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► **To cite this version:**

| A Sigal, Josep Batle. Semi-classical estimation of the fine-structure constant. 2022. hal-03644063

HAL Id: hal-03644063

<https://hal.science/hal-03644063v1>

Preprint submitted on 18 Apr 2022

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Semi-classical estimation of the fine-structure constant

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Abstract

In the present work we propose a semi-classical model that links well-known phenomena of quantum field theory, such as the Casimir force and the vacuum polarization. We assume for the Casimir effect in dielectrics, that the virtual pairs statistically created in the vacuum near the plates induce image dipoles in the latter, interacting with each other. Taking the limit of the dielectric constants of the plates to the value corresponding to the vacuum, we arrive at an equality that provides, on the one hand, a framework of unconventional physical interpretation for the Casimir effect through vacuum polarization and, on the other hand, a possible way for a theoretical estimation of the fine-structure constant. Regarding the latter, an example result of a first approximation to the problem is shown, with which we find a numerical value for the aforementioned fundamental constant, with an error of 44 parts per million with respect to the 2018 CODATA experimental recommended value.

Keywords: Casimir effect, vacuum polarization, fine-structure constant.

1. Motivation

Eugene Wigner, in his historical essay of 1960 [1], asserted that “The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve”. This is particularly true when comparing the validity of the predictions of quantum electrodynamics (QED) with the experimental value for the Lamb shift. The Lamb shift currently provides a measurement of the fine-structure constant α to better than one part in a million, allowing a precision test of quantum electrodynamics.

Complementarily, there is a flip side to the previous coin. Namely, what are the odds of obtaining the right (in the experimental sense) value of a physical constant from first principles? Is this perchance a contradiction in terms, and oxymoron? After all, the very same definition of *constant* implies that it must be independent of the mathematical machinery behind the theory, only to be found or contrasted experimentally. In other words, to what extent the consistent mathematical manipulation of numerical values can infer the correct value (again in the experimental sense) of a

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physical constant? The estimation of a physical constant in terms of universal mathematical values is perhaps one of the most intriguing philosophical conundrums in theoretical physics. Notice that we completely disregard all “cabbalistic” (in Sommerfeld’s own words) efforts to find combinations of numbers that may possess striking similarities with either α or $1/\alpha$.

In the present contribution, we shall embark on the particular endeavor of consistently obtaining the aforementioned fine-structure constant α , by contrasting two physical theories explaining the same phenomenon, and thus retrieving α from first principles.

Several attempts have been made in the past to such purpose, already beginning with Sommerfeld, who coined the definition of the fine-structure constant in 1916, as well as the use of the Greek letter alpha to denote it. It is well known that the fine-structure constant α , which characterizes the strength of the electromagnetic interaction, is one of the most fundamental constants of physics. It is crucial central in all calculations of quantum electrodynamics and electroweak theory. Sommerfeld introduced it in the early attempts to account for the degeneracy of the Bohr’s atom levels due to relativistic effects [2], i.e., the fine and hyperfine structures of atomic energy levels. Furthermore, the precision in its experimental measurement gives account of the consistency of the theoretical framework used. See Ref. [3] for a historical account of all attempts to determine it up to recent times.

“All good theoretical physicists”, Richard Feynman once remarked [4], “put this number up on their wall and worry about it. Is it related to π or perhaps to the base of natural logarithms? Nobody knows. It’s one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding.”

The first attempt to conciliate the Casimir effect, to be discussed later, with the traditional electrostatic force was carried out by Casimir himself in 1953, shortly after his seminal work of 1948 [5]. His “mousetrap” model of the electron aimed at explaining the stability of charged particles from the balancing between the attractive quantum vacuum force and the self-energy repulsion between charges [6]. However, the model however had to be rejected, for Boyer proved that the Casimir force for a spherical shell was repulsive [7]. What is of relevance to the present work is that Casimir’s mousetrap model retrieved for the first time a means to compute the fine-structure constant itself from first principles, albeit with the wrong sign and with several orders of magnitude of difference.

