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ABSTRACT

We investigate the structure of Wigner distribution functions of energy eigenstates of quartic and sextic anharmonic oscillators. The corresponding Moyal equations are shown to be solvable, revealing new properties of Schrödinger eigenfunctions of these oscillators. In particular, they provide alternative approaches to their determination and corresponding eigenenergies, away from the conventional differential equation method, which usually relies on the not so widely known bi- and tri-confluent Heun functions.

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I. INTRODUCTION

Since its introduction in 1932,¹ the Wigner distribution function has fueled a vast amount of research on a variety of topics over a time span of almost a century. The main theoretical question is how quantum mechanics can be consistently expressed in terms of classical phase space, a topic that has been extensively explored by many workers (see, e.g., Refs. 2 and 3). Multiple applications of the Wigner function have arisen in diverse fields such as quantum chemistry, quantum electronics, quantum optics as well as signal processing, speech processing, and biological spectograms (see, e.g., Refs. 4 and 5).

The harmonic oscillator Wigner distribution function has been exactly calculated and widely studied over a period of several decades (see, for example, the early work of Bartlett and Moyal).⁶ However, to the best knowledge of the author, there seems to be no attention given to the standard Q^4 or Q^6 anharmonic oscillators. This work is aimed to bridge this gap and to draw some consequences on the Schrödinger eigenfunctions of these oscillators.

Anharmonic oscillators are non-trivial quantum systems with one degree of freedom Q and conjugate momentum P , satisfying the canonical commutation relation $[Q, P] = i$ (we have for simplicity put $\hbar = 1, m = 1$), with dynamics given by the Hamiltonian

$$H_n = \frac{P^2}{2} + \frac{\omega^2}{2}Q^2 + \frac{1}{2}Q^n \quad \text{with } n = 4, 6. \quad (1)$$

So far, attempts to obtain exact closed form eigenfunctions and energy eigenvalues have not been very successful. The standard Schrödinger approach, which factorizes the “main” asymptotic behavior at infinity as the exponential of some power of the Schrödinger variable x representing the operator Q , has led to differential equations for higher transcendental functions, the Bi-Confluent (BCH) and Tri-Confluent Heun (TCH) functions,⁷ respectively, for the Q^6 and Q^4 oscillators (see, for example, some recent works such as Refs. 8 and 9).

In this paper, we undertake to look at these two problems from the point of view of the Wigner distribution function. In Sec. II, we describe the general structure of the so-called “Moyal bracket” equation and its “partner,” which are partial differential operators in classical phase space. Their forms will be explicitly determined, in Sec. III, for a standard Hamiltonian of the type

$$H(P, Q) = \frac{P^2}{2} + V(Q), \quad (2)$$

where $V(Q)$ is a potential term, which is assumed to be smooth, concave, symmetric $V(Q) = V(-Q)$, and infinite at infinite distances. Sections IV and V are devoted to the Wigner distribution functions of the quartic and the sextic oscillators, their properties, and the use of these results to obtain, at least in principle, the energy eigenvalues and the Schrödinger eigenfunctions. The remarkable point is that the Wigner distribution function leads to unexpected unconventional approaches to reach these energy eigenvalues and Schrödinger eigenfunctions based on integral relations for the quartic and sextic oscillator Schrödinger eigenfunctions. This is our main result. A conclusion follows with some perspectives in view of formulating quantum mechanics in the framework of the continuous representation theory of the canonical commutation relations in Hilbert space, as advocated in Ref. 10.

II. GENERAL FEATURES OF THE “MOYAL-BRACKET” AND ITS “PARTNER” FOR ONE DEGREE OF FREEDOM SYSTEM

Originally, the question raised by Wigner¹ is whether, for a given quantum state of this system $|\psi\rangle$, there is a distribution function on classical phase space $(p, q) \in \mathbb{R}^2$ such that the average value of a quantum observable can be evaluated as an average of the corresponding classical observable with this distribution function. He came up with an answer having the following expression:

$$W_\psi(p, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \psi^* \left(q - \frac{1}{2}\tau \right) e^{-i\tau p} \psi \left(q + \frac{1}{2}\tau \right), \quad (3)$$

where $\psi(x)$ is the Schrödinger wave function of the state $|\psi\rangle$ and $W_\psi(p, q)$ is the corresponding phase space distribution function.

Time evolution was laboriously established by Moyal in his seminal paper of 1949,¹¹ as an equation of motion for time dependent $W_\psi(p, q; t)$ [which contains now time dependent Schrödinger wavefunction $\psi(x, t)$ in Eq. (3)],

$$\frac{\partial}{\partial t} W_\psi(p, q; t) = 2 \sin \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) W_\psi(p, q; t). \quad (4)$$

The right-hand side of this equation is dubbed “Moyal Bracket” of the pair of functions on phase space $[H(p, q), W_\psi(p, q; t)]$, as a generalization of the Poisson bracket and should be understood as follows. The sine function is replaced by the power series expansion of its argument,

$$\frac{1}{2} \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right].$$

The action of partial derivatives

$$\left(\frac{\partial}{\partial q_H}, \frac{\partial}{\partial p_H} \right), \quad \left[\text{respectively,} \quad \left(\frac{\partial}{\partial q_W}, \frac{\partial}{\partial p_W} \right) \right],$$

on $H(p, q)$ [respectively, $(W_\psi(p, q; t))$] is to be first calculated. Then, in the obtained result, one sets $q_H = q_W = q$ as well as $p_H = p_W = p$ so that the end expression is just a function of $(p, q; t)$.

Alternatively, instead of the Moyal bracket, one may use, for example, the \ast -product introduced in Ref. 12. However, in this paper, we choose to keep Moyal’s notation to facilitate the connection to his seminal work.¹¹

To get an idea of how the Moyal bracket works as a sum of odd powers of Poisson brackets, we may write down a few first powers of the Poisson bracket of $H(p, q) W_\psi(p, q; t)$ as follows:

- zeroth order: $H(p, q) W_\psi(p, q; t)$,
- first order (or ordinary Poisson bracket):

$$\begin{aligned} & \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) W_\psi(p, q; t) \\ &= \frac{\partial H(p, q)}{\partial q} \frac{\partial W_\psi(p, q; t)}{\partial p} - \frac{\partial H(p, q)}{\partial p} \frac{\partial W_\psi(p, q; t)}{\partial q}, \end{aligned} \quad (5)$$

- second order:

$$\begin{aligned} & \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] \left(\frac{\partial H(p, q)}{\partial q} \frac{\partial W_\psi(p, q; t)}{\partial p} - \frac{\partial H(p, q)}{\partial p} \frac{\partial W_\psi(p, q; t)}{\partial q} \right) \\ &= \left(\frac{\partial^2 H(p, q)}{\partial q^2} \frac{\partial^2 W_\psi(p, q; t)}{\partial p^2} - 2 \frac{\partial^2 H(p, q)}{\partial q \partial p} \frac{\partial^2 W_\psi(p, q; t)}{\partial q \partial p} + \frac{\partial^2 H(p, q)}{\partial p^2} \frac{\partial^2 W_\psi(p, q; t)}{\partial q^2} \right). \end{aligned} \quad (6)$$

We can then see that, by recursion, the n th order application of the Poisson bracket to a function product $H(p, q) W_\psi(p, q; t)$ is given by

$$\sum_{l=0}^n (-1)^l \frac{n!}{l!(n-l)!} \frac{\partial^n H(p, q)}{\partial p^l \partial q^{n-l}} \frac{\partial^n W_\psi(p, q; t)}{\partial q^l \partial p^{n-l}}. \quad (7)$$

However, it was realized later on that the Moyal equation of motion is not sufficient to determine the structure of the Wigner distribution function (see, e.g., Ref. 2). In particular, for stationary states of the system, the Moyal equation is a time independent equation

$$2 \sin \frac{1}{2} \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) W_\psi(p, q) = 0. \quad (8)$$

Then, in this situation, it has been suggested that the Wigner distribution function also obeys a “partner” equation (see, for example, Ref. 13)

$$2 \cos \frac{1}{2} \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) W_\psi(p, q) = 2\epsilon W_\psi(p, q), \quad (9)$$

where ϵ is the energy eigenvalue of the stationary eigenstate $|\psi\rangle$.

