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Stabilization of Heterogeneous Quasilinear Traffic Flow System with Disturbances

Lina Guan, Liguo Zhang and Christophe Prieur

Abstract—In this paper, we investigate the problem of boundary stabilization for a heterogeneous quasilinear traffic flow system with disturbances in the congested regime. The $H^2$ integral input-to-state stability of multi-type traffic system described by first-order quasilinear hyperbolic partial differential equations is obtained in closed loop with a boundary controller. The control is at the inlet boundary of the considered road section and is optimally designed and computed for the linearized system. Making use of the backstepping transformation, the integral input-to-state stability of the quasilinear system is derived by mapping the transformed quasilinear system into an integral input-to-state stable target system for which a strict Lyapunov function is constructed. By simulations, we illustrate that the linear controller designed by using backstepping method locally stabilizes the quasilinear system.

Index Terms—Heterogeneous quasilinear traffic flow system, disturbances, backstepping, integral input-to-state stability, Lyapunov function.

I. INTRODUCTION

High traffic demand causes the slow velocity of traffic stream as a result of the interaction between vehicles. The extreme traffic congestion will happen when the traffic demand approaches the capacity of the intersections along the road. Traffic congestion has some negative effects: wasting time, delays, air pollution, wasting fuel, frequent vehicle repairs and replacements, road rage and high chance of collisions.

Aiming to understand and develop an optimal transport network with efficient movement of vehicles and minimal traffic congestion problem, traffic flow is studied to understand the interactions between travelers (drivers and their vehicles) and infrastructure (highways, signage and traffic control devices). Congestion propagates upstream from a traffic bottleneck and depends on the upstream traffic flow and density in the direction of propagation. Several equilibrium, frequent lane changes, overtaking and platoon dispersion probably happen in the congested traffic on account of the interplay between different types of vehicles and drivers [20]. In this paper, we develop a control law to reject disturbances (high traffic demand) for the congested traffic on a freeway, which involves several types of vehicles characterized by some properties including vehicle size, drivers’ style of vehicle operation and reaction time and other factors. There are many macroscopic traffic flow models for heterogeneous traffic. In [14], an extended speed gradient (SG) model is used to study the mixed traffic flow system. A new car-following model is presented in [23]. In [19], a continuum multi-type traffic model is introduced on the basis of the three-dimensional flow–concentration surface. Paper [10] studies a two-type vehicle heterogeneous traffic model to acquire overtaking and creeping traffic flows. In [18], the extended macroscopic $N$-type Aw-Rascle (AR) traffic model is used for heterogeneous traffic by using area occupancy. This paper introduces this nonlinear model considering vehicle size, area occupancy, road width of the heterogeneous traffic to study the problem of disturbances rejection in the congested traffic. The concept of area occupancy is introduced for measuring heterogeneous traffic concentration in [17] and [2].

In [15], the data-driven optimal controller is designed for connected and autonomous vehicles (CAVs) in a mixed-traffic situation (specifically, including heterogeneous human-driven vehicles). In [21], for a comparison of the ecological potential of variable speed limits (VSLs) and signalized access control, the dynamics of the system is modeled by microscopic traffic simulator (SUMO) and controlled by a nonlinear model predictive control (NMPC) framework with an artificial neural network (ANN) to predict the fuel consumption. The interaction between CAV and human-driven vehicle (HDV) dynamics is investigated, and a complete CAV control input and the feasible conditions of a platoon formation is presented in [16] (see also [24]). By using the linear component of a nonlinear dynamical system to define a warm start for a model-free, policy gradient method, the nonlinear model is guaranteed to converge to the (nearly) global optimal controller in [22]. Paper [29] designs the reinforcement learning controllers to remove the congested traffic for $2 \times 2$ quasilinear Aw-Rascle-Zhang (ARZ) partial differential equations (PDEs) model by using the proximal policy optimization. This paper uses the backstepping method to obtain the integral input-to-state stability (iISS) of the quasilinear system with the application of the designed control law for the corresponding linearized system. This method has been used in many research. More recently, by using the backstepping, paper [27] designs an observer to estimate traffic states of the nonlinear ARZ traffic flow model. Paper [3] designs a controller for the underactuated cascade network of interconnected PDEs systems. Paper [8] designs an output feedback boundary control to solve the stop and go traffic problem of linearized two-type AR traffic flow system.

In this paper, extending the control problem of the single vehicle-type traffic system in paper [11], we solve the problem of the stabilization for a multi-type traffic flow system of first-order hyperbolic quasilinear partial differential equations (PDEs) in the congested regime, with disturbances and actuation at the inlet boundary and capacity drop in the downstream boundary of a considered road segment. The optimal controller which is designed for the linearized system by backstepping method is applied to locally stabilize the quasilinear system around a nonuniform equilibrium. The iISS of the quasilinear system is derived by making use of the Lyapunov method to analyze the iISS of the target system, which is mapped into the quasilinear system by a backstepping transformation.

