(s)) ds, t > 0 \text{ and } \Gamma(z)(0) = \tilde{u}_0, \quad (7)$$

and F defined by

$$F(\tilde{u}) = -3(d-2)\tilde{u}^2 - (d-2)|x|^2 \tilde{u}^3. \quad (8)$$

In the below, we aim to prove that once T small enough, Γ satisfies the following properties:

(H1): Γ maps B_0 into itself.

(H2): Γ is a contraction mapping on B_0 i.e. there exists $\lambda \in (0, 1)$ such that

$$\|\Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2)\|_{X^T} \leq \lambda \|\tilde{u}_1 - \tilde{u}_2\|_{X^T}, \forall \tilde{u}_1, \tilde{u}_2 \in B_0.$$

- *Proof of (H1):* Let $\tilde{u} \in B_0$ arbitrarily, we derive from (8) and the fact $\|\tilde{u}\|_{X^T} \leq 1$

$$\left\| (1 + |x|^2) F(\tilde{u}(s)) \right\|_{L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})} \leq C \left(\|\tilde{u}(s)\|_{L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})}^2 + \|\tilde{u}(s)\|_{L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})}^3 \right) \leq C \|\tilde{u}(s)\|_{L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})},$$

Applying Lemma 2.1 with $\alpha = 2$ and $t - s \in (0, 1)$, we get

$$\left\| e^{(t-s)\Delta} F(\tilde{u}(s)) \right\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} \leq C_1 \|\tilde{u}\|_{X^T},$$

which yields

$$\left\| \Gamma(\tilde{u}) - e^{t\Delta} \tilde{u}_0 \right\|_{X^T} = \left\| \int_0^t e^{(t-s)\Delta} F(\tilde{u}(s)) ds \right\|_{X^T} \leq C_1 \sqrt{T} \|\tilde{u}\|_{X^T}, \quad (9)$$

for all $\tilde{u} \in B_0$. Taking $T \leq \left(\frac{1}{C_1}\right)^2$, we get

$$\left\| \Gamma(\tilde{u}) - e^{t\Delta} \tilde{u}_0 \right\|_{X^T} \leq \|\tilde{u}\|_{X^T}, \quad \forall \tilde{u} \in B_0.$$

In addition, since \tilde{u} is radially symmetric, so $F(\tilde{u})$ defined as in (8) is and the convolution in (2) saves the symmetry that leads $\Gamma(\tilde{u}(t))$ is radially symmetric for all $t \in [0, T]$. Finally, we conclude (H1).

- *Proof of (H2)*: Let $\tilde{u}_1, \tilde{u}_2 \in B_0$, and we write

$$\begin{aligned} F(\tilde{u}_1(s)) - F(\tilde{u}_2(s)) &= -3(d-2)(\tilde{u}_1(s) - \tilde{u}_2(s))(\tilde{u}_1(s) + \tilde{u}_2(s)) \\ &\quad - (d-2)|x|^2(\tilde{u}_1(s) - \tilde{u}_2(s))(\tilde{u}_1^2(s) + \tilde{u}_1(s)\tilde{u}_2(s) + \tilde{u}_2^2(s)). \end{aligned}$$

Since $\|\tilde{u}_1\|_{X^T} \leq 1$ and $\|\tilde{u}_2\|_{X^T} \leq 1$, we derive

$$\left\| F(\tilde{u}_1(s)) - F(\tilde{u}_2(s)) \right\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} \leq C \|\tilde{u}_1(s) - \tilde{u}_2(s)\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})}, \quad \forall s \in [0, T].$$

Regarding to **Lemma 2.1**, we have

$$\left\| e^{(t-s)\Delta} (F(\tilde{u}_1(s)) - F(\tilde{u}_2(s))) \right\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} \leq C \|\tilde{u}_1(s) - \tilde{u}_2(s)\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} \leq C \|\tilde{u}_1 - \tilde{u}_2\|_{X^T},$$

for all $t - s > 0, s \in (0, t)$. Thus, we derive from (6) that for all $t \in (0, T]$

$$\left\| \Gamma(\tilde{u}_1)(t) - \Gamma(\tilde{u}_2)(t) \right\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} \leq C \|\tilde{u}_1 - \tilde{u}_2\|_{X^T} \left(\int_0^t ds \right) \leq C_2 T \|\tilde{u}_1 - \tilde{u}_2\|_{X^T},$$

which yields

$$\left\| \Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2) \right\|_{X^T} \leq C_2 T \|\tilde{u}_1 - \tilde{u}_2\|_{X^T}, \quad \forall \tilde{u}_1, \tilde{u}_2 \in B_0.$$

Finally, we choose $C_2 \sqrt{T} \leq \frac{1}{2}$ i.e. $T \leq \left(\frac{1}{2C_2}\right)^2$ then

$$\left\| \Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2) \right\|_{X^T} \leq \frac{1}{2} \|\tilde{u}_1 - \tilde{u}_2\|_{X^T}, \quad \forall \tilde{u}_1, \tilde{u}_2 \in B_0,$$

which concludes (H2).