At the moment, there exists some theoretical justifications for the appearance of this partner equation beyond the symmetry $\sin - \cos$ (see Refs. 13–15). From now on, we shall call these two partial differential operators in (p, q) acting on $W_\psi(p, q; t)$,

$$\text{SIN}(p, q) W_\psi(p, q; t) = \left\{ 2 \sin \frac{1}{2} \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) \right\} W_\psi(p, q) \quad (10)$$

and

$$\text{COS}(p, q) W_\psi(p, q; t) = \left\{ 2 \cos \frac{1}{2} \left[\frac{\partial}{\partial q_H} \frac{\partial}{\partial p_W} - \frac{\partial}{\partial p_H} \frac{\partial}{\partial q_W} \right] H(p, q) \right\} W_\psi(p, q), \quad (11)$$

to remind the reader of their basic structure and origin. $\text{SIN}(p, q)$ [respectively, $\text{COS}(p, q)$] are given by the standard sine (respectively cosine) series of powers of the Poisson bracket differential operator (7).

III. GENERAL FEATURES OF CANONICAL HAMILTONIAN DYNAMICS

In this section, we work out the forms of the operators $\text{SIN}(p, q)$ and $\text{COS}(p, q)$, for a typical Hamiltonian

$$H(P, Q) = \frac{P^2}{2} + V(Q), \quad (12)$$

under various changes of variables.

First, in the original (p, q) coordinates, as given by Dahl in Ref. 13, we have

$$\text{SIN}(p, q) = -p \frac{\partial}{\partial q} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m+1)!} \frac{1}{2^{2m}} \frac{\partial^{2m+1} V(q)}{\partial q^{2m+1}} \frac{\partial^{2m+1}}{\partial p^{2m+1}} \quad (13)$$

and

$$\text{COS}(p, q) = p^2 - \frac{1}{4} \frac{\partial^2}{\partial q^2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{1}{2^{2m-1}} \frac{\partial^{2m} V(q)}{\partial q^{2m}} \frac{\partial^{2m}}{\partial p^{2m}}. \quad (14)$$

Next, consider the partial Fourier transform of the time independent Wigner distribution function

$$\tilde{W}_\psi(k, q) = \int_{-\infty}^{\infty} dp e^{ipk} W_\psi(p, q). \quad (15)$$

From Eq. (3), we see that

$$\tilde{W}_\psi(k, q) = \psi\left(q - \frac{k}{2}\right) \psi\left(q + \frac{k}{2}\right). \quad (16)$$

Consequently, the operators $\text{SIN}(p, q)$ and $\text{COS}(p, q)$ in the variables (q, k) through the substitutions

$$\frac{\partial}{\partial p} \rightarrow ik \quad \text{and} \quad p \rightarrow i \frac{\partial}{\partial k} \quad (17)$$

will have the following expressions:

$$\begin{aligned} \text{SIN}(p, q) &= \widetilde{\text{SIN}}(k, q) = \left(-\frac{\partial^2}{\partial q \partial k} + \left(V\left(q + \frac{1}{2}k \right) - V\left(q - \frac{1}{2}k \right) \right) \right), \\ \text{COS}(p, q) &= \widetilde{\text{COS}}(k, q) = -\left(\frac{\partial^2}{\partial k^2} + \frac{1}{4} \frac{\partial^2}{\partial q^2} \right) + \left(V\left(q + \frac{1}{2}k \right) + V\left(q - \frac{1}{2}k \right) \right). \end{aligned} \quad (18)$$

A further change of variables

$$q = \frac{1}{2}(x + y) \quad \text{and} \quad k = (x - y) \quad (19)$$

shows that $\text{SIN}(p, q)$ and $\text{COS}(p, q)$ appear as the difference and sum of Schrödinger Hamiltonians in x and y variables,

$$\begin{aligned} \widetilde{\text{SIN}}(k, q) &= \widetilde{\text{SIN}}(x, y) = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \right) - \left(-\frac{1}{2} \frac{\partial^2}{\partial y^2} + V(y) \right), \\ \widetilde{\text{COS}}(k, q) &= \widetilde{\text{COS}}(x, y) = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \right) + \left(-\frac{1}{2} \frac{\partial^2}{\partial y^2} + V(y) \right). \end{aligned} \quad (20)$$

They both act successively on

$$\widetilde{W}_\psi(k, q) = \psi\left(q - \frac{k}{2} \right) \psi\left(q + \frac{k}{2} \right) = \overline{W}_\psi(x, y) = \psi(x) \psi(y), \quad (21)$$

which is consistent with the fact that these partial differential operators are separable in (x, y) . With these (x, y) variables, one gets back to the more usual description in terms of the density matrix of von Neumann equation (see, for example, Ref. 17).

To uncover new features of the Wigner distribution functions for

$$V_n(Q) = \frac{\omega^2}{2} Q^2 + \frac{1}{2} Q^n, \quad (22)$$

where $n = 4, 6$, we undertake to solve the corresponding equations (8) and (9).

It is evident that we have the commutation relation

$$[\text{SIN}(p, q), \text{COS}(p, q)] = 0. \quad (23)$$

The remarkable feature is that the operator $\text{SIN}(p, q)$ is *superintegrable* for these two anharmonic oscillators. This means that there exists another set of variables for which they are also separable (see, e.g., Ref. 16). We shall go first to this new set of variables to obtain the general form of this solution. Then, we use the partner equation (10), as suggested by many authors to attempt to fully determine the Wigner distribution function. To the best of our knowledge, such computation has not been undertaken so far. Only the standard harmonic oscillator has been studied in details since the early days of the formulation of Moyal and Groenevold, thanks to the possibility to solve fully the two equations (8) and (9).

IV. QUARTIC OSCILLATOR WIGNER DISTRIBUTION FUNCTION AND CONSEQUENCES

In this section, we derive the quartic oscillator Wigner distribution function associated with its time independent eigenstates. As known, the eigenfunctions of the Schrödinger quartic oscillator $\psi(x, \epsilon_n)$ of energy ϵ_n form a real valued orthonormal basis in $\mathcal{L}^2(\mathbb{R})$, with the inner product

$$\int_{\mathbb{R}} dx \psi(x, \epsilon_n) \psi(x, \epsilon_m) = \delta_{n,m}.$$

They are solutions of the Schrödinger equation

$$H_4(x) \psi(x, \epsilon_n) = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{\omega^2}{2} x^2 + \frac{1}{2} x^4 \right) \psi(x, \epsilon_n) = \epsilon_n \psi(x, \epsilon_n). \quad (24)$$

As parity is a good quantum number, there are two disjoint sets of eigenfunctions that are, respectively, even and odd under space reflection around the coordinate origin. The spectrum of $H_4(x)$ is real, discrete, non-degenerate, and positive for $\omega \in \mathbb{R}$.