This paper is organized as follows. In Section II, the quasilinear system and control problem are formulated. In Section III, the main result is given, such that the controller designed for linearized system locally stabilizes the quasilinear system in the $H^2$ sense. Moreover, the proof of this result is given. The simulations are in Section IV.
Some concluding remarks are shown in Section V. The introduced multi-type traffic flow model in [18] that represents the dynamics of a heterogeneous traffic on a road segment with road length $L$ and the number of vehicle types $N$ is as follows,

$$\partial_t p_i(x, t) + \partial_x \left( p_i(x, t) v_i(x, t) \right) = 0,$$ (1)

with the independent space variable $x \in [0, L]$ and the independent time variable $t \in (0, \infty)$. The density $p_i(x, t)$ is the number of vehicle type $i$ passing road section per unit length and the velocity $v_i(x, t)$ is the average speed of vehicle type $i$ passing location $x$ in unit time, with the index of vehicle type $i = 1, 2, \cdots, N$. The area occupancy $A_o(p) = A_o(\rho) = \frac{a_o}{\rho}$ with $a = (a_1, a_2, \cdots, a_N)^T$ and $\rho = (\rho_1, \rho_2, \cdots, \rho_N)^T$, $a_i$ is the occupied surface per vehicle for type $i$ and $W$ is the road width. Area occupancy $0 < A_o \leq 1$ describes the percentage of road space that is occupied by all the vehicle types on the considered road section. The relaxation time $\tau_i$ describes the driving behavior of drivers for vehicle type $i$. The coefficients $a_i$ and $\tau_i$ characterize the type of $i$.

In the model (1)-(2), the traffic pressure $p_i(A_o)$ of vehicle type $i$ is (see [8]), $p_i(A_o) = v_i^M \left( A_o / A_o^M \right)^{\gamma_i - 1}$, $i = 1, 2, \cdots, N$, where the free-flow velocity $v_i^M$ and the maximum area occupancy $0 < A_o^M \leq 1$ respectively describe the maximal velocity of vehicle type $i$ in the free regime and the maximum percentage of occupied surface for vehicle type $i$ in the congested regime, when only vehicle type $i$ is on the road. The constant $\gamma_i > 1$ is the pressure exponent of type $i$. In [7], the equilibrium speed-Ao relationship of vehicle type $i = 1, 2, \cdots, N$ is $V_{e,i}(A_o) = v_i^M \left( A_o / A_o^M \right)^{\gamma_i - 1}$. The decreasing function $V_{e,i}(A_o)$ of area occupancy $A_o$ represents the desired velocity of the drivers. Define a numoniform driver, for $x \in [0, L]$, $u^*(x) = (\rho_1^*(x), v_1^*(x), \cdots, \rho_N^*(x), v_N^*(x))^T$ with $\rho_i^*, v_i^* \in C^1([0, L]; \mathbb{R})$. Denoting $u = (\rho_1, v_1, \cdots, \rho_N, v_N)^T$ and $\bar{u} = u - u^*$, the system (1)-(2) is rewritten as, for all $x \in [0, L]$, $t \in (0, +\infty)$,

$$\partial_t \bar{u}(x, t) + \bar{F}(\bar{u}, u^*(x)) \partial_x \bar{u}(x, t) = \bar{G} \left( \bar{u}, u^*(x) \right) \bar{u}(x, t),$$ (3)

where

$$\bar{F}(\bar{u}, u^*(x)) = \begin{bmatrix}
\bar{F}_{11}(\bar{u}, u^*(x)) & \bar{F}_{12}(\bar{u}, u^*(x)) & \cdots & \bar{F}_{1N}(\bar{u}, u^*(x)) \\
\bar{F}_{21}(\bar{u}, u^*(x)) & \bar{F}_{22}(\bar{u}, u^*(x)) & \cdots & \bar{F}_{2N}(\bar{u}, u^*(x)) \\
\vdots & \vdots & \ddots & \vdots \\
\bar{F}_{N1}(\bar{u}, u^*(x)) & \bar{F}_{N2}(\bar{u}, u^*(x)) & \cdots & \bar{F}_{NN}(\bar{u}, u^*(x))
\end{bmatrix},$$

with for $i, j = 1, 2, \cdots, N$,

$$\bar{F}_{ij}(\bar{u}, u^*(x)) = \begin{cases}
\rho_i^*(x) & \text{if } j = i, \\
0 & \text{if } j \neq i.
\end{cases}$$

The actuation signal vector $U \in C^0([0, \infty); \mathbb{R}^{2N-m})$ with a coefficient matrix $\Theta \in \mathcal{M}_{2N-2m}(\mathbb{R})$ is implemented to reduce the unknown and bounded disturbances $\bar{d}(t) \in C^0([0, \infty); \mathbb{R}^N)$ by the on-ramp metering at the upstream boundary of the considered road segment and $\Pi_{NL}(\bar{u}(0, t)) \in C^2([0, \infty); \mathbb{R}^N)$. The well-posedness of the closed loop system (3), (5)-(6) in the $H^2$ space is under a necessary condition that the initial conditions $\bar{u}(0, 0) = \bar{u}_0(\cdot) \in \mathcal{M}_{2N-2m}(\mathbb{R})$ satisfy the following second-order compatibility conditions,

$$A_1 \bar{u}_0(0) = d(0) + \Theta U(0) - \Pi_{NL}(\bar{u}_0(0)),$$ (7)

$$B_1 \bar{u}_0(L) = 0,$$ (8)

$$A_1 \left( -\bar{F}(\bar{u}_0(0), u^*(0)) \bar{u}_0(0) + \bar{G}(\bar{u}_0(0), u^*(0)) \bar{u}_0(0) \right).$$ (9)
\[ \widetilde{G}_{ij}(\widetilde{u}, u^*(x)) = \begin{cases} 
\frac{1}{T_i} \delta_{ii} + v'_i(x) \sigma_{ii}(\rho) \rho_i^*(x) - \delta_{ii}(\rho) v''_i(x) & \text{if } j = i, \\
0 & \text{if } j \neq i. 
\end{cases} \]  

