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On High-Gain Observer Design for Nonlinear Systems with Delayed Output Measurements

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Abstract

In this paper, we propose a high-gain observer design for nonlinear systems with time-varying delayed output measurements. Based on a recent high-gain like observer design method, called HG/LMI observer, a larger bound of the time-delay is allowed compared to that obtained by using the standard high-gain methodology. Such a HG/LMI observer adopts a lower value of the tuning parameter, which results in the reduction of the value of the observer gain, and an increase in the maximum bound of the delay required to ensure exponential convergence. Indeed, an explicit relation between the maximum bound of the delay and the observer tuning parameter is derived by using a Lyapunov-Krasovskii functional jointly with the Halanay inequality. Such a relation shows clearly the superiority of HG/LMI observer design methodology. An application to nonlinear systems with sampled measurements is provided. Furthermore, the proposed methodology is extended to systems with nonlinear outputs. This extension provides more general synthesis conditions and encompasses the linear case as a particular situation. Finally, two numerical examples are proposed to illustrate the performance of the proposed observer design procedure, and comparison to standard approaches is also provided.

Key words: Nonlinear systems, high-gain observer, delayed output measurements, linear matrix inequalities (LMIs), Lyapunov-Krasovskii functionals.

1 Introduction

For many decades, the interest of automatic control community to nonlinear observers continues to grow because of their crucial role in the design of control schemes, namely trajectory tracking, fault diagnosis, and health monitoring (Parisini, 1997), (Alcorta-Garcia and Frank, 1997), (Gao and Ho, 2006). Due to the introduction of new technologies and the complexity of novel industrial infrastructures, the use of nonlinear ob-

servers has been emerged in modern applications such as synchronization of multi-agent systems, cyber-attacks detection, and control of cyber-physical systems (Zhu and Basar, 2011), (Teixeira *et al.*, 2010), (An and Liu, 2014). More recently, data-driven or learning-based neuro-adaptive observers have been introduced, building a synergy between data and model-based methods (Chakrabarty *et al.*, 2019), (Koga *et al.*, 2019), (Liu *et al.*, 2018). Despite the various methodologies of nonlinear observer in the literature, namely the extended Kalman observer Kalman (1960), Luenberger observer (Luenberger, 1971), (Huong *et al.*, 2019), high-gain observer (Gauthier and Kupka, 1994), sliding mode observer (Alessandri, 2003), and LMI-based observers (Zemouche and Boutayeb, 2013), (Mazenc *et al.*, 2017), for instance, these solutions are not general and can be applied only on a specific class of systems. The high-gain observer is particularly interesting due to its easy implementation because it depends on only

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one single tuning parameter, which requires a specific condition, to ensure exponential convergence. Despite this simplicity of implementation, the high-gain observer is far from being a perfect solution to nonlinear estimation, and it has three limitations that should be highlighted: 1) numerical problems because for high dimensional systems due to the high values of the observer gain; 2) the presence of peaking phenomenon; and 3) the high sensitivity to output disturbances (measurement noise, delayed outputs, sampled data,...). Many research activities have been interested in this research area aiming to propose solutions overcoming such drawbacks of the high-gain observer (Zemouche *et al.*, 2019), (Alessandri and Rossi, 2015), (Astolfi and Marconi, 2015), (Khalil, 2017), (Boizot *et al.*, 2010), (Nguyen and Trinh, 2016), (Trinh *et al.*, 2004). In this paper, we focus only on the use of high-gain methodologies for systems with delayed outputs. Indeed, such a problem is complex when the goal is to provide an observer with a maximum allowable value of the time-delay, which is the main motivation of this paper. Several efficient solutions, based on high-gain methodology, have been proposed in the literature to cope with this issue (Ahmed-Ali *et al.*, 2009), (Van Assche *et al.*, 2011). Since systems with sampled output measurements can be rewritten in an equivalent form as systems with delayed outputs, similar results have been proposed in (Ahmed-Ali *et al.*, 2013a), (Ahmed-Ali *et al.*, 2013b), (Bouraoui *et al.*, 2015), (Zhang and Shen, 2017), (Ahmed-Ali *et al.*, 2014).

In (Ahmed-Ali *et al.*, 2009), the authors proposed a cascade high-gain observer based on the Lyapunov method in which explicit relations between the delay and the number of cascade observers have been proposed. Subsequently, an extension to high-gain observer design with time varying-delay and sampled data cases where the delay is sufficiently small has been presented in Van Assche *et al.* (2011). Although the maximum value of the allowable delay is improved, however, it still remains small due to the high value of the tuning parameter required by the standard high-gain observer. On the other hand, the maximum bound of the delay is obtained by solving an ordinary differential equation that depends on the design tuning parameter. The proposed work in this paper has been motivated by this issue, namely establishing a high-gain like design method with a lower tuning parameter, which leads to a higher maximum value of the delay. To this end, we proposed an observer design method, which covers the standard high-gain observer, used in the above papers, as a particular case. This is based on the exploitation of the LMI-based approach combined with the standard high-gain technique, called HG/LMI observer, which allows introducing a compromise index j_s . This index j_s offers the possibility to adjust the value of the tuning high-gain parameter ensuring exponential convergence for larger values of delay. The convergence analysis is performed by using a Lyapunov-Krasovskii functional. To reach less conservative bounds, the Halanay

inequality is applied on the integral term containing the derivative of the error, instead of developing strong upper bounds to make it vanish from the Lyapunov analysis. The obtained results show explicitly the effectiveness of the proposed HG/LMI observer-based technique, due to the mathematical relation between the tuning parameter of the observer and the maximum bound of the delay. An extension to nonlinear systems with sampled measurements is given as an application of the result.

