Stabilization of Heterogeneous Quasilinear Traffic Flow System with Disturbances

Lina Guan, Liguo Zhang and Christophe Prieur

***Abstract*—In this paper, we investigate the problem of boundary stabilization for a heterogeneous quasilinear traffic flow system with disturbances in the congested regime. The H^2 integral input-to-state stability of multi-type traffic system described by first-order quasilinear hyperbolic partial differential equations is obtained in closed loop with a boundary controller. The control is at the inlet boundary of the considered road section and is optimally designed and computed for the linearized system. Making use of the backstepping transformation, the integral input-to-state stability of the quasilinear system is derived by mapping the transformed quasilinear system into an integral input-to-state stable target system for which a strict Lyapunov function is constructed. By simulations, we illustrate that the linear controller designed by using backstepping method locally stabilizes the quasilinear system.**

***Index Terms*—Heterogeneous quasilinear traffic flow system, disturbances, backstepping, integral input-to-state stability, Lyapunov function.**

I. Introduction

High traffic demand causes the slow velocity of traffic stream as a result of the interaction between vehicles. The extreme traffic congestion will happen when the traffic demand approaches the capacity of the intersections along the road. Traffic congestion has some negative effects: wasting time, delays, air pollution, wasting fuel, frequent vehicle repairs and replacements, road rage and high chance of collisions.

Aiming to understand and develop an optimal transport network with efficient movement of vehicles and minimal traffic congestion problem, traffic flow is used to study the interactions between travellers (drivers and their vehicles) and infrastructure (highways, signage and traffic control devices). Congestion propagates upstream from a traffic bottleneck and depends on the upstream traffic flow and density in the direction of propagation. Several equilibriums, frequent lane changes, overtaking and platoon dispersion probably happen in the congested traffic on account of the interplay between different types of vehicles and drivers [20]. In this paper, we develop a control law to reject disturbances (high traffic demand) for the congested traffic on a freeway, which involves several types of vehicles characterized by some properties including vehicle size, drivers' style of vehicle operation and reaction time and other factors. There are many macroscopic traffic flow models for heterogeneous traffic. In [14], an extended speed gradient (SG) model is used to study the mixed traffic flow system. A new car-following model is presented in [23]. In [19], a continuum multi-type traffic model is introduced on the basis of a three-dimensional flow–concentration surface. Paper [10] studies a two-type vehicle heterogeneous traffic model to acquire overtaking and creeping traffic flows. In [18], the extended macroscopic N-type Aw-Rascle (AR) traffic model is used for heterogeneous traffic by using area occupancy. This paper introduces this nonlinear model considering vehicle size, area occupancy, road width of the heterogeneous traffic to study the problem of disturbances rejection in the congested traffic. The concept of area occupancy is introduced for measuring heterogeneous traffic concentration in [17] and [2].

In [15], the data-driven optimal controller is designed for connected and autonomous vehicles (CAVs) in a mixed-traffic situation (specifically, including heterogeneous human-driven vehicles). In [21], for a comparison of the ecological potential of variable speed limits (VSLs) and signalized access control, the dynamics of the system is modeled by microscopic traffic simulator (SUMO) and is controlled by a nonlinear model predictive control (NMPC) framework with an artificial neural network (ANN) to predict the fuel consumption. The interaction between CAV and human-driven vehicle (HDV) dynamics is investigated, and a complete CAV control input and the feasible conditions of a platoon formation is presented in [16] (see also [24]). By using the linear component of a nonlinear dynamical system to define a warm start for a model-free, policy gradient method, the nonlinear model is guaranteed to converge to the (nearly) global optimal controller in [22]. Paper [29] designs the reinforcement learning controllers to remove the congested traffic for 2×2 quasilinear Aw-Rascle-Zhang (ARZ) partial differential equations (PDEs) model by using the proximal policy optimization. This paper uses the backstepping method to obtain the integral input-to-state stability (iISS) of the quasilinear system with the application of the designed control law for the corresponding linearized system. This method has been used in many research. More recently, by using the backstepping, paper [27] designs an observer to estimate traffic states of the nonlinear ARZ traffic flow model. Paper [3] designs a controller for the underactuated cascade network of interconnected PDEs systems. Paper [8] designs an output feedback boundary control to solve the stop and go traffic problem of linearized two-type AR traffic flow system.

In this paper, extending the control problem of the single vehicle-type traffic system in paper [11], we solve the problem of the stabilization for a multi-type traffic flow system of first-order hyperbolic quasilinear partial differential equations (PDEs) in the congested regime, with disturbances and actuation at the inlet boundary and capacity drop in the downstream boundary of a considered road segment. The optimal controller which is designed for the linearized system by backstepping method is applied to locally stabilize the quasilinear system around a nonuniform equilibrium. The iISS of the quasilinear system is derived by making use of the Lyapunov method to analyze the iISS of the target system, which is mapped into the quasilinear system by a backstepping transformation.

This paper is organized as follows. In Section II, the quasilinear system and control problem are formulated. In Section III, the main result is given, such that the controller designed for linearized system locally stabilizes the quasilinear system in the H^2 sense. Moreover, the proof of this result is given. The simulations are in Section IV.

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This work is supported by the National Natural Science Foundation of China (NSFC, Grant No. 61873007), the Beijing Natural Science Foundation (Grant No. 1182001), and a research grant from project PHC CAI YUANPEI under grant number 44029QD, and by MIAI @ Grenoble Alpes (ANR-19-P3IA-0003).

Some concluding remarks are shown in Section V.

Notation. $\mathcal{M}_{n,n}(\mathbb{R})$ denotes the set of $n\times n$ real matrices. $[A]_{i,j}$ denotes the entry of matrix A in the i-th row and the j-th column. For a function $\varphi=(\varphi_1,\cdots,\varphi_n)^\top:[0,L]\to\mathbb{R}^n$, $\|\varphi\|_\infty=\sup_{x\in[0,L]}|\varphi(x)|$. For a matrix $M\in\mathcal{M}_{n,n}$, $|M|=\sup\{|M\varphi|:\varphi\in\mathbb{R}^n,|\varphi|=1\}$, $\|M\|_\infty=\sup\{|M(x,\xi)|:x,\xi\in[0,L]\}$, the definition of L^2-norm is $\|\varphi\|_{L^2}=\sqrt{\int_0^L(\varphi_1^2(\xi)+\cdots+\varphi_n^2(\xi))\mathrm{d}\xi}$, H^2-norm is $\|\varphi\|_{H^2}=\|\varphi\|_{L^2}+\|\partial_x\varphi\|_{L^2}+\|\partial_{xx}\varphi\|_{L^2}$.

II. MULTI-TYPE QUASILINEAR HYPERBOLIC TRAFFIC FLOW SYSTEM AND PROBLEM STATEMENT

A. Multi-type traffic model

The introduced multi-type traffic flow model in [18] that represents the dynamics of a heterogeneous traffic on a road segment with road length L and the number of vehicle types N is as follows,

$$\partial_t\rho_i(x,t)+\partial_x\Big(\rho_i(x,t)v_i(x,t)\Big)=0, \tag{1}$$

$$\partial_t\Big(v_i(x,t)+p_i(Ao)\Big)+v_i(x,t)\partial_x\Big(v_i(x,t)+p_i(Ao)\Big)=\frac{V_{e,i}(Ao)-v_i(x,t)}{\tau_i}, \tag{2}$$

with the independent space variable $x\in[0,L]$ and the independent time variable $t\in[0,\infty)$. The density $\rho_i(x,t)$ is the number of vehicle type i passing road section per unit length and the velocity $v_i(x,t)$ is the average speed of vehicle type i passing location x in unit time, with the index of vehicle type $i=1,2,\cdots,N$. The area occupancy $Ao(\rho)$ is $Ao(\rho)=\frac{a^\top\rho}{W}$, with $a=(a_1,a_2,\cdots,a_N)^\top$ and $\rho=(\rho_1,\rho_2,\cdots,\rho_N)^\top$, a_i is the occupied surface per vehicle for type i and W is the road width. Area occupancy $0<Ao\le 1$ describes the percentage of road space that is occupied by all the vehicle types on the considered road section. The relaxation time τ_i describes the driving behavior of drivers for vehicle type i. The coefficients a_i and τ_i characterize the vehicle type i.