(4)

In the compatibility conditions, (8) and (10) naturally derived from the physical meaning, but (7) and (9) are artificially derived from the designed feedback control law and they rigorously require the specific values of the initial conditions. Keeping the effect of stabilization of the control law, a modification of the control law is done in the boundary conditions (5)-(6), so that there is no requirement of any specific values on the initial conditions. To more specific, inspired by [9], the modified boundary conditions extend the controller as follows:

\[ A_1 \vec{u}(0, t) = d(t) + \Theta \dot{U}(t) + w_1(t) + w_2(t) - \Pi_{NL} \vec{u}(0, t), \]

(11)

\[ B_1 \vec{u}(L, t) = 0, \]

(12)

where \( w_1, w_2 \in C^1([0, \infty); \mathbb{R}^N) \) are the solutions of the following system:

\[ \begin{align*}
\dot{w}_1 &= -d_1 I_{N \times N} w_1, \\
\dot{w}_2 &= -d_2 I_{N \times N} w_2,
\end{align*} \]

(13)

with the constant \( N \)-dimensional diagonal positive definite matrices \( d_1 I_{N \times N}, d_2 I_{N \times N} \) (\( d_1 \neq d_2 \)). Under the modification of the control law, we can obtain the following compatibility conditions,

\[ A_1 \vec{u}_0(0) = d(0) + \Theta \dot{U}(0) + w_1(0) + w_2(0) - \Pi_{NL} \vec{u}_0(0), \]

(14)

\[ B_1 \vec{u}_0(L) = 0, \]

(15)

\[ A_1 \left( -\widetilde{F}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) + \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) \right) = d(0) + \Theta \dot{U}(0) + (d_1 I_{N \times N} w_1(0) - d_2 I_{N \times N} w_2(0)) - \Pi_{NL} \frac{d\Pi_{NL}}{dt} \]

\[ \times \left( -\widetilde{F}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) + \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) \right), \]

(16)

\[ B_1 \left( \dot{F}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) + \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) \right) = B_1 \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0). \]

(17)

For the purpose of eliminating the compatibility conditions (7) and (9), the initial conditions of \( w_1, w_2 \) satisfy, \( w_1(0) + w_2(0) = g_1(\vec{u}_0), -d_1 I_{N \times N} w_1(0) + d_2 I_{N \times N} w_2(0) \) = \( g_2(\vec{u}_0), \) with

\[ g_1(\vec{u}_0) = A_1 \vec{u}_0(0) - d(0) - \Theta \dot{U}(0) + \Pi_{NL} \vec{u}_0(0), g_2(\vec{u}_0) = \]

\[ A_1 \left( -\widetilde{F}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) + \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) \right) - d(0) - \Theta \dot{U}(0) + \Pi_{NL} \frac{d\Pi_{NL}}{dt} \]

\[ \times \left( -\widetilde{F}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) + \vec{G}(\vec{u}_0(0), u^*(0)) \vec{u}_0(0) \right). \]

(18)

In order to verify the compatibility conditions (7) and (9), select

\[ w_1(0) = -\frac{g_2(\vec{u}_0) + d_2 I_{N \times N} g_1(\vec{u}_0)}{d_1 - d_2}, \]

(19)

\[ w_2(0) = \frac{d_1 I_{N \times N} g_1(\vec{u}_0) + g_2(\vec{u}_0)}{d_1 - d_2}. \]

C. State transformations

For deriving the form of characteristic values of the quasilinear system (3), (11)-(12), (13) and making the analysis easier, we handle a transformation for the state \( \vec{u} \). Defining an invertible transformation \( \vec{R}(x, t) = \Phi(x) \vec{u}(x, t) \) with \( x \in [0, L], t \in [0, +\infty), \) from \( \vec{u} \) to the new variables \( \vec{R} = \left( \vec{R}^+, \vec{R}^- \right)^\top : [0, L] \times [0, +\infty) \to \mathbb{R}^{2N} \) with \( \vec{R}^+ : [0, L] \times [0, +\infty) \to \mathbb{R}^{2N}, \vec{R}^- : [0, L] \times [0, +\infty) \to \mathbb{R}^{2N}, \) \( A_{NL}(\vec{R}, x) = \left( R^+(0, t), R^-(0, t) \right)^\top, \) the system (3), (11)-(12), (13) is mapped into the following simplified system, for all \( x \in [0, L], t \in [0, +\infty), \)

\[ \begin{align*}
\partial_t R(x, t) &= \Lambda(x) \partial_x R(x, t) - \Sigma(x) R(x, t) + \Lambda_{NL} \partial_x R(x, t) \\
&= \Sigma_{NL}(\vec{R}, x) \vec{R}(x, t),
\end{align*} \]