Compared to the short version of this manuscript presented in the conference paper (Adil *et al.*, 2020), the present paper contains the following contributions, in addition to the detailed proofs of the main results:

- a general design method for nonlinear systems with multi-nonlinearities based on the proposed HG/LMI observer framework is presented. The standard high-gain observer is covered as a particular solution.
- extension to systems with nonlinear outputs is established by using the mean value theorem and suitable, but not conservative assumptions.
- extension to nonlinear systems with sampled measurements is developed.
- a numerical observer design procedure is added to show how the observer parameters are computed.
- extensive numerical evaluations are provided. The theoretical results are illustrated through two numerical examples, with comparisons between the proposed HG/LMI approach, the standard high-gain observer design, and the high-gain observer method proposed by (Van Assche *et al.*, 2011).

The extension to systems with nonlinear outputs is particularly a new contribution and a non-straightforward extension. The generalization is based on the use of the differential mean value theorem and some judicious mathematical arrangements. The extension leads to more general design conditions, which can be reduced easily to those of the linear case. This extension is simple to implement, and therefore it is appropriate for applications to real-world models.

The rest of the paper is organized as follows. In Section 2, the problem is formulated and the class of systems under consideration is presented. The high gain observer is then recalled and its convergence is analyzed in Section 3. Section 4 presents and discusses the proposed observer design strategy using the HG/LMI observer with a lower value tuning parameter highlighted with the observer design procedure. Extensions to systems with sampled measurements and systems with nonlinear outputs are also provided in Section 4. Two numerical examples with comparisons to the standard high-gain observer and the observer presented in (Van Assche *et al.*, 2011) are given in Section 5. Finally, we end the paper with a conclusion summarizing the main contributions and some future works.

2 Preliminaries and Problem Formulation

In this section, we introduce some useful preliminaries and formulate the estimation problem.

2.1 Preliminaries

We will recall some lemmas, which are necessary for the mathematical developments given in the next sections.

Lemma 1 (Zemouche and Boutayeb (2013))

Considering the function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the two following items are equivalent

- Ψ is γ_Ψ -Lipschitz with respect to its argument, i.e.:

$$\|\Psi(X) - \Psi(Y)\| \leq \gamma_\Psi \|X - Y\|, \forall X, Y \in \mathbb{R}^n.$$

- For all $i, j = 1, \dots, n$, there exist functions $\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, such that $\forall X, Y \in \mathbb{R}^n$

$$\Psi(X) - \Psi(Y) = \sum_{i=1}^n \sum_{j=1}^n \psi_{ij} H_{ij}(X - Y),$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}},$$

where

$$\psi_{ij} \triangleq \psi(X^{Y_{j-1}}, X^{Y_j}) \quad \text{and} \quad H_{ij} = v_n(i)v_n^\top(j),$$

$v_n(i)$ is the i^{th} vector of the canonical basis of \mathbb{R}^n .

We also recall some useful inequalities exploited in the proof of some results.

Lemma 2 (Jensen's Inequality (Gu, 2000)) For any constant symmetric and positive definite matrix $M \in \mathbb{R}^{n \times n}$, scalars t_1, t_2 and vector function $v : [t_1, t_2] \rightarrow \mathbb{R}^n$, then the following inequality holds

$$\left(\int_{t_1}^{t_2} v(\beta) d\beta \right)^T M \left(\int_{t_1}^{t_2} v(\beta) d\beta \right) \leq (t_2 - t_1) \left(\int_{t_1}^{t_2} v^\top(\beta) M v(\beta) d\beta \right).$$

Lemma 3 (Young's Inequality (Nguyen and Trinh, 2016))

Let X and Y be two matrices of appropriate dimensions. Then, for every invertible matrix S and scalar $\mu > 0$, we have

$$X^\top Y + Y^\top X \leq \mu X^\top S X + \frac{1}{\mu} Y^\top S^{-1} Y.$$

2.2 Problem formulation

We consider the class of nonlinear systems described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + f_1(x_1) \\ x_3 + f_2(x_1, x_2) \\ \vdots \\ x_{n-1} + f_{n-1}(x_1, x_2, x_{n-1}) \\ f_n(x_1, \dots, x_n) \end{bmatrix} \\ y &= x_1(t - \tau(t)), \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system and $y(t) \in \mathbb{R}$ is the measured output. We assume that $\tau(t)$ is a known time-varying delay satisfying

$$0 < \tau(t) \leq \tau_M.$$

The functions $f_i : \mathbb{R}^i \rightarrow \mathbb{R}$, $i=1, \dots, n$, satisfy the following Lipschitz property:

$$|f_i(x_1 + \Delta_1, \dots, x_i + \Delta_i) - f_i(x_1, \dots, x_i)| \leq \sum_{j=1}^i k_j |\Delta_j|$$

where k_j is the Lipschitz constant and $\Delta_j \in \mathbb{R}, \forall j = 1, \dots, n$.

For simplicity of the presentation, system (1) can be rewritten under the following form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) \\ y(t) = Cx(t - \tau(t)), \end{cases} \quad (2)$$

where the matrices A and C are defined by

$$C = [1 \ 0 \ \dots \ 0], \quad (A)_{i,j} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{if } j \neq i + 1, \end{cases}$$

and

$$f(x(t)) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_{n-1}(x_1, x_2, x_{n-1}) \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

Let us introduce the following Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L[y(t) - C\hat{x}(t - \tau(t))], \quad (3)$$

where $\hat{x}(t)$ represents the state estimation and L is the observer gain.

The dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ is given as follows

$$\dot{e}(t) = Ae(t) + [f(x(t)) - f(\hat{x}(t))] - LCe(t - \tau(t)). \quad (4)$$

The objective consists in designing a high-gain observer for system (1) that provides stability of the estimation error. We also provide an expression of the maximum bound of the allowable delay and the design parameter under which the proposed observer converges exponentially.

To establish the exponential convergence, the following lemma is useful.

