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CLASSIFICATION OF LOG SMOOTH TORIC DEL PEZZO PAIRS

ACHIM NAPAME

ABSTRACT. We give a description of all log-Fano pairs (X, D) where X is a smooth toric surface and D a reduced simple normal crossing divisor such that the pair (X, D) is equivariant.

1. INTRODUCTION

The Enriques-Kodaira classification gives a classification of complex compact surfaces by using their Kodaira dimension. Nonsingular projective minimal surfaces with Kodaira dimension $-\infty$ have an important position in this problem of classification, they correspond in the MMP-terminology to *Mori's fiber spaces* (see [Mat02, Theorem 1.5.5]). Del Pezzo surfaces form a class of surfaces with kodaira dimension $-\infty$.

A *toric variety* X of dimension n is an irreducible variety that contains a torus $T \simeq (\mathbb{C}^*)^n$ as a dense open subset, together with an action of T on X that extends the natural action of T on itself. We say that a toric surface is *log del Pezzo* if it has at worst log-terminal singularities and its anticanonical divisor is a \mathbb{Q} -Cartier ample divisor. According to [Dais17, Proposition 4.2], there is a one-to-one correspondence between toric log del Pezzo surfaces and LDP-polygons which are convex lattice polygons containing the origin in their interior and having only primitive vertices. The classification of toric log del Pezzo surfaces is an open problem. There are results on the classification according to the index of the anticanonical divisor (see [CLS11, Section 8.3] for index one and [KKN10] for index two).

Motivated by *Itaka's philosophy*, in this paper we are interested by the question of classification of log smooth toric del Pezzo pairs. A pair (X, D) is called a *log smooth del Pezzo pair* if X is a smooth surface and D a simple normal crossing divisor such that $-(K_X + D)$ is ample. Moreover, we say that (X, D) is *toric* if X is toric and the sheaf $T_X(-\log D)$ is an equivariant sheaf which means that D is an invariant divisor under the action of the torus T of X .

Theorem 1.1. *Let X be a complete smooth toric surface such that $\mathrm{rk} \mathrm{Pic}(X) \geq 3$. For any reduced invariant divisor D of X , the pair (X, D) is not toric log del Pezzo.*

According to this Theorem, to find the classification of log smooth toric del Pezzo pairs, it suffices to consider complete smooth toric surfaces with Picard rank one or two. The variety $\mathbb{P}^2 = \mathbb{C}\mathbb{P}^2$ is the unique smooth toric surface of Picard rank one. We denote by $[x_0 : x_1 : x_2]$ the homogeneous coordinates of \mathbb{P}^2 and D_i the divisor defined by $x_i = 0$.

Theorem 1.2. *If $X = \mathbb{P}^2$, then the log smooth pair (X, D) is toric log del Pezzo if and only if $D \in \{D_0, D_1, D_2\} \cup \{D_0 + D_1, D_0 + D_2, D_1 + D_2\}$.*

Every smooth toric surface of Picard rank two is of the form $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ with $r \in \mathbb{N}$ (we refer to [CLS11, Theorem 7.3.7] and [Kle88]). We denote by $\pi : X \rightarrow \mathbb{P}^1$ the projection on the base \mathbb{P}^1 . Let $E_0 = \{0\}$ and $E_\infty = \{\infty\}$ be two divisors of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (the identification come from $[x_0 : x_1] \mapsto x_1/x_0$). Let \mathcal{F}_0 and \mathcal{F}_1 be two rank one vector bundles over \mathbb{P}^1 whose sheaf of sections

are respectively given by $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(r)$. We denote by s_i a section of \mathcal{F}_i . The irreducible T -invariant divisors of X are given by

$$D_0 = \{s_0 = 0\}, \quad D_1 = \pi^{-1}(E_0), \quad D_2 = \{s_1 = 0\} \quad \text{and} \quad D_3 = \pi^{-1}(E_\infty).$$

If $D = \sum_{i \in I} D_i$ with $I \subset \{0, 1, 2, 3\}$ such that $\text{card}(I) \geq 3$, then $-(K_X + D)$ is not ample by Proposition 3.1. We have here the classification of toric log del Pezzo pairs for toric surfaces with Picard rank two.