In the model (1)-(2), the traffic pressure $p_i(Ao)$ of vehicle type i is (see [8]), $p_i(Ao)=v_i^M\left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i}$, $i=1,2,\cdots,N$, where the free-flow velocity v_i^M and the maximum area occupancy $0<Ao_i^M\le 1$ respectively describe the maximal velocity of vehicle type i in the free regime and the maximum percentage of occupied surface for vehicle type i in the congested regime, when only vehicle type i is on the road. The constant $\gamma_i>1$ is the pressure exponent of type i. In [7], the equilibrium speed-Ao relationship of vehicle type $i(=1,2,\cdots,N)$ is $V_{e,i}(Ao)=v_i^M-p_i(Ao)=v_i^M\left(1-\left(\frac{Ao}{Ao_i^M}\right)^{\gamma_i}\right)$. The decreasing function $V_{e,i}(Ao)$ of area occupancy Ao represents the desired velocity of the drivers. Define a nonuniform equilibrium, for $x\in[0,L]$, $u^*(x)=(\rho_1^*(x),v_1^*(x),\cdots,\rho_N^*(x),v_N^*(x))^\top$ with $\rho_i^*,v_i^*\in\mathcal{C}^2([0,L];\mathbb{R})$. Denoting $u=(\rho_1,v_1,\cdots,\rho_N,v_N)^\top$ and $\widetilde{u}=u-u^*$, the system (1)-(2) is rewritten as, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$\partial_t\widetilde{u}(x,t)+\widetilde{F}(\widetilde{u},u^*(x))\partial_x\widetilde{u}(x,t)=\widetilde{G}(\widetilde{u},u^*(x))\widetilde{u}(x,t), \tag{3}$$

where

$$\widetilde{F}(\widetilde{u},u^*(x))=\begin{bmatrix}\widetilde{F}_{11}(\widetilde{u},u^*(x)) & \widetilde{F}_{12}(\widetilde{u},u^*(x)) & \cdots & \widetilde{F}_{1N}(\widetilde{u},u^*(x))\\ \widetilde{F}_{21}(\widetilde{u},u^*(x)) & \widetilde{F}_{22}(\widetilde{u},u^*(x)) & \cdots & \widetilde{F}_{2N}(\widetilde{u},u^*(x))\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{F}_{N1}(\widetilde{u},u^*(x)) & \widetilde{F}_{N2}(\widetilde{u},u^*(x)) & \cdots & \widetilde{F}_{NN}(\widetilde{u},u^*(x))\end{bmatrix},$$

with for $i,j=1,2,\cdots,N$,

$$\widetilde{F}_{ij}(\widetilde{u},u^*(x))=\begin{cases}\begin{bmatrix}v_i^*(x) & \rho_i^*(x)\\ 0 & v_i^*(x)-\rho_i^*(x)\delta_{ii}(\rho)\end{bmatrix}, & \text{if } j=i,\\ \begin{bmatrix}0 & 0\\ (v_i^*(x)-v_j^*(x))\delta_{ij}(\rho) & -\rho_j^*(x)\delta_{ij}(\rho)\end{bmatrix}, & \text{if } j\neq i,\end{cases}$$

and

$$\widetilde{G}(\widetilde{u},u^*(x))=\begin{bmatrix}\widetilde{G}_{11}(\widetilde{u},u^*(x)) & \widetilde{G}_{12}(\widetilde{u},u^*(x)) & \cdots & \widetilde{G}_{1N}(\widetilde{u},u^*(x))\\ \widetilde{G}_{21}(\widetilde{u},u^*(x)) & \widetilde{G}_{22}(\widetilde{u},u^*(x)) & \cdots & \widetilde{G}_{2N}(\widetilde{u},u^*(x))\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{G}_{N1}(\widetilde{u},u^*(x)) & \widetilde{G}_{N2}(\widetilde{u},u^*(x)) & \cdots & \widetilde{G}_{NN}(\widetilde{u},u^*(x))\end{bmatrix},$$

with for $i,j=1,2,\cdots,N$, $\widetilde{G}_{ij}$ is defined in (4) at the top of next page. Therein, for $i,j=1,2,\cdots,N$,

$$\delta_{ij}(\rho)=\partial_{\rho_j}p_i(Ao)=\frac{v_i^M\gamma_ia_j}{Ao_i^MW}\left(\frac{Ao}{Ao_i^M}\right)^{\gamma_i-1},$$

$$\sigma_{ij}(\rho)=\partial_{\rho_j}\delta_{ij}(\rho)=\frac{v_i^M\gamma_i(\gamma_i-1)a_j^2}{(Ao_i^MW)^2}\left(\frac{Ao}{Ao_i^M}\right)^{\gamma_i-2}.$$

Assume that the system (3) is strictly hyperbolic, for all $u^*\in\mathbb{R}^{2N}([0,L])$, the matrix $\widetilde{F}(0,u^*)$ has $2N$ real distinct nonzero eigenvalues. Given $2N$ eigenvalues, $\forall x\in[0,L]$, $\lambda_1(x)>\lambda_2(x)>\cdots>\lambda_m(x)>0>-\lambda_{m+1}(x)>\cdots>-\lambda_{2N}(x)$, of $\widetilde{F}(0,u^*)$ ($\lambda_i\in\mathcal{C}^1([0,L])$, $i=1,\ldots,2N$, m is the number of positive eigenvalues and $0\le m<2N$). For $x\in[0,L]$, define $\Lambda(x)=\mathrm{diag}\{\lambda_1(x),\cdots,\lambda_m(x),-\lambda_{m+1}(x),\cdots,-\lambda_{2N}(x)\}$, therein $-\lambda_{m+1}(x),\cdots,-\lambda_{2N}(x)<0$ means that the traffic wave moves upstream in the congested regime.

B. Problem statement

The motivation of control is to reduce the traffic congestion on a road segment, with disturbances at the inlet boundary, in the presence of constant density and capacity drop in the downstream boundary. The diagram of control model is presented in Figure 1. As proven in [12], the problem is equivalent to computing the boundary control U in the space H^2 such that system (3) converges to zero equilibrium, with the following boundary conditions, for all $t\in[0,+\infty)$,

$$A_1\widetilde{u}(0,t)=d(t)+\Theta U(t)-\Pi_{NL}(\widetilde{u}(0,t)), \tag{5}$$

$$B_1\widetilde{u}(L,t)=0, \tag{6}$$

where

$$A_1=\mathrm{diag}\left\{[v_1^*(0),\rho_1^*(0)],\ldots,[v_N^*(0),\rho_N^*(0)]\right\}\in\mathcal{M}_{N,2N},$$

$$B_1=\mathrm{diag}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\ldots,\begin{bmatrix}1&0\\0&0\end{bmatrix}\right\}\in\mathcal{M}_{2N,2N},$$

$$\Pi_{NL}(\widetilde{u}(0,t))=\left[\widetilde{\rho}_1(0,t)\widetilde{v}_1(0,t),\cdots,\widetilde{\rho}_N(0,t)\widetilde{v}_N(0,t)\right]^\top.$$

The actuation signal vector $U\in\mathcal{C}^0([0,\infty);\mathbb{R}^{2N-m})$ with a coefficient matrix $\Theta\in\mathcal{M}_{N,2N-m}(\mathbb{R})$ is implemented to reduce the unknown and bounded disturbances $d\in\mathcal{C}^2([0,\infty);\mathbb{R}^N)$ by the on-ramp metering at the upstream boundary of the considered road segment and $\Pi_{NL}(\widetilde{u}(0,t))\in\mathcal{C}^2([0,\infty);\mathbb{R}^N)$. The well-posedness of the closed loop system (3), (5)-(6) in the H^2 space is under a necessary condition that the initial conditions $\widetilde{u}(\cdot,0)=\widetilde{u}_0(\cdot)\in H^2([0,L];\mathbb{R}^{2N})$ satisfy the following second-order compatibility conditions,

$$A_1\widetilde{u}_0(0)=d(0)+\Theta U(0)-\Pi_{NL}(\widetilde{u}_0(0)), \tag{7}$$

$$B_1\widetilde{u}_0(L)=0, \tag{8}$$

$$A_1\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right)$$

$$\widetilde{G}_{ij}(\widetilde{u},u^*(x))=\begin{cases}\begin{bmatrix}v_i^{*\prime}(x) & \rho_i^{*\prime}(x)\\ \frac{1}{\tau_i}\delta_{ii}(\rho)+v_i^*(x)\sigma_{ii}(\rho)\rho_i^{*\prime}(x)-\delta_{ii}(\rho)v_i^{*\prime}(x) & \frac{1}{\tau_i}+v_i^{*\prime}(x)+\sum_{j=1,j\neq i}^{N}\delta_{ij}(\rho)\rho_j^{*\prime}(x)\end{bmatrix}, & \text{if } j=i,\\ \begin{bmatrix}0 & 0\\ \frac{1}{\tau_i}\delta_{ij}(\rho)+v_i^*(x)\sigma_{ij}(\rho)\rho_j^{*\prime}(x)-\delta_{ij}(\rho)v_j^{*\prime}(x) & -\delta_{ij}(\rho)\rho_j^{*\prime}(x)\end{bmatrix}, & \text{if } j\neq i.\end{cases} \tag{4}$$

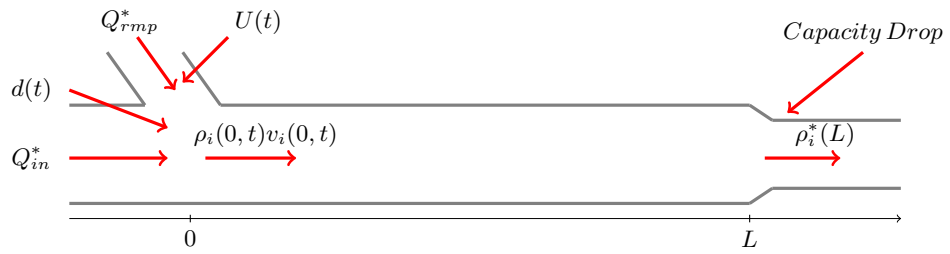


Fig. 1. Multi-type vehicles traffic on a road with on-ramp metering and disturbances at the inlet boundary and a capacity drop in the downstream boundary.

