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Calogero-Moser systems with periodic and doubly periodic interaction potentials and loop algebras

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Abstract: For a finite number of particles on the line pairwise interacting with $1/r^2$ potential, the positions are given by the eigenvalues of some time dependant matrix. Infinite periodic or doubly periodic replication of the particles yields Calogero-Moser systems with periodic or doubly periodic interaction potential. We are thus lead to consider matrices of infinite order, which are identified with Fourier series with matrix coefficients, depending on an additionnal parameter. These distributional loops of matrices (tori of matrices in the doubly periodic case) are shown to obey simple (partial) differential equations, which allow us to determine them explicitly. Thus we obtain the already known solution of the Calogero-Moser system on the circle, and give a new insight for the system on the torus.

I. INTRODUCTION

Calogero-Moser systems were introduced long ago and were proven to be completely integrable by Lax pairs techniques, see Ref. 1. The systems with $\frac{1}{r^2}$ and $\frac{1}{\sin^2 r}$ interaction potentials received in Ref. 2 an enlightening geometric interpretation through Marsden-Weinstein reduction of geodesic flows on Lie algebras or Lie groups. Though, the system with a Weierstrass \mathcal{P} function interaction potential remains less understood. The two particle system has been solved in Ref. 3, and the only known integration scheme in the general case was provided by Krichever in Ref. 4, through the solitonic solutions of K-P equation.

The present work starts from the solution of the system with $\frac{1}{r^2}$ interaction potential and proceeds by infinite replication. Calogero and Franoise had previously used a duplication scheme to produce new integrable systems and their solutions from the system on the line in Ref. 5. Here we have in mind the eulerian development of $\frac{1}{\sin^2}$ and proceed by infinite replication by translations of the system on the line. The corresponding matrices are infinite dimensional and are interpreted as distributional elements of some loop algebra. The striking fact is that these distributions are just elementary differential operators. This allows us to give a new proof of the solution of the system with $\frac{1}{\sin^2 r}$ interaction potential.

Then, we apply a doubly infinite replication to investigate the system with Weierstrass \mathcal{P} function interaction potential, which leads us to a partial differential operator on the space of doubly periodic functions with values in \mathbb{C}^n . Though this operator is very simple looking and much similar to the preceding one, it is indeed much more involved to determine its spectrum and this is beyond the author's reach. We connect this problem to some finite difference equation in a space of simply periodic analytic functions.

II. INFINITE REPLICATION

It is known that the Calogero-Moser system on the real line with $\frac{1}{r^2}$ pairwise interaction potential can be integrated in the following way [5].

Let x_1, x_2, \dots, x_n denote the initial positions of the particles,

and y_1, \dots, y_n their initial velocities.

Let

$$A = \begin{pmatrix} ix_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & ix_n \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} iy_1 & \frac{1}{x_2-x_1} & \dots & \frac{1}{x_n-x_1} \\ \frac{1}{x_1-x_2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{x_n-x_{n-1}} \\ \frac{1}{x_1-x_n} & \dots & \frac{1}{x_{n-1}-x_n} & iy_n \end{pmatrix}$$

so that $[A, B] = M = \begin{pmatrix} 0 & -i & \dots & -i \\ -i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -i \\ -i & \dots & -i & 0 \end{pmatrix}$

The position of particles at time t are then the eigenvalues of $A + tB$, divided by i .

We also recall that the positions at time t of the Calogero-Moser on the circle with pairwise interaction potential $1/\sin^2 \frac{r}{2}$ (also called Sutherland system) are the arguments of the eigenvalues of the matrix $\exp(A).\exp(tC)$,

where $C = (c_{k,l})$ with $c_{k,l} = \frac{i}{e^{i(x_l-x_k)}-1}$ for $k \neq l$ and $c_{k,k} = iy_k$,

Our first aim is to show that the solution of the system on the circle can be deduced from a system on the line with infinitely many particles.