A. Time independent Moyal's equation

The time independent Moyal's equation for a state $|\psi\rangle$ is

$$\text{SIN}(p, q) W_\psi(p, q) = 0,$$

where

$$\text{SIN}(p, q) = \left(-p \frac{\partial}{\partial q} + (\omega^2 q + 2q^3) \frac{\partial}{\partial p} - \frac{q}{2} \frac{\partial^3}{\partial p^3} \right). \quad (25)$$

We now show that the operator $\text{SIN}(p, q)$ is separable in two distinct sets of variables. First, in the change of variables $(p, q) \rightarrow (k, q)$, the $\text{SIN}(p, q)$ operator becomes $\widetilde{\text{SIN}}(k, q)$,

$$\widetilde{\text{SIN}}(k, q) = (-i) \left(\frac{\partial^2}{\partial k \partial q} - (\omega^2 + 2q^2) q k - \frac{1}{2} q k^3 \right). \quad (26)$$

It acts on functions $\widetilde{W}_\psi(k, q)$, defined by

$$\widetilde{W}_\psi(k, q) = W_\psi(p, q). \quad (27)$$

Now, we introduce new variables (u, v) defined by

$$u = \left(q^2 + \frac{k^2}{4} \right), \quad v = \left(q^2 - \frac{k^2}{4} \right). \quad (28)$$

Then, Eq. (10) turns into

$$\widehat{\text{SIN}}(u, v) \widehat{W}_\psi(u, v) = 0, \quad (29)$$

since $\widetilde{\text{SIN}}(k, q) = \widehat{\text{SIN}}(u, v)$ and $\widetilde{W}_\psi(k, q) = \widehat{W}_\psi(u, v)$, with

$$\widehat{\text{SIN}}(u, v) = i\sqrt{u^2 - v^2} \left\{ \left(-\frac{\partial^2}{\partial u^2} + 2u + \omega^2 \right) - \left(-\frac{\partial^2}{\partial v^2} \right) \right\}. \quad (30)$$

It is clear that an elementary solution of Eq. (29) is of the form

$$\begin{pmatrix} \cosh zv \\ \sinh zv \end{pmatrix} \text{Ai} \left(\gamma u + \frac{z^2 + \omega^2}{\gamma^2} \right), \quad (31)$$

made up from products of the Airy function of first kind (decaying toward 0 at $+\infty$) and hyperbolic functions ($\cosh x$ or $\sinh x$ according to even/odd requirement), in which z is a real free parameter.

The corresponding $\text{COS}(p, q)$ operator in (u, v) variables is

$$\widehat{\text{COS}}(u, v) = -\frac{1}{2} \left(\frac{\partial}{\partial u} + u \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2v \frac{\partial^2}{\partial u \partial v} \right) + \frac{1}{2} (\omega^2 u + (4u^2 - 2v^2)). \quad (32)$$

This form is not very useful, but we give it for completeness. As we shall see, it will be more convenient to use the form of $\widehat{\text{COS}}(u, v)$ in (x, y) variables because it is simply the sum of the two Schrödinger Hamiltonians of the quartic oscillator, as shown in Eq. (20).

B. General form of the Wigner distribution function

This general solution $W_\psi(u, v)$ of the time independent Moyal equation in the variables (u, v) [Eqs. (29) and (30)] is then a linear superposition of elementary solutions over $z \in \mathbb{R}$,

$$\widehat{W}_\psi(u, v) = \int_{-\infty}^{\infty} dz \phi(z) \begin{pmatrix} \cosh zv \\ \sinh zv \end{pmatrix} \text{Ai} \left(\gamma u + \frac{z^2 + \omega^2}{\gamma^2} \right), \quad (33)$$

where $\gamma^3 = 2$, the superposition parameter is $z \in \mathbb{R}$, and $\phi(z)$ is a yet unknown superposition density.

Now, it is illuminating to go back, respectively, to (k, q) variables and then to (x, y) variables in this solution

$$\widehat{W}_\psi(u, v) = \widetilde{W}_\psi(k, q) = \overline{W}_\psi(x, y) = \psi(x) \psi(y). \quad (34)$$

Then, re-expressing (u, v) in terms of (x, y) ,

$$u = \frac{x^2 + y^2}{2} \quad \text{and} \quad v = xy, \tag{35}$$

we end up with a nice symmetric integral expression in terms of quartic anharmonic oscillator Schrödinger eigenfunctions,

$$\psi(x, \epsilon) \psi(y, \epsilon) = \int_{-\infty}^{\infty} dz \phi(z) \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai} \left(\frac{x^2 + y^2 + z^2 + \omega^2}{y^2} \right). \tag{36}$$

It is evident that the choice of the factor $\cosh xyz$ (respectively, $\sinh xyz$) is dictated by the parity of the eigenfunction $\psi(x)$ [or equivalently $\psi(y)$].

C. Determination of the superposition density $\phi(z)$

By assumption, $\psi(x, \epsilon_n)$ is an eigenfunction of the Schrödinger Hamiltonian $H_4(x)$ [see Eq. (24)]. Then, we have the following theorem:

Theorem 4.1. *As $\psi(x)$ [or $\psi(y)$] is an eigenfunction of the Schrödinger Hamiltonian $H_4(x)$, $\phi(z)$ turns out to be also an eigenfunction of $H_4(x)$,*

$$H_4(x)\phi(z) = \left(-\frac{1}{2} \frac{d^2}{dz^2} + \frac{\omega^2}{2} z^2 + \frac{1}{2} z^4 \right) \phi(z) = \epsilon \phi(z). \tag{37}$$

This implies that

$$\phi(z) = \mu(\epsilon_n) \psi(z, \epsilon_n), \tag{38}$$

where $\mu(\epsilon_n)$ is a proportionality constant. Consequently, we have the integral identity

$$\psi(x, \epsilon_n) \psi(y, \epsilon_n) = \mu(\epsilon_n) \int_{-\infty}^{\infty} dz \psi(z, \epsilon_n) \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai} \left(\frac{x^2 + y^2 + z^2 + \omega^2}{y^2} \right). \tag{39}$$

Thus, the $\psi(x, \epsilon_n)$ verifies an integral identity of the type verified by zonal functions such as Legendre polynomials (see Ref. 18). This identity has been established way back in 1974 in a different context (see Refs. 19 and 20). It has been recently rediscovered by Hallnäs and Langmann in Ref. 21, who were not aware of Refs. 19 and 20. However, the underlying group structure is yet to be determined. It must be pointed out that this structure has emerged from an entirely different context, which is that of continuous representation of the canonical commutation relations pioneered by Klauder.²² There are some (deceptive) similarities between the two formalisms even if the final result is the same. The careful reader can certainly notice that the key formulas for a change of crucial variables are not the same: here, e.g., Eq. (28) is different from Eq. (IV 2a) of Ref. 20. We plan to investigate this problem in a future work.

Proof. The proof is based on three technical lemmas. □

Lemma 4.2. *The multiplicative action of the polynomial $z^2(x^2 + y^2)$ on the factor $\text{Ai}(T)$ can be written as a differential operator in z as*

$$-\frac{1}{2} z^2 (x^2 + y^2) \text{Ai}(T) = \frac{1}{2} \left(-\frac{d^2}{dz^2} + z^2(\omega^2 + z^2) + \frac{d}{z dz} \right) \text{Ai}(T), \tag{40}$$

where

$$T = \left(\frac{\omega^2 + x^2 + y^2 + z^2}{y^2} \right)$$

and $y^3 = 2$.

Proof. Calculate the right-hand side of Eq. (40), and the result is exactly the left-hand side. □

Lemma 4.3. *The action of the quartic Hamiltonian in x on the elementary solution of the Moyal bracket operator in the variables (x, y, z) is as follows:*

$$\begin{aligned}
 H_4(x) \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai}(T) &= -\frac{1}{2} \{z^2(x^2 + y^2) + x^2y^2\} \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai}(T) \\
 &+ \left\{ -xyz \begin{pmatrix} \sinh xyz \\ \cosh xyz \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \right\} \gamma \text{Ai}'(T),
 \end{aligned} \tag{41}$$

where $\text{Ai}'(T)$ is the derivative of $\text{Ai}(T)$ with respect to T .