Other models followed with time, directly or indirectly related to the Casimir effect. Carazza and Guidetti [8], regarded the electron as a charged sphere that completely absorbs any radiation with a wave number less than a certain critical value, the attraction of the Casimir force balancing the mutual repulsion of the different parts of a charge distribution. With this equilibrium condition, they obtained an acceptable estimate for the value of the fine-structure constant.

In pursuit of addressing Feynman’s original quest, we shall first revisit two important effects, namely, i) the Casimir effect, which is one of the most thoroughly investigated effects of quantum field theory (QFT), and ii) the vacuum polarization. These two processes will be studied in connection with the fine-structure constant. With the revision of the previous physical effects at hand, we will be in a position to present the model of the present work, which shall involve an alleged dipolar

interaction between *image* dipoles, induced by the virtual pairs statistically created in vacuum. The main result for the estimation of α will be thus presented and finally some conclusions will be drawn.

2. The Casimir effect and the vacuum polarization, revisited

Since its first theoretical prediction, the Casimir force has been generalized and experimentally corroborated for different geometries, types and configurations of the material bodies that attract each other when they are at very small distances [9,10]. The mere fact that it arises from the perturbation by the boundaries of the zero-point vacuum energy, makes it have a fundamental significance, not only for the study and understanding of the quantum vacuum itself, but also for the most varied areas of physics, ranging from the non-trivial topology of space-time and other cosmological applications, to nanoscopic physics and elementary particle physics [11]. Traditionally in physics textbooks, discussions regarding the zero-point energy of the vacuum have been avoided by simply redefining the zero of energy. However, it is precisely the *changes* in the zero-point energy that are observable, being the Casimir effect one such notable instance.

This quantum effect is named after Casimir who first predicted the attractive force per unit area between two conducting plane-parallel plates in vacuum [5]:

$$\frac{F}{A} = -\frac{\hbar c \pi^2}{240 a^4} \quad (1)$$

where \hbar is Planck's constant, c is the speed of light and a is the distance between the plates.

Less than ten years later, Lifshitz generalized this force for the case of dielectric plates and also studied its temperature dependence [12]. In particular, for two identical dielectric plates and temperature $T = 0$, he derived the following analytical expression:

$$\frac{F_d}{A} = -\frac{\hbar c \pi^2}{240 a^4} \frac{(\varepsilon-1)^2}{(\varepsilon+1)^2} \varphi(\varepsilon) \quad (2)$$

where ε is the relative permittivity of the plates and the function $\varphi(\varepsilon)$ arises from the integration of all the Fourier components of the electromagnetic radiation between the plates. In the asymptotic limit $\varepsilon \rightarrow \infty$, we have $\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2} \varphi(\varepsilon) \rightarrow 1$ and recover the equation (1) applied to the case of metal plates. And when $\varepsilon \rightarrow 1$, in the vacuum limit, the force (2) goes to zero, even though the function limit $\lim_{\varepsilon \rightarrow 1} \varphi(\varepsilon) = \varphi_0$ is not zero and can be computed analytically.

Why is the dependence on Casimir effect with the fine-structure constant relevant? As can be seen in (1) and (2), in principle there is no such dependence. At least directly, Casimir forces do not depend on the electron charge. As Jaffe proposes [13] and then Karnassnigg picks up the point [14], this effect can be deduced without need of zero-point energies. They consider the latter only as a heuristic starting point for simpler calculations, showing that Casimir forces can be calculated without reference to vacuum. Furthermore, Karnassnigg ensures that the Casimir standard result for ideal

metal plates can be assumed at the limit of $\alpha \rightarrow \infty$, and that for finite values of α the result will depend on the nature of the plates. This is based on the fact that there are no perfect conductors in nature.