Theorem 1.3. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ with $r \in \mathbb{N}$. Then :*

- (1) $-K_X$ or $-(K_X + D_0)$ are ample if and only if $r \in \{0, 1\}$.
- (2) If $D \in \{D_1, D_3, D_0 + D_1, D_0 + D_3\}$, $-(K_X + D)$ is ample if and only if $r = 0$.
- (3) If $D \in \{D_2, D_2 + D_1, D_2 + D_3\}$, $-(K_X + D)$ is ample if and only if $r \in \mathbb{N}$.
- (4) if $D \in \{D_0 + D_2, D_1 + D_3\}$, $-(K_X + D)$ is not ample for any $r \in \mathbb{N}$.

In the first part of this paper, we will recall some properties about toric varieties and we will give the characterization of ample divisors on toric surfaces. In the second part (Section 3), we will describe the family of log smooth toric del Pezzo pairs.

Acknowledgments. I would like to thank my advisor Carl TIPLER for our discussions on this subject.

2. AMPLE DIVISORS ON TORIC SURFACES

2.1. Toric varieties. Let N be a rank n lattice and $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual with pairing $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$. The lattice N is the lattice of one-parameter subgroups of $N \otimes_{\mathbb{Z}} \mathbb{C}^*$. For $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we define $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. A fan Σ in $N_{\mathbb{R}}$ is a set of rational strongly convex polyhedral cone in $N_{\mathbb{R}}$ such that:

- Each face of a cone in Σ is also a cone in Σ ;
- The intersection of two cones in Σ is a face of each.

Definition 2.1 ([CLS11, Definition 1.2.16]). A cone σ in $N_{\mathbb{R}}$ is *smooth* if its minimal generators form part of a \mathbb{Z} -basis of N .

By [CLS11, Definition 3.1.18], a fan Σ is *smooth* if every cone σ in Σ is smooth. A fan Σ is *complete* if

$$\bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}.$$

For $\sigma \in \Sigma$, let $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ where $\mathbb{C}[S_\sigma]$ is the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

If $\sigma, \sigma' \in \Sigma$, we have $U_\sigma \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$. We denote by X_Σ the toric variety associated to a fan Σ ; the variety X_Σ is obtained by gluing affine charts $(U_\sigma)_{\sigma \in \Sigma}$.

The variety X_Σ is normal and its torus is $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Let $\Sigma(k)$ be the set of k -dimensional cones of Σ . There is a bijective correspondence between cones $\sigma \in \Sigma$ and T -orbits $O(\sigma)$ in X_Σ ; moreover we have $\dim O(\sigma) = \dim(X_\Sigma) - \dim(\sigma)$.

For any ray $\rho \in \Sigma(1)$, there is a Weil divisor D_ρ defined as the Zariski closure of $O(\rho)$ in X_Σ . We denote by $u_\rho \in N$ the minimal generator of $\rho \in \Sigma(1)$. By [CLS11, Theorem 4.1.3], if $N_{\mathbb{R}} = \text{Span}(u_\rho : \rho \in \Sigma(1))$ then

$$(1) \quad \text{card}(\Sigma(1)) = \dim(X_\Sigma) + \text{rk Cl}(X_\Sigma).$$

Let X be the toric variety associated to a fan Σ . We say that X is *smooth* (resp. *complete*) if and only if Σ is smooth (resp. complete). We know the expression of the canonical divisor K_X .

Theorem 2.2 ([CLS11, Theorem 8.2.3]). *The canonical divisor of X is the torus invariant divisor*

$$K_X = - \sum_{\rho \in \Sigma(1)} D_\rho .$$

A Cartier divisor of X is ample, if it satisfies the toric Kleiman Criterion.