Lemma 4 (Halanay (1966)) *If there exist a positive Lyapunov-Krasovskii functional $V(e(t))$ such that*

$$\frac{d}{dt}V(e(t)) \leq -\alpha V(e(t)) + \beta \sup_{s \in [t-\tau_M, t]} V(e(s)),$$

where $\alpha > \beta > 0$, then there exist two scalars $\eta > 0$ and $\delta > 0$ such that

$$V(e(t)) \leq \eta e^{-\delta(t-t_0)} \quad \text{for all } t \geq t_0.$$

3 Preliminary Results Based on Standard High-Gain Observer

This section is devoted to the preliminary results shared into several intermediate results. Indeed, after some preliminary and useful mathematical lemmas and propositions related to delay systems theory, we propose a new LMI-based observer design procedure. The latter provides conditions on the maximum upper bound of the allowed delay guaranteeing exponential convergence of the observer. The design technique is based on the use of the standard high-gain observer methodology. First, as in (Gauthier and Kupka, 1994), the observer gain L is written as follows:

$$L \triangleq T(\theta)K; \quad \theta \geq 1, \quad (5)$$

where

$$T(\theta) := \text{diag}(\theta, \dots, \theta^n) \quad \text{and} \quad K \in \mathbb{R}^n.$$

Moreover, the estimation error is transformed into

$$\bar{e} := T^{-1}(\theta)e,$$

where $T^{-1}(\theta)$ is the inverse of $T(\theta)$ given by

$$T^{-1}(\theta) := \text{diag}\left(\frac{1}{\theta}, \dots, \frac{1}{\theta^n}\right).$$

The dynamics of the transformed error is given by

$$\dot{\bar{e}}(t) = \theta(A - KC)\bar{e}(t) + T^{-1}(\theta)\Delta f - \theta KC(\bar{e}(t - \tau(t)) - \bar{e}(t)), \quad (6)$$

with

$$\Delta f := f(x) - f(x - T(\theta)\bar{e}).$$

From the fact that $\theta \geq 1$ and by using the Lipschitz condition (2.2), it was shown in (Alessandri and Rossi, 2013) that there exists a positive constant k_f , independent of θ , such that

$$\|T^{-1}(\theta)\Delta f\| \leq k_f\|\bar{e}\|. \quad (7)$$

Before presenting the main theorem corresponding to the use of the standard high-gain observer, we will introduce first a series of intermediate results. Such intermediate results will improve the clarity and readability of the first main theorem.

Lemma 5 *Let f be a nonlinear function satisfying (7). Then for any symmetric and positive definite matrix P and a vector \bar{e} of appropriate dimensions, we have*

$$\mathcal{H}e\left\{\left(T^{-1}(\theta)\Delta f\right)^\top \bar{e}\right\} \leq 2k_f\lambda_{\max}(P)\bar{e}^\top \bar{e}, \quad (8)$$

where $\mathcal{H}e\{\mathcal{S}\} := \mathcal{S} + \mathcal{S}^\top$.

PROOF. The proof is omitted. It is straightforward by applying the well-known Cauchy-Schwarz inequality and the fact that $\theta \geq 1$.

Proposition 6 *Let Y be an arbitrary matrix of appropriate dimension. Define Γ as*

$$\Gamma \triangleq \bar{e}^\top(t)Y^\top C\left(\bar{e}(t) - \bar{e}(t - \tau(t))\right) + \left(\bar{e}(t) - \bar{e}(t - \tau(t))\right)^\top C^\top Y \bar{e}(t),$$

where $\bar{e}(t)$ is the transformed estimation error satisfying (6). Then there exist three positive scalars $\mu_i > 0$, $i = 1, \dots, 3$ such that Γ satisfies the following inequality:

$$\begin{aligned} \Gamma &\leq \frac{1}{\mu_1} \bar{e}^\top(t)Y^\top Y \bar{e}(t) \\ &+ \mu_1 \left(1 + \frac{1}{\mu_2}\right) \tau_M \theta^2 \int_{t-\tau_M}^t (\bar{e}_2(s))^2 ds \\ &+ \mu_1 \left(1 + \frac{1}{\mu_3}\right) (1 + \mu_2) k_1^2 \tau_M \int_{t-\tau_M}^t (\bar{e}_1(s))^2 ds \\ &+ \mu_1 (1 + \mu_2) (1 + \mu_3) \tau_M K_1^2 \theta^2 \int_{t-\tau_M}^t (\bar{e}_1(s - \tau(s)))^2 ds \end{aligned} \quad (9)$$

where \bar{e}_1 and \bar{e}_2 are the first and second components of the estimation error vector \bar{e} , and K_1 is the first component of K .

PROOF. By applying the Young inequality, given in Lemma 3, on Γ , we obtain

$$\begin{aligned} \Gamma &\leq \mu_1 \left(\bar{e}(t) - \bar{e}(t - \tau(t)) \right)^\top C^\top C \left(\bar{e}(t) - \bar{e}(t - \tau(t)) \right) \\ &\quad + \frac{1}{\mu_1} \bar{e}^\top(t) Y^\top Y \bar{e}(t), \end{aligned} \quad (10)$$

for any given scalar $\mu_1 > 0$. By using the Newton-Leibniz integration formula

$$\bar{e}(t) - \bar{e}(t - \tau(t)) = \int_{t-\tau(t)}^t \dot{\bar{e}}(s) ds,$$

inequality (10) can be rewritten as follows:

$$\Gamma \leq \mu_1 \mathbb{J}(t) + \frac{1}{\mu_1} \bar{e}^\top(t) \mathcal{Y}^\top \mathcal{Y} \bar{e}(t). \quad (11)$$

where

$$\mathbb{J}(t) \triangleq \left(\int_{t-\tau(t)}^t \dot{\bar{e}}(s) ds \right)^\top C^\top C \left(\int_{t-\tau(t)}^t \dot{\bar{e}}(s) ds \right).$$