$$= \dot{d}(0)+\Theta\dot{U}(0)-\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\Big|_{t=0}$$
$$\times\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right), \tag{9}$$
$$B_1\widetilde{F}(\widetilde{u}_0(L),u^*(L))\widetilde{u}_0'(L)=B_1\widetilde{G}(\widetilde{u}_0(L),u^*(L))\widetilde{u}_0(L). \tag{10}$$

In the compatibility conditions, (8) and (10) naturally derived from the physical meaning, but (7) and (9) are artificially derived from the designed feedback control law and they rigorously require the specific values of the initial conditions. Keeping the effect of stabilization of the control law, a modification of the control law is done in the boundary conditions (5)-(6), so that there is not requirement of any specific values on the initial conditions. To more specific, inspired by [9], the modified boundary conditions extend the controller as follows:

$$A_1\widetilde{u}(0,t)=d(t)+\Theta U(t)+w_1(t)+w_2(t)-\Pi_{NL}(\widetilde{u}(0,t)), \tag{11}$$
$$B_1\widetilde{u}(L,t)=0, \tag{12}$$

where $w_1,w_2\in\mathcal{C}^1([0,\infty);\mathbb{R}^N)$ are the solutions of the following system:

$$\dot{w}_1=-d_1I_{N\times N}w_1,\ \dot{w}_2=-d_2I_{N\times N}w_2, \tag{13}$$

with the constant N-dimension diagonal positive definite matrices $d_1I_{N\times N},d_2I_{N\times N}$ ($d_1\neq d_2$). Under the modification of the control law, we can obtain the following compatibility conditions,

$$A_1\widetilde{u}_0(0)=d(0)+\Theta U(0)+w_1(0)+w_2(0)-\Pi_{NL}(\widetilde{u}_0(0)), \tag{14}$$
$$B_1\widetilde{u}_0(L)=0, \tag{15}$$
$$A_1\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right)$$
$$= \dot{d}(0)+\Theta\dot{U}(0)+(-d_1I_{N\times N}w_1(0)-d_2I_{N\times N}w_2(0))-\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\Big|_{t=0}$$
$$\times\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right), \tag{16}$$
$$B_1\widetilde{F}(\widetilde{u}_0(L),u^*(L))\widetilde{u}_0'(L)=B_1\widetilde{G}(\widetilde{u}_0(L),u^*(L))\widetilde{u}_0(L). \tag{17}$$

For the purpose of eliminating the compatibility conditions (7) and (9), the initial conditions of w_1, w_2 satisfy, $w_1(0)+w_2(0)=g_1(\widetilde{u}_0),-(d_1I_{N\times N}w_1(0)+d_2I_{N\times N}w_2(0))=g_2(\widetilde{u}_0)$, with $g_1(\widetilde{u}_0)=A_1\widetilde{u}_0(0)-d(0)-\Theta U(0)+\Pi_{NL}(\widetilde{u}_0(0)),g_2(\widetilde{u}_0)=A_1\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right)-\dot{d}(0)-\Theta\dot{U}(0)+\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\Big|_{t=0}\left(-\widetilde{F}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0'(0)+\widetilde{G}(\widetilde{u}_0(0),u^*(0))\widetilde{u}_0(0)\right)$. In order to verify the compatibility conditions (7) and (9), select

$$w_1(0)=-\frac{g_2(\widetilde{u}_0)+d_2I_{N\times N}g_1(\widetilde{u}_0)}{d_1-d_2}, \tag{18}$$
$$w_2(0)=\frac{d_1I_{N\times N}g_1(\widetilde{u}_0)+g_2(\widetilde{u}_0)}{(d_1-d_2)}. \tag{19}$$

C. State transformations

For deriving the form of characteristic values of the quasilinear system (3), (11)-(12), (13) and making the analysis easier, we handle a transformation for the state $\widetilde{u}$. Defining an invertible transformation $\bar{R}(x,t)=\Phi(x)\widetilde{u}(x,t)$ with $x\in[0,L]$, $t\in[0,+\infty)$, from $\widetilde{u}$ to the new variables $\bar{R}=(\bar{R}^+,\bar{R}^-)^\top:[0,L]\times[0,+\infty)\to\mathbb{R}^{2N}$ with $\bar{R}^+:[0,L]\times[0,+\infty)\to\mathbb{R}^m$, $\bar{R}^-:[0,L]\times[0,+\infty)\to\mathbb{R}^{2N-m}$, $\bar{R}_{in}(t)=(\bar{R}^+(0,t),\bar{R}^-(L,t))^\top$, $\bar{R}_{out}(t)=(\bar{R}^+(L,t),\bar{R}^-(0,t))^\top$, the system (3), (11)-(12), (13) is mapped into the following simplified system, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$\partial_t\bar{R}(x,t)+\Lambda(x)\partial_x\bar{R}(x,t)-\Sigma(x)\bar{R}(x,t)+\Lambda_{NL}(\bar{R},x)\partial_x\bar{R}(x,t)$$
$$=\Sigma_{NL}(\bar{R},x)\bar{R}(x,t), \tag{20}$$
$$\bar{R}_{in}(t)=K_P\bar{R}_{out}(t)+\Gamma_0\Big(d(t)+\Theta U(t)\Big)+\Gamma_0(w_1(t)+w_2(t))$$
$$-\Gamma_0\Pi_{NL}(\Phi^{-1}(0)\bar{R}(0,t)), \tag{21}$$

where $\Lambda(x)=\bar{\Lambda}(0,x)$, $\Sigma(x)=\partial_{\bar{R}}\bar{\Sigma}(0,x)$, $\Lambda_{NL}(\bar{R},x)=\bar{\Lambda}(\bar{R},x)-\Lambda(x)$, $\Sigma_{NL}(\bar{R},x)=\bar{\Sigma}(\bar{R},x)-\Sigma(x)$, with

$$\bar{\Lambda}(\bar{R},x)=\Phi(x)\widetilde{F}\left(\Phi^{-1}(x)\bar{R},u^*(x)\right)\Phi^{-1}(x),$$
$$\bar{\Sigma}(\bar{R},x)=\Phi(x)\widetilde{G}\left(\Phi^{-1}(x)\bar{R},u^*(x)\right)\Phi^{-1}(x)$$
$$-\Phi(x)\widetilde{F}\left(\Phi^{-1}(x)\bar{R},u^*(x)\right)\Phi^{-1}(x)',$$

the main diagonal elements of matrix $\Sigma\in\mathcal{C}([0,L];\mathcal{M}_{2N,2N}(\mathbb{R}))$ are zeros, and $K_P\in\mathcal{M}_{2N,2N}(\mathbb{R}),\Gamma_0\in\mathcal{M}_{2N,N}(\mathbb{R})$ are given gain matrices. The well-posedness of the system (20)-(21) in the H^2 space is under a necessary condition that the initial condition $\bar{R}(\cdot,0)=\bar{R}_0(\cdot)\in H^2([0,L];\mathbb{R}^{2N})$ satisfies the following second-order compatibility condition,

$$\bar{R}_{in}(0)=K_P\bar{R}_{out}(0)+\Gamma_0\left(d(0)+\Theta U(0)\right)$$
$$+\Gamma_0(w_1(0)+w_2(0))-\Gamma_0\Pi_{NL}(\Phi^{-1}(0)\bar{R}_0(0)), \tag{22}$$

$$
\begin{aligned}
&([M_i^1]_{1\le i\le m},[M_j^2]_{m+1\le j\le 2N})^\top\\
&=K_P([M_i^2]_{1\le i\le m},[M_j^1]_{m+1\le j\le 2N})^\top+\Gamma_0\Big(\dot d(0)+\Theta\dot U(0)\Big)\\
&+\Gamma_0(-d_1I_{N\times N}w_1(0)-d_2I_{N\times N}w_2(0))-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde u}(\Phi^{-1}(0)\Big|_{t=0}\\
&\times\Phi^{-1}(0)\Big(-\big(\Lambda(0)+\Lambda_{NL}(\bar R_0(0),0)\big)\bar R_0'(0)\\
&+\big(\Sigma(0)+\Sigma_{NL}(\bar R_0(0),0)\big)\bar R_0(0)\Big),
\end{aligned}\tag{23}
$$

with

$$
\begin{aligned}
M^1=&-\big(\Lambda(0)+\Lambda_{NL}(\bar R_0(0),0)\big)\bar R_0'(0)\\
&+\big(\Sigma(0)+\Sigma_{NL}(\bar R_0(0),0)\big)\bar R_0(0),\\
M^2=&-\big(\Lambda(L)+\Lambda_{NL}(\bar R_0(L),L)\big)\bar R_0'(L)\\
&+\big(\Sigma(L)+\Sigma_{NL}(\bar R_0(L),L)\big)\bar R_0(L).
\end{aligned}
$$

Since for $x\in[0,L]$, $\Lambda_{NL}(0,x)=0$, $\Sigma_{NL}(0,x)=0$ and $\Pi_{NL}(0)=\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde u}(0)=0$, then 0 is an equilibrium of the quasilinear system (20)-(21). For the sake of conveniently analyzing the applicability of designed controller for the linearized system, the quasilinear system (20)-(21) is written as the linear part plus the nonlinear terms, done in the next section.