Let us recall the Eulerian expansion:

$$\frac{1}{\sin^2 r} = \sum_{k \in \mathbb{Z}} \frac{1}{(r - k\pi)^2}$$

According to this formula, one can write the hamiltonian

$$H = \frac{1}{2} \sum_i y_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{1}{4 \sin^2 \frac{x_j - x_i}{2}}$$

in the form

$$\begin{aligned} H &= \frac{1}{2} \sum_i y_i^2 + \frac{1}{2} \sum_{i \neq j} \sum_{k \in \mathbb{Z}} \frac{1}{(x_j - x_i - 2k\pi)^2} \\ &= \frac{1}{2} \sum_i y_i^2 + \frac{1}{2} \sum_{i \neq j} \lim_{N \rightarrow +\infty} \frac{1}{2N} \sum_{k, l \in [-N, N]} \frac{1}{[(x_j + 2k\pi) - (x_i + 2l\pi)]^2} \end{aligned}$$

Writing f as $f = \begin{pmatrix} \vdots \\ f_{-1} \\ f_0 \\ f_1 \\ \vdots \end{pmatrix}$, one gets

$$\tilde{A}f = \begin{pmatrix} \vdots \\ -2i\pi f_{-1} + Af_{-1} \\ Af_0 \\ 2i\pi f_1 + Af_1 \\ \vdots \end{pmatrix} = f' + Af \text{ where the prime designates}$$

derivation with respect to the variable θ , and

$$\tilde{B}f = \begin{pmatrix} \vdots \\ \dots + P_0 f_{-1} + P_{-1} f_0 + P_{-2} f_1 + \dots \\ \dots + P_1 f_{-1} + P_0 f_0 + P_{-1} f_1 + \dots \\ \dots + P_2 f_{-1} + P_1 f_0 + P_0 f_1 + \dots \\ \vdots \end{pmatrix} = \Lambda f$$

where $\Lambda(\theta) = \dots + P_{-1}e^{-2i\pi\theta} + P_0 + P_1e^{2i\pi\theta} + \dots$

We now proceed to the computation of Λ .

The equation $[\tilde{A}, \tilde{B}] = \tilde{M}$ can be rewritten in the following way:

$$\forall k \in \mathbb{Z} \quad AP_k - P_k(A - 2ik\pi I) = M$$

Multiplying each member by $e^{2ik\pi\theta}$, we get after summation:

$$A \sum_{k \in \mathbb{Z}} P_k e^{2ik\pi\theta} - \sum_{k \in \mathbb{Z}} P_k e^{2ik\pi\theta} A + \sum_{k \in \mathbb{Z}} 2ik\pi P_k e^{2ik\pi\theta} = M \sum_{k \in \mathbb{Z}} e^{2ik\pi\theta}$$

Thus $[A, \Lambda(\theta)] + \Lambda'(\theta) = M\delta_0$

In order to solve this equation on the circle, we first solve the associated equation on the line:

$$[A, \Lambda(\theta)] + \Lambda'(\theta) = M \sum_{k \in \mathbb{Z}} \delta_k \text{ (cf Poisson summation formula)}$$

and we'll retain the 1-periodic solutions.

The associated homogeneous equation is

$$\Lambda'(\theta) = -[A, \Lambda(\theta)]$$

the general solution of which is $\Lambda(\theta) = e^{-\theta A} K e^{\theta A}$ (K constant matrix).

The solution of the complete equation is obtained by varying the constant K , which satisfies the equation $e^{-\theta A} K' e^{\theta A} = M \sum_{k \in \mathbb{Z}} \delta_k$

Therefore, K is constant on each interval $]k, k+1[$ ($k \in \mathbb{Z}$) and has Heaviside singularities at integer points. All that we need is the value of K on $]0, 1[$. In order to get this value, we just write

$$B = P_0 = \int_0^1 \Lambda(\theta) d\theta = \int_0^1 e^{-\theta A} K e^{\theta A} d\theta$$

Writing $K = (K_{k,l})_{1 \leq k, l \leq n}$, we get

$$e^{-\theta A} K e^{\theta A} = (e^{i\theta(x_l - x_k)} K_{kl})_{1 \leq k, l \leq n}$$

and thus $iy_k = B_{k,k} = K_{k,k}$ and for $k \neq l$

$$\frac{1}{x_l - x_k} = B_{k,l} = \frac{e^{i(x_l - x_k)} - 1}{i(x_l - x_k)} K_{k,l}$$

which yields $K_{k,l} = \frac{i}{e^{i(x_l - x_k)} - 1}$.