Another key ingredient to be discussed is the vacuum polarization. The vacuum polarization is another well-known process of quantum electrodynamics that will serve as a tool for our model. This process describes the production of virtual electron-positron pairs through the application of electric and magnetic fields large enough to, precisely, polarize the vacuum [15]. The first discussions by Heisenberg and Dirac [16] on this phenomenon gave rise to Uehling [17] to calculate, to first order of the coupling constant $\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}$ (e is the elementary charge), the corrections to the coulombic electrostatic potential between two test charges in vacuum [18]. The consequences of vacuum polarization begin to appear when the field approaches the value known as the Schwinger limit, first derived by Sauter [19] and raised earlier by Bohr (as Sauter himself acknowledges in his paper, thanking Heisenberg for telling him about Bohr's hypothesis):

$$E_S = \frac{m_e^2 c^3}{e\hbar} \approx 1.32 \times 10^{18} \text{ V/m} \quad (3)$$

where m_e is the mass of the electron.

Above this limit, the electromagnetic field becomes non-linear. In the computations of these nonlinear corrections of the fields in vacuum, Schwinger calculated the probability per unit space-time volume of creating a virtual pair through a constant electric field strength E . Schwinger's formula [20] for the rate of creation of virtual pairs is expressed as:

$$\frac{dN}{dVdt} = \frac{\alpha}{\pi^2} E^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi \frac{E_S}{E}} \quad (4)$$

The previous formulas will be used as the main inputs for our model, shown in the following.

3. The model

With the above formulae, we are now in a position to present the model of this work, which involves an alleged dipolar interaction between *image* dipoles, induced by the virtual pairs statistically created in vacuum between two plates.

Let us sketch the different stages of the model:

1. Firstly, the use the method of images in electrostatics will be applied to the case of the Casimir force in dielectrics, thus contributing in this way to a derivation and understanding of the Casimir effect in an unconventional way. Roughly speaking, we shall derive a classical analogue of the Casimir effect due to the interaction of charge images. The functional dependence with respect to the distance between plates, as well as the the dielectric constants in the media, will be recovered.

2. Secondly, the rate of production of pairs of electrons and positrons beyond a critical value will be taken into account.

3. Finally, the conciliation between the quantum and classical approaches *in the limit of vanishing plates, that is, no disruption of the zero-point energy of the electromagnetic vacuum caused by the mere presence of the plates* (the limit of the relative permittivity of the plates tending to unity) shall retrieve a good numerical approximation of the fine-structure constant.

Imagine that an electron-positron pair is created in the vacuum between the plates (see [Figure 1a](#)). In pursuit of energy-momentum conservation, the probability of creating a pair increases right next to the plates, where the nuclei are. Now, using the method of image charges, an electron-positron pair in front of the dielectric plate creates an image dipole $p = qd$ therein, where d is the effective distance between the charges and q is given by:

$$q = -e \frac{(\varepsilon-1)}{(\varepsilon+1)} \quad (5)$$

In a heuristic way, one can imagine the force between the dielectric plates as due to the interaction between the image dipoles induced in them by the virtual pairs statistically created in the vacuum (see [Figure 1b](#)). At this point, before going on, it is pertinent to make some observations. First, one could reasonably criticize this model arguing that the tiny lifetime of the virtual pairs, which can be estimated by the uncertainty relation $\Delta t \approx \hbar/4m_e c^2$, makes their mutual interaction impossible at the characteristic distances of the Casimir force. However, as we will see in what follows, the proposed dipolar interactions have a statistical nature, so we should not think the model in terms of an isolated virtual pair, but rather in terms of a statistical collection of them. Furthermore, the theoretical approximation of the fine-structure constant, which is one of the objectives of this work, will be independent of the distance between the plates a . Secondly, one could also argue that this electrostatic description of the dipolar interaction is inappropriate because a dipole moment as the one assumed here, requires rather rigid and stable spatial structures between the charges, which does not apply to the case of a virtual electron-positron pair. Nevertheless, as we will also see later and we already mentioned above, what will be used is an effective distance or average distance between the charges of the virtual pairs considered.