III. Local H^2 iISS of the quasilinear model

Inspired by [9], the problem of local H^2 iISS of the first-order hyperbolic quasilinear system (3) and (13) is studied in this section. We will show that the controller derived for linearized system by using backstepping method locally stabilizes this quasilinear system.

Concerning the linearized system, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$
\partial_tR(x,t)+\Lambda(x)\partial_xR(x,t)=\Sigma(x)R(x,t),\tag{24}
$$

$$
R_{in}(t)=K_PR_{out}(t)+\Gamma_0\left(d(t)+\Theta U(t)\right),\tag{25}
$$

the designed control law U is derived by using the following backstepping transformation, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$
Z(x,t)=R(x,t)-\int_x^L K(x,\xi)R(\xi,t)\,\mathrm{d}\xi=\mathcal{K}[R],\tag{26}
$$

where $K(x,\xi)=\begin{bmatrix}0&0\\G^1(x,\xi)&G^2(x,\xi)\end{bmatrix}\in\mathcal{M}_{2N,2N}(\mathcal{C}^2(\mathbb{T}))$, with suitable $G^1\in\mathcal{M}_{2N-m,m}(\mathcal{C}^2(\mathbb{T}))$ and $G^2\in\mathcal{M}_{2N-m,2N-m}(\mathcal{C}^2(\mathbb{T}))$ defined in the domain $\mathbb{T}=\big\{(x,\xi)\in\mathbb{R}^2\big|\,0\le x\le\xi\le L\big\}$ as in paper [12]. The kernels $G^1(x,\xi)$, $G^2(x,\xi)$ satisfy an $2N\times 2N$ first-order hyperbolic PDEs, whose well-posedness is shown in [1, D.6]. The corresponding iISS target system is, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$
\begin{aligned}
\partial_tZ(x,t)+\Lambda(x)\partial_xZ(x,t)=&\Sigma_1(x)Z(x,t)\\
&+\int_x^L C_1(x,\xi)Z(\xi,t)\,\mathrm{d}\xi,
\end{aligned}\tag{27}
$$

$$
Z_{in}(t)=K_PZ_{out}(t)+K_I\int_0^t Z_{out}(\sigma)\,\mathrm{d}\sigma+\Gamma_0d(t),\tag{28}
$$

where $Z=(Z^+,Z^-)^\top$, with $Z^+:[0,L]\times[0,+\infty)\to\mathbb{R}^m$, $Z^-:[0,L]\times[0,+\infty)\to\mathbb{R}^{2N-m}$, $Z_{in}(t)=(Z^+(0,t),Z^-(L,t))^\top$, $Z_{out}(t)=(Z^+(L,t),Z^-(0,t))^\top$, the matrices $\Sigma_1\in\mathcal{M}_{2N,2N}(\mathcal{C}^2([0,L]))$, $C_1\in\mathcal{M}_{2N,2N}(\mathcal{C}^2(\mathbb{T}))$ and $K_I=\begin{bmatrix}K_I^{11}&K_I^{12}\\\mathbf{0}_{(2N-m),m}&\mathbf{0}_{(2N-m),(2N-m)}\end{bmatrix}$, with $K_I^{11}\in\mathcal{M}_{m,m}(\mathbb{R})$, $K_I^{12}\in\mathcal{M}_{m,2N-m}(\mathbb{R})$ are given. The controller for the linearized system (24)-(25) is derived as follows, for all $t\in(0,+\infty)$,

$$
\begin{aligned}
U(R)=&(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta\int_0^t\big(K_I^{11}R^+(L,\sigma)+K_I^{12}R^-(0,\sigma)\big)\,\mathrm{d}\sigma\\
&-(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta K_I^{12}\int_0^t\int_0^L\Big[G^1(0,\xi)R^+(\xi,\sigma)\\
&+G^2(0,\xi)R^-(\xi,\sigma)\Big]\,\mathrm{d}\xi\,\mathrm{d}\sigma-(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta\Gamma_1\\
&\times\int_0^L\Big[G^1(0,\xi)R^+(\xi,t)+G^2(0,\xi)R^-(\xi,t)\Big]\,\mathrm{d}\xi,
\end{aligned}\tag{29}
$$

where the coefficient matrices Γ_1, Γ_2 are given, and the matrix $\overline\Theta$ is in $\mathcal{M}_{2N-m,m}$ and such that $\overline\Theta\Gamma_2\Theta$ is invertible. So the controller in terms of the original variable $\widetilde u$ in (11) is, for all $t\in(0,+\infty)$,

$$
\begin{aligned}
U(t)=&(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta\int_0^t\bigg(\Big[K_I^{11}\quad 0\Big]\Phi(L)\widetilde u(L,\sigma)\\
&+\Big[0\quad K_I^{12}\Big]\Phi(0)\widetilde u(0,\sigma)\bigg)\,\mathrm{d}\sigma-(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta K_I^{12}\\
&\times\int_0^t\int_0^L\Big[G^1(0,\xi)\quad G^2(0,\xi)\Big]\Phi(\xi)\widetilde u(\xi,\sigma)\,\mathrm{d}\xi\,\mathrm{d}\sigma\\
&-(\overline\Theta\Gamma_2\Theta)^{-1}\overline\Theta\Gamma_1\int_0^L\Big[G^1(0,\xi)\quad G^2(0,\xi)\Big]\Phi(\xi)\widetilde u(\xi,t)\,\mathrm{d}\xi.
\end{aligned}\tag{30}
$$

The formula of control law U in (29) for the linearized system (24)-(25) is derived by using backstepping method and the proof of iISS of the corresponding target system (27)-(28) is obtained in [12]. In this paper, we derive the iISS of the quasilinear system (3), (11)-(12), (13) with the state feedback law (30) as follows:

Theorem 1: *Considering the system (3), (13) with the extended boundary conditions (11)-(12), the initial conditions $\widetilde u_0\in H^2([0,L];\mathbb{R}^{2N})$, $w_1(0),w_2(0)\in\mathbb{R}^N$ verifying (18)-(19), and the control law (30). Then, for every $\alpha>0$, there exist positive constants δ,c,b such that, for any d satisfying $\dot d,\ddot d\in L^2[0,\infty)$, if $\|\widetilde u_0\|_{H^2}\le\delta$, it holds that for all $t\in[0,+\infty)$,*

$$
\begin{aligned}
&\|\widetilde u(\cdot,t)\|_{H^2}^2+|w_1(t)|^2+|w_2(t)|^2\le ce^{-\alpha t}\big(\|\widetilde u_0\|_{H^2}^2+|w_1(0)|^2\\
&+|w_2(0)|^2\big)+b\int_0^t(|\dot d(s)|^2+|\ddot d(s)|^2)\,\mathrm{d}s.
\end{aligned}\tag{31}
$$

Proof. As a result of the invertible transformation $\Phi(x)$, $x\in[0,L]$, the system (3), (11)-(12), (13) has the same dynamical behavior as the system (20)-(21). Therefore, in order to prove the iISS of the quasilinear system (3), (11)-(12), (13), we firstly prove the iISS of the quasilinear system (20)-(21) after applying the control law U in (29), which is designed for the linearized system (24)-(25). The quasilinear system (20)-(21) is mapped into a target system $\bar Z$ by using the direct transformation $\bar Z=\mathcal{K}[\bar R]$ and the inverse transformation $\bar R=\mathcal{L}[\bar Z]$ with $\mathcal{C}^2(\mathbb{T})$ kernel functions. Differentiating twice with respect to x in these transformations, it is shown that the H^2 norm of $\bar Z$ is equivalent to the H^2 norm of $\bar R$. So local iISS of the system $\bar R$ is same as local iISS of the target system. The equations of the target system $\bar Z$ is, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$
\begin{aligned}
\partial_t\bar Z(x,t)+\Lambda(x)\partial_x\bar Z(x,t)=&\Sigma_1(x)\bar Z(x,t)\\
&+\int_x^L C_1(x,\xi)\bar Z(\xi,t)\,\mathrm{d}\xi+F_3[\bar Z,\partial_x\bar Z]+F_4[\bar Z],
\end{aligned}\tag{32}
$$

$$
\bar Z_{in}(t)=K_P\bar Z_{out}(t)+\bar X(t)+\Gamma_0(w_1(t)+w_2(t)),\tag{33}
$$

$$
\bar X(t)=K_I\int_0^t\bar Z_{out}(\sigma)\,\mathrm{d}\sigma+\Gamma_0\Big(d(t)-\bar\Pi_{NL}(\bar Z(0,t))\Big),\tag{34}
$$