Theorem 2.3 (Toric Kleiman Criterion, [CLS11, Theorem 6.3.13]). *Let D be a Cartier divisor on a complete toric variety X . Then D is ample if and only if $D \cdot C > 0$ for all torus-invariant irreducible curves $C \subset X$.*

2.2. Description of complete toric surfaces. We assume that $N = \mathbb{Z}^2$, $M = \mathbb{Z}^2$ and the pairing $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ is the dot product. Let Σ be a smooth complete fan in \mathbb{R}^2 and X the toric surface associated to Σ . We denote by T the torus of X . There is a family of primitive vectors $\{u_i \in N : 0 \leq i \leq n-1\}$ with $n \geq 3$ such that

$$\Sigma = \{0\} \cup \{\text{Cone}(u_i) : 0 \leq i \leq n-1\} \cup \{\text{Cone}(u_i, u_{i+1}) : 0 \leq i \leq n-1\}$$

where $u_n = u_0$. For all $i \in \{0, \dots, n-1\}$, we set $\rho_i = \text{Cone}(u_i)$, $\sigma_i = \text{Cone}(u_i, u_{i+1})$ and D_i the divisor corresponding to the ray ρ_i . As Σ is smooth, we can assume that $\det(u_i, u_{i+1}) = \pm 1$. From now on, we will assume that $\det(u_i, u_{i+1}) = 1$. For all $i \in \{0, \dots, n-1\}$, there is $\gamma_i \in \mathbb{Z}$ such that

$$(2) \quad u_{i-1} - \gamma_i u_i + u_{i+1} = 0 .$$

The number γ_i is $\det(u_{i-1}, u_{i+1})$.

2.3. Ample divisors of X . In this part, if $L = \sum_i a_i D_i$ is a Cartier divisor of X , we will give a condition on the numbers a_i which ensures that L is ample.

By [CLS11, Proposition 6.4.4], for all $i \in \{0, \dots, n-1\}$, we have

$$(3) \quad \begin{cases} D_i \cdot D_i = -\gamma_i \\ D_k \cdot D_i = 1 & \text{if } k \in \{i-1, i+1\} \\ D_k \cdot D_i = 0 & \text{if } k \notin \{i-1, i, i+1\} \end{cases} .$$

Proposition 2.4. *A Cartier divisor $L = \sum_i a_i D_i$ is ample if and only if for all $i \in \{0, \dots, n-1\}$, $a_{i+1} + a_{i-1} - \gamma_i a_i > 0$.*

Proof. The set of torus-invariant irreducible curve of X is $\{D_i : 0 \leq i \leq n-1\}$. By (3), we have

$$(4) \quad L \cdot D_i = a_{i-1} + a_{i+1} - \gamma_i a_i .$$

By using the toric Kleiman Criterion (Theorem 2.3), we deduce that L is ample if and only if $a_{i+1} + a_{i-1} - \gamma_i a_i > 0$ for all $i \in \{0, \dots, n-1\}$. \square

Let $L = \sum_i a_i D_i$ be a Cartier divisor of X . The polytope corresponding to L is given by

$$(5) \quad P = \{m \in \mathbb{Z}^2 : \langle m, u_i \rangle \geq -a_i \text{ for } i \in \{0, \dots, n-1\}\}$$

and the facet of P perpendicular to the vector u_i is given by

$$(6) \quad P_i = \{m \in \mathbb{Z}^2 : \langle m, u_i \rangle = -a_i\} \cap P .$$

The polytope P is a polygon; hence any vertex of P is precisely the intersection of two facets. If for all $i \in \{0, \dots, n-1\}$, $P_i \neq \emptyset$, then $P_i \cap P_{i+1}$ contains only one point; we set $\{m_i\} = P_i \cap P_{i+1}$, we have

$$P = \text{Conv}(m_i : 0 \leq i \leq n-1)$$

and P_i is the edge having m_{i-1} and m_i for extremities.

Remark 2.5. The point m_i is the solution of the equations $\langle m_i, u_i \rangle = -a_i$ and $\langle m_i, u_{i+1} \rangle = -a_{i+1}$.