Then, taking account the above, we return to the model. We know that the potential energy for this dipole-dipole interaction ([Figure 1b](#)) is given by:

$$V(a, \theta_1, \theta_2, \vartheta) = -\frac{p^2}{4\pi\varepsilon_0} \frac{1}{a^3} (\cos \vartheta - 3 \cos \theta_1 \cos \theta_2) \quad (6)$$

where θ_1, θ_2 are the angles of the dipoles with respect to the axis connecting their centers, and ϑ is the relative torsion angle between their orientations. Note that since the pairs are created near the plates, the distance between the interacting dipoles is indistinguishable from the distance a between the plates. In our case, we want to calculate the average $\langle V(a, \theta_1, \theta_2, \vartheta) \rangle$ for all possible orientations of the dipoles. In order to calculate it, we start from (6), rewriting it as follows:

$$V(a, \theta_1, \theta_2, \vartheta) = V_0(a)f(\Omega) \quad (7)$$

where $V_0(a) = -\frac{p^2}{4\pi\epsilon_0} \frac{1}{a^3}$ and $f(\Omega) = (\cos \vartheta - 3 \cos \theta_1 \cos \theta_2)$.

In the calculation of the average we must be careful, since the angular average for “free” dipoles, in which all angles are equally probable, is annulled. In the same way that Keesom force is derived [21], a weighted average must be computed, where the Boltzmann factor $e^{xf(\Omega)}$, with $x = -\frac{V_0(a)}{kT}$, weights those angles with lower energy. Unlike the derivation of Keesom force, in our case we will not do the approximation $|x| \ll 1$. Instead we will need to evaluate the exact integral and then take the limit $|x| \rightarrow \infty$ ($T \rightarrow 0$) in the result of the integration. We then have to solve the following integral:

$$\langle V(a) \rangle = V_0(a) \frac{\int_0^{2\pi} d\vartheta \int_0^\pi \int_0^\pi f(\Omega) e^{xf(\Omega)} \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2}{\int_0^{2\pi} d\vartheta \int_0^\pi \int_0^\pi e^{xf(\Omega)} \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2} \quad (8)$$

which can be rewritten as:

$$\langle V(a) \rangle = V_0(a) \frac{d}{dx} \log \left[\int_0^{2\pi} d\vartheta \int_0^\pi \int_0^\pi e^{xf(\Omega)} \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 \right] \quad (9)$$

It can be verified that the integral in (9), which we call $g(x)$, is given by:

$$g(x) = \int_0^{2\pi} d\vartheta \int_0^\pi \int_0^\pi e^{xf(\Omega)} \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 = \frac{8\pi I_0(x) Shi(3x)}{3x} \quad (10)$$

where $I_0(x)$ is the modified Bessel function of the first kind and zero order, while $Shi(x) = \int_0^x \frac{\sinh t}{t} dt$ is the hyperbolic sine integral. It is easy to corroborate that:

$$\lim_{x \rightarrow \infty} \left[\frac{d}{dx} \log(g(x)) \right] = 4 \quad (11)$$

In short we obtain this simple result:

$$\langle V(a, \theta_1, \theta_2, \vartheta) \rangle_{T=0} = 4V_0(a) = -4 \frac{p^2}{4\pi\epsilon_0} \frac{1}{a^3} \quad (12)$$

Using $p = qd$ and (5) in the equation (12), we can write the following expression for the average dipole-dipole interaction force per unit area A :

$$\frac{Fd}{A} = -\frac{1}{A} \frac{d}{da} \langle V(a) \rangle_{T=0} = -\alpha \frac{\hbar c (\epsilon-1)^2}{a^4 (\epsilon+1)^2} \frac{12d^2}{A} \quad (13)$$

where $\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}$ is the well-known fine-structure constant, also called coupling constant, as we mentioned above. Following our argument, we impose the equality between (13) and Lifshitz's expression (2):

$$-\frac{\hbar c \pi^2 (\epsilon-1)^2}{240a^4 (\epsilon+1)^2} \varphi(\epsilon) = -\alpha \frac{\hbar c (\epsilon-1)^2}{a^4 (\epsilon+1)^2} \frac{12d^2}{A} \quad (14)$$

The major issue in the equation (14) is that we can eliminate from the equality both the image charge factor $\frac{(\varepsilon-1)^2}{(\varepsilon+1)^2}$ and the distance between the plates a . This allows us to take the limit $\varepsilon \rightarrow 1$, corresponding to an entire vacuum space (without confinement) and thus find out the following equality for α :

$$\alpha = \frac{A_{\text{eff}}}{d_{\text{eff}}^2} \frac{\pi^2}{2880} \varphi_0 \quad (15)$$