where

$$
\begin{aligned}
&F_3[\bar Z,\partial_x\bar Z]=-\mathcal{K}[\big(\Lambda(x)+F_1[\bar Z]\big)\partial_x\bar Z]+\Lambda(x)\partial_x\bar Z,\\
&F_4[\bar Z]=-\mathcal{K}_{12}[\bar Z]-\mathcal{K}[\big(\Lambda(x)+F_1[\bar Z]\big)\mathcal{K}_1[\mathcal{L}[\bar Z]]+\Sigma(x)\mathcal{L}[\bar Z]\\
&\qquad+F_2[\bar Z]\mathcal{L}[\bar Z]],\\
&\bar\Pi_{NL}(\bar Z(0,t))=\Pi_{NL}(\Phi^{-1}(0)\bar R(0,t)),\\
&\mathcal{K}_{12}[\bar Z]=\Sigma_1(x)\bar Z(x,t)+\int_x^L C_1(x,\xi)\bar Z(\xi,t)\,\mathrm{d}\xi,
\end{aligned}
$$

$$\mathcal{K}_1[\bar{Z}] = \int_x^L \partial_x K(x,\xi)\bar{Z}(\xi,t)\,\mathrm{d}\xi - K(x,x)\bar{Z}(x,t),$$
$$F_1[\bar{Z}] = \Lambda_{NL}(\mathcal{L}[\bar{Z}],x),$$
$$F_2[\bar{Z}] = \Sigma_{NL}(\mathcal{L}[\bar{Z}],x),$$

with $\bar{Z} = (\bar{Z}^+, \bar{Z}^-)^\top$, $\bar{Z}^+ : [0,L]\times[0,+\infty) \to \mathbb{R}^m$, $\bar{Z}^- : [0,L]\times[0,+\infty) \to \mathbb{R}^{2N-m}$, $\bar{Z}_{in}(t) = (\bar{Z}^+(0,t), \bar{Z}^-(L,t))^\top$, $\bar{Z}_{out}(t) = (\bar{Z}^+(L,t), \bar{Z}^-(0,t))^\top$. Next, the local stability of system (32)-(34) in the space H^2 is analyzed by using the Lyapunov approach. It includes analyzing the growth of $\|\bar{Z}\|_{L^2}$, $\|\eta\|_{L^2}$ and $\|\zeta\|_{L^2}$ (by definition $\eta = \partial_t\bar{Z}$ and $\zeta = \partial_{tt}\bar{Z}$).

The following Lyapunov function candidate is introduced for the stability analysis of system (32)-(34), for all $x\in[0,L]$, $t\in[0,+\infty)$, $\bar{V}\big(\bar{Z}(x,t),\bar{X}(t),\eta(x,t),\zeta(x,t)\big) = \bar{V}_1+\bar{V}_2+\bar{V}_3$, where

$$\bar{V}_1 = \int_0^L \Big(\bar{Z}^\top(x,\cdot)\bar{\mathcal{P}}_{11}(x)\bar{Z}(x,\cdot) + \bar{Z}^\top(x,\cdot)\bar{\mathcal{P}}_{12}(x)\bar{X}(\cdot) + \bar{X}^\top(\cdot)\bar{\mathcal{P}}_{12}^\top(x)\bar{Z}(x,\cdot) + \bar{X}^\top(\cdot)\bar{P}_{22}\bar{X}(\cdot)\Big)\,\mathrm{d}x, \tag{35}$$

$$\bar{V}_2 = \int_0^L \eta^\top(x,\cdot)\bar{\mathcal{P}}_3(x)\eta(x,\cdot)\,\mathrm{d}x, \tag{36}$$

$$\bar{V}_3 = \int_0^L \zeta^\top(x,\cdot)\bar{\mathcal{P}}_4[\bar{Z}](x)\zeta(x,\cdot)\,\mathrm{d}x, \tag{37}$$

with

$$\bar{\mathcal{P}}_{11}(x) \triangleq \bar{P}_{11}\mathrm{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\},$$
$$\bar{\mathcal{P}}_{12}(x) \triangleq \bar{P}_{12}\mathrm{diag}\left\{e^{-\frac{\mu}{2}x}I_m, e^{\frac{\mu}{2}x}I_{2N-m}\right\},$$
$$\bar{\mathcal{P}}_3(x) \triangleq \bar{P}_3\mathrm{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\},$$

$\bar{\mathcal{P}}_4[\bar{Z}](x)$ is defined in [9, Lemma 5.2], $\bar{P}_{11}$, $\bar{P}_3$ are $2N\times 2N$ diagonal positive-definite matrices, $\bar{P}_{22}$ is a $2N\times 2N$ symmetric positive-definite matrix, and $\bar{P}_{12}$ is a $2N\times 2N$ matrix.

Take the time derivative of $\bar{V}_1$ along the solutions to (32)-(34), use integration by parts, and define $\dot{\bar{V}}_1 = \dot{\bar{V}}_{1L} + \dot{\bar{V}}_{1NL}$, where $\dot{\bar{V}}_{1L}$ is the time derivative of $\bar{V}_1$ along the linear target system (27)-(28), computed as follows

$$\begin{aligned}
\dot{\bar{V}}_{1L} &= -\bar{Z}^\top(x,t)\Lambda(x)\bar{\mathcal{P}}_{11}(x)\bar{Z}(x,t)\Big|_0^L \\
&+ \int_0^L \bar{Z}^\top(x,t)\left(\Lambda(x)\bar{\mathcal{P}}_{11}(x)\right)_x\bar{Z}(x,t)\,\mathrm{d}x \\
&- \Big(\bar{Z}^\top(x,t)\Lambda(x)\bar{\mathcal{P}}_{12}(x)\bar{X}(t) + \bar{X}^\top(t)\bar{\mathcal{P}}_{12}^\top(x)\Lambda(x)\bar{Z}(x,t)\Big)\Big|_0^L \\
&+ \int_0^L \Big(\bar{Z}^\top(x,t)\left(\Lambda(x)\bar{\mathcal{P}}_{12}(x)\right)_x\bar{X}(t) \\
&+ \bar{X}^\top(t)\Big(\bar{\mathcal{P}}_{12}^\top(x)\Lambda(x)\Big)_x\bar{Z}(x,t)\Big)\,\mathrm{d}x \\
&+ \int_0^L \Big(\Big(\Sigma_1(x)\bar{Z}(x,t) + \int_x^L C_1(x,\xi)\bar{Z}(\xi,t)\,\mathrm{d}\xi\Big)^\top \\
&(\bar{\mathcal{P}}_{11}(x)\bar{Z}(x,t) + \bar{\mathcal{P}}_{12}(x)\bar{X}(t)) \\
&+ (\bar{Z}^\top(x,t)\bar{\mathcal{P}}_{11}(x) + \bar{X}^\top(t)\bar{\mathcal{P}}_{12}^\top(x)) \\
&\Big(\Sigma_1(x)\bar{Z}(x,t) + \int_x^L C_1(x,\xi)\bar{Z}(\xi,t)\,\mathrm{d}\xi\Big)\Big)\,\mathrm{d}x \\
&+ \int_0^L \Big((\bar{Z}^\top(x,t)\bar{\mathcal{P}}_{12}(x) + \bar{X}^\top(t)\bar{P}_{22})\Big(K_I\bar{Z}_{out}(t) + \Gamma_0\dot{d}(t)\Big) \\
&+ \Big(K_I\bar{Z}_{out}(t) + \Gamma_0\dot{d}(t)\Big)^\top(\bar{\mathcal{P}}_{12}^\top(x)\bar{Z}(x,t) + \bar{P}_{22}\bar{X}(t))\Big)\,\mathrm{d}x,
\end{aligned} \tag{38}$$

and where $\dot{\bar{V}}_{1NL}$ is defined as:

$$\begin{aligned}
\dot{\bar{V}}_{1NL} &= \int_0^L \Big(\Big(F_3[\bar{Z},\partial_x\bar{Z}] + F_4[\bar{Z}]\Big)^\top\bar{\mathcal{P}}_{11}(x)\bar{Z}(x,t) \\
&+ \bar{Z}^\top(x,t)\bar{\mathcal{P}}_{11}(x)\Big(F_3[\bar{Z},\partial_x\bar{Z}] + F_4[\bar{Z}]\Big)\Big)\,\mathrm{d}x \\
&+ \int_0^L \Big(\bar{Z}^\top(x,t)\bar{\mathcal{P}}_{12}(x)\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big) \\
&+ \Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)^\top\bar{\mathcal{P}}_{12}^\top(x)\bar{Z}(x,t) \\
&+ \Big(F_3[\bar{Z},\partial_x\bar{Z}] + F_4[\bar{Z}]\Big)^\top\bar{\mathcal{P}}_{12}(x)\bar{X}(t) \\
&+ \bar{X}^\top(t)\bar{\mathcal{P}}_{12}^\top(x)\Big(F_3[\bar{Z},\partial_x\bar{Z}] + F_4[\bar{Z}]\Big)\Big)\,\mathrm{d}x \\
&+ L\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)^\top\bar{P}_{22}\bar{X}(t) \\
&+ L\bar{X}^\top(t)\bar{P}_{22}\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big).
\end{aligned} \tag{39}$$