(20)

\[ \begin{align*}
\dot{R}_{in}(t) &= K_{P} \dot{R}_{out}(t) + \Gamma_0 \left( d(t) + \Theta \dot{U}(t) \right) + \Gamma_0 (w_1(t) + w_2(t)) \\
&- \Gamma_0 \Pi_{NL} \Phi^{-1}(\vec{R}(0, t)),
\end{align*} \]

(21)

where \( \Lambda(x) = \vec{\Lambda}(0, x), \Sigma(x) = \vec{\Sigma}(2, \vec{X})(x), \Lambda_{NL}(\vec{R}, x) = \vec{\Lambda}(\vec{R}, x) - \vec{\Lambda}(0, x), \Sigma_{NL}(\vec{R}, x) = \vec{\Sigma}(\vec{R}, x) - \vec{\Sigma}(0, x), \) with

\[ \vec{\Lambda}(\vec{R}, x) = \Phi(x) \vec{F} \left( \Phi^{-1}(x) \vec{R}, u^*(x) \right) \Phi^{-1}(x), \]

\[ \vec{\Sigma}(\vec{R}, x) = \Phi(x) \vec{G} \left( \Phi^{-1}(x) \vec{R}, u^*(x) \right) \Phi^{-1}(x), \]

the main diagonal elements of matrix \( \Sigma \in C([0, L]; \mathcal{M}_{2N,2N}(\mathbb{R})) \) are zeros, and \( K_{P} \in \mathcal{M}_{2N,2N}(\mathbb{R}), \) \( \Gamma_0 \in \mathcal{M}_{2N,2N}(\mathbb{R}) \) are given gain matrices. The well-posedness of the system (20)-(21) in the \( H^2 \) space is under a necessary condition that the initial condition \( \vec{R}(:, 0) = \vec{R}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}) \) satisfies the following second-order compatibility condition,

\[ \begin{align*}
\dot{R}_{in}(0) &= K_{P} \dot{R}_{out}(0) + \Gamma_0 (d(0) + \Theta \dot{U}(0)) \]

\[ + \Gamma_0 (w_1(0) + w_2(0)) - \Gamma_0 \Pi_{NL} \Phi^{-1}(0) \vec{R}_0(0),
\end{align*} \]

(22)
\[
\left[ \begin{array}{c}
[M_1^2]_{1 \leq i \leq m}, [M_2^2]_{m+1 \leq j \leq 2N}^T
\end{array} \right] + K_P \left[ \begin{array}{c}
[M_2^2]_{1 \leq i \leq m}, [M_1^2]_{m+1 \leq j \leq 2N}^T
\end{array} \right] + \Gamma_0 \left( \begin{array}{c}
d(0) + \Theta \dot{U}(0)
\end{array} \right) \\
+ \Gamma_0 \left( -d_1 I_{x_1 N} w_1(0) - d_2 I_{x_2 N} w_2(0) \right) - \Gamma_0 \frac{\partial}{\partial \sigma} \left( \Phi^{-1}(0) \right) \\
\times \Phi^{-1}(0) \left( -\left( \Phi(0) + \Lambda_{NL} (\Phi(0), 0) \right) \Phi(0), \right) \\
+ \left( \Sigma(0) + \Sigma_{NL} (\Phi(0), 0) \right) \frac{\partial}{\partial \sigma} \left( \Phi^{-1}(0) \right)
\],
\] (23)

with
\[
M^1 = - \left( \Phi(0) + \Lambda_{NL} (\Phi(0), 0) \right) \Phi(0), \\
M^2 = \left( \Phi(0) + \Lambda_{NL} (\Phi(0), 0) \right) \Phi(0), \\
\]

Since for \( x \in [0, L] \), \( \Lambda_{NL}(x, 0) = 0 \), \( \Sigma_{NL}(x, 0) = 0 \) and \( \Pi_{NL} = 0 \), then \( 0 \) is an equilibrium of the quasi-linear system (20)-(21).

For the sake of conveniently analyzing the applicability of designed controller for the linearized system, the quasi-linear system (20)-(21) is written as the linear part plus the nonlinear terms, done in the next section.

III. LOCAL \( H^2 \) iISS OF THE QUASILINEAR MODEL

Inspired by [9], the problem of local \( H^2 \) iISS of the first-order hyperbolic quasi-linear system (3) and (13) is studied in this section. We will show that the controller derived for linearized system by using backstepping method locally stabilizes this quasi-linear system.