Thus K is the matrix C of the Kazhdan-Kostant-Sternberg theory [5].

$$\text{and } \Lambda(\theta) = e^{-\theta A} C e^{\theta A}.$$

Recall that the positions of the particle at time t are, in the finite case, the eigenvalues of $A + tB$ divided by i . We conjecture that in the case at hand, they are the eigenvalues of $\frac{d}{d\theta} + (A + t\Lambda(\theta))id$ (divided by i), acting on the space of 1-periodic functions.

Since Λ is continuous except at integer points where it has Heaviside singularities, we look for the 1-periodic, differentiable except at integer points, but everywhere continuous solutions of this equation.

On $]0, 1[$, this equation reads

$$f' + (A + e^{-\theta A} C e^{\theta A})f = \lambda f$$

Set $f(\theta) = e^{\lambda\theta} e^{-\theta A} g(\theta)$. The preceding differential equation is equivalent to

$$g'(\theta) + tCg(\theta) = 0$$

or

$$f(\theta) = e^{\lambda\theta} e^{-\theta A} e^{-t\theta C} f_0$$

with constant f_0 .

This equation is satisfied by a non zero, 1-periodic and continuous function if only if $e^{-\lambda}$ is an eigenvalue of $e^{-A} e^{-tC}$

Thus, the eigenvalues of $\tilde{A} + t\tilde{B}$ are the (pure imaginary) complex numbers the exponentials of which are eigenvalues of $e^{tC} e^A$, which are the same as the eigenvalues of $e^A e^{tC}$.

This agrees with the solution given in Ref. 2.

IV. DOUBLY PERIODIC POTENTIAL

IV.1. The Operator

We apply the same scheme when the pairwise interaction potential is the Weierstrass \mathcal{P} function,

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(k,l) \neq (0,0)} \left(\frac{1}{(z - 2k\pi - 2il\omega\pi)^2} - \frac{1}{(2k\pi + 2il\omega\pi)^2} \right)$$

We will use to this purpose matrices of infinite order indexed by $(\mathbb{Z} \times \mathbb{Z})^2$ instead of \mathbb{Z}^2 , and the underlying Lie algebra will be the complexification of $u(n)$, that is $gl_n(\mathbb{C})$.

Vectors of $\mathbb{C}^{\mathbb{Z} \times \mathbb{Z}}$ will be identified with \mathbb{C}^n -valued doubly periodic functions (with periods 1 and $1/\omega$ with respect to variables θ and ϕ),

$$f(\theta, \phi) = \sum_{k,l} \xi_{k,l} e^{2ik\pi\theta} e^{2il\pi\omega\phi}$$

The matrix \tilde{A} has now the block decomposition

$$\tilde{A} = (A_{klk'l'})_{k,l,k',l' \in \mathbb{Z}}$$

with $A_{klk'l'} = A + (2ik\pi + 2l\omega\pi)I$ if $k = k'$ and $l = l'$ and $A_{k,l,k',l'} = 0$ otherwise. It is identified with the operator

$$\begin{aligned} (\tilde{A}f)(\theta, \phi) &= Af(\theta, \phi) + \sum_{k,l} \xi_{k,l} (2ik\pi - 2l\omega\pi) e^{2ik\pi\theta} e^{2il\pi\omega\phi} \\ &= Af(\theta, \phi) + \frac{\partial f}{\partial \theta} - \frac{1}{i} \frac{\partial f}{\partial \phi} \\ &= Af(z) + 2 \frac{\partial f}{\partial \bar{z}} \end{aligned}$$

(by writing $z = \theta + i\phi$)

In the same way, we identify \tilde{B} with the multiplication operator $f \mapsto \Lambda f$ where Λ is a doubly periodic $gl_n(\mathbb{C})$ -valued function.