From system (32)-(34), we derive, for all $x\in[0,L]$, $t\in[0,+\infty)$,

$$\partial_t\eta(x,t) + \Lambda(x)\partial_x\eta(x,t) = \Sigma_1(x)\eta(x,t) + \int_x^L C_1(x,\xi)\eta(\xi,t)\,\mathrm{d}\xi + F_5[\bar{Z},\partial_x\eta] + F_6[\bar{Z},\eta,\partial_x\bar{Z}], \tag{40}$$

$$\eta_{in}(t) = K_P\eta_{out}(t) + \dot{\bar{X}}(t) + \Gamma_0(-d_1I_{N\times N}w_1(t) - d_2I_{N\times N}w_2(t)), \tag{41}$$

$$\dot{\bar{X}}(t) = K_I\bar{Z}_{out}(t) + \Gamma_0\Big(\dot{d}(t) - \frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big), \tag{42}$$

where

$$\begin{aligned}
&F_5[\bar{Z},\partial_x\eta] = \mathcal{K}[(\Lambda(x)+F_1[\bar{Z}])\partial_x\eta] + \Lambda(x)\partial_x\eta, \\
&F_6[\bar{Z},\eta,\partial_x\bar{Z}] = \mathcal{K}[\big(\Lambda(x)+F_1[\bar{Z}]\big)\mathcal{K}_1[\mathcal{L}[\eta]] - \Sigma(x)\mathcal{L}[\eta] \\
&- F_2[\bar{Z}]\mathcal{L}[\eta] + F_{12}[\eta](\partial_x\bar{Z} + \mathcal{K}_1[\mathcal{L}[\bar{Z}]]) - F_{22}[\eta]\mathcal{L}[\bar{Z}]] - \mathcal{K}_{12}[\eta], \\
&F_{12}[\eta] = \frac{\mathrm{d}}{\mathrm{d}t}F_1[\bar{Z}], \\
&F_{22}[\eta] = \frac{\mathrm{d}}{\mathrm{d}t}F_2[\bar{Z}].
\end{aligned}$$

Take the time derivative of $\bar{V}_2$ along the solutions to (32)-(34), use integration by parts, and define $\dot{\bar{V}}_2 = \dot{\bar{V}}_{2L} + \dot{\bar{V}}_{2NL}$, where $\dot{\bar{V}}_{2L}$ is the time derivative of $\bar{V}_2$ along the linear target system (27)-(28),

$$\begin{aligned}
\dot{\bar{V}}_{2L} &= -\eta^\top(x,t)\Lambda(x)\bar{\mathcal{P}}_3(x)\eta(x,t)\Big|_0^L + \int_0^L \eta^\top(x,t) \\
&\left(\Lambda(x)\bar{\mathcal{P}}_3(x)\right)_x\eta(x,t)\,\mathrm{d}x + \int_0^L \Big(\Big(\Sigma_1(x)\eta(x,t) \\
&+ \int_x^L C_1(x,\xi)\eta(\xi,t)\,\mathrm{d}\xi\Big)^\top\bar{\mathcal{P}}_3(x)\eta(x,t) + \eta^\top(x,t)\bar{\mathcal{P}}_3(x) \\
&\Big(\Sigma_1(x)\eta(x,t) + \int_x^L C_1(x,\xi)\eta(\xi,t)\,\mathrm{d}\xi\Big)\Big)\,\mathrm{d}x,
\end{aligned} \tag{43}$$

and where $\dot{\bar{V}}_{2NL}$ is defined as:

$$\begin{aligned}
\dot{\bar{V}}_{2NL} &= (K_P\dot{\bar{Z}}_{out}(t))^\top\bar{E}_1\bar{P}_3\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big) \\
&+ \Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)^\top\bar{P}_3\bar{E}_1(K_P\dot{\bar{Z}}_{out}(t)) \\
&+ \Big(K_I\bar{Z}_{out}(t) + \Gamma_0\Big(\dot{d}(t) - \frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big) \\
&- \Gamma_0(d_1I_{N\times N}w_1(t) + d_2I_{N\times N}w_2(t))\Big)^\top\bar{E}_1\bar{P}_3 \\
&\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big) \\
&+ \Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)^\top\bar{E}_1\bar{P}_3
\end{aligned}$$

$$\Big(K_I\bar{Z}_{out}(t)+\Gamma_0\Big(\dot{d}(t)-\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)$$
$$-\Gamma_0(d_1I_{N\times N}w_1(t)+d_2I_{N\times N}w_2(t))\Big)$$
$$+\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)^\top\bar{E}_1\bar{P}_3$$
$$\Big(-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\Big)$$
$$+\int_0^L\Big(\Big(F_5[\bar{Z},\partial_x\eta]+F_6[\bar{Z},\eta,\partial_x\bar{Z}]\Big)^\top\bar{\mathcal{P}}_3(x)\eta(x,t)$$
$$+\eta^\top(x,t)\bar{\mathcal{P}}_3(x)\Big(F_5[\bar{Z},\partial_x\eta]+F_6[\bar{Z},\eta,\partial_x\bar{Z}]\Big)\Big)\,\mathrm{d}x. \tag{44}$$

Since the term $\dot{\bar{V}}_{1L}+\dot{\bar{V}}_{2L}$ is analyzed in [12, Theorem 1], which is for the iISS of the linear target system (27)-(28), there exist positive constants c_1, b_1, a_1, a_2, a_3, such that

$$\dot{\bar{V}}_{1L}+\dot{\bar{V}}_{2L}\le -c_1(\bar{V}_1+\bar{V}_2)+b_1|\dot{d}|^2+a_1(w_1^2+w_2^2)$$
$$+a_2\|\bar{Z}\|_\infty(\bar{V}_1+\bar{V}_2)+a_3\|\eta\|_\infty(\bar{V}_2+\bar{V}_3). \tag{45}$$

Now analyze the remaining items $\dot{\bar{V}}_{1NL}$ and $\dot{\bar{V}}_{2NL}$. There exist positive constants k_1, k_2, k_3, k_4, k_5, k_6, k_7 such that

$$\dot{\bar{V}}_{1NL}\le k_1\int_0^L\left(\left|F_3[\bar{Z},\partial_x\bar{Z}]\right|+\left|F_4[\bar{Z}]\right|\right)\left|\bar{Z}\right|\,\mathrm{d}x$$
$$+k_2\int_0^L\Big(\left|\bar{Z}\right|\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$+\left(\left|F_3[\bar{Z},\partial_x\bar{Z}]\right|+\left|F_4[\bar{Z}]\right|\right)\left|\bar{X}(t)\right|\Big)\,\mathrm{d}x$$
$$+k_3\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|\left|\bar{X}(t)\right|,$$

$$\dot{\bar{V}}_{2NL}\le k_4\left|\dot{\bar{Z}}_{out}(t)\right|\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$+k_5\Big(\left|K_I\bar{Z}_{out}(t)+\Gamma_0\dot{d}(t)\right|+\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$+|\Gamma_0(d_1I_{N\times N}w_1(t)+d_2I_{N\times N}w_2(t))|\Big)\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$+k_6\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|^2$$
$$+k_7\int_0^L\Big(\left|F_5[\bar{Z},\partial_x\eta]\right|+\left|F_6[\bar{Z},\eta,\partial_x\bar{Z}]\right|\Big)\left|\eta\right|\,\mathrm{d}x.$$

Applying [9, Lemma B.2], there exists $\delta_1>0$, and positive constants k_8, k_9, k_{10}, h_1, h_2, h_3, h_4 such that for all $\bar{Z}$ satisfying $\|\bar{Z}\|_\infty\le\delta_1$, it holds $\left|\Gamma_0\bar{\Pi}_{NL}(\bar{Z}(0,t))\right|\le k_8|\bar{Z}(0,t)|$, $\|\bar{Z}\|_{L^2}^2+\left|\bar{X}(t)\right|^2\le k_9\bar{V}_1$, $\|\eta\|_{L^2}^2\le k_{10}\bar{V}_2$, and

$$\int_0^L\left(\left|F_3[\bar{Z},\partial_x\bar{Z}]\right|+\left|F_4[\bar{Z}]\right|\right)\left|\bar{Z}\right|\,\mathrm{d}x\le h_1k_9(\|\bar{Z}\|_\infty+\|\eta\|_\infty)\bar{V}_1,$$
$$\int_0^L\left|\bar{Z}\right|\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|\,\mathrm{d}x$$
$$\le h_2(k_9\|\bar{Z}\|_\infty\bar{V}_1+k_{10}\|\eta\|_\infty\bar{V}_2),$$
$$\int_0^L\left(\left|F_3[\bar{Z},\partial_x\bar{Z}]\right|+\left|F_4[\bar{Z}]\right|\right)\left|\bar{X}(t)\right|\,\mathrm{d}x\le h_3k_9(\|\bar{Z}\|_\infty+\|\eta\|_\infty)\bar{V}_1,$$
$$\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|\left|\bar{X}(t)\right|\le h_4(k_9\|\bar{Z}\|_\infty\bar{V}_1+k_{10}\|\eta\|_\infty\bar{V}_2).$$

We deduce, for all $\bar{Z}$ satisfying $\|\bar{Z}\|_\infty\le\delta_1$,

$$\dot{\bar{V}}_{1NL}\le(h_1+h_3)k_9(\|\bar{Z}\|_\infty+\|\eta\|_\infty)\bar{V}_1$$
$$+(h_2+h_4)(k_9\|\bar{Z}\|_\infty\bar{V}_1+k_{10}\|\eta\|_\infty\bar{V}_2). \tag{46}$$