In the above equation (15), we have three unknown parameters: 1) $\varphi_0 = \lim_{\varepsilon \rightarrow 1} \varphi(\varepsilon)$, that is, the asymptotic value of the function $\varphi(\varepsilon)$ for vacuum; 2) A_{eff} , the effective area corresponding to an “average” or effective virtual pair –remember that in our model we consider the interaction force between dipoles induced by the virtual pairs, on average–; 3) d_{eff} , the average or effective distance or separation between the electron and the positron in a virtual pair created in vacuum.

In the following, we will analyze and compute these three parameters as accurately as possible.

1) φ_0 , the asymptotic limit for vacuum of the function $\varphi(\varepsilon)$

This parameter can be calculated exactly. For that, Lifshitz’s calculations must be reviewed because in his article [12], the value of φ_0 is expressed only approximately, without an explicit derivation of it. Assuming static dielectric constants ε and temperature $T = 0$, he reaches this intermediate expression for identical dielectric plates:

$$\frac{F}{A} = -\frac{\hbar c}{32\pi^2 a^4} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \frac{1}{\frac{(s+p)^2}{(s-p)^2} e^{x-1}} + \frac{1}{\frac{(s+\varepsilon p)^2}{(s-\varepsilon p)^2} e^{x-1}} \right\} dp dx, \quad s = \sqrt{\varepsilon - 1 + p^2} \quad (16)$$

where the variable $x \sim ap \frac{\omega}{c}$ is associated with the oscillation frequencies ω of the stationary waves within the plates and $p = \sqrt{k_y^2 + k_z^2}$ is the modulus of the two-dimensional wave vector associated with the wave numbers k_y and k_z , corresponding to the parallel directions to the plates.

Recalling equation (2), with (16) we have the following expression for the function $\varphi(\varepsilon)$:

$$\varphi(\varepsilon) = \frac{15}{2\pi^4} \frac{(\varepsilon+1)^2}{(\varepsilon-1)^2} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \frac{1}{\frac{(s+p)^2}{(s-p)^2} e^{x-1}} + \frac{1}{\frac{(s+\varepsilon p)^2}{(s-\varepsilon p)^2} e^{x-1}} \right\} dp dx, \quad s = \sqrt{\varepsilon - 1 + p^2} \quad (17)$$

It is important to note that the final result we are interested in, that is $\lim_{\varepsilon \rightarrow 1} \varphi(\varepsilon) = \varphi_0$, does not change at all if we first take the approximation $\frac{(s+p)^2}{(s-p)^2} e^x - 1 \cong \frac{(s+p)^2}{(s-p)^2} e^x$. In this way we can separate the integrals in (17), obtaining:

$$\int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \frac{1}{\frac{(s+p)^2}{(s-p)^2} e^{x-1}} + \frac{1}{\frac{(s+\varepsilon p)^2}{(s-\varepsilon p)^2} e^{x-1}} \right\} dp dx \cong \int_0^\infty x^3 e^{-x} dx \int_1^\infty \frac{1}{p^2} \left\{ \frac{(s-p)^2}{(s+p)^2} + \frac{(s-\varepsilon p)^2}{(s+\varepsilon p)^2} \right\} dp \quad (18)$$

It is easy to show that $\int_0^\infty x^3 e^{-x} dx = 6$, while the integral on p is more complicated but can be solved with elementary functions. The limit $\varepsilon \rightarrow 1$ of the resulting function of the integral over p is zero. But in (17) we have the factor $\frac{(\varepsilon+1)^2}{(\varepsilon-1)^2}$. The limit that interests us converges to the following finite value:

$$\lim_{\varepsilon \rightarrow 1} \left[\frac{(\varepsilon+1)^2}{(\varepsilon-1)^2} \int_1^\infty \frac{1}{p^2} \left\{ \frac{(s-p)^2}{(s+p)^2} + \frac{(s-\varepsilon p)^2}{(s+\varepsilon p)^2} \right\} dp \right] = \frac{23}{30} \quad (19)$$

Then, taking the limit $\varepsilon \rightarrow 1$ in (17), we finally obtain:

$$\lim_{\varepsilon \rightarrow 1} \varphi(\varepsilon) = \varphi_0 = \frac{3 \cdot 23}{2\pi^4} \quad (20)$$

We shall insist in the fact that the approximation carried out in (18) is not necessary and that it is done so for the sake of simplicity. The calculation of φ_0 without this approximation is slightly more involved but leads to the exact same result given by (20).