We recall that a lattice M defines a measure ν on $M_{\mathbb{R}}$ as the pull-back of the Haar measure on $M_{\mathbb{R}}/M$. The measure ν is translation invariant and satisfies $\nu(M_{\mathbb{R}}/M) = 1$. Let $\text{vol}(P_i)$ be the volume of P_i with respect to the measure determined by the lattice

$$\Lambda = M \cap \{m \in \mathbb{Z}^2 : \langle m, u_i \rangle = -a_i\}$$

in the affine span of Λ .

Proposition 2.6. *Let P be the polytope corresponding to a Cartier divisor $L = \sum_i a_i D_i$ such that for all $i \in \{0, \dots, n-1\}$, $P_i \neq \emptyset$. We have*

$$\text{vol}(P_i) = |a_{i+1} + a_{i-1} - \gamma_i a_i| .$$

We first observe that :

Lemma 2.7. *If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two point of \mathbb{Z}^2 , then*

$$\text{card}(\{tx + (1-t)y : t \in [0; 1]\} \cap \mathbb{Z}^2) - 1 = \gcd(|x_1 - y_1|, |x_2 - y_2|) .$$

Proof. We can assume that $(x_1, x_2) = (0, 0)$. Let $A = \{ty : t \in [0; 1]\} \cap \mathbb{Z}^2$. If $y_1 = 0$, then $\text{card}(A) = y_2 + 1$ and when $y_2 = 0$, $\text{card}(A) = y_1 + 1$. For the case $y_1 \neq 0$ and $y_2 \neq 0$, we can reduce the study to the case where $y_1 > 0$ and $y_2 > 0$.

First case : We assume that $\gcd(y_1, y_2) = 1$. If $t = 0$ or $t = 1$, then $ty \in A$. If there is $t \in]0; 1[$ such that $ty \in A$, then $t = \frac{p}{q}$ with $p, q \in \mathbb{N}$ such that $1 \leq p < q$. As $\frac{py_1}{q}, \frac{py_2}{q} \in \mathbb{N}$, we deduce that q divides y_1 and y_2 . This is in contradiction with the fact that $\gcd(y_1, y_2) = 1$. Thus, we deduce that $\text{card}(A) = 2$.

Second case : $\gcd(y_1, y_2) \geq 2$. We have $ty \in A$ if and only if $t = \frac{k}{\gcd(y_1, y_2)}$ with $k \in \{0, 1, \dots, \gcd(y_1, y_2)\}$. Thus, $\text{card}(A) = \gcd(y_1, y_2) + 1$. \square

Proof of Proposition 2.6. We have $\text{vol}(P_i) = \text{card}(P_i \cap \mathbb{Z}^2) - 1$. We write $u_i = \alpha_i e_1 + \beta_i e_2$ with $\alpha_i, \beta_i \in \mathbb{Z}$. The equations $\langle m_i, u_i \rangle = -a_i$ and $\langle m_i, u_{i+1} \rangle = -a_{i+1}$ give $m_i = \begin{pmatrix} a_{i+1} \beta_i - a_i \beta_{i+1} \\ -a_{i+1} \alpha_i + a_i \alpha_{i+1} \end{pmatrix}$. We have

$$\overrightarrow{m_{i-1} m_i} = \begin{pmatrix} \beta_i(a_{i+1} + a_{i-1}) - a_i(\beta_{i+1} + \beta_{i-1}) \\ -\alpha_i(a_{i+1} + a_{i-1}) + a_i(\alpha_{i-1} + \alpha_{i+1}) \end{pmatrix} .$$

The equalities $\det(u_{i-1}, u_i) = \det(u_i, u_{i+1}) = 1$ give

$$(7) \quad \alpha_i(\beta_{i-1} + \beta_{i+1}) = \beta_i(\alpha_{i-1} + \alpha_{i+1}) .$$

First case : We assume that $\alpha_i = 0$. As $\gcd(\alpha_i, \beta_i) = 1$, we deduce that $\alpha_{i-1} + \alpha_{i+1} = 0$ and by the relation $\det(u_{i-1}, u_i) = 1$, we have $\alpha_{i-1} \beta_i = 1$, i.e $\alpha_{i-1} = \beta_i = \pm 1$. Hence,

$$\gamma_i = \alpha_{i-1} \beta_{i+1} - \alpha_{i+1} \beta_{i-1} = \beta_i \beta_{i+1} + \beta_i \beta_{i-1} = \beta_i(\beta_{i-1} + \beta_{i+1}) .$$