Proof. Calculate the second derivative on the product of the hyperbolic function with $\text{Ai}(T)$ and regroup terms in $\text{Ai}(T)$ as well as terms in $\text{Ai}'(T)$. \square

Lemma 4.4. A straightforward identity is

$$\frac{d^2}{dx^2} \begin{pmatrix} \sinh xyz \\ \cosh xyz \end{pmatrix} = y^2 z^2 \begin{pmatrix} \sinh xyz \\ \cosh xyz \end{pmatrix}. \tag{42}$$

Proof. Apply $H_4(x)$ to the left-hand side of Eq. (36). Its action under the z -integration sign can be rewritten successively, thanks to the three previous lemmas as $H_4(z)$ acting now on $\phi(z)$. This implies then that $\phi(z, \epsilon_n)$ is proportional to $\psi(z, \epsilon_n)$. \square

Collecting the statements of these lemmas, it is easy to see that Eq. (39) is established.

D. Consequences of the Schrödinger eigenfunction normalization $\|\psi\|^2 = 1$

In Eq. (39), where $\psi(x)$ is understood as $\psi(x, \epsilon_n)$, set $x = y$ and integrate over $x \in \mathbb{R}$. This yields

$$\int_{\mathbb{R}} dx \psi(x, \epsilon_n) F_0(x) = \frac{1}{\mu_n}, \tag{43}$$

where $\mu_n = \mu(\epsilon_n)$ for shortness and

$$F_0(x) = \int_{\mathbb{R}} du \begin{pmatrix} \cosh xu^2 \\ \sinh xu^2 \end{pmatrix} \text{Ai}\left(\gamma u^2 + \frac{x^2 + \omega^2}{y^2}\right). \tag{44}$$

This implies that

$$F_0(x) = \sum_{n=0}^{\infty} \frac{1}{\mu_n} \psi(x, \epsilon_n). \tag{45}$$

We now derive a homogeneous integral equation for the eigenfunctions $\psi_n(x)$ with the kernel

$$K(x, y) = \sum_{n=0}^{\infty} \frac{1}{\mu_n^2} \psi(x, \epsilon_n) \psi(y, \epsilon_n). \tag{46}$$

Divide both sides of Eq. (39) by μ_n and sum over $n \in \mathbb{N}$. The left-hand side is just the kernel $K(x, y)$ and the right-hand side is its expression,

$$K(x, y) = \int_{\mathbb{R}} dz F_0(z) \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai}\left(\frac{x^2 + y^2 + z^2 + \omega^2}{y^2}\right). \tag{47}$$

$K(x, y)$ is evidently positive, real, symmetric, and square integrable.

Moreover, its trace is

$$\text{Tr } K = \int_{\mathbb{R}} dx K(x, x) = (F_0, F_0) < \infty. \tag{48}$$

Let $F_m(x)$ be

$$F_m(x) = \int_{\mathbb{R}} dy K^m(x, y) F_0(y) = \sum_{n=0}^{\infty} \frac{1}{\mu_n^{2m+1}} \psi(x, \epsilon_n). \tag{49}$$

Iterated kernels K^m for $m = 1, 2, \dots$ can be seen to have this simple expression

$$K^m(x, y) = \int_{\mathbb{R}} dz F_{m-1}(z) \begin{pmatrix} \cosh xyz \\ \sinh xyz \end{pmatrix} \text{Ai}\left(\frac{x^2 + y^2 + z^2 + \omega^2}{y^2}\right), \quad (50)$$

which corresponds to

$$K^m(x, y) = \sum_{n=0}^{\infty} \frac{1}{\mu_n^{2m}} \psi(x, \epsilon_n) \psi(y, \epsilon_n). \quad (51)$$

Consequently, we also have

$$\text{Tr } K^m = \int_{\mathbb{R}} dx K(x, x) = (F_{m-1}, F_0) < \infty. \quad (52)$$

The set of $\{F_m(x)\}$ for $m \in \mathbb{N}$ form a non-orthogonal set in $\mathcal{L}^2(\mathbb{R})$ and plays a key role in the search for a solution of the quartic oscillator problem. Equation (52) may be used to normalize $K^m(x, y)$ in Eq. (51), and the result may be viewed as a particular m^{th} -order density matrix of the quartic oscillator.

E. Toward solving the quartic anharmonic oscillator problem

The integral identity verified by the Schrödinger eigenfunction of the quartic anharmonic oscillator opens the way to a new approach leading to the spectrum and stationary wavefunctions of this oscillator via an integral formalism in contrast to the standard differential approach.

Then, using Eq. (49), we may construct the following infinite numerical system of equations in μ_n :

$$\begin{aligned} (F_0, F_0) &= \frac{1}{\mu_0^2} + \frac{1}{\mu_2^2} + \dots + \frac{1}{\mu_n^2} + \dots, \\ (F_1, F_0) &= \frac{1}{\mu_0^4} + \frac{1}{\mu_2^4} + \dots + \frac{1}{\mu_n^4} + \dots, \\ &\vdots \\ (F_{n-1}, F_0) &= \frac{1}{\mu_0^{2n}} + \frac{1}{\mu_2^{2n}} + \dots + \frac{1}{\mu_n^{2n}} + \dots, \\ &\vdots \end{aligned} \quad (53)$$

The left-hand side of these equations is known; therefore, these equations can be, in principle, solved to get the μ_n . Of course, this task will require some technical steps from systems of infinite equations, in particular, infinite systems of Van der Monde equations.

By construction, $F_m(x)$ in Eq. (49) has the expansion in terms of the $\psi(x, \epsilon_n)$ given by

$$F_m(x) = \sum_{n=0}^{\infty} \frac{1}{\mu_n^{2m+1}} \psi(x, \epsilon_n), \quad (54)$$

where $m = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$. As the integral equation eigenvalues μ_n are determined, one can proceed to obtain the Schrödinger eigenfunctions $\psi(x, \epsilon_n)$ by inverting the linear infinite system of Eq. (54). Again we must deal with an infinite system of linear equations, so this procedure is subjected to the ability to find the actual way of solving an infinite system of linear equations.

Next, it is also possible to obtain the Schrödinger energy eigenvalues ϵ_n related to $\psi_n(x)$ by solving the infinite linear system of equations once the μ_n have been determined,

$$(F_{m-1}, HF_0) = \sum_{n=0}^{\infty} \frac{1}{\mu_n^{2m}} \epsilon_n \quad \text{for } m \in \mathbb{N}. \quad (55)$$

So, provided there exists a practical way to solve these three sets of infinite equations (59)–(61), the problem of the quartic oscillator can be considered as solved. The Wigner function approach allows a new look at the problem by transferring the old differential problem into an integral problem via a remarkable functional identity for the Schrödinger eigenfunctions $\psi(x, \epsilon_n)$. The present approach does lead to the correct asymptotic behavior with the factor $\text{Ai}\left(\frac{\omega^2 + x^2 + y^2 + z^2}{y^2}\right)$ and does suggest that $\psi(x, \epsilon_n)$ is some linear combination of the $F_m(x)$ functions, which are all defined by integrals and do show the correct asymptotic behavior at infinity.

Remark. [Appendix A](#) gives a short review of the conventional approach to the stationary Schrödinger quartic oscillator problem by detailing its connection to the so-called Tri-Confluent Heun (TCH) function. Identity (39) may be re-expressed as an identity for (TCH) functions.

V. SEXTIC OSCILLATOR WIGNER DISTRIBUTION FUNCTION AND CONSEQUENCES

Consider now the sextic oscillator Schrödinger problem with Hamiltonian $H_6(x)$,

$$H_6(x)\psi(x, \epsilon_n) = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{\omega^2}{2} x^2 + \frac{1}{2} x^6\right)\psi(x, \epsilon_n) = \epsilon_n \psi(x, \epsilon_n). \quad (56)$$

The real eigenfunctions $\psi(x, \epsilon_n)$ ($n \in \mathbb{N}$) form a $\mathcal{L}^2(\mathbb{R})$ orthonormal basis with the standard inner product. They split into two subsets of x -even and x -odd functions. The aim of this section is to derive the Wigner distribution function for the sextic oscillator. The approach is analogous to Sec. IV E.