From Jensen Inequality introduced in Lemma 2 and the fact that $C\bar{e}(s) = \bar{e}_1(s)$ and

$$\dot{\bar{e}}_1(s) = \left(\theta \bar{e}_2(s) + \frac{1}{\theta} \Delta f_1 - \theta K_1 \bar{e}_1(s - \tau(s)) \right), \quad (12)$$

we obtain an upper bound of $\mathbb{J}(t)$ as follows:

$$\begin{aligned} \mathbb{J}(t) &\leq \tau(t) \int_{t-\tau(t)}^t \dot{\bar{e}}(s)^\top C^\top C \dot{\bar{e}}(s) ds, \\ &\leq \tau_M \int_{t-\tau_M}^t (\dot{\bar{e}}_1(s))^2 ds, \\ &= \tau_M \int_{t-\tau_M}^t \left(\theta \bar{e}_2(s) + \frac{1}{\theta} \Delta f_1 - \theta K_1 \bar{e}_1(s - \tau(s)) \right)^2 ds, \\ &\leq \left(1 + \frac{1}{\mu_2} \right) \tau_M \theta^2 \int_{t-\tau_M}^t (\bar{e}_2(s))^2 ds \\ &\quad + \left(1 + \mu_2 \right) \tau_M \int_{t-\tau_M}^t \left(\frac{1}{\theta} \Delta f_1 - \theta K_1 \bar{e}_1(s - \tau(s)) \right)^2 ds \\ &\leq \left(1 + \frac{1}{\mu_2} \right) \tau_M \theta^2 \int_{t-\tau_M}^t (\bar{e}_2(s))^2 ds \\ &\quad + \left(1 + \frac{1}{\mu_3} \right) \left(1 + \mu_2 \right) k_1^2 \tau_M \int_{t-\tau_M}^t (\bar{e}_1(s))^2 ds \\ &\quad + \left(1 + \mu_2 \right) \left(1 + \mu_3 \right) \tau_M K_1^2 \theta^2 \int_{t-\tau_M}^t (\bar{e}_1(s - \tau(s)))^2 ds \end{aligned} \quad (13)$$

where μ_2 and μ_3 come from the application of Young inequality. By substituting (13) in (11), inequality (9) is inferred. This ends the proof of Proposition 6.

Lemma 7 Let $X(t)$ be a positive and continuous function. Let $\tau(\cdot)$ be a positive time-delay with $\tau(t) \leq \tau_M$. Then the following inequality holds:

$$\int_{t-\tau(t)}^t X(s - \tau(s)) ds \leq \tau_M \sup_{s \in [t-2\tau_M, t]} \{X(s)\}. \quad (14)$$

PROOF. The proof is straightforward. Indeed, if $s \in [t - \tau_M, t]$ and $\tau(s) \geq 0$, then we have

$$s - \tau(s) \in [t - 2\tau_M, t].$$

Proposition 8 Consider the Lyapunov function $V_1(\bar{e}(t)) \triangleq \bar{e}^\top(t) P \bar{e}(t)$, where $P = P^\top > 0$. Then the derivative of V_1 along the trajectories of (6) satisfies the following inequality:

$$\begin{aligned} \frac{d}{dt} V_1 &\leq \theta \bar{e}^\top(t) \left[A^\top P + P A - C^\top \mathcal{Y} - \mathcal{Y}^\top C \right. \\ &\quad \left. + \frac{1}{\mu_1} \mathcal{Y}^\top \mathcal{Y} \right] \bar{e}(t) + 2k_f \lambda_{\max}(P) \bar{e}^\top(t) \bar{e}(t) \\ &\quad + \mu_1 \left(1 + \frac{1}{\mu_2} \right) \tau_M \theta^3 \int_{t-\tau_M}^t (\bar{e}_2(s))^2 ds \\ &\quad + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{s \in [t-2\tau_M, t]} V_1(s), \end{aligned} \quad (15)$$

where

$$\begin{aligned} \varpi(\mu_1, \mu_2, \mu_3) &\triangleq \frac{\mu_1}{\lambda_{\min}(P)} \left[\left(1 + \frac{1}{\mu_3} \right) \left(1 + \mu_2 \right) k_1^2 \theta \right. \\ &\quad \left. + \left(1 + \mu_2 \right) \left(1 + \mu_3 \right) K_1^2 \theta^3 \right], \end{aligned} \quad (16)$$

and

$$\mathcal{Y} = K^\top P.$$

PROOF. The proof can be obtained straightforwardly from Lemma 5, Proposition 6, and Lemma 7.

Lemma 9 Let us define the function

$$\vartheta(z) \triangleq \int_{t-\tau_M}^t \int_{\xi}^t z(s) ds d\xi,$$

where $z(s) \in \mathbb{R}^+$ and $\tau_M > 0$ is a scalar constant. Then the two following equations hold:

$$\vartheta(z) \leq \tau_M \int_{t-\tau_M}^t z(s) ds. \quad (17a)$$

$$\frac{d}{dt} \vartheta(z(t)) = \tau_M z(t) - \int_{t-\tau_M}^t z(s) ds. \quad (17b)$$

PROOF. The first inequality (17a) is obvious and can be obtained by integrating with respect to ξ and the fact that $t - s \leq \tau_M$ for $s \in [t - \tau_M, t]$.

Now we are ready to state the preliminary result summarized in the following theorem providing new LMI conditions ensuring exponential convergence of the observer.

Theorem 10 *Assume there exist a positive definite matrix P , a matrix \mathcal{Y} of appropriate dimension and real constants $\mu_i > 0, i = 1, \dots, 3, \lambda > 0, \tau_M > 0$ such that the following conditions hold*

$$\begin{bmatrix} \text{He}\{PA - \mathcal{Y}^\top C\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} \\ & & -\mu_1 \end{bmatrix} \leq 0, \quad (18)$$

$$\theta > \max\left(1, \frac{2k_f \lambda_{\max}(P)}{\lambda}\right), \quad (19)$$

$$\tau_M \leq \min(\tau_1, \tau_2) \quad (20)$$

where

$$\tau_1 \triangleq \sqrt{\frac{[\lambda\theta - 2k_f \lambda_{\max}(P)]}{\lambda_{\max}(P) \varpi(\mu_1, \mu_2, \mu_3)}} \quad (21)$$

$$\tau_2 \triangleq \frac{1}{\mu_1 \left(1 + \frac{1}{\mu_2}\right) \theta^2 + \frac{[\lambda\theta - 2k_f \lambda_{\max}(P)]}{\lambda_{\max}(P)}} \quad (22)$$

with

$$\varpi(\mu_1, \mu_2, \mu_3) \triangleq \frac{\mu_1}{\lambda_{\min}(P)} \left[\left(1 + \frac{1}{\mu_3}\right) (1 + \mu_2) k_1^2 \theta + (1 + \mu_2) (1 + \mu_3) K_1^2 \theta^3 \right], \quad (23)$$

$$\mathcal{R} = [0 \quad 1 \quad 0_{1 \times n-2}], \quad (24)$$

$$K = P^{-1} \mathcal{Y}^\top = [K_1 \dots K_n]^\top. \quad (25)$$

Then the observer (3) converges exponentially and the observer gain is given by $L = T(\theta)K$.