Concerning the linearized system, for all \( x \in [0, L], t \in [0, +\infty) \),
\[
\partial_t R(x, t) + \Lambda(x) \partial_x R(x, t) = \Sigma(x) R(x, t),
\]
(24)
\[
R_{in}(t) = K_R R_{out}(t) + I \left( d(t) + \theta \dot{U}(t) \right),
\]
(25)
the desired control law \( U \) is derived by using the following backstepping transformation, for all \( x \in [0, L], t \in [0, +\infty) \),
\[
Z(x, t) = R(x, t) - \int_0^t K(x, \xi) R(\xi, t) \, d\xi = K[R],
\]
(26)
where
\[
K(x, \xi) = \left[ \begin{array}{cc}
0 & G_1^1(x, \xi) \\
G_2^2(x, \xi) & 0
\end{array} \right] \in M_{2N, 2N}(\mathbb{C}^2(T)),
\]
with suitable \( G_1^1 \in M_{2N, 2N}(\mathbb{C}^2(T)) \) and \( G_2^2 \in M_{2N, 2N}(\mathbb{C}^2(T)) \) defined in the domain \( T = \{ (x, \xi) \in \mathbb{R}^2, 0 \leq x \leq \xi \leq L \} \) as in paper [12]. The kernels \( G_1^1(x, \xi), G_2^2(x, \xi) \) satisfy an \( 2N \times 2N \) first-order hyperbolic PDEs, whose well-posedness is shown in [1, 6, D]. The corresponding iISS target system is, for all \( x \in [0, L], t \in [0, +\infty) \),
\[
\partial_t Z(x, t) + \Lambda(x) \partial_x Z(x, t) = \Sigma_1(x) Z(x, t),
\]
(27)
\[
Z_{in}(t) = K_P Z_{out}(t) + K_I \int_0^t \Sigma_1(x, \xi) Z(\xi, t) \, d\xi,
\]
(28)
where
\[
Z = (Z^+, Z^-)^T, \quad \text{with} \ Z^+ : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^m, \ Z^- : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^{2N - m}, \ Z(t) = (Z^+(t), Z^-(t))^T, \ Z_{out}(t) = (Z^+(t), Z^-(t))^T, \ \\
\Sigma_1 \in M_{2N, 2N}(\mathbb{C}(0, L)), \ C_1 \in M_{2N, 2N}(\mathbb{C}^2(T)) \text{ and } K_I = \left[ \begin{array}{c}
K_I^{11}, K_I^{21}
\end{array} \right], \text{ with } K_I^{11} \in M_{m, m}(\mathbb{R}), K_I^{21} \in M_{m, 2N - m}(\mathbb{R}) \text{ are given.}
\]

The controller for the linearized system (24)-(25) is derived as follows, for all \( t \in [0, +\infty) \),
\[
U(R) = (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma,
\]
\[
- (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
+ G^2(0, \xi) R^+(0, \xi) \right) d\xi - (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
- (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
+ (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
\left( \begin{array}{c}
\Theta \Gamma_2 \Theta^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\end{array} \right) R(t)
\],
(29)
where the coefficient matrices \( \Gamma_1, \Gamma_2 \) are given, and the matrix \( \Theta \) is in \( M_{2N - m, m} \), and such that \( \Theta \Gamma_2 \Theta \) is invertible. So the controller in terms of the original variable \( \tilde{u} \) in (11) is, for all \( t \in [0, +\infty) \),
\[
U(t) = (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( \left[ \begin{array}{c}
K_I^{11}
\end{array} \right] \Phi(\sigma) \tilde{u}(L, \sigma) \right) d\sigma - (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
\left( \begin{array}{c}
\Theta \Gamma_2 \Theta^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\end{array} \right) \Phi(\sigma) \tilde{u}(L, \sigma) + \left[ \begin{array}{c}
\tilde{u}(L, \sigma)
\end{array} \right] \Phi(\sigma) \tilde{u}(L, \sigma) \right) d\sigma - (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
\left( \begin{array}{c}
\Theta \Gamma_2 \Theta^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\end{array} \right) \Phi(\sigma) \tilde{u}(L, \sigma) d\sigma - (\Theta \Gamma_2 \Theta)^{-1} \Theta \int_0^t \left( K_I^{11} R^+(L, \sigma) + K_I^{21} R^-(0, \sigma) \right) d\sigma
\]
\[
\]
\( K_1[K] = \int_x \partial_x K(x, \xi) \bar{Z}(\xi, t) \, d\xi - K(x, x) \bar{Z}(x, t), \)
\( F_1[\bar{Z}] = \Lambda_{NL}(\bar{L}[\bar{Z}], x), \)
\( F_2[\bar{Z}] = \Sigma_{NL}(\bar{L}[\bar{Z}], x), \)

with \( \bar{Z} = (Z^+, Z^-)^\top, Z^+ : [0, L] \times [0, +\infty) \to \mathbb{R}^m, Z^- : [0, L] \times [0, +\infty) \to \mathbb{R}^{2N-m}, \bar{Z}_{in}(t) = (Z^+(0, t), Z^-(L, t))^\top \), \( \bar{Z}_{out}(t) = (Z^+(L, t), Z^-(0, t))^\top \). Next, the local stability of system (32)-(34) in the space \( H^2 \) is analyzed by using the Lyapunov approach. It includes analyzing the growth of \( \|\bar{Z}\|_{L^2}, \|\eta\|_{L^2} \) and \( \|\xi\|_{L^2} \) (by definition \( \eta = \partial_t \bar{Z}, \xi = \partial_t \bar{Z} \)).