2) A_{eff} , the effective area of a virtual pair.

To approximate this parameter, we will use the Schwinger formula (4) for the probability, per unit space-time volume, of creating virtual pairs by applying an electric field E . For the limit case $E = E_S$, for which the effects of vacuum polarization begin to be expressed, equation (4) remains:

$$\frac{dN}{dV dt} = \frac{1}{\pi^2 \lambda_e^4} \sum_{n=1}^\infty \frac{1}{n^2} e^{-n\pi} = \frac{1}{\pi^2 \lambda_e^4} \text{Li}_2(e^{-\pi}) \quad (21)$$

where $\lambda_e = \frac{\hbar}{m_e c}$ is the reduced Compton wavelength of the electron and Li_2 is the Euler's dilogarithm function.

Now, before continuing, we must make some remarks. First, it is observed that the formula (21) in the Schwinger limit case becomes independent of the elementary charge e . This, in principle, contributes to a possible non-recursive closed numerical expression for the computation of the fine-structure constant in (15). Besides, it is important to emphasize that this probability of creating a pair in vacuum is expressed per unit of space-time. This is so because one cannot conceive the creation-annihilation quantum process without understanding that the existence of virtual pairs is linked to the existence of an inherent relativistic space-time.

Considering the above, let us denote in (21) the space-time volume unit $dVdt$ as dS^4 . Then, let us approximate the area A_{eff} needed for the creation of a single virtual pair ($dN = 1$) as follows:

$$A_{\text{eff}} \cong dS^2 = \sqrt{dS^4} = \frac{\pi}{\sqrt{\text{Li}_2(e^{-\pi})}} \lambda_e^2 \quad (22)$$

Regarding the physical justification to identify this Schwinger “space-time area” $\sqrt{dVdt} = dS^2$ with the effective area A_{eff} that we are looking for in our vision of the Casimir effect, we propose two equivalent arguments. The first is noting that the conventional derivation of the Casimir effect through QFT begins with standing waves in vacuum which, as is well known, have a functional symmetry of the spatial variables and the temporal variable. The presence of the plates imposes boundary conditions in the direction z perpendicular to them. Only then the symmetry is broken and both the z - wave modes and the frequencies of the waves are quantized. Taking into account that the area of the plates is large compared to the separation between them, in the sum over all the possible excitation modes for the calculation of the vacuum energy, only the two parallel directions x - y are considered, which are integrated as a continuous, together with the discrete sum of the frequencies. The divergence of the latter requires the use of the well-known mathematical gadget of zeta-regularization, which allows dealing with infinities [5].

The second justification can be given by a recent work based on the exact loop quantum gravity (LQG) solution for a spherically symmetric vacuum space-time, where the expression of the Casimir force is recovered without the need for regularization or renormalization techniques [22]. This is possible due to the quantization of space-time in LQG followed by the assumption at the end of their derivation in which the separation of the vertices of the spin network of the quantum space-time is small.

With either of the above options it follows that, in principle, the area for the Casimir effect is actually a “space-time area”. Only when vacuum is confined in a given region of space-time, the symmetry breaking forces this Casimir area to be only a spatial one. In our case, even if the region is confined, we cannot assume a large area because we are dealing with virtual pairs that act on an area whose scale is the Compton wavelength. But in addition, at the last we take the vacuum limit of the dielectric material plates, with which we have an unconfined problem, so the consideration of a space-time area, as we assume in (22), is then quite appropriate.

3) d_{eff} , the electron-positron effective distance of a virtual pair.