M having the same meaning as before, Λ satisfies the partial derivative equation

$$[A, \Lambda] + 2 \frac{\partial \Lambda}{\partial \bar{z}} = M \delta_O$$

where O denotes the origin of the torus.

As before, we start by solving the equation on \mathbb{C}

$$[A, \Lambda] + 2 \frac{\partial \Lambda}{\partial \bar{z}} = M \sum_{k,l} \delta_{k+il/\omega}$$

rather than the equation on the torus, and then we'll go back to the initial equation.

The solution of the homogeneous equation are the functions of the form

$$z \mapsto e^{-\frac{\bar{z}A}{2}} K e^{\frac{\bar{z}A}{2}}$$

where K is any holomorphic matrix valued entire function.

We solve the complete equation by varying the "constant" K , which satisfies the equation

$$2e^{-\frac{\bar{z}A}{2}} \frac{\partial K}{\partial \bar{z}} e^{\frac{\bar{z}A}{2}} = M \sum_{k,l} \delta_{k+il/\omega}$$

$$\frac{\partial K}{\partial \bar{z}} = \frac{1}{2} \sum_{\Omega \in \mathcal{R}} \delta_{\Omega} e^{\frac{\bar{\Omega}A}{2}} M e^{-\frac{\bar{\Omega}A}{2}}$$

$$\text{where } \mathcal{R} = \mathbb{Z} + \frac{i}{\omega} \mathbb{Z}$$

The solutions of this equation are the meromorphic functions whose poles are simple and are the points of the network \mathcal{R} , the residue at Ω being $\frac{1}{2\pi} e^{\frac{\bar{\Omega}A}{2}} M e^{-\frac{\bar{\Omega}A}{2}}$.

But Λ is also doubly periodic, and therefore

$$\forall \Omega \in \mathcal{R} \quad e^{-(\bar{z}+\bar{\Omega})A/2} K(z+\Omega) e^{(\bar{z}+\bar{\Omega})A/2} = e^{-\bar{z}A/2} K(z) e^{\bar{z}A/2}$$

$$K(z+\Omega) = e^{\bar{\Omega}A/2} K(z) e^{-\bar{\Omega}A/2}$$

Therefore:

the diagonal coefficients of K are pole free (because the diagonal coefficients of M are null) elliptic functions, and thus they are constant. In the same way as above, writing $B = \iint_{[0,1]^2} \Lambda$, we get $K_{k,k} = iy_k$;

and the off-diagonal coefficients $K_{k,l}$ of K are meromorphic functions which satisfy

$$K_{k,l}(z + \Omega) = e^{\bar{\Omega}i(x_k - x_l)/2} K_{k,l}(z)$$

and with simple poles $\Omega \in \mathcal{R}$ and residues $\frac{1}{2i\pi} e^{\bar{\Omega}i(x_k - x_l)}$.

As we prefer to deal with (simply) periodic functions, we set

$$P(z) = e^{-zA/2} K(z) e^{zA/2}$$

One have for all $\Omega \in \mathcal{R}$,

$$\begin{aligned} P(z + \Omega) &= e^{-(z+\Omega)A/2} e^{\bar{\Omega}A/2} K(z) e^{-\bar{\Omega}A/2} e^{(z+\Omega)A/2} \\ &= e^{-(\Omega - \bar{\Omega})A/2} P(z) e^{(\Omega - \bar{\Omega})A/2} \end{aligned}$$

Thus P is periodic with period 1 and

$$P(z + i/\omega) = e^{-i/\omega A} P(z) e^{i/\omega A}$$

The diagonal coefficients of P are those of K . They are constants: $P_{k,k} = iy_k$.