A. Time independent Moyal's equation

The computed Moyal bracket operator for the sextic oscillator on phase space (p, q) is

$$\text{SIN}(p, q) = -p \frac{\partial}{\partial q} + (\omega^2 + 3q^4)q \frac{\partial}{\partial p} - \frac{5}{2}q^3 \frac{\partial^3}{\partial p^3} + \frac{3}{16}q \frac{\partial^5}{\partial p^5}. \quad (57)$$

Now, by going to the variables (k, q) , the $\text{SIN}(p, q)$ operator becomes $\widehat{\text{SIN}}(k, q)$,

$$\widehat{\text{SIN}}(k, q) = (-i) \left(\frac{\partial^2}{\partial k \partial q} - (\omega^2 + 3q^4)qk - \frac{5}{2}q^3k^3 - \frac{3}{16}qk^5 \right). \quad (58)$$

It acts on functions $\widetilde{W}_\psi(k, q)$, defined by

$$\widetilde{W}_\psi(k, q) = W_\psi(p, q). \quad (59)$$

We now go over the variables (u, v) , already introduced previously [see Eq. (28)],

$$u = \left(q^2 + \frac{k^2}{4} \right), \quad v = \left(q^2 - \frac{k^2}{4} \right), \quad (60)$$

and then, Moyal's equation (10) for the sextic oscillator takes the form

$$\widehat{\text{SIN}}(u, v) \widehat{W}_\psi(u, v) = 0, \quad (61)$$

since $\widehat{\text{SIN}}(k, q) = \widehat{\text{SIN}}(u, v)$ and $\widetilde{W}_\psi(k, q) = \widehat{W}_\psi(u, v)$, with

$$\widehat{\text{SIN}}(u, v) = i\sqrt{u^2 - v^2} \left\{ \left(-\frac{\partial^2}{\partial u^2} + 4u^2 \right) - \left(-\frac{\partial^2}{\partial v^2} + v^2 \right) + \omega^2 \right\}. \quad (62)$$

The partner time independent $\widehat{\text{COS}}(u, v)$ operator is

$$\widehat{\text{COS}}(u, v) = -\frac{1}{2} \left(\frac{\partial}{\partial u} + u \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2v \frac{\partial^2}{\partial u \partial v} \right) + \frac{1}{2} (\omega^2 u + u(4u^2 - 3v^2)). \quad (63)$$

It is given for completeness and shall not be used.

B. General form of the Wigner distribution function

Thus, the time independent Moyal's equation for the sextic oscillator is therefore doubly *separable* in (u, v) and in (x, y) coordinates. This makes this equation *superintegrable* in the sense of Ref. 16. It admits elementary solutions of the form of a product of Weber (or parabolic cylinder) functions $D_\lambda(z)$ (see Ref. 24), decreasing at infinity ($z \rightarrow \infty$), solution of

$$\left(\frac{d^2}{dz^2} - \frac{z^2}{4} + \left(\lambda + \frac{1}{2} \right) \right) D_\lambda(z) = 0. \quad (64)$$

Consequently, an elementary solution of Eq. (61) is of the form $D_\nu(2u) D_\lambda(\sqrt{2}v)$ subjected to the condition

$$\lambda = 2\nu + \beta^*, \tag{65}$$

where $\beta^* = \frac{1}{2}(1 + \omega^2)$. Thus, the general solution of Eq. (29) is a linear superposition of elementary solutions $D_\nu(2u) D_\lambda(\sqrt{2}v)$.

As we move later to the Schrödinger variables (x, y) , via Eq. (35), space reflection symmetry in (x, y) for the Schrödinger eigenfunction should be respected. This requires λ to be a non-negative integer $\lambda = m \in \mathbb{N}$. $D_\lambda(v\sqrt{2})$ becomes a Hermite function $D_m(xy\sqrt{2})$,³⁰ which fulfills this requirement. Consequently, $\nu = \nu(m) = \frac{1}{2}(m - \beta^*)$ and

$$\widehat{W}_\psi(u, v) = \sum_{m \in \mathbb{N}} C_m D_{\nu(m)}(2u) D_m(\sqrt{2}v), \tag{66}$$

where C_m are the superposition coefficients.

Going to (x, y) variables, we have now

$$\psi(x, \epsilon_n) \psi(y, \epsilon_n) = \sum_{m \in \mathbb{N}} C_m(\epsilon_n) D_{\nu(m)}(x^2 + y^2) D_m(xy\sqrt{2}), \tag{67}$$

in which $C_m = C_m(\epsilon_n)$ depends on ϵ_n , the eigenvalue of $H_6(x)$ for the eigenfunction $\psi_n(x)$.

It remains to determine the linear superposition coefficient $C_m(\epsilon_n)$, which may be done, in principle, by making use of the “partner” operator (63). However, as in the case of the quartic oscillator, this form is cumbersome and we shall turn to its variant in the (x, y) variables in Subsection V C.

C. Determination of the superposition coefficients $C_m(n)$

Analogously to the case of the quartic oscillator, we shall apply the sextic anharmonic Schrödinger Hamiltonian $H_6(x)$ to both sides of Eq. (67) to determine the $C_m(\epsilon_n)$. The result is given by the following theorem:

Theorem 5.1. *The superposition coefficients $C_m(\epsilon_n)$ of Eq. (67) fulfill the following recursion relation:*

$$2(m + 1)(m + 2) C_{m+2}(\epsilon_n) + 2\epsilon C_m(\epsilon_n) + [m + (2 - \beta^*)] C_{m-2}(\epsilon_n) = 0, \tag{68}$$

with the two first terms being given, respectively, for n even and odd.

The proof of this theorem relies on the following two technical lemmas:

Lemma 5.2. *The action of $H_6(x)$ on an elementary solution of the Moyal bracket is given by*

$$\begin{aligned} & H_6(x) D_{\nu(m)}(x^2 + y^2) D_m(xy\sqrt{2}) \\ &= \left[-x^2 y^2 + \left(m + \frac{1}{2}\right) \right] (x^2 + y^2) D_{\nu(m)}(x^2 + y^2) D_m(xy\sqrt{2}) \\ &+ D'_{\nu(m)}(x^2 + y^2) \left(D_m(\sqrt{2}xy) + 2\sqrt{2}xy D'_m(xy\sqrt{2}) \right), \end{aligned} \tag{69}$$

where $D'_\lambda(z)$ is the z -derivative of $D_\lambda(z)$.

Proof. The proof consists in computing the second x -derivative of the product $D_{\nu(m)}(x^2 + y^2) D_m(xy\sqrt{2})$ and reordering the result. \square

The next step consists in transforming this result using the recursion relations of the parabolic cylinder functions, see Ref. 24. First, we apply the following relation:

$$z D_\lambda(z) = D_{\lambda+1} + \lambda D_{\lambda-1}(z), \tag{70}$$

respectively, on the variable $(x^2 + y^2)$ and then on $(xy\sqrt{2})$, and the first term in the right-hand side of Eq. (69) can be written now as

$$\begin{aligned} & \left[-x^2 y^2 + \left(m + \frac{1}{2}\right) \right] (x^2 + y^2) D_{\nu(m)}(x^2 + y^2) D_m(xy\sqrt{2}) \\ &= -\frac{1}{2} [D_{\nu(m)+1}(x^2 + y^2) + \nu(m) D_{\nu(m)-1}(x^2 + y^2)] [D_{m+2}(xy\sqrt{2}) + m(m - 1) D_{m-2}(xy\sqrt{2})]. \end{aligned} \tag{71}$$