PROOF. Consider the following Lyapunov-Krasovskii functional

$$V(t) = V(\bar{e}(t)) = V_1(\bar{e}(t)) + \theta V_2(\bar{e}(t)), \quad (26)$$

where

$$V_1(t) = \bar{e}^\top(t) P \bar{e}(t),$$

and

$$V_2(t) = \int_{t-\tau_M}^t \int_{\xi}^t (\bar{e}_2(s))^2 ds d\xi.$$

From Schur lemma (Boyd *et al.*, 1994), LMI (18) is equivalent to

$$A^\top P + PA - C^\top \mathcal{Y} - \mathcal{Y}^\top C + \frac{1}{\mu_1} \mathcal{Y}^\top \mathcal{Y} + \tau_M \mathcal{R}^\top \mathcal{R} < -\lambda I.$$

Hence, by substituting this inequality in the derivative of $V(\bar{e}(t))$ and from Proposition 8 and Lemma 9, we get the following inequality:

$$\begin{aligned} \frac{d}{dt} V(t) &\leq -\left(\theta\lambda - 2k_f \lambda_{\max}(P)\right) \bar{e}^\top(t) \bar{e}(t) \\ &\quad - \theta \tau_M \left(\frac{1}{\tau_M} - \mu_1 \left(1 + \frac{1}{\mu_2}\right) \theta^2\right) \int_{t-\tau_M}^t (\bar{e}_2(s))^2 ds \\ &\quad + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{s \in [t-2\tau_M, t]} V_1(s), \end{aligned}$$

for any positive scalars $\mu_i, i = 1, \dots, 3$, where $\varpi(\mu_1, \mu_2, \mu_3)$ is defined in (16). From (22), we deduce that

$$\frac{1}{\tau_M} - \mu_1 \left(1 + \frac{1}{\mu_2}\right) \theta^2 > 0.$$

Then by applying Lemma 9, Lemma 7, and since $V_1(s) \leq V(s)$, we obtain

$$\begin{aligned} \frac{d}{dt} V(t) &\leq -\left(\theta\lambda - 2k_f \lambda_{\max}(P)\right) \bar{e}^\top(t) \bar{e}(t) \\ &\quad - \theta \left(\frac{1}{\tau_M} - \mu_1 \left(1 + \frac{1}{\mu_2}\right) \theta^2\right) V_2(t) \\ &\quad + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{s \in [t-2\tau_M, t]} V(s). \end{aligned}$$

Finally, from (19) and the inequality below

$$-\bar{e}^\top(t) \bar{e}(t) \leq -\frac{1}{\lambda_{\max}(P)} V_1(t),$$

we get

$$\frac{d}{dt} V(t) \leq -\alpha V(t) + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{[t-2\tau_M, t]} V(s),$$

where

$$\alpha \triangleq \min\left(\frac{\theta\lambda - 2k_f \lambda_{\max}(P)}{\lambda_{\max}(P)}, \frac{1}{\tau_M} - \mu_1 \left(1 + \frac{1}{\mu_2}\right) \theta^2\right)$$

$\varpi(\mu_1, \mu_2, \mu_3)$ is defined in (16). From (21) and (22) we have $\varpi(\mu_1, \mu_2, \mu_3) < \alpha$. Consequently, we deduce from Lemma 4 that there exist two positive scalars η and δ such that

$$V(\bar{e}(t)) \leq \eta e^{-\delta(t-t_0)}, \quad \forall t \geq t_0,$$

which means that the estimation error is exponentially stable to zero. This ends the proof.

Remark 11 All the scalar positive quantities $\mu_i, i = 1, 3$ come from the application of Young inequality appropriately. The variable μ_1 is considered as a decision variable in the LMI (18), while the μ_2 and μ_3 appear in (21)-(22). They should be fixed adequately to increase the value of the tolerated τ_M . However, since they are involved nonlinearly, it is difficult to optimize their computation. To simplify the computation of τ_1 and τ_2 , we chose as values of μ_2 and μ_3 those often used in the Young inequality, namely $\mu_2 = \mu_3 = 1$. Therefore, (21)-(22) are simplified as follows:

$$\tau_1 \triangleq \sqrt{\frac{\lambda_{\min}(P) [\lambda\theta - 2k_f\lambda_{\max}(P)]}{4\mu_1\theta\lambda_{\max}(P) [k_1^2 + K_1^2\theta^2]}} \quad (27)$$

$$\tau_2 \triangleq \frac{1}{2\mu_1\theta^2 + \frac{[\lambda\theta - 2k_f\lambda_{\max}(P)]}{\lambda_{\max}(P)}}. \quad (28)$$

Remark 12 In the particular case where the system contains only one nonlinearity in the last component ($f_i \equiv 0, i = 1, \dots, n-1$), then Theorem 10 with $k_1 = 0$ can be applied to get larger values of the tolerate τ_M . Indeed, in such a situation, the dynamics of \bar{e}_1 in (12) does not contain the term $\frac{1}{\theta}\Delta f_1$.