The off-diagonal coefficients of P are meromorphic with simple poles on the network and residue at zero: $\frac{1}{2i\pi}$

They satisfy

$$P_{k,l}(z + 1) = P_{k,l}(z) \text{ and } P_{k,l}(z + i/\omega) = e^{\frac{x_k - x_l}{\omega}} P_{k,l}(z) \quad (1)$$

Let us recall the definition of Riemann's theta functions (see for instance Ref. 6)

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z) = \sum_{m \in \mathbb{Z}} e^{i\pi[\tau(m-a)^2 + 2(m-a)(z-b)]}$$

where $Im(\tau) > 0$ and $a, b \in \mathbb{C}$. This defines an entire function which satisfies the fundamental relation

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z + p\tau + q) = e^{i\pi(-2pz - p^2\tau + 2pb - 2aq)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z)$$

$\forall p, q \in \mathbb{Z}$. and the zeros of $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ are simple and located at the $(a - \frac{1}{2})\tau + b - \frac{1}{2} + \tau b + q$, $p, q \in \mathbb{Z}$.

Set $\tau = i/\omega$. The function $q = \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\vartheta \begin{bmatrix} a' \\ b' \end{bmatrix}}$ satisfies

$$q(z + \frac{i}{\omega}p + q) = e^{i\pi(2\frac{p}{\omega}(b-b') - 2(a-a')q)} q(z)$$

This matches with (1) if we choose a, b, a', b' such that

$$a - a' = 0 \quad \text{and} \quad b - b' = -i \frac{x_k - x_l}{2\pi\omega}$$

and the location of poles matches if $a' = b' = \frac{1}{2}$.

Then $\frac{P_{k,l}}{q}$ is an entire function with at most one simple pole in the fundamental cell (since theta functions have just one simple zero in the fundamental cell), thus it is constant and equal to its value at zero.

Hence,

$$\begin{aligned} P_{k,l}(z) &= \frac{1}{2i\pi} \times \frac{\vartheta' \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0)}{\vartheta \begin{bmatrix} 1/2 \\ \frac{1}{2} - i \frac{x_k - x_l}{2\pi\omega} \end{bmatrix} (0)} \times \frac{\vartheta \begin{bmatrix} 1/2 \\ \frac{1}{2} - i \frac{x_k - x_l}{2\pi\omega} \end{bmatrix} (z)}{\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z)} \\ &= \frac{1}{2i\pi} \times \frac{\vartheta_1'(0)}{\vartheta_1(i \frac{x_k - x_l}{2\pi\omega})} \times \frac{\vartheta_1(z + i \frac{x_k - x_l}{2\pi\omega})}{\vartheta_1(z)} \end{aligned}$$

where we have set $\vartheta_1 = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ (Jacobi's notation).

We have another way for expressing the off-diagonal coefficients of P .

Let $\delta \in]0, 2\pi[$.

Consider the series $f(z) = \sum_{j \in \mathbb{Z}} \frac{e^{-j\delta}}{1 - e^{-2j\pi/\omega} e^{2i\pi z}}$.

This series converges normally on all compact of $\mathbb{C} \setminus \mathcal{R}$ and defines a meromorphic function with period 1 that verifies $f(z + \frac{i}{\omega}) = e^\delta f(z)$. Its only poles are simple and are the points of \mathcal{R} . Its residue at 0 is $\frac{1}{-2i\pi}$.

In the same way, for $\delta \in]-2\pi, 0[$, the series

$f(z) = \sum_{j \in \mathbb{Z}} \frac{e^{-j\delta}}{1 - e^{2j\pi/\omega} e^{-2i\pi z}}$ defines a meromorphic function with period 1 that verifies $f(z + \frac{i}{\omega}) = e^\delta f(z)$. Its residue at 0 is $\frac{1}{2i\pi}$.

Suppose that $x_1 < x_2 < \dots < x_n < x_1 + 2\pi$. Then one can write

$$\begin{aligned} \text{if } k < l \quad P_{k,l}(z) &= \sum_{j \in \mathbb{Z}} \frac{e^{-j(x_k - x_l)/\omega}}{1 - e^{2j\pi/\omega} e^{-2i\pi z}} \\ \text{if } k > l \quad P_{k,l}(z) &= - \sum_{j \in \mathbb{Z}} \frac{e^{-j(x_k - x_l)/\omega}}{1 - e^{-2j\pi/\omega} e^{2i\pi z}} \end{aligned}$$

IV.2. The spectrum

We conjecture that the positions of the particles at time t are the eigenvalues of the operator

$$A + 2\frac{\partial}{\partial \bar{z}} + t\Lambda(z)$$

acting on the space of doubly periodic functions (with periods 1 and i/ω).