By Lemma 2.7, we have

$$\begin{aligned} \text{vol}(P_i) &= \gcd(|\beta_i(a_{i+1} + a_{i-1}) - a_i(\beta_{i+1} + \beta_{i-1})|, 0) \\ &= \gcd(|(a_{i+1} + a_{i-1}) - a_i \beta_i(\beta_{i+1} + \beta_{i-1})|, 0) \\ &= \gcd(|a_{i+1} + a_{i-1} - a_i \gamma_i|, 0) \\ &= |a_{i+1} + a_{i-1} - \gamma_i a_i| \end{aligned}$$

The case $\beta_i = 0$ is similar to the case $\alpha_i = 0$.

Second case : $\alpha_i \beta_i \neq 0$ and $\alpha_{i-1} + \alpha_{i+1} = 0$. By (7), we have $\beta_{i-1} + \beta_{i+1} = 0$. Hence, $u_{i-1} = -u_{i+1}$ and $\gamma_i = 0$. Thus,

$$\begin{aligned} \text{vol}(P_i) &= \gcd(|\beta_i(a_{i-1} + a_{i+1})|, |\alpha_i(a_{i-1} + a_{i+1})|) \\ &= |a_{i-1} + a_{i+1}| \gcd(|\beta_i|, |\alpha_i|) \\ &= |a_{i-1} + a_{i+1}| \end{aligned}$$

Third case : $\alpha_i \beta_i \neq 0$ and $\alpha_{i-1} + \alpha_{i+1} \neq 0$. As $\gcd(\alpha_i, \beta_i) = 1$, by (7) we have $\alpha_{i-1} + \alpha_{i+1} = \alpha_i$ and $\beta_{i-1} + \beta_{i+1} = \beta_i$. Hence,

$$\text{vol}(P_i) = \gcd(|\beta_i(a_{i+1} + a_{i-1} - a_i)|, |\alpha_i(a_{i+1} + a_{i-1} - a_i)|) = |a_{i+1} + a_{i-1} - a_i|$$

To finish the proof, we must show that $\gamma_i = 1$. We have

$$\begin{aligned} \gamma_i &= \alpha_{i-1} \beta_{i+1} - \alpha_{i+1} \beta_{i-1} \\ &= \alpha_{i-1}(\beta_i - \beta_{i-1}) - \beta_{i-1}(\alpha_i - \alpha_{i-1}) \\ &= \alpha_{i-1} \beta_i - \beta_{i-1} \alpha_i \\ &= 1 \end{aligned}$$

Hence, we get the proof. \square

With this proof, we observe that : if P is the polytope corresponding to the polarized toric surface (X, L) , then $L \cdot D_i = \text{vol}(P_i)$. It is a particular case of the result of [Dan78, Section 11].

3. SMOOTH TORIC LOG DEL PEZZO PAIRS

Let X be a smooth complete toric surface associated to a fan Σ . If D is a reduced T -invariant Weil divisor of X , we will give a description of all pairs (X, D) such that $-(K_X + D)$ is ample. By (1), we have

$$\text{card}(\Sigma(1)) = 2 + \text{rk}(\text{Pic}(X)) .$$

We keep the notations of the previous section.

3.1. Analysis. Let Δ be a subset of $\{0, \dots, n-1\}$ and $D = \sum_{i \in \Delta} D_i$.

Proposition 3.1. *Let X be a complete smooth toric surface. If $\text{card}(\Delta) \geq 3$, then $-(K_X + D)$ is not ample.*