Then, using the second recursion relation for the parabolic cylinder function (see also in Ref. 24),

$$2 D'_\lambda(z) = \lambda D_{\lambda-1}(z) - D_{\lambda+1}(z), \tag{72}$$

combined with the first recursion relation, one can put the second term in the right-hand side of Eq. (69) under the form

$$\begin{aligned} & D'_{v(m)}(x^2 + y^2) \left(D_m(xy\sqrt{2}) + 2\sqrt{2}xy D'_m(xy\sqrt{2}) \right) \\ &= \frac{1}{2} \left(v(m) D_{v(m)-1}(x^2 + y^2) - D_{v(m)+1}(x^2 + y^2) \right) \\ &+ \left(D_m(xy\sqrt{2}) + (xy\sqrt{2}) [m D_{m-1}(xy\sqrt{2}) - D^{m+1}(xy\sqrt{2})] \right). \end{aligned} \tag{73}$$

Putting these results together, we arrive at the following result:

$$\begin{aligned} H_6(x) \psi(x\epsilon_n) \psi(y, \epsilon_n) &= - \sum_{m=0}^{\infty} C_m(\epsilon_n) \\ &\times \left\{ m(m-1) D_{v(m)+1}(x^2 + y^2) D_{m-2}(xy\sqrt{2}) + v(m) D_{v(m)-1}(x^2 + y^2) D_{m+2}(xy\sqrt{2}) \right\}, \end{aligned} \tag{74}$$

where $v(m) = \frac{1}{2}(m - \beta^*)$ and $\beta^* = \frac{1}{2}(1 + \omega^2)$, as given by Eq. (65).

Proof of Theorem 5.1. Since

$$H_6(x) \psi(x, \epsilon_n) \psi(y, \epsilon_n) = \epsilon_n \psi(x, \epsilon_n) \psi(y, \epsilon_n), \tag{75}$$

we get, after shifting and relabeling the m -summation in Eq. (74), the result of Eq. (68).

D. Toward the solution of the sextic oscillator via the three-way recursion relation

As we have seen, the solution of time independent Moyal's equation gives an expression of the product of Schrödinger eigenfunctions $\psi(x, \epsilon_n) \psi(y, \epsilon_n)$ of the sextic oscillator as a discrete sum over a product of a Hermite function of $(xy\sqrt{2})$ and a parabolic cylinder function of $(x^2 + y^2)$. The asymptotic behavior of $\psi(x, \epsilon_n) \psi(y, \epsilon_n)$ is correctly described by this function product. The main point is that the old Schrödinger differential equation problem is now shifted to the determination a discrete infinite sequence of coefficients $C_m(\epsilon_n)$ satisfying an infinite set of linear homogeneous equations (68).

A few observations are in order.

- (a) The existence of a nontrivial solution $C_m(\epsilon_n)$ requires that the infinite determinant of this system of linear homogeneous equations should vanish, i.e.,

$$\begin{vmatrix} 2\epsilon & 0 & 2.1.2 & 0 & 0 & \dots \\ 0 & 2\epsilon & 0 & 2.2.3 & 0 & \dots \\ (4 - \beta^*) & 0 & 2\epsilon & 0 & 2.3.4 & \dots \\ 0 & (4 - \beta^*) & 0 & 2\epsilon & 0 & 2.4.5 \\ 0 & 0 & (6 - \beta^*) & 0 & 2\epsilon & 0 \\ \vdots & \vdots & \vdots & \ddots & & \end{vmatrix} = 0. \tag{76}$$

In principle, this equation yields the energy eigenvalues ϵ_n of the sextic oscillator.

- (b) As a solution of a linear homogeneous system of equations, the $C_m(\epsilon_n)$ depend on one constant, which can be computed using the normalization condition $\|\psi_n\|^2 = 1$.

E. An identity for the Schrödinger eigenfunction product analogous to the one for the quartic oscillator

As generally known, the $C_m(\epsilon)$, obeying a three-way recursion relation, is a polynomial of the real variable ϵ with the two first ones $C_0(\epsilon)$ and $C_1(\epsilon)$ given by “boundary” conditions. One possible way to obtain these polynomials makes use of their generating function $G(t, \epsilon)$ defined by

$$G(t, \epsilon) = \sum_{m=0}^{\infty} t^m C_m(\epsilon). \tag{77}$$

Then, for given ϵ , one can retrieve $C_m(\epsilon)$ through Cauchy's integral

$$C_m(\epsilon) = \frac{1}{2i\pi} \oint \frac{dt}{t^{m+1}} G(t, \epsilon), \tag{78}$$

with $G(t, \epsilon)$ given by the following lemma:

Lemma 5.3. The generating function $G(t, \epsilon)$ of the polynomials $C_m(\epsilon)$ verifies the differential equation

$$2 \frac{d^2}{dt^2} G(t, \epsilon) + t^3 \frac{d}{dt} G(t, \epsilon) + \{(4 - \beta^*)t^2 + 2\epsilon\} G(t, \epsilon) = 0. \tag{79}$$

Proof. Multiply the recursion relation by t^m and sum over m . Then, identify in the sum terms that arise from the first and second t -derivatives of $G(t, \epsilon)$. Finally, regroup the terms to obtain Eq. (79).

Using (78), we can now put the expansion (66) under the form

$$\psi(x, \epsilon_n) \psi(y, \epsilon_n) = \frac{1}{2i\pi} \oint \frac{dt}{t} G(t, \epsilon_n) \mathcal{K}(x^2 + y^2, xy\sqrt{2}; t), \tag{80}$$

where

$$\mathcal{K}(x^2 + y^2, xy\sqrt{2}; t) = \sum_{m \in \mathbb{N}} \frac{1}{t^m} D_{v(m)}(x^2 + y^2) D_m(xy\sqrt{2}). \tag{81}$$

Equation (80) [with (81)] has the same structure as Eq. (36). The quartic oscillator superposition density $\phi(z)$ is finally found to be proportional to the eigenfunction $\psi(z, \epsilon)$ of $H_4(x)$. Here, for the sextic oscillator, as shown by Eq. (80), it is $G(t, \epsilon_n)$, which is the superposition density. However, the three-way permutation symmetry between (x, y, t) no longer exists in Eq. (80). Only a permutation symmetry between (x, y) persists. Yet we do have the following remarkable result on the nature of $G(z, \epsilon_n)$:

Proposition 5.4. The generating function $G(t, \epsilon_n)$ of the polynomials $C_m(\epsilon_n)$ is related to the eigenfunction $\psi(z, \underline{\epsilon})$ of a sextic oscillator with parameter $\underline{\omega}^2$ and eigenenergy $\underline{\epsilon}$ by

$$G(t, \epsilon_n) = \underline{\mu}_n \exp\left(\frac{z^4}{4}\right) \psi(z, \underline{\epsilon}_n), \tag{82}$$

where $\underline{\mu}_n$ is a proportionality factor and $\underline{\epsilon}_n = \pm i\epsilon_n$.