Notice that the upper bound of the delay can be very small for high values of θ . This means that for relatively important delays, the considered observer cannot guarantee the exponential convergence of the estimation error. Indeed, an exponential observer with a high value of θ tolerates small upper bound τ_M of the delay. In the next section, to overcome this problem, we propose to use the high-gain like-observer with lower tuning parameter introduced in (Zemouche *et al.*, 2019).

4 HG/LMI Observer based Design

In this section, we extend the HG/LMI observer methodology developed in (Zemouche *et al.*, 2019) for nonlinear systems to system (1) with the objective of improving the allowable value of τ_M while ensuring exponential convergence of the observer.

4.1 HG/LMI-based transformation

From the LPV/LMI method in (Zemouche and Boutayeb, 2013), each nonlinear component Δf_i in (6) can be rewritten under the following form (Zemouche *et al.*, 2019):

$$\Delta f_i = \sum_{j=1}^{i-j_i} \theta^j \psi_{ij} \bar{e}_j + \sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} \bar{e}_{k_i(j)},$$

where

$$k_i(j) = i - (j_i - j), \quad 0 \leq j_i \leq i.$$

It follows that Δf is written as

$$\Delta f = \underbrace{\sum_{i=1}^n \sum_{j=1}^{i-j_i} \theta^j \psi_{ij} v_n(i) \bar{e}_j}_{\Delta f_1} \quad \text{for HG} \\ + \underbrace{\sum_{i=1}^n \sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} v_n(i) \bar{e}_{k_i(j)}}_{\text{for LPV/LMI}}$$

Therefore, the error dynamics (6) is rewritten as follows:

$$\dot{\bar{e}}(t) = \theta(\mathcal{A}(\Psi^\theta) - KC)\bar{e}(t) + T^{-1}(\theta)\Delta f_1 - \theta KC(\bar{e}(t - \tau(t)) - \bar{e}(t)), \quad (29)$$

with

$$\mathcal{A}(\Psi^\theta) = A + \sum_{i=1}^n \sum_{j=1}^{j_i} \psi_{ij}^\theta v_n(i) e_n k_i(j),$$

$$\Psi^\theta = \begin{pmatrix} \psi_{11}^\theta \\ \vdots \\ \psi_{1j_1}^\theta \\ \psi_{21}^\theta \\ \vdots \\ \psi_{2j_2}^\theta \\ \vdots \\ \psi_{nj_n}^\theta \end{pmatrix} \in \mathbb{R}^{\sum_{i=1}^n j_i},$$

and

$$\psi_j^\theta = \frac{\psi_{ik_i(j)}}{\theta^{1+(j_i-j)}}.$$

Define the convex bounded set

$$\mathcal{H}_{j_s}^\sigma = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^n j_i} : \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \leq \Phi_{ij} \leq \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\},$$

for which the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{j_s}^\sigma} = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^n j_i} : \Phi_{ij} \in \left\{ \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}}, \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\} \right\}.$$

where $\bar{\gamma}_{\gamma_{ik_i(j)}} \geq 0$ and $\underline{\gamma}_{\gamma_{ik_i(j)}} \leq 0$ are, respectively, the lower and the upper bounds of the bounded parameter

$\psi_{ik_i(j)}$. Then, for two positive scalars σ_1, σ_2 , we have

$$\sigma_1 \leq \sigma_2 \implies \mathcal{H}_{j_s}^{\sigma_1} \supset \mathcal{H}_{j_s}^{\sigma_2}.$$

It follows that

$$\lim_{\sigma \rightarrow +\infty} \mathcal{H}_{j_s}^{\sigma} = \{0\}.$$

On the other hand, we can show that there exists a constant k_{j_s} independent from θ such that the following inequality holds:

$$\|T^{-1}(\theta)\Delta f_1\| \leq \frac{k_{j_s}}{\theta^{j_s}} \|\bar{e}\|. \quad (30)$$

4.2 HG/LMI synthesis conditions

This section is devoted to the main theorem, which provides sufficient synthesis conditions guaranteeing exponential convergence of the estimation error. The design is based on the use of the HG/LMI technique for the class of nonlinear systems with delayed outputs given in (1).

Theorem 13 *Assume there exist a positive definite matrix P , a matrix \mathcal{Y} of appropriate dimension and real constants $\mu_i > 0, i = 1, \dots, 3, \lambda > 0, \tau_M > 0$ such that the following conditions hold*

$$\begin{bmatrix} \mathcal{H}e\left\{PA(\Psi) - \mathcal{Y}^\top C\right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 I \end{bmatrix} \leq 0, \quad (31)$$

$$\theta > \max\left(\sigma, \sqrt[1+j_s]{\frac{2k_{j_s}\lambda_{\max}(P)}{\lambda}}\right), \quad (32)$$

$$\tau_M \leq \min(\tau_1, \tau_2) \quad (33)$$

where

$$\tau_1 \triangleq \sqrt{\frac{\left[\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)\varpi(\mu_1, \mu_2, \mu_3)}} \quad (34)$$

$$\tau_2 \triangleq \frac{1}{\mu_1\left(1 + \frac{1}{\mu_2}\right)\theta^2 + \frac{\left[\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)}} \quad (35)$$

with

$$\varpi(\mu_1, \mu_2, \mu_3) \triangleq \frac{\mu_1}{\lambda_{\min}(P)} \left[\left(1 + \frac{1}{\mu_3}\right) \left(1 + \mu_2\right) k_1^2 \theta + \left(1 + \mu_2\right) \left(1 + \mu_3\right) K_1^2 \theta^3 \right], \quad (36)$$

$$\mathcal{R} = \begin{bmatrix} 0 & 1 & 0_{1 \times n-2} \end{bmatrix}, \quad (37)$$

$$K = P^{-1}\mathcal{Y}^\top = [K_1 \dots K_n]^\top. \quad (38)$$

Then the observer (3) converges exponentially and the observer gain is given by $L = T(\theta)K$.