Now, we continue on the proof of the proposition, since X^T is a Banach space and Γ is a contractive map from B_0 to itself, so we apply Banach fixed point theorem that there uniquely exists $\tilde{u} \in B_0$ such that

$$\tilde{u}(t) = \Gamma(\tilde{u})(t), \quad \forall t \in [0, T],$$

and we have the estimate

$$\|\tilde{u}\|_{X^T} \leq \|e^{t\Delta} \tilde{u}_0\|_{X^T} + \|\Gamma(\tilde{u}) - e^{t\Delta} \tilde{u}_0\|_{X^T} \leq \|\tilde{u}_0\|_{L_{1+|\cdot|^2}^\infty(\mathbb{R}^{d+2})} + 1. \quad (10)$$

In particular, by the parabolic regularity of the semi-group $e^{t\Delta}$, we improve that $\tilde{u}(t) \in C^2(\mathbb{R}^{d+2})$, $\forall t \in (0, T)$ and it satisfies

equation (5) for all $(x, t) \in \mathbb{R}^{d+2} \times (0, T)$ point-wise, and we derive from (10)

$$|\tilde{u}(x, t)| \leq \frac{\|\tilde{u}_0\|_{L_{1+|x|^2}^\infty} + 1}{1 + |x|^2}, \forall (x, t) \in \mathbb{R}^{d+2} \times (0, T].$$

Finally, we get the conclusion of the Proposition.

Consequently, **Proposition 2.2** implies the following result:

Proposition 2.3: Let $d \geq 1$ be an integer number and initial choice $u_0 \in L_{1+r^2}^\infty(\mathbb{R}_+)$. Then, there exists $T = T(u_0)$ such that problem (1) has a unique solution on $[0, T]$ and $u(t) \in L_{1+|x|^2}^\infty(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$, $\forall t \in (0, T]$. In particular, we have the following estimate

$$|u(r, t)| \leq \frac{C(u_0)}{1 + r^2}, \forall (r, t) \in \mathbb{R}_+ \times (0, T).$$

Proof: Let $u_0 \in L_{1+r^2}^\infty(\mathbb{R}_+)$, then, it follows from the extension (4) that $\tilde{u}_0 \in L_{rad}^\infty(\mathbb{R}^{d+2}) \cap L_{1+|x|^2}^\infty(\mathbb{R}^{d+2})$. Applying **Proposition 2.2**, we obtain the existence and the uniqueness of the solution \tilde{u} to equation (5) on $[0, T(u_0)]$ and $\tilde{u}(t) \in L_{rad}^\infty(\mathbb{R}_+) \cap L_{1+|x|^2}^\infty(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$, $\forall t \in (0, T]$, then, the problem (1) and (8) are equivalent in the radially symmetric setting, this leads to the existence and the uniqueness of u and also the conclusion the proposition completely follows.

Remark 2.4: We can repeat the proof of Propositions 2.2 and 2.3, to establish the local existence and uniqueness in spaces $L_{1+r^\alpha}^\infty(\mathbb{R}_+)$ where $\alpha \geq \frac{2}{3}$. Since the main difficulty is to handle the hugeness of nonlinear term $r^2 u^3$ at infinity. However, once $u \in L_{1+r^\alpha}^\infty(\mathbb{R}_+)$, $\alpha \geq \frac{2}{3}$ it follows that the nonlinearity is controlled well by

$$r^2 u^3 = \frac{r^2}{(1 + r^\alpha)^3} \left((1 + r^\alpha) u \right)^3 \leq \|u\|_{L_{1+r^\alpha}^\infty}^3 (1 + r^\alpha)^{\frac{2}{\alpha} - 3},$$

since $\frac{2}{\alpha} - 3 \leq 0$. We kindly refer the readers to check the details of the general results.

III. CONCLUSION

As we showed in **Proposition 2.3**, the local Cauchy problem in $L_{1+r^2}^\infty(\mathbb{R}_+)$ is completely solved (also in $L_{1+r^\alpha}^\infty(\mathbb{R}_+)$, $\alpha \geq \frac{2}{3}$ as in **Remark 2.4**). In comparison to the result proved in Donninger and Schörkhuber, 2019 where the authors considered the problem in $C_0^\infty(\mathbb{R}_+)$, our result is better. The technique of the proof replies to the idea given by Biernat and Seki, 2019 that extended the original problem to the radially symmetric one in \mathbb{R}^{d+2} , and then, we established a new property that the semi-group $e^{t\Delta}$, $t > 0$ reserves the polynomial decays showed in Lemma 2.1, then the existence and uniqueness follows by the route map based on Banach fixed point theorem.