Let λ be some (pure imaginary) complex number and f some doubly periodic function satisfying

$$Af + 2\frac{\partial f}{\partial \bar{z}} + t\Lambda f = \lambda f$$

Set $f(z) = e^{\frac{\lambda \bar{z}}{2}} e^{-\bar{z}A/2} g(z)$

The preceding p.d.e. becomes

$$2e^{\frac{\lambda \bar{z}}{2}} e^{-\bar{z}A/2} \frac{\partial g}{\partial \bar{z}} + te^{-\bar{z}A/2} K(z) e^{\bar{z}A/2} e^{\frac{\lambda \bar{z}}{2}} e^{-\bar{z}A/2} g(z) = 0$$

which amounts to

$$2\frac{\partial g}{\partial \bar{z}} + tK(z)g(z) = 0$$

Recall that K is holomorphic except at the points of the network

\mathcal{R} . Set $g(z) = e^{-t\bar{z}K(z)/2} h(z)$.

The preceding equation amounts to

$\frac{\partial h}{\partial \bar{z}} = 0$, except at the points of the network.

Therefore, except at these points, f takes the form

$$f(z) = e^{\lambda \bar{z}/2} e^{-\bar{z}A/2} e^{-t\bar{z}K(z)/2} h(z)$$

where h is holomorphic.

The function f is doubly periodic. This means that for all Ω in the network,

$$f(z + \Omega) = f(z)$$

Since

$$\begin{aligned} f(z + \Omega) &= e^{\lambda(\bar{z} + \bar{\Omega})/2} e^{-\bar{z}A/2} e^{-\bar{\Omega}A/2} e^{-t(\bar{z} + \bar{\Omega})K(z + \Omega)/2} h(z + \Omega) \\ &= e^{\lambda(\bar{z} + \bar{\Omega})/2} e^{-\bar{z}A/2} e^{-t(\bar{z} + \bar{\Omega})K(z)/2} e^{-\bar{\Omega}A/2} h(z + \Omega) \end{aligned}$$

we get

$$h(z + \Omega) = e^{-\lambda\bar{\Omega}/2} e^{\bar{\Omega}A/2} e^{t\bar{\Omega}K(z)/2} h(z)$$

Set then $h(z) = e^{tzK(z)/2} k(z)$ and recall that $K(z + \Omega) = e^{\bar{\Omega}A/2} K(z) e^{-\bar{\Omega}/2}$

The preceding condition reads

$$e^{\bar{\Omega}A/2} e^{t(z + \Omega)K(z)/2} e^{-\bar{\Omega}A/2} k(z + \Omega) = e^{-\lambda\bar{\Omega}/2} e^{\bar{\Omega}A/2} e^{t\bar{\Omega}L(z)/2} e^{tzK(z)/2} k(z)$$

which simplifies to

$$k(z + \Omega) = e^{-\lambda\bar{\Omega}/2} e^{\bar{\Omega}A/2} e^{t(\bar{\Omega} - \Omega)K(z)/2} k(z)$$

Set then $k(z) = e^{-\lambda z/2} e^{zA/2} \phi(z)$. The preceding condition reads thus

$$e^{-\lambda(z + \Omega)/2} e^{(z + \Omega)A/2} \phi(z + \Omega) = e^{-\lambda\bar{\Omega}/2} e^{\bar{\Omega}A/2} e^{t(\bar{\Omega} - \Omega)K(z)/2} e^{-\lambda z/2} e^{zA/2} \phi(z)$$

which simplifies to

$$\phi(z + \Omega) = e^{\lambda(\Omega - \bar{\Omega})/2} e^{-(z + \Omega - \bar{\Omega})A/2} e^{t(\bar{\Omega} - \Omega)K(z)/2} e^{zA/2} \phi(z)$$

Therefore we look for analytic functions on $\mathbb{C} \setminus \mathcal{R}$ satisfying