PROOF. Consider the following Lyapunov-Krasovskii functional

$$V(t) = V_1(t) + \theta V_2(t),$$

where

$$V_1(t) = \bar{e}^\top(t)P\bar{e}(t),$$

and

$$V_2(t) = \int_{t-\tau_M}^t \int_{\xi}^t (\bar{e}_2(s))^2 ds d\xi.$$

From Schur lemma (Boyd *et al.*, 1994), LMI (31) is equivalent to

$$\begin{aligned} \mathcal{A}(\Psi^\sigma)^\top P + P\mathcal{A}(\Psi^\sigma) - C^\top \mathcal{Y} - \mathcal{Y}^\top C + \frac{1}{\mu_1} \mathcal{Y}^\top \mathcal{Y} \\ + \tau_M \mathcal{R}^\top \mathcal{R} \leq -\lambda I. \end{aligned}$$

By analogy to the proof of Theorem 10, the derivative of V along the trajectories of (29) satisfies

$$\begin{aligned} \frac{d}{dt} V(t) \leq & -\left(\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right) \bar{e}^\top(t)\bar{e}(t) \\ & - \theta \left(\frac{1}{\tau_M} - \mu_1\left(1 + \frac{1}{\mu_2}\right)\theta^2\right) V_2(t) \\ & + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{s \in [t-2\tau_M, t]} V(s) \end{aligned}$$

for any positive scalars $\mu_i, i = 1, \dots, 3$, where $\varpi(\mu_1, \mu_2, \mu_3)$ is defined in (16).

Therefore

$$\frac{d}{dt} V(t) \leq -\alpha V(t) + \varpi(\mu_1, \mu_2, \mu_3) \tau_M^2 \sup_{[t-2\tau_M, t]} V(s),$$

with

$$\alpha \triangleq \min\left(\frac{\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}}{\lambda_{\max}(P)}, \frac{1}{\tau_M} - \mu_1\left(1 + \frac{1}{\mu_2}\right)\theta^2\right).$$

From (34) and (35) we have $\varpi(\mu_1, \mu_2, \mu_3) < \alpha$. Hence, Lemma 4 allows concluding exponential convergence to zero of the estimation error. This ends the proof.

Remark 14 *The previous proof shows the role of the "compromise index" j_s . It allows reducing the tuning parameter value of the observer. Since the expression of the delay depends on the tuning parameter, reducing this later increments the value of the maximum delay τ_M compared to the standard high-gain observer.*

Remark 15 *One important property of the standard high gain observers is the fast exponential convergence. However, such fast convergence has several drawbacks: high sensitivity to measurement noise; peaking phenomenon; and time-delay in the output. Indeed, high values of delay affect the convergence of the observer. It is important to keep in mind that the standard high-gain observer is a particular solution to the proposed methodology. It corresponds exactly to the case $j_s = 0$. This means that if for a given value of the delay τ , the standard high-gain observer converges, then also the proposed HG/LMI based observer converges with the same convergence rate. The proposed observer offers the possibility to adjust the values of j_s and θ to have a good tradeoff between ensuring analytically the exponential convergence, with adjustable convergence rate, and allowing large values of the delay, while standard high-gain observer does not have this possibility. A large value of the delay could make the standard high-gain observer diverge, while the proposed observer can converge by increasing the index j_s to adjust the value of θ . This analysis comes from the interpretation of the design conditions given in Theorem 10 and Theorem 13, and supported by the numerical results provided in Section 5. Furthermore, by adjusting θ and j_s , the proposed observer is able to avoid the peaking phenomenon, to reduce the sensitivity to high-frequency measurement noise, and to enhance the convergence rate if necessary.*

4.3 Application to sampled-data case

Theorem 13 can be applied straightforwardly to the case of systems with sampled output measurements. The output is sampled at instants t_k satisfying

$$0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$$

with $\lim_{t \rightarrow +\infty} t_k = +\infty$. In this case, the sampling period $\tau_k = t - t_k$ is positive with $\tau_k \leq \tau_M, \forall k \geq 0$. To apply the results of the previous sections, we write the sampled-output as a delayed-output, where the delay satisfies all the required conditions. Indeed, the output $y(t_k)$ can be written as

$$y(t_k) = y(t - \tau(t))$$

with $\tau(t) = t - t_k$. For all $t \in [t_k, t_{k+1}]$, we have

$$0 < \tau(t) \leq \tau_M.$$

Hence, Theorem 13 can be applied to build an observer for system (2) based on the sampled-measurements

$y(t_k)$. This application is summarized in the following corollary.

Corollary 16 *Let us consider the following observer (39) corresponding to system (2):*

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + f(\hat{x}(t)) + L(y(t_k) - C\hat{x}(t_k)), \\ t &\in [t_k, t_{k+1}[, \end{aligned} \quad (39)$$

where L is given by (5). Assume there exist a positive definite matrix P and a matrix \mathcal{Y} of appropriate dimensions and real constants $\mu_i > 0, i = 1, \dots, 3, \lambda > 0$, and $\tau_M > 0$ such that the conditions (31)-(42) of Theorem 13 hold. Then the observer (39) converges exponentially.

4.4 Extension to systems with nonlinear output

This section provides an extension of the result to nonlinear output case. Hence we consider system (1) with output measurement

$$y(t) = h(x_1(t - \tau(t))) \quad (40)$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotonic nonlinear function. That is we assume that there exists $0 < \delta \leq 1$ such that

$$\frac{\partial h}{\partial z}(z) \geq \delta, \quad \forall z \in \mathbb{R}. \quad (41)$$

Notice that without condition (41), we lose local observability of the system. Assume also that h is γ_h -Lipschitz. Without loss of generality, we assume that

$$\gamma_h \triangleq \max_{z \in \mathbb{R}} \left(\left| \frac{\partial h}{\partial z}(z) \right| \right) = 1. \quad (42)$$

Indeed, if (42) is not satisfied, then instead of $y(t)$, we can use as measurement the new output

$$y_h(t) \triangleq \frac{y(t)}{\max_{z \in \mathbb{R}} \left(\left| \frac{\partial h}{\partial z}(z) \right| \right)} \triangleq \bar{h}(x_1(t - \tau(t))) \quad (43)$$

with

$$\bar{h}(x_1(t - \tau(t))) \triangleq \frac{h(x_1(t - \tau(t)))}{\max_{z \in \mathbb{R}} \left(\left| \frac{\partial h}{\partial z}(z) \right| \right)}$$

satisfying

$$\max_{z \in \mathbb{R}} \left(\left| \frac{\partial \bar{h}}{\partial z}(z) \right| \right) = 1.$$

Now consider the observer corresponding to (1) and (40) as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L[y(t) - h(\hat{x}_1(t - \tau(t)))], \quad (44)$$

where \hat{x} represents the state estimation and L is the observer gain.