IV. REFERENCES

- [1] A. Actor, Classical solutions of SO(2)-Yang-Mills theories, Rev. Mod. Phys. 51 (1979) 461-525.
- [2] P. Biernat and Y. Seki, Type II blow-up mechanism for supercritical harmonic map heat flow, Int. Math. Res. Not. IMRN. 2 (2019) 407-456.
- [3] R. Donninger, Stable self-similar blowup in energy supercritical yang-mills theory, Mathematische Zeitschrift. 278 (2012) 1005-1032.

- [4] R. Donniger, and B. Schörkhuber, Stable blowup for the supercritical Yang-Mills heat flow, *Journal Differential Geometry*. 113(1) (2019) 55-94.
- [5] A. Gastel, Singularities of the first kind in the harmonic map and Yang-Mills heat flows, *Mathematische Zeitschrift*. 242(1) (2002) 47-62.
- [6] J. F. Grotowski, Finite time blow-up for the Yang-Mills heat flow in higher dimensions, *Mathematische Zeitschrift*. 237(2) (2001) 321-333.
- [7] S. Klainerman, and M. Machedon, Finite energy solutions of the Yang-Mills equations in \mathbb{R}^{3+1} , *Annals of Mathematics*. 142(1) (1995) 39-119.
- [8] S. J. Oh, Gauge choice for the Yang-Mills equations using the Yang-Mills heat flow and local well-posedness in H^1 , *Journal of Hyperbolic Differential Equations*. 11(1) (2014) 1-108.
- [9] P. Quittner and P. Souplet, *Superlinear parabolic problems*, Birkhäuser Advanced Birkhäuser Verlag. Basel, 2007.
- [10] T. Tao, Local well-posedness of the Yang-Mills equation in the temporal gauge below the energy norm, *Journal Differential Equation*. 189(2) (2003) 366-382.
- [11] G. 't Hooft, 50 years of Yang-Mills theory, *World Scientific Publishing Co., Pte. Ltd. Hackensack*, 2005.
- [12] O. Dumitrescu, Equivariant solutions of the Yang-Mills equations, *Stud. Cerc. Mat.* 34(4) (1982) 329-333.
- [13] H. Kozono, Y. Maeda, and H. Naito, Global solution for the Yang-Mills gradient flow on 4-manifolds, *Nagoya Mathematical Journal*. 139 (1995) 93-128.
- [14] J. Rade, On the Yang-Mills heat equation in two and three dimensions, *Journal für die Reine und angewandte Mathematik*. 431 (1992) 123-163.
- [15] A. Schlatter, Global existence of the Yang-Mills flow in four dimensions, *Journal für die Reine und angewandte Mathematik*. 479 (1996) 133-148.
- [16] M. Struwe, The Yang-Mills flow in four dimensions, *Calculus of Variations and Partial Differential Equations*. 2(2) (1994) 123-150.
- [17] B. Weinkove, Singularity formation in the Yang-Mills flow, *Calculus of Variations and Partial Differential Equations*. 19(2) (2004) 211-220.
- [18] S. K. Donaldson, and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press. New York, 1990.
- [19] M.C. Hong, G. Tian, Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections, *Math. Ann.* 330 (2004) 441-472.
- [20] A. Schlatter, M. Struwe, and A. Shadi Tahvildar-Zadeh. Global existence of the equivariant Yang-Mills heat flow in four space dimensions, *American Journal of Mathematics*. 120(1) (1998) 117-128.
- [21] A. Schlatter, Long-time behaviour of the Yang-Mills flow in four dimensions, *Annals of Global and Geometry*. 15(1) (1997) 1-25.
- [22] C. Kelleher, and J. Streets, Entropy, stability, and Yang-Mills flow, *Communications in Contemporary Mathematics*. 18(2) (2016) 1550032.
- [23] P. Bizon, Formation of singularities in Yang-Mills equations, *Acta Physica Polonica B*. 33(7) (2002) 1893-1922.

- [24] Z. Chen, and Y. Zhang, Stabilities of homothetically shrinking Yang-Mills solitons, *Trans. Amer. Math. Soc.* 367(7) (2015) 5015–5041.
- [25] C. Kelleher, and J. Streets, Singularity formation of the Yang-Mills flow, *Ann. Inst. H. Poincaré Anal. Non Linéaire.*, 35(6) (2018) 1655–1686.
- [26] G. Tian, Gauge theory and calibrated geometry I, *Annals of Mathematics.* 151 (2000) 193–268.
- [27] Chen, Y., Shen, C.-L.: Monotonicity formula and small action regularity for Yang-Mills flows in higher dimensions. *Calc. Var. PDEs* 2, 389–403 (1994)
- [28] M. Atiyah, and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philosophical Transactions of the Royal Society A.* 308 (1982) 524–615.