Applying [9, Lemma B.3], there exist positive constants $\delta_2\le\delta_1$, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11} such that for all $\bar{Z}$ satisfying $\|\bar{Z}\|_\infty+\|\eta\|_\infty\le\delta_2$, $\|\zeta\|_{L^2}\le k_{11}\bar{V}_3^{1/2}$, $k_{11}>0$, it holds

$$\left|\dot{\bar{Z}}_{out}(t)\right|\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|\le h_5\|\eta\|_\infty(\bar{V}_2+\bar{V}_3),$$
$$\Big(\left|K_I\bar{Z}_{out}(t)+\Gamma_0\dot{d}(t)\right|+\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$+|\Gamma_0(d_1I_{N\times N}w_1(t)+d_2I_{N\times N}w_2(t))|\Big)\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|$$
$$\le h_6\|\bar{Z}\|_\infty\bar{V}_2+h_6\|\bar{Z}\|_\infty(\bar{V}_1+\bar{V}_2)+h_7\left|\dot{d}(t)\right|^2+h_8(w_1^2+w_2^2),$$
$$\left|\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\eta(0,t)\right|^2\le h_9\|\bar{Z}\|_\infty\bar{V}_2,$$
$$\int_0^L\Big(\left|F_5[\bar{Z},\partial_x\eta]\right|+\left|F_6[\bar{Z},\eta,\partial_x\bar{Z}]\right|\Big)\left|\eta\right|\,\mathrm{d}x$$
$$\le h_{10}\|\bar{Z}\|_\infty(\bar{V}_2+\bar{V}_3)+h_{11}\|\bar{Z}\|_\infty\bar{V}_2\left(\|\bar{Z}\|_\infty+\|\eta\|_\infty+1\right),$$

and therefore

$$\dot{\bar{V}}_{2NL}\le k_4h_5\|\eta\|_\infty(\bar{V}_2+\bar{V}_3)$$
$$+(k_4h_5+k_5h_6+k_6h_9+k_7h_{10}+k_7h_{11})\|\bar{Z}\|_\infty\bar{V}_2$$
$$+k_5\Big(h_6\|\bar{Z}\|_\infty(\bar{V}_1+\bar{V}_2)+h_7\left|\dot{d}(t)\right|^2+h_8(w_1^2+w_2^2)\Big)$$
$$+k_7h_{10}\|\bar{Z}\|_\infty\bar{V}_3+k_7h_{11}\bar{V}_2\left(\|\bar{Z}\|_\infty+\|\eta\|_\infty\right). \tag{47}$$

Taking the time derivative in (40), we obtain the following equation for ζ,

$$\partial_t\zeta(x,t)+(\Lambda(x)+F_1[\bar{Z}])\partial_x\zeta(x,t)=F_7[\bar{Z},\zeta,\partial_x\zeta]$$
$$+F_8[\bar{Z},\eta,\zeta,\partial_x\bar{Z},\partial_x\eta], \tag{48}$$
$$\zeta_{in}(t)=K_P\zeta_{out}(t)+\ddot{\bar{X}}(t)+\Gamma_0(d_1^2w_1(t)+d_2^2w_2(t)), \tag{49}$$
$$\ddot{\bar{X}}(t)=K_I\dot{\bar{Z}}_{out}(t)+\Gamma_0\Big(\ddot{d}(t)-\frac{\mathrm{d}^2\Pi_{NL}}{\mathrm{d}\bar{Z}^2}(\bar{Z}(0,t))\eta^2(0,t)$$
$$-\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\bar{Z}}(\bar{Z}(0,t))\zeta(0,t)\Big), \tag{50}$$

where

$$F_7[\bar{Z},\zeta,\partial_x\zeta]=-\mathcal{K}[(\Lambda(x)+F_1[\bar{Z}])(\partial_x\zeta+\mathcal{K}_1[\mathcal{L}[\zeta]])]$$
$$+(\Lambda(x)+F_1[\bar{Z}])(\partial_x\zeta+\mathcal{K}_1[\mathcal{L}[\zeta]]),$$
$$F_8[\bar{Z},\eta,\zeta,\partial_x\bar{Z},\partial_x\eta]=-(\Lambda(x)+F_1[\bar{Z}])\mathcal{K}_1[\mathcal{L}[\zeta]]$$
$$+\mathcal{K}[(\Sigma(x)+F_2[\bar{Z}])\mathcal{L}[\zeta]-2F_{12}[\bar{Z}](\partial_x\eta+\mathcal{K}_1[\mathcal{L}[\eta]])$$
$$+2F_{22}[\bar{Z}]\mathcal{L}[\eta]+F_{23}\mathcal{L}[\bar{Z}]-F_{13}(\partial_x\bar{Z}+\mathcal{K}_1[\mathcal{L}[\bar{Z}]])],$$
$$F_{13}[\bar{Z},\eta,\zeta]=\frac{\mathrm{d}^2\Lambda_{NL}(\mathcal{L}[\bar{Z}],x)}{\mathrm{d}t^2},$$
$$F_{23}[\bar{Z},\eta,\zeta]=\frac{\mathrm{d}^2\Sigma_{NL}(\mathcal{L}[\bar{Z}],x)}{\mathrm{d}t^2}.$$

Take the time derivative of $\bar{V}_3$ along the solutions to (32)-(34), apply [9, Lemma 5.2], and define $\dot{\bar{V}}_3=\dot{\bar{V}}_{3L}+\dot{\bar{V}}_{3NL}$, where $\dot{\bar{V}}_{3L}$ is the time derivative of $\bar{V}_{3L}$ along the linear target system (27)-(28),

$$\dot{\bar{V}}_{3L}=-\zeta^\top(x,t)\Big(\bar{\mathcal{P}}_4[\bar{Z}](x)(\Lambda(x)+F_1[\bar{Z}])\Big)\zeta(x,t)\Big|_0^L$$
$$+\int_0^L\zeta^\top(x,t)\Big(\bar{\mathcal{P}}_4[\bar{Z}](x)(\Lambda(x)+F_1[\bar{Z}])\Big)_x\zeta(x,t)\,\mathrm{d}x$$
$$+\int_0^L\zeta^\top(x,\cdot)\left(\bar{\mathcal{P}}_4[\bar{Z}](x)\right)_t\zeta(x,\cdot),$$

and

$$\dot{\bar{V}}_{3NL}=\int_0^L\Big(\Big(F_7[\bar{Z},\zeta,\partial_x\zeta]+F_8[\bar{Z},\eta,\zeta,\partial_x\bar{Z},\partial_x\eta\Big)^\top$$

$$\bar{\mathcal{P}}_4[\bar{Z}](x)\zeta(x,t)+\zeta^\top(x,t)\bar{\mathcal{P}}_4[\bar{Z}](x)\Big(F_7[\bar{Z},\zeta,\partial_x\zeta]$$
$$+F_8[\bar{Z},\eta,\zeta,\partial_x\bar{Z},\partial_x\eta]\Big)\Big)\,\mathrm{d}x.$$

According to [9, Proposition 5.4], there exist positive constants c_2, h_{12}, h_{13}, h_{14}, h_{15}, h_{16}, h_{17}, h_{18}, h_{19}, h_{20} such that

$$-\zeta^\top(x,t)\Big(\bar{\mathcal{P}}_4[\bar{Z}](x)(\Lambda(x)+F_1[\bar{Z}])\Big)\zeta(x,t)\Big|_0^L$$
$$\le h_{12}\|\zeta\|_\infty(\bar{V}_1+\bar{V}_2+\bar{V}_3)+h_{13}(w_1^2+w_2^2)$$
$$+h_{14}\|\eta\|_\infty(\bar{V}_2+\bar{V}_3)+h_{15}|\ddot{d}|^2+h_{16}\|\zeta\|_\infty\bar{V}_2^2,$$
$$\int_0^L\zeta^\top(x,t)\Big(\bar{\mathcal{P}}_4[\bar{Z}](x)\left(\Lambda(x)+F_1[\bar{Z}]\right)\Big)_x\zeta(x,t)\,\mathrm{d}x$$
$$\le -c_2\bar{V}_3+h_{17}(\|\bar{Z}\|_\infty+\|\eta\|_\infty)\bar{V}_3,$$
$$\int_0^L\zeta^\top(x,\cdot)\left(\bar{\mathcal{P}}_4[\bar{Z}](x)\right)_t\zeta(x,\cdot)\,\mathrm{d}x\le h_{18}\bar{V}_3\|\eta\|_\infty,$$
$$\int_0^L\zeta^\top(x,t)\bar{\mathcal{P}}_4[\bar{Z}](x)\Big(F_7[\bar{Z},\zeta,\partial_x\zeta]+F_8[\bar{Z},\eta,\zeta,\partial_x\bar{Z},\partial_x\eta]\Big)\,\mathrm{d}x$$
$$\le h_{19}\bar{V}_2\|\zeta\|_\infty\left(\|\bar{Z}\|_\infty+\|\eta\|_\infty\right)$$
$$+h_{20}\left(\|\eta\|_\infty\bar{V}_3+\|\zeta\|_\infty\bar{V}_2+\|\bar{Z}\|_\infty\bar{V}_3\right).$$