Proof. In Appendix B, we have shown that the eigenfunction $\psi(x, \underline{\epsilon}_n)$ of a sextic oscillator Hamiltonian with parameter $\underline{\omega}^2$ and energy $\underline{\epsilon}_n$ is of the form $\psi(x, \underline{\epsilon}_n) = \exp\left(-\frac{x^4}{4}\right) \phi(x, \underline{\epsilon}_n)$, where $\phi(x, \underline{\epsilon}_n)$ verifies the equation

$$-\frac{d^2}{dx^2} \phi(x, \underline{\epsilon}) + 2x^3 \frac{d}{dx} \phi(x, \underline{\epsilon}) + ((\underline{\omega}^2 + 3)x^2 - 2\underline{\epsilon}) \phi(x, \underline{\epsilon}) = 0. \tag{83}$$

Under the scaling transformation $t = \underline{\lambda} x$, $G(t, \epsilon_n) = \underline{G}(z, \pm i\epsilon_n)$ and Eq. (79) takes the form of Eq. (83) (see Appendix B), provided that $\underline{\lambda}^4 = -4$. This allows us to make the identification

$$(\underline{\omega}^2 + 3) = (8 - 2\beta^*) \quad \text{or} \quad \underline{\omega}^2 + \omega^2 = 4. \tag{84}$$

as well as $\underline{\epsilon} = \pm i\epsilon$. Hence Eq. (82) is established. Note that (84) expresses a kind of duality between the two sextic oscillators of parameters ω^2 and $\underline{\omega}^2$. The “self-dual” case would be for $\underline{\omega}^2 = \omega^2 = 2$. \square

The product of two eigenfunctions of the sextic oscillator can be now expressed as

$$\psi(x, \epsilon_n) \psi(y, \epsilon_n) = \frac{\underline{\mu}_n}{2i\pi} \oint \frac{dz}{z} \psi(z, \pm i\epsilon_n) \exp\left(\frac{z^4}{4}\right) \underline{\mathcal{K}}(x^2 + y^2, xy\sqrt{2}; z), \tag{85}$$

where

$$\underline{\mathcal{K}}(x^2 + y^2, xy\sqrt{2}; z) = \sum_{m \in \mathbb{N}} \frac{1}{(\underline{\lambda} z)^m} D_{v(m)}(x^2 + y^2) D_m(xy\sqrt{2}). \tag{86}$$

Of course, it is understood that on the left-hand side of Eq. (85), the eigenfunctions $\psi(x, \epsilon_n)$ are related to ω^2 , whereas on the right-hand side of Eq. (85), we have $\psi(z, \pm i\epsilon)$ related to $\underline{\omega}^2$, which verifies (84).

We refrain from starting an in-depth study of this relation, which is susceptible to lead to a new approach to finding the actual Schrödinger eigenfunctions of the sextic oscillator and prefer to postpone it for future work.

F. The quasi-exactly solvable cases of the sextic oscillator problem

As it is well-known, the sextic oscillator is found to be exactly solvable for a particular choice of its parameters in the sense²⁶ that the eigenfunctions can be expressed analytically in terms of elementary functions, and the corresponding energy eigenvalues are zeros of an algebraic equation (see, e.g., Refs. 23, 27, and 28). Following Appendix B, the eigenfunctions of the sextic oscillator Hamiltonian (56) with parameter ω^2 and eigenvalue ϵ are proportional to a Bi-Confluent Heun (BCH) function of x^2 [see equations in Proposition B.1, respectively for even and odd stationary states].

For a BCH function to be reduced to a polynomial of order m , two conditions should be met (see Ref. 29).

- (a) The arrest of the defining power series of the BCH function at order $2m$ (respectively, $2m + 1$) for even state (respectively, odd state) yields an algebraic equation of order m in ϵ , from which the energy eigenvalues ϵ can be computed.
- (b) The coefficient $(\gamma - \alpha - 2)$ in equation (B2) should be an even non-negative integer.

Realization of condition (b), respectively, for even and odd states given by Eqs. (10) and (11) uses the α and γ parameters of the BCH functions $N(-\frac{1}{2}, 0, -\frac{\omega^2}{2}, -\epsilon\sqrt{2}; \frac{x^2}{\sqrt{2}})$ and $N(\frac{1}{2}, 0, -\frac{\omega^2}{2}, -\epsilon\sqrt{2}; \frac{x^2}{\sqrt{2}})$. This yields

$$\begin{aligned} -\frac{\omega^2}{2} + \frac{1}{2} - 2 &= 2I_e, \\ -\frac{\omega^2}{2} - \frac{1}{2} - 2 &= 2I_o, \end{aligned}$$

where I_e and I_o are non-negative integers as required. Consequently,

$$\begin{aligned} \omega^2 &= -(4I_e + 3) = -4J_e + 1, \\ \omega^2 &= -(4I_o + 5) = -4J_o - 1, \end{aligned} \tag{87}$$

where $J_{e,o} = (I_{e,o} + 1)$. This is precisely what Bender and Dunne stated in Ref. 28 in a slightly different context.

The question is how the quasi-exactly solvable features shows up at the level of the Wigner distribution function of the sextic oscillator. Following the values of ω^2 in Eq. (87), we deduce the following values of the index $\nu(m)$ of the parabolic cylinder function in the expansion (67), respectively, for $m = 2m'$ (even state) and for $m = 2m'' + 1$ (odd state):

$$\nu(2m') = m' + J_e - \frac{1}{2}, \tag{88}$$

$$\nu(2m'' + 1) = m'' + J_o + \frac{1}{2}, \tag{89}$$

after substitution of the corresponding $\beta_e^* = 1 - 2J_e$ and $\beta_o^* = -2J_o$ in the definition of $\nu(m)$ above Eq. (66). This means that only half integer indexed parabolic functions will be present in the expansion (67). They can be alternatively rewritten in terms of the modified Bessel functions $K_\sigma(x)$, where σ is also a half integer (see Ref. 24).

A last question one may ask is whether the expansion in (67) can contain only Hermite functions. Or equivalently can $\nu(m)$ take non-negative integer values? The answer is no. Here is why. Again we separate the cases with even and odd states.

- For even states, we have $m = 2m'$; therefore, by definition, $\nu(2m') = \frac{1}{2}(2m' - \beta_e^*)$. If $\nu(2m')$ is to be a non-negative integer, we must have $\beta_e^* = -2L_e$ where $L_e \in \mathbb{N}$. This implies that $\omega^2 = (2\beta_e^* - 1) = -2L_e - 1$. However, according to Eq. (87), this value of ω^2 is for Schrödinger odd state. Hence, there is a contradiction with respect to both sides of Eq. (67).
- For odd states, we have $m = 2m'' + 1$; therefore, by definition, $\nu(2m'' + 1) = \frac{1}{2}(2m'' + 1 - \beta_o^*)$. If $\nu(2m')$ is to be a non-negative integer, we must have $\beta_o^* = (-2L_o - 1)$ where $L_o \in \mathbb{N}$. This implies that $\omega^2 = (2\beta_o^* - 1) = -2(L_o + 1)$. However, according to Eq. (87), this value of ω^2 is for Schrödinger even state. Hence, there is a contradiction with respect to both sides of Eq. (67).

VI. CONCLUSION AND PERSPECTIVES

In this work, we have obtained explicit representations for the Wigner distribution functions for the quartic and sextic anharmonic oscillators. The main point is that their determining equations are superintegrable, which has allowed a representation with Schrödinger stationary energy eigenfunctions under the form of an equation characteristic of zonal spherical functions, a theory developed by Berezin and Gelfand in the 1960s. It would be of interest to probe further in this direction to see whether a group theoretical basis exists. As a laboratory for simple quantum mechanical systems, these two oscillators do have interesting features and properties, which may help in exploring new ideas about formulating quantum mechanics in phase space. The continuous representation theory of the canonical commutation relations in Hilbert space, for example, may be an appropriate candidate. This study also lays grounds and motivation for investigating the ways to solve infinite equations systems, which are probably related to the subject of Hill's determinant. We hope to tackle these problems in the future.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX A: CONVENTIONAL APPROACH TO THE Q^4 OSCILLATOR

In the conventional approach to the Q^4 oscillator, an eigenfunction of the $H_4(x)$ Hamiltonian with $\underline{\omega}^2$ and energy $\underline{\epsilon}$ is assumed to be a function product of the form $\psi(x, \underline{\epsilon}) = \exp\left(-\frac{x^3}{3} - \frac{\omega^2}{2}x\right) \phi(x, \underline{\epsilon})$. Then, $\phi(x, \underline{\epsilon})$ is the solution of the equation

$$-\frac{d^2}{dx^2} \phi(x, \underline{\epsilon}) + (2x^2 + \underline{\omega}^2) \frac{d}{dx} \phi(x, \underline{\epsilon}) + \left(2x - \frac{\omega^4}{4} - 2\underline{\epsilon}\right) \phi(x, \underline{\epsilon}) = 0. \quad (A1)$$