Proof. By Theorem 2.2, we have

$$-(K_X + D) = \sum_{i \in \Delta'} D_i$$

where $\Delta' = \{0, \dots, n-1\} \setminus \Delta$. Let P be the polytope corresponding to $-(K_X + D)$. We set $\partial P = P_0 \cup P_1 \cup \dots \cup P_{n-1}$. By (6), for all $i \in \Delta$, $0 \in P_i$. Hence, $0 \in \partial P$ is a vertex. According to Proposition 2.4 and Proposition 2.6, if P is the polytope corresponding to an ample divisor, then all vertex v of P is precisely the intersection of two facets P_j and P_{j+1} for some $j \in \{0, \dots, n-1\}$. With this fact, we deduce that $-(K_X + D)$ is not ample. \square

Proposition 3.2. *Let X be a complete smooth toric surface such that $\text{rk Pic}(X) \geq 3$. If $\text{card}(\Delta) \in \{1, 2\}$, then $-(K_X + D)$ is not ample.*

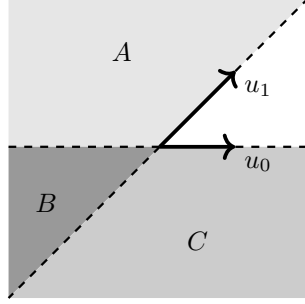
Proof. We start with the case $\text{card} \Delta = 1$. We assume that $D = D_0$. We have

$$-(K_X + D) \cdot D_{n-1} = 1 - \gamma_{n-1} \quad \text{and} \quad -(K_X + D) \cdot D_1 = 1 - \gamma_1 .$$

If $-(K_X + D)$ is ample, then $\gamma_{n-1} \leq 0$ and $\gamma_1 \leq 0$. Let $A = \{\beta u_1 - \alpha u_0 : \alpha, \beta \geq 0\}$, $B = \{-\alpha u_0 - \beta u_1 : \alpha, \beta \geq 0\}$ and $C = \{\alpha u_0 - \beta u_1 : \alpha, \beta \geq 0\}$. As X is complete, the fan Σ of X satisfies

$$\bigcup_{\sigma \in \Sigma} \sigma = A \cup B \cup C \cup \text{Cone}(u_0, u_1).$$

As $\gamma_1 = \det(u_0, u_2)$ and $\gamma_{n-1} = \det(u_{n-2}, u_0)$, we deduce that $u_2 \in B$ and $u_{n-2} \in A$.



When $n \geq 5$, this is in contradiction with the fact that if $u_2 \in B$, then u_{n-2} is in B or C . Thus, we deduce that $-(K_X + D)$ is not ample.

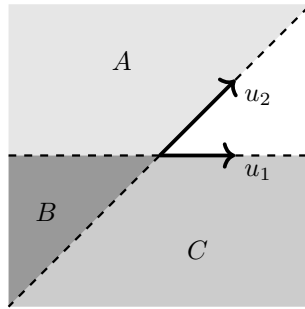
We now assume that $\text{card}(\Delta) = 2$. After renumbering the indices, we can assume that $D = D_1 + D_j$ with $j \in \{2, \dots, n-1\}$.

First case. We assume that $j \in \{3, \dots, n-1\}$. Let P be the polytope of $-(K_X + D)$. As $0 \in P_1$ and $0 \in P_j$, we deduce that 0 is a vertex of P . Hence, for $k \in \{2, \dots, j-1\}$, we have $\text{vol}(P_k) = 0$. By Propositions 2.4 and 2.6, we deduce that $-(K_X + D)$ is not ample.

Second case. We assume that $D = D_1 + D_2$. We have

$$-(K_X + D) \cdot D_3 = 1 - \gamma_3 \quad \text{and} \quad -(K_X + D) \cdot D_0 = 1 - \gamma_0.$$

Let $A = \{-\alpha u_1 + \beta u_2 : \alpha, \beta \geq 0\}$, $B = \{-\alpha u_1 - \beta u_2 : \alpha, \beta \geq 0\}$ and $C = \{\alpha u_1 - \beta u_2 : \alpha, \beta \geq 0\}$.



If $-(K_X + D)$ is ample, then $\gamma_3 \leq 0$ and $\gamma_0 \leq 0$. As $\gamma_3 = \det(u_2, u_4)$ and $\gamma_0 = \det(u_{n-1}, u_1)$, we deduce that $u_4 \in C$ and $u_{n-1} \in A$. If $n \geq 6$, this situation contradicts the positioning order of vectors u_i . If $n = 5$, we have $u_4 \in A$ and $u_4 \in C$, this is not possible.