The following Lyapunov function candidate is introduced for the stability analysis of system (32)-(34), for all \( x \in [0, L], t \in [0, +\infty), \)
\( V_1(\bar{Z}(x, t), \bar{X}(t), \eta(x, t), \xi(x, t)) = \bar{V}_1 + \bar{V}_2 + \bar{V}_3, \)
where
\[
\bar{V}_1 = \int_0^L \left( Z^+(x, \cdot) \bar{P}_{11}(x) \bar{Z}(x, \cdot) + Z^-(x, \cdot) \bar{P}_{12}(x) \xi(x) \right) \, dx,
\]
\[
\bar{V}_2 = \int_0^L \eta^+(x) \bar{P}_{21}(x) \eta(x, \cdot) \, dx,
\]
\[
\bar{V}_3 = \int_0^L \xi^-(x) \bar{P}_{31}(\bar{Z}) \xi(x, \cdot) \, dx,
\]
with \( \bar{P}_{11}(x) = \bar{P}_{11}(x), \bar{P}_{12}(x) = \bar{P}_{12}(x), \bar{P}_{31}(\bar{Z}) = \bar{P}_{31}(\bar{Z}) \).

\( \bar{P}_{11}(\bar{Z}) \) is defined in [9, Lemma 5.2]. \( \bar{P}_{11}, \bar{P}_3 \) are \( 2N \times 2N \) diagonal positive-definite matrices, \( \bar{P}_{22} \) is a \( 2N \times 2N \) symmetric positive-definite matrix, and \( \bar{P}_{12} \) is a \( 2N \times 2N \) matrix.

Take the time derivative of \( \bar{V}_1 \) along the solutions to (32)-(34), use integration by parts, and define \( \dot{\bar{V}}_1 = \dot{\bar{V}}_{1L} + \dot{\bar{V}}_{1NL} \), where \( \dot{\bar{V}}_{1L} \) is the time derivative of \( \bar{V}_1 \) along the linear target system (27)-(28), computed as follows:
\[
\dot{\bar{V}}_{1L} = -Z^+(x, t) \Lambda(x) \bar{P}_{11}(x) \bar{Z}(x, t) \bigg|_0^L + \int_0^L \left( Z^+(x, t) \left( \Lambda(x) \bar{P}_{11}(x) \right) \bar{Z}(x, t) \right) \, dx
\]
\[
+ \left( Z^-(x, t) \left( \Lambda(x) \bar{P}_{12}(x) \right) \xi(x) \right) \bigg|_0^L + \int_0^L \left( Z^+(x, t) \left( \Lambda(x) \bar{P}_{12}(x) \right) \xi(x) \right) \, dx
\]
\[
+ \left( \Sigma_1(x) \bar{Z}(x, t) + \int_x^L C_1(x, \xi) \bar{Z}(\xi, t) \, d\xi \right) \bigg|_0^L + \int_x^L \left( \Sigma_1(x) \bar{Z}(x, t) + \int_x^L C_1(x, \xi) \bar{Z}(\xi, t) \, d\xi \right) \, dx
\]
\[
+ \left( \Sigma_1(x) \bar{Z}(x, t) + \int_x^L C_1(x, \xi) \bar{Z}(\xi, t) \, d\xi \right) \, dx
\]
\[
+ \left( \Sigma_1(x) \bar{Z}(x, t) + \int_x^L C_1(x, \xi) \bar{Z}(\xi, t) \, d\xi \right) \, dx
\]
\[
+ \left( K_1 \dot{Z}_{out}(t) + \Gamma_{od}(t) \right) \left( \bar{P}_{12}(x) \bar{Z}(x, t) + \bar{P}_{22} \dot{\bar{X}}(t) \right) \, dx
\]
\[
(38)
\]
\[
\begin{align*}
&\left(K_1\dot{Z}_{\text{out}}(t) + \Gamma_0 \left[ d(t) - \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right] \right) \\
&- \Gamma_0(d_1 I_N x N w_1(t) + d_2 I_N x N w_2(t)) + \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t)) \\
&+ \int_0^L \left( \left(F_5[Z,\partial_Z \eta] + F_6[Z,\eta,\partial_z \dot{Z}] \right) P_3(x) \right) \mathrm{d}x \\
&+ \eta^\top(x,t) P_3(x) \left( F_5[Z,\partial_z \eta,\partial_x \eta] + F_6[Z,\eta,\partial_z \dot{Z}] \right) \mathrm{d}x.
\end{align*}
\] (44)

Since the term \( \dot{V}_1 + \dot{V}_2 \) is analyzed in [12, Theorem 1], which is for the iSS of the linear target system (27)-(28), there exist positive constants \( c_1, b_1, a_1, a_2, a_3 \), such that
\[
\dot{V}_1 \leq -c_1(V_1 + V_2) + b_1\|d\|^2 + a_2(w_1^2 + w_2^2) + a_2\|Z\|_\infty(V_1 + V_2) + a_3\|\eta\|_\infty(V_1 + V_2).
\] (45)

Now analyze the remaining items \( \dot{V}_{1NL} \) and \( \dot{V}_{2NL} \). There exist positive constants \( k_1, k_2, k_3, k_4, k_5, k_6, k_7 \) such that
\[
\dot{V}_{1NL} \leq k_1 \int_0^L \left( \left| F_5[Z,\partial_z \eta] + |F_4[Z]| \right| \right) \mathrm{d}x \\
+ k_2 \int_0^L \left( \left| F_5[Z,\partial_z \dot{Z}] + |F_4[Z]| \right| \right) \mathrm{d}x \\
+ k_3 \left| \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right| \left| \dot{X}(t) \right|,
\] (46)