This is the last ingredient we need to complete our mental model and arrive at a theoretical approximation of the fine-structure constant. It is known that the length scale for a virtual electron-positron pair is the Compton wavelength, but to achieve a more precise value, we refer to the work of Leuchs et al [23]. They propose a semi-classical dispersion model that allows to derive the correct orders of magnitude of the electric permittivity ϵ_0 and the magnetic permeability μ_0 of the vacuum, shedding light on their physical meanings.

As a first qualitative approach, they use the same length scale r both for the effective volume in the polarization $P = \frac{e^2}{m_e \omega_0^2 r^3} E$ induced by an electric field E , and for the magnetic dipole moment $M = \frac{e^2 r^2}{m_e} B$ induced by a magnetic field B .

Later, in a more quantitative approach, they suggest taking into account the actual charge distribution in the virtual pair. Thus, for the computation of the induced magnetic dipole moment, instead of using r^2 , we must use the average orbital radius needed for the diamagnetic term $\langle \rho^2 \rangle$, where ρ is the radial variable in cylindrical coordinates. Considering a solid spherical volume $V = \frac{4}{3}\pi R^3$ with radius R and constant probability density for the electron and positron charges, we apply the Langevin's formalism of diamagnetism. With a spherically symmetric distribution, we have $\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3}\langle r^2 \rangle$ and then $\langle \rho^2 \rangle = \frac{2}{3}\langle r^2 \rangle$. For our uniform charge case, a simple calculation shows that $\langle \rho^2 \rangle = \frac{2}{5}R^2$. Combining all the above and imposing $\tilde{\epsilon}_0 \tilde{\mu}_0 = 1/c^2$ for the approximate parameters of permittivity $\tilde{\epsilon}_0$ and permeability $\tilde{\mu}_0$, it is easy to show that $R = \sqrt{\frac{5}{2}}\lambda_e$. So, let us approximate the average or effective electron-positron distance as:

$$d_{\text{eff}} \cong R = \sqrt{\frac{5}{2}}\lambda_e \quad (23)$$

We recall that (23) is only a first approximation and the problem remains open for more precise approaches where, for example, it is considered that the charge distribution is probably centered around the origin [23] and, on the other hand, the QFT corrections to the relativistic result for the gyromagnetic ratio are also taken into account.

4. Results

With the aim of retrieving a numerical value for the fine-structure constant, we have calculated (20), and approximated (22) and (23), the three parameters required. Putting it in (15), and taking the inverse (number best known to physicists), we have:

$$\frac{1}{\alpha} \cong \frac{4800\pi\sqrt{\text{Li}_2(e^{-\pi})}}{23} \cong 137.042144475 \quad (24)$$

This value has a relative error of 44 parts per million when compared to $\frac{1}{\alpha_{\text{exp}}} = 137.035999084$ (value recommended by CODATA 2018 [24], with a relative standard uncertainty of 0.15 parts per billion). Let us recall that in order to derive, not *calculate*, the value of α in terms of universal mathematical constants, we had to confront two complementary approaches for the same problem. To whether or not one can do so within the scope of the same theory alone (as done so far in most cases in the literature), and reminiscent of Gödel's incompleteness theorems, is beyond the scope of the present work.

5. Discussion

We can summarize the present contribution with two main conclusions, as well as some additional considerations. First, on the basis of a simple model of dipolar interaction between the image dipoles induced in vacuum by the virtual pairs that are statistically created, we find a numerical approximation for the fine-structure constant with a relative error of 44 parts per million when compared to the 2018 CODATA experimental recommended value.

Secondly, the model used for the fine-structure approach sheds light on the nature of the Casimir effect, which can be interpreted, in an unconventional way, as a consequence of the vacuum polarization with boundary conditions, without the need to wield the zeta function regularization.