ϕ is periodic with period 1 and

$$\phi(z + i/\omega) = e^{i\lambda/\omega} e^{-(z + 2i/\omega)A/2} e^{-it/\omega K(z)} e^{zA/2} \phi(z) = e^{i\lambda/\omega} L(z) \phi(z) \quad (2)$$

where $L(z) = e^{-(z + 2i/\omega)A/2} e^{-it/\omega K(z)} e^{zA/2} = e^{-i/\omega A} e^{-it/\omega P(z)}$

We conjecture that the correct condition is that ϕ , which coincides with f on the real axis, should have no singularity there.

Since ϕ is then holomorphic on the domain $|Im(z)| < 1$, we can write $\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j e^{2i\pi j z}$, this expansion being valid in the same domain.

Since $L(x - \frac{i}{2})$ is a smooth 1-periodic function of the real variable x , we can write $L(x - \frac{i}{2}) = \sum_{j \in \mathbb{Z}} L_j e^{2i\pi j x}$.

Condition (2) is satisfied if and only if it is satisfied on the line $\mathbb{R} - \frac{i}{2}$ (because only analytic functions are involved), and this reads:

$$\forall x \in \mathbb{R} \quad e^{-\pi j} \phi_j = e^{i\lambda} \sum_{k \in \mathbb{Z}} L_{j-k} e^{\pi k} \phi_k$$

that is

$e^{-i\lambda} = e^x$ is an eigenvalue of the infinite matrix $T = (T_{j,k})$ (block decomposition) with $T_{j,k} = e^{\pi(j+k)} L_{j-k}$.

Our problem is to express the Fourier expansion of $x \mapsto L(x - i/2)$, which is beyond our reach in the general case.

Though, some remarks can be made.

ϑ_1 is odd, therefore

$$P_{k,l}(-z) = -P_{l,k}(z) \quad \text{if } k \neq l$$

The series expansion of P yields also

$$P_{k,l}(\bar{z}) = -\overline{P_{l,k}(z)}$$

For $x \in \mathbb{R}$, we have for $k \neq l$: $\overline{P_{k,l}(x - i/2)} = -P_{l,k}(\overline{x - i/2}) = -P_{l,k}(x + i/2) = -e^{x_l - x_k} P_{l,k}(x - i/2)$.

and this holds also for $k = l$ since $P_{k,k}$ is a pure imaginary number.

Therefore, $e^{-iA/\omega} P(x - i/2)$ is skew-hermitian, and thus, $-itP/\omega$ is hermitian with respect to the hermitian form with matrix e^{-iA} . Thus so is $e^{-itP/\omega}$ and thus $L = e^{-iA/\omega} e^{-itP/\omega}$ is hermitian (at $x - i/2$ for real x).

Therefore for all integer j ,

$$\begin{aligned} L_j &= \int_0^1 e^{-2i\pi j x} L(x - \frac{i}{2}) dx = \int_0^1 e^{-2i\pi j x} [\bar{L}(x - \frac{i}{2})]^\dagger dx \\ &= \overline{\int_0^1 e^{2i\pi j x} L(x - \frac{i}{2}) dx} = [\bar{L}_{-j}]^\dagger \end{aligned}$$

where the dagger denotes transposition, and thus the infinite matrix T is hermitian.

Numerical simulations confirms our conjectures. Quite surprisingly, finite submatrices of small order of the infinite matrix T give very accurate results. Using a submatrix of order 14, in the two particle case at time 1, we got a four digit agreement with the positions given by Runge-Kutta integration.

CONCLUSION

Our infinite replication scheme has succeeded in re-investigating the Calogero-Moser system with $\frac{1}{\sin^2 r}$ potential, but made the system with elliptic potential more mysterious than ever by connecting it to a new finite difference equation. Many questions remain open. Does this problem have a close form solution? How is it connected with the solution given by Krichever⁶ through K-P equation? And is there a geometric interpretation of the system analogous to the one given by Kazhdan, Kostant and Sternberg⁵ for $\frac{1}{\sin^2 r}$ potential?

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