Applying [9, Lemma B.3], there exist positive constants \( \delta_i \leq \delta_i, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11} \) such that for all \( Z \) satisfying \( \|\dot{Z}\|_\infty + \|\eta\|_\infty \leq \delta_i \), the result holds
\[
\dot{V}_{2NL} \leq k_4 \left| \dot{Z}_{\text{out}}(t) \right| \left| \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right| \\
+ k_5 \left( K_1 \dot{Z}_{\text{out}}(t) + \Gamma_0 d(t) + \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right) \\
+ \left| \Gamma_0(d_1 I_N x N w_1(t) + d_2 I_N x N w_2(t)) \right| \left| \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right| \\
+ k_6 \left( \Gamma_0 \frac{\partial N L}{\partial Z}(Z(0),t)\eta(0,t) \right)^2 \\
+ k_7 \int_0^L \left( \left| F_5[Z,\partial_z \eta] + |F_4[Z]| \right| \right) \mathrm{d}x
\] (47)

Taking the time derivative in (40), we obtain the following bound for \( \zeta \):
\[
\dot{\zeta}(x,t) + \left( \Lambda(x) + F_1[Z] \right) \zeta(x,t) = F_7[Z,\zeta,\partial_x \zeta] + F_8[Z,\eta,\zeta,\partial_x Z,\partial_x \eta]
\] (48)

where
\[
F_7[Z,\zeta,\partial_x \zeta] = -K_1[Z](Z(0)) \partial_x \zeta + K_1[Z] \zeta
\] (49)

Taking the time derivative of \( V_3 \) along the solutions to (32)-(34), apply [9, Lemma 5.2], and define \( \dot{V}_3 = \dot{V}_{3NL} + \dot{V}_{3NL} \), where \( \dot{V}_{3NL} \) is the time derivative of \( V_{3NL} \) along the linear target system (27)-(28).
\[
\dot{V}_3 = -\zeta^\top \left( x,t \right) \left( \dot{P}_4[Z](x) \Lambda(x) + F_1[Z] \right) \zeta(x,t) \\
+ \int_0^L \zeta^\top \left( x,t \right) \left( \dot{P}_4[Z](x) \Lambda(x) + F_1[Z] \right) \zeta(x,t) \mathrm{d}x \\
+ \int_0^L \zeta^\top \left( x,\cdot \right) \left( \dot{P}_4[Z](x) \right) \zeta(x,\cdot) \mathrm{d}x
\] (50)

We deduce, for all \( Z \) satisfying \( \|\dot{Z}\|_\infty \leq \delta_i \),
\[
\dot{V}_{1NL} \leq (h_1 + h_3)k_9(\|\dot{Z}\|_\infty + \|\eta\|_\infty) + (h_2 + h_4)(k_9\|\dot{Z}\|_\infty V_1 + k_{10}\|\eta\|_\infty V_2).
\] (51)
$$\mathcal{P}_4[\bar{Z}(x)](\zeta(x,t) + \frac{\xi}{\zeta(x,t)}\mathcal{P}_4[\bar{Z}(x)](F_1[\bar{Z}, \zeta, \partial_x \zeta] + F_8[\bar{Z}, \eta, \zeta, \partial_x \bar{Z}, \partial_x \eta])\right)dx.$$ 

According to [9, Proposition 5.4], there exist positive constants $c_2$, $h_{12}$, $h_{13}$, $h_{14}$, $h_{15}$, $h_{16}$, $h_{17}$, $h_{18}$, $h_{19}$, $h_{20}$ such that

$$-\zeta^T(x,t)\left(\mathcal{P}_4[\bar{Z}(x)](\bar{A}(x) + F_1[\bar{Z}])\right)(\zeta(x,t))\right)_0^L \leq h_{12}[\|\zeta\|_{\infty}(V_1 + V_2 + V_3) + h_{13}(w_1 + w_2^2) + h_{14}[\|\bar{Z}\|_{+}(V_2 + V_3)] + h_{15}d^2] + h_{16}[\|\zeta\|_{\infty}V_2^2], 

$$

$$\int_0^L \zeta^T(x,t)\left(\mathcal{P}_4[\bar{Z}(x)]\left(\bar{A}(x) + F_1[\bar{Z}])\right)(\zeta(x,t)\right)dx \leq -c_2\bar{V}_3 + h_{17}[\|\bar{Z}\|_{\infty} + \|\eta\|_{\infty}] + h_{18}\bar{V}_3, 

$$

$$\int_0^L \zeta^T(x,t)(\mathcal{P}_4[\bar{Z}(x)]\zeta(x,t)\right)dx \leq h_{19}\bar{V}_3[\|\bar{z}\|_{\infty} + \|\eta\|_{\infty}] + h_{20}[\|\eta\|_{\infty}V_3 + \|\bar{z}\|_{\infty}V_2 + \|\bar{Z}\|_{\infty}V_3].$$