Much criticism can be derived from our approach. Quantitatively, the main source of inaccuracies is given by the ratio $\frac{A_{\text{eff}}}{d_{\text{eff}}^2}$. As a study focused in obtaining order of magnitudes coincidences, our results are fairly acceptable. Taken as such, with no approximations made, it is rather astonishing that a simple approach as the present one can reproduce the value of the fine structure constant with such an incredible accuracy. Indeed, why would one simply accept without further elaboration, for instance, that the applied electric field strength ought to be exactly the one corresponding to $E_S = \frac{m_e^2 c^3}{e\hbar}$ and not some fraction of it? Certainly that would be the case if the vacuum polarization acted in a Heaviside step function fashion. In the same line of thought, the very same value of E_S is thought to be extremely difficult to reach experimentally, if not impossible.

Other instances admit a more convincing explanation. In the case of considering the distance between vanishing plates as the one between induced image charges, the creation of pairs is evidently more favored nearby the presence of matter. Also, further contributions to the interacting potential between charge images are identically cancelled in the limit $\epsilon \rightarrow 1$ for additional powers of $\frac{(\epsilon-1)^2}{(\epsilon+1)^2}$.

On the qualitative side, it is plain that we “mimic” the action of the Casimir effect by a model of induced interacting dipoles, which, in turn, are produced by pairs of electrons and positrons that appear due to the action of a vacuum-stripping electric field. Should one consider the creation of *virtual* pairs instead? That would involve considering extremely small time intervals. Would still be possible for the previous approximations to hold?

In addition to the heuristic and statistical nature of the current contribution, the model must be understood from the point of view of several limits. Namely, the limit of the Schwinger field for the creation of matter per unit of space-time volume; the limit of uncertainty for the Heisenberg principle in its form given by the energy and the lifetime of the particles; the limit to the vacuum for the space occupied by the plates of the Casimir effect; and, finally, the thermodynamic limit of zero temperature in the statistical expression of the dipolar interaction potential that we propose. The use of all these limits altogether results in a more-than-acceptable numerical value that we obtain for the fine-structure constant in a first approximation to the problem, justified by the semi-classical nature of the model.

Summing up, we hope to have shed new light on the classical interpretation of the Casimir effect and, at least from orders of magnitude, to have opened a research path to obtain the precise numerical value of the fine structure constant from first principles.

Acknowledgments

AS would like to thank FaMAFyC from Universidad Nacional de Córdoba for supporting research. JB acknowledges fruitful discussions with Maria del Mar Batle and Regina Batle. JB received no funding for the present research.

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Figures and Figures Captions

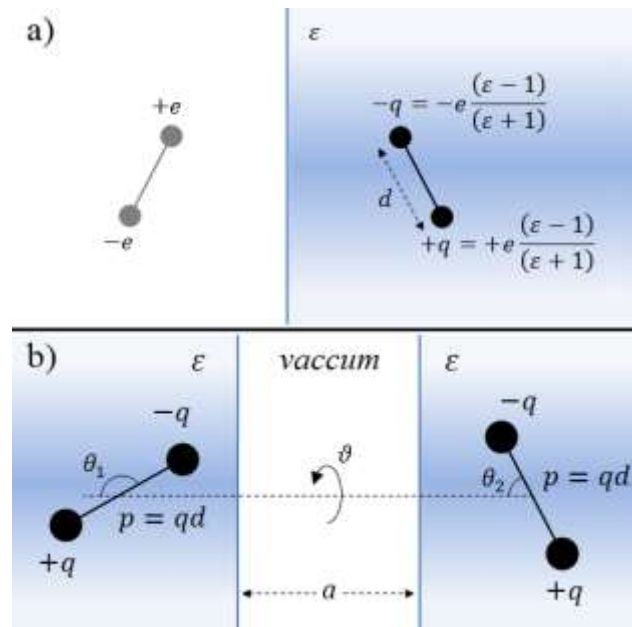


Figure 1. a) An electron-positron pair with charges $\pm e$ created in the vicinity of the dielectric plate generates an image dipole with charges $\pm q = \pm e \frac{(\epsilon - 1)}{(\epsilon + 1)}$ and effective distance d . b) Dipole-dipole interaction between two image dipoles $p = qd$ generated on the dielectric plates. θ_1 and θ_2 are their angles with respect to the axis connecting their centers, while ϑ is the relative torsion angle.