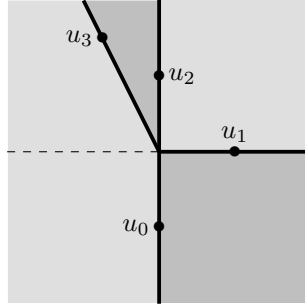
Thus, we deduce that $-(K_X + D)$ is not ample. \square

If $\Delta \neq \emptyset$, according to Propositions 3.1 and 3.2, it is enough to study the positivity of $-(K_X + D)$ when $\text{rk Pic}(X) \in \{1, 2\}$ and $\text{card}(\Delta) \in \{1, 2\}$. The Propositions 3.1 and 3.2 prove Theorem 1.1.

3.2. Proofs of Theorems 1.2 and 1.3. Let (e_1, e_2) be the standard basis of \mathbb{Z}^2 . The rays of the fan of \mathbb{P}^2 are the half-line generated by $u_1 = e_1$, $u_2 = e_2$ and $u_0 = -(e_1 + e_2)$. The divisors D_i of \mathbb{P}^2 defined in Theorem 1.2 correspond to the divisors associated to the ray $\text{Cone}(u_i)$.

Proof of Theorem 1.2. We have the linear equivalence $D_1 \sim D_0$ and $D_2 \sim D_0$. By Theorem 2.2, we have $K_X = -(D_0 + D_1 + D_2)$, i.e. $-K_X \sim 3D_0$. As D_0 is ample, we deduce that $-(K_X + D)$ is not ample if and only if $D = D_1 + D_2 + D_3$. \square

We now assume that $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ with $r \in \mathbb{N}$. The rays of the fan of X are the half lines generated by the vectors $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 + r e_2$ and $u_0 = -e_2$.



The numbers γ_i defined in (2) are given by

$$(8) \quad \gamma_0 = -r \quad \gamma_1 = 0 \quad \gamma_2 = r \quad \gamma_3 = 0 .$$

By Proposition 2.4, the divisor $L = a_0 D_0 + a_1 D_1 + a_2 D_2 + a_3 D_3$ is ample if and only if

$$a_0 + a_2 > 0 \quad , \quad a_1 + a_3 > r a_2 \quad , \quad a_1 + a_3 > -r a_0$$

if and only if

$$(9) \quad a_0 + a_2 > 0 \quad \text{and} \quad a_1 + a_3 > r a_2 .$$

We have the linear equivalence of divisors

$$(10) \quad D_1 \sim D_3 \quad \text{and} \quad D_2 \sim D_0 - r D_3 .$$

The divisor D_i of X corresponding the ray $\text{Cone}(u_i)$ defined here correspond to those defined in Theorem 1.3.

Proof of Theorem 1.3. As $-K_X = D_0 + D_1 + D_2 + D_3$, by (10), we have

$$\begin{aligned} -K_X &\sim 2D_0 + (2-r)D_3 \\ -(K_X + D_0) &\sim D_0 + (2-r)D_3 \\ -(K_X + D_3) &\sim 2D_0 + (1-r)D_3 \\ -(K_X + D_0 + D_3) &\sim D_0 + (1-r)D_3 \end{aligned}$$

and

$$\begin{aligned} -(K_X + D_2) &\sim D_0 + 2D_3 \\ -(K_X + D_2 + D_3) &\sim D_0 + D_3 \\ -(K_X + D_2 + D_0) &\sim 2D_3 \\ -(K_X + D_1 + D_3) &\sim 2D_0 - rD_3 \end{aligned}$$

If $a_1 = a_2 = 0$, the condition (9) becomes $a_0 > 0$ and $a_3 > 0$. This allows us to conclude. \square

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UNIV BREST, UMR CNRS 6205, LABORATOIRE DE MATHÉMATIQUES DE BRETAGNE ATLANTIQUE, FRANCE

Email address: achim.napame@univ-brest.fr