Therefore there exist positive constants $\delta_1 \leq \delta_2$, $c_3$ and $c_4$ satisfying $c_3 + c_4 < c_2$ such that, for all $Z$ satisfying $\|\bar{Z}\|_{\infty} + \|\eta\|_{\infty} + \|\zeta\|_{\infty} \leq \delta_3$, it holds

$$\hat{V}_{3L} \leq (c_2 + c_3)(\bar{V}_1 + \bar{V}_2 + \bar{V}_3) + h_{15}[d^2] + h_{13}(w_1^2 + w_2^2),$$

$$\hat{V}_{3NL} \leq c_4(\bar{V}_1 + \bar{V}_2).$$

(51)

Combining (51), (52) with (45), (46), (47), and letting $c = c_3 + c_4 - c_2 - c_1$ and $a = a_1 + k_5h_8 + h_{13}$, we get, for all $\|\bar{Z}\|_{\infty} + \|\eta\|_{\infty} + \|\zeta\|_{\infty} \leq \delta_2$,

$$\hat{V} \leq -c\hat{V} + (b_1 + k_5h_7)[d^2] + h_{15}[d^2] + a(w_1^2 + w_2^2).$$

(53)

By defining $\hat{W} = \hat{V} + \frac{h_2}{2}(\frac{a_1^2}{w_1} + \frac{a_2^2}{w_2})$, we derive

$$\hat{W} \leq -c\hat{W} + (b_1 + k_5h_7)[d^2] + h_{15}[d^2] + (a - b_4)(w_1^2 + w_2^2).$$

Choosing $a < b_4$, for some positive constant $c_5$, we obtain

$$\hat{W} \leq -c_5\hat{W} + (b_1 + k_5h_7)[d^2] + h_{15}[d^2].$$

(54)

For sufficiently small $\hat{W}(0)$, $\hat{W}$ exponentially converge to zero. If $\|\bar{Z}\|_{L^2} + \|\eta\|_{L^2} + \|\zeta\|_{L^2}$ is sufficiently small, $\hat{V}$ is equivalent to the $H^2$ norm of $\bar{Z}$. We deduce (31) by a standard application of comparison lemma. This concludes the proof of Theorem 1.

Furthermore, under the assumption of Theorem 1, coming back to (1)-(2), we deduce that the quasilinear system $u$ is locally iISS around the nonlinear equilibrium $u^* \in$ the space $H^2$.

IV. SIMULATIONS

In consideration of verifying the applicability of the control law $U$ in (29) to locally stabilize the quasilinear system $u$ around the nonlinear equilibrium $u^*$ on a considered road section (road length: 1km, width: 6.5m), the values of traffic parameters for two vehicle classes are chosen as in [12], see Table I. Let

$$K_P = \begin{bmatrix} 0 & 0 & 0 & -7.85 \\ 0 & 0 & 6.85 \\ -5.67 & -5.09 & 7.15 & 0 \end{bmatrix} \times 10^{-5},$$

$$K_D^1 = \begin{bmatrix} -20 & 30 & 30 \\ -24 & -7 & 26 \\ -10 & 20 & -30 \end{bmatrix} \times 10^{-5}, K_D^2 = \begin{bmatrix} 60 \\ 30 \\ 20 \end{bmatrix} \times 10^{-5},$$

and $d(t) = \left[ \begin{array}{c} 275e^{-30t} \cos 30\pi t \\ 8.75e^{-20t} \cos 30\pi t \end{array} \right]$, illustrated for simulation by rush hours. The optimal controller $U$ not only stabilizes the linearized system (24)-(25), but also stabilizes the quasilinear system (20)-(21). Figure 3 gives the numerical simulations of the quasilinear two vehicle-class traffic model with the controller $U$. It is obvious that the designed controller for the linearized system stabilizes the quasilinear plant system. However, given a large initial condition

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
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<tr>
<td>Relaxation time</td>
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<td>s</td>
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<td>Pressure exponent</td>
<td>$\gamma_1$</td>
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<td>Free-flow velocity</td>
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<td>Maximum AO</td>
<td>$A_{\theta}^+$</td>
<td>0.5</td>
<td>L</td>
</tr>
<tr>
<td>Occupied surface per vehicle</td>
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<td>m^2</td>
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<tr>
<td>$\rho^l_1$ (0)</td>
<td>110</td>
<td>m^-1</td>
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<tr>
<td>$\rho^l_2$ (0)</td>
<td>70</td>
<td>m^-1</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3. From left top to right down: time and space evolutions of the state components $v_1, v_2, \rho_2$ to (1)-(2).

$u(x,0), x \in [0,L]$, namely the initial condition given by (55) by replacing 0.05 by 0.1, that is sufficiently “far” from the equilibrium $u^*(x)$, the corresponding solution to the same closed-loop system does not converge to the equilibrium. It is consistent with the locality of the stability as described in Theorem 1, in other words, the initial state should be in a neighborhood of the equilibrium (in $H^2$ norm) in order to guarantee the iISS of the multi-type quasilinear traffic system.

V. CONCLUSION

The problem of boundary stabilization of a multi-type quasilinear traffic flow system with disturbances was solved by actuation at the inlet boundary of the considered road segment. The applicability of the control law, which was designed for the corresponding linearized system by using the backstepping method, has been proven for the locally iISS of the quasilinear system. It would be of interest to extend this result to design an observer and combine them to obtain an observer-based output feedback controller. The extension of this result to other first-order hyperbolic systems would be also interesting.

REFERENCES