Therefore there exist positive constants $\delta_3\le\delta_2$, c_3 and c_4 satisfying $c_3+c_4<c_2$ such that, for all $\bar{Z}$ satisfying $\|\bar{Z}\|_\infty+\|\eta\|_\infty+\|\zeta\|_\infty\le\delta_3$, it holds

$$\dot{\bar{V}}_{3L}\le(-c_2+c_3)(\bar{V}_1+\bar{V}_2+\bar{V}_3)+h_{15}|\ddot{d}|^2+h_{13}(w_1^2+w_2^2),\tag{51}$$

$$\dot{\bar{V}}_{3NL}\le c_4(\bar{V}_1+\bar{V}_2+\bar{V}_3).\tag{52}$$

Combining (51), (52) with (45), (46), (47), and letting $c=c_3+c_4-c_2-c_1$ and $a=a_1+k_5h_8+h_{13}$, we get, for all $\|\bar{Z}\|_\infty+\|\eta\|_\infty+\|\zeta\|_\infty\le\delta_2$,

$$\dot{\bar{V}}\le-c\bar{V}+(b_1+k_5h_7)|\dot{d}|^2+h_{15}|\ddot{d}|^2+a(w_1^2+w_2^2).\tag{53}$$

By defining $\bar{W}=\bar{V}+\frac{b_4}{2}(\frac{w_1^2}{d_1}+\frac{w_2^2}{d_2})$, we derive

$$\dot{\bar{W}}\le-c\bar{V}+(b_1+k_5h_7)|\dot{d}|^2+h_{15}|\ddot{d}|^2+(a-b_4)(w_1^2+w_2^2).$$

Choosing $a<b_4$, for some positive constant c_5, we obtain

$$\dot{\bar{W}}\le-c_5\bar{W}+(b_1+k_5h_7)|\dot{d}|^2+h_{15}|\ddot{d}|^2.\tag{54}$$

For sufficiently small $\bar{W}(0)$, $\bar{W}$ exponentionally converge to zero. If $\|\bar{Z}\|_{L^2}+\|\eta\|_{L^2}+\|\zeta\|_{L^2}$ is sufficiently small, $\bar{V}$ is equivalent to the H^2 norm of $\bar{Z}$. We deduce (31) by a standard application of comparison lemma. This concludes the proof of Theorem 1. □

Furthermore, under the assumption of Theorem 1, coming back to (1)-(2), we deduce that the quasilinear plant system u is locally iISS around the nonuniform equilibrium u^* in the space H^2.

IV. Simulations

In consideration of verifying the applicability of the control law U in (29) to locally stabilize the quasilinear system u around the nonuniform equilibrium u^* on a considered road section (road length: 1km, width: 6.5m), the values of traffic parameters for two vehicle classes are chosen as in [12], see Table I. Let

$$K_P=\begin{bmatrix}0&0&0&-7.85\\0&0&0&6.85\\0&0&0&-41.88\\-5.67&-5.09&7.15&0\end{bmatrix}\times10^{-5},$$

$$K_I^{11}=\begin{bmatrix}-20&30&30\\-24&-7&26\\-10&20&-30\end{bmatrix}\times10^{-5},K_I^{12}=\begin{bmatrix}60\\30\\20\end{bmatrix}\times10^{-5},$$

Name	Symbol	Value	Unit
Relaxation time	τ_1	30	s
	τ_2	60	s
Pressure exponent	γ_1	2.5	1
	γ_2	2	1
Free-flow velocity	v_1^M	80	$\frac{km}{h}$
	v_2^M	60	$\frac{km}{h}$
Maximum AO	Ao_1^M	0.9	1
	Ao_2^M	0.85	1
Occupied surface per vehicle	a_1	10	m^2
	a_2	42	m^2
equilibrium density at the inlet	$\rho_1^*(0)$	110	$\frac{veh}{km}$
	$\rho_2^*(0)$	70	$\frac{veh}{km}$
equilibrium velocity at the inlet	$v_1^*(0)$	50	$\frac{km}{h}$
	$v_2^*(0)$	25	$\frac{km}{h}$

TABLE I
SELECTED VALUES OF PARAMETERS FOR TWO VEHICLE CLASSES.

$$\Gamma_0=\begin{bmatrix}0&0.0469\\0.0156&-0.0625\\0.0332&0.2041\\0&0\end{bmatrix},\Gamma_1=\begin{bmatrix}-0.785\\1.0467\\-4.2039\end{bmatrix},$$
$$\Gamma_2=\begin{bmatrix}0&0.0469\\0&-0.0625\\0.0332&0.2051\end{bmatrix}.$$

The values of parameters K_P, K_I^{11}, K_I^{12}, Γ_0, Γ_1, Γ_2 were obtained by seeking the optimal controller U in paper [12] in order to minimize the likelihood of congested traffic. Letting $\Theta=\begin{bmatrix}0.2\\0.8\end{bmatrix}\times10^{-5},\bar{\Theta}=\begin{bmatrix}1&1&1\end{bmatrix}$, we derive the optimal controller for the linearized system (24)-(25), see Figure 2. Given the initial condition, for

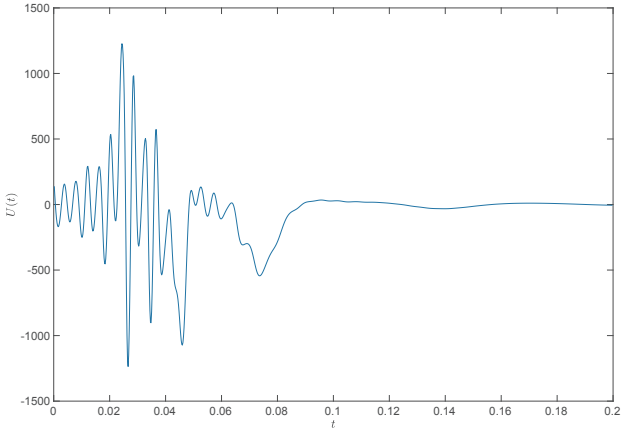


Fig. 2. Time evolution of controller (30).

$x\in[0,L]$

$$u(x,0)=\begin{bmatrix}\rho_1^*(x)+0.05\rho_1^*(x)\cos12\pi x\\v_1^*(x)+0.05v_1^*(x)\sin12\pi x\\\rho_2^*(x)+0.05\rho_2^*(x)\cos12\pi x\\v_2^*(x)+0.05v_2^*(x)\sin12\pi x\end{bmatrix},\tag{55}$$

and $d(t)=\begin{bmatrix}275e^{-30t}\cos30\pi t\\87.5e^{-20t}\cos30\pi t\end{bmatrix}$, illustrated for simulation by rush hours. The optimal controller U not only stabilizes the linearized system (24)-(25), but also stabilizes the quasilinear system (20)-(21). Figure 3 gives the numerical simulations of the quasilinear two vehicle-class traffic model with the controller U. It is obvious that the designed controller for the linearized system stabilizes the quasilinear plant system. However, given a large initial condition

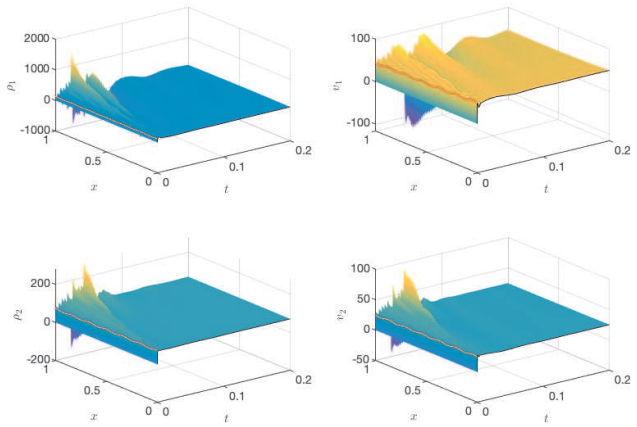


Fig. 3. From left top to right down: time and space evolutions of the state components ρ_1, v_1, ρ_2, v_2 to (1)-(2).

$u(x,0)$, $x \in [0, L]$, namely the initial condition given by (55) by replacing 0.05 by 0.1, that is sufficiently "far" from the equilibrium $u^*(x)$, the corresponding solution to the same closed-loop system does not converge to the equilibrium. It is consistent with the locality of the stability as described in Theorem 1, in other words, the initial state should be in a neighborhood of the equilibrium (in H^2 norm) in order to guarantee the iISS of the multi-type quasilinear traffic system.

V. Conclusion

The problem of boundary stabilization of a multi-type quasilinear traffic flow system with disturbances was solved by actuation at the inlet boundary of the considered road segment. The applicability of the control law, which was designed for the corresponding linearized system by using the backstepping method, has been proven for the locally iISS of the quasilinear system. It would be of interest to extend this result to design an observer and combine them to obtain an observer-based output feedback controller. The extension of this result to other first-order hyperbolic systems would be also interesting.

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