We now show that $\phi(x, \underline{\epsilon})$ is, in fact, a Tri-Confluent-Heun (TCH) function. The canonical form of such TCH function $y(\alpha, \beta, \gamma; z)$ is given in Ref. 7 as the solution of the canonical TCH equation

$$\frac{d^2}{dz^2} y(z) - (\gamma + 3z^2) \frac{d}{dz} y(z) + [\alpha + (\beta - 3)z] y(z) = 0, \quad (A2)$$

where $y(z)$ stands for $y(\alpha, \beta, \gamma; z)$ for ease of notation. A scaling transformation $x = \lambda'z$ with $\lambda' = \left(\frac{3}{2}\right)^{\frac{1}{3}}$ shows that

$$\phi(x, \underline{\epsilon}) = y(\alpha, \beta, \gamma; z) \quad (A3)$$

is a special TCH function with parameters

$$\alpha = \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(2\underline{\epsilon} + \frac{\omega^4}{4}\right), \quad \beta = 6, \quad \text{and} \quad \gamma = \left(\frac{3}{2}\right)^{\frac{1}{3}} \underline{\omega}^2. \quad (A4)$$

Polynomial solutions are also given in pages 262–263 of Ref. 7. They depend on a parameter d related to β by $\beta = 3(d + 1)$. Hence, in our case, $d = 1$, which means that there are only second order polynomial solutions in z with unfortunately complex energies

$$\underline{\epsilon} = \frac{1}{2} \left(-\frac{\omega^4}{4} \pm i \sqrt{\frac{\omega^2}{\lambda'^3}} \right), \quad (A5)$$

therefore physically not relevant.

APPENDIX B: CONVENTIONAL APPROACH TO THE Q^6 OSCILLATOR

In the conventional approach to the Q^6 oscillator, an eigenfunction of the $H_6(x)$ Hamiltonian with $\underline{\omega}^2$ and energy $\underline{\epsilon}$ is assumed to be a function product of the form $\psi(x, \underline{\epsilon}) = \exp\left(-\frac{x^4}{4}\right) \phi(x, \underline{\epsilon})$. Then, $\phi(x, \underline{\epsilon})$ is the solution to the equation

$$-\frac{d^2}{dx^2} \phi(x, \underline{\epsilon}) + 2x^3 \frac{d}{dx} \phi(x, \underline{\epsilon}) + ((\underline{\omega}^2 + 3)x^2 - 2\underline{\epsilon}) \phi(x, \underline{\epsilon}) = 0. \quad (B1)$$

Proposition B.1. $\phi(x, \underline{\epsilon})$ is, in fact, a Bi-Confluent Heun (BCH) function $N(\alpha, \beta, \delta, \gamma; t)$ of variable t and special values of parameters $(\alpha, \beta, \delta, \gamma)$ in the notation of Ref. 7. One obtains the following:
for even states,

$$\psi(x, \underline{\epsilon}) = \exp\left(-\frac{x^4}{4}\right) N\left(-\frac{1}{2}, 0, -\frac{\omega}{2}, -\underline{\epsilon}\sqrt{2}; \frac{x^2}{\sqrt{2}}\right),$$

and for odd states,

$$\psi(x, \epsilon) = x \exp\left(-\frac{x^4}{4}\right) N\left(\frac{1}{2}, 0, -\frac{\omega}{2}, -\epsilon\sqrt{2}; \frac{x^2}{\sqrt{2}}\right).$$

Proof. The canonical BCH equation (see Refs. 7 and 25) verified by $N(\alpha, \beta, \delta, \gamma; x)$ is

$$x \frac{d^2}{dx^2} N(x) + (1 + \alpha - \beta x - 2x^2) \frac{d}{dx} N(x) + \left\{ (\gamma - 2 - \alpha)x - \frac{1}{2} [\delta + \beta(1 + \alpha)] \right\} N(x) = 0. \quad (\text{B2})$$

Here, $N(x)$ is a short notation for $N(\alpha, \beta, \delta, \gamma; x)$, which is the unbounded solution at $x \rightarrow \infty$. □

- For even parity eigenstate, we write $\phi_e(x) = \widehat{\phi}_e^e(x^2)$, which, as a function $\widehat{\phi}_e^e(\xi)$ of $\xi = x^2$, obeys the equation

$$\xi \frac{d^2}{d\xi^2} \widehat{\phi}_e^e(\xi) + \left(\frac{1}{2} - \xi^2\right) \frac{d}{d\xi} \widehat{\phi}_e^e(\xi) + \left(\frac{\epsilon}{2} - \frac{\omega^2 + 3}{4} \xi\right) \widehat{\phi}_e^e(\xi) = 0, \quad (\text{B3})$$

which under the scaling transform $\xi = \zeta \sqrt{2}$ as $\widehat{\phi}_e^e(\xi) = \overline{\phi}_e^e(\zeta)$ becomes

$$\zeta \frac{d^2}{d\zeta^2} \overline{\phi}_e^e(\zeta) + \left(\frac{1}{2} - 2\zeta^2\right) \frac{d}{d\zeta} \overline{\phi}_e^e(\zeta) + \left(\frac{\epsilon}{\sqrt{2}} - \frac{\omega^2 + 3}{2} \zeta\right) \overline{\phi}_e^e(\zeta) = 0. \quad (\text{B4})$$

Hence, in view of (B2), $\overline{\phi}_e^e(\zeta)$ is the BCH function $N(-\frac{1}{2}, 0, -\frac{\omega^2}{2}, -\epsilon\sqrt{2}; \frac{\zeta^2}{\sqrt{2}})$.

- A similar treatment for odd parity eigenstate, where we assume $\phi_o(x) = x \widehat{\phi}_e^o(x^2)$, leads to the equation

$$\xi \frac{d^2}{d\xi^2} \widehat{\phi}_e^o(\xi) + \left(\frac{3}{2} - \xi^2\right) \frac{d}{d\xi} \widehat{\phi}_e^o(\xi) + \left(\frac{\epsilon}{2} - \frac{\omega^2 + 5}{4} \xi\right) \widehat{\phi}_e^o(\xi) = 0, \quad (\text{B5})$$

which under the scaling transform $\xi = \zeta \sqrt{2}$ as $\widehat{\phi}_e^o(\xi) = \overline{\phi}_e^o(\zeta)$ becomes

$$\zeta \frac{d^2}{d\zeta^2} \overline{\phi}_e^o(\zeta) + \left(\frac{3}{2} - 2\zeta^2\right) \frac{d}{d\zeta} \overline{\phi}_e^o(\zeta) + \left(\frac{\epsilon}{\sqrt{2}} - \frac{\omega^2 + 5}{2} \zeta\right) \overline{\phi}_e^o(\zeta) = 0. \quad (\text{B6})$$

Hence, in view of (B2), $\overline{\phi}_e^o(\zeta)$ is the BCH function $N(\frac{1}{2}, 0, -\frac{\omega^2}{2}, -\epsilon\sqrt{2}; \frac{\zeta^2}{\sqrt{2}})$.

As in the usual approach to determine the eigenfunctions of a Hamiltonian (1), one seek to prevent the growth of the auxiliary wave function $\phi(x, \epsilon)$ at infinity to dominate the convergence factor $\exp(-\frac{x^4}{4})$ by demanding that $\phi(x, \epsilon)$ be a polynomial. Such a procedure leads to Heun polynomials, which forces ω to take special discrete negative integer values, and the sextic oscillator problem is said to be quasi-exactly soluble (QES). Although Heun polynomials have been studied earlier,^{7,29} their properties relevant for the sextic oscillator have been established only much later by Bender and Dunne.²⁸

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