From the differential mean value theorem, there exists $z(t) \in Co(x_1(t - \tau(t)), \hat{x}_1(t - \tau(t)))$ such that

$$h(x_1(t - \tau(t))) - h(\hat{x}_1(t - \tau(t))) = \theta \frac{\partial h}{\partial z}(z) C \bar{e}(t - \tau(t)),$$

where $Co(x_1(t - \tau(t)), \hat{x}_1(t - \tau(t)))$ is the convex hull defined by $x_1(t - \tau(t))$ and $\hat{x}_1(t - \tau(t))$. Then, the dynamics of the transformed error, $\bar{e}(t)$, is given by

$$\begin{aligned} \dot{\bar{e}}(t) = & \theta \left(A - \frac{\partial h}{\partial z}(z) K C \right) \bar{e}(t) + T^{-1}(\theta) \Delta f \\ & + \theta \frac{\partial h}{\partial z}(z) K C \left(\bar{e}(t) - \bar{e}(t - \tau(t)) \right). \end{aligned} \quad (45)$$

By using the HG/LMI technique, the error dynamics (45) can be written under the form:

$$\begin{aligned} \dot{\bar{e}}(t) = & \theta \left(\mathcal{A}(\Psi^\theta) - \frac{\partial h}{\partial z}(z) K C \right) \bar{e}(t) + T^{-1}(\theta) \Delta f_1 \\ & + \theta \frac{\partial h}{\partial z}(z) K C \left(\bar{e}(t) - \bar{e}(t - \tau(t)) \right), \end{aligned} \quad (46)$$

Following the same steps than the previous section, will lead to the following theorem.

Theorem 17 *Assume there exist a positive definite matrix P , a matrix \mathcal{Y} of appropriate dimension and real constants $\mu_i > 0, i = 1, \dots, 3, \lambda > 0, \tau_M > 0$ such that the following conditions hold*

a) *The following LMI conditions (47)-(48) are satisfied:*

$$\begin{bmatrix} \mathcal{H}e \left\{ P \mathcal{A}(\Psi) - \mathcal{Y}^\top C \right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0, \quad (47)$$

$$\begin{bmatrix} \mathcal{H}e \left\{ P \mathcal{A}(\Psi) - \delta \mathcal{Y}^\top C \right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0; \quad (48)$$

- b) θ satisfies (32), subject to (47)-(48);
c) τ_M satisfies, (33)-(35), subject to (47)-(48);
d) The observer gain matrix K is given by $K = P^{-1} \mathcal{Y}^\top$.

Then the observer (44) converges exponentially.

PROOF. The proof follows exactly the same steps of the previous section. On the other hand, assuming (42)

is important and allows using the same developments established in Theorem 13 with slight modifications. Indeed, due to (42), the term $\frac{\partial h}{\partial z}(z)$ appears only in the matrix block $\mathcal{H}e \left\{ P \mathcal{A}(\Psi) - \frac{\partial h}{\partial z}(z) \mathcal{Y}^\top C \right\}$. Since this matrix is convex in $\frac{\partial h}{\partial z}(z)$, then from the convexity principle, it is sufficient to solve the LMIs with $\max_{z \in \mathbb{R}} \left(\frac{\partial h}{\partial z}(z) \right) = 1$ and $\min_{z \in \mathbb{R}} \left(\frac{\partial h}{\partial z}(z) \right) = \delta$. Hence the two LMIs (47)-(48).

Remark 18 *The linear case can be deduced straightforwardly from the nonlinear output case by taking $\delta = 1$. Indeed, $h(\cdot)$ is linear if and only if $\frac{\partial h}{\partial z}(z) \equiv \text{Constant}$. $\frac{\partial h}{\partial z}(z)$ is identically constant if and only if*

$$\max_{z \in \mathbb{R}} \left(\frac{\partial h}{\partial z}(z) \right) = \min_{z \in \mathbb{R}} \left(\frac{\partial h}{\partial z}(z) \right).$$

From (42), $\frac{\partial h}{\partial z}(z)$ is constant if $\delta = 1$. In this case, LMIs (47) and (48) are identical, and then reduced to (47) only, which corresponds to the linear case.

4.5 Numerical design procedure

This section is devoted to a numerical observer design procedure. Due to the presence of several decision variables as observer parameters, in the previous theorems, a well-structured numerical design procedure will help the users to implement the proposed methodology. The proposed design procedure is based on the use of the gridding method. We introduce a bijective change of variables $\varrho = \frac{\sigma}{1-\sigma}$, ($\sigma = \frac{\varrho}{1+\varrho}$) where the new variable $\varrho \in]\frac{1}{2}, 1]$. The proposed procedure allows obtaining a lower design parameter, θ , and a larger upper bound on the delay, ϱ , provided by Theorem 13 (which is applicable also on Corollary 16). To solve LMI (31), we use Matlab LMI Toolbox and YALMIP. Furthermore, LMI (31) are always feasible (Zemouche *et al.*, 2019), however they have an infinite number of solutions and depend on σ (or equivalently, on ϱ). Then the gridding method will return the solution giving a lower value of θ , and a larger bound on ϱ . On the other hand, it is worth noting that LMI (31) depends on τ_M , which is computed by (33)-(35) after solving the (31). Then, to solve the LMI (31) independently from τ_M , we need to introduce a new variable $\bar{\tau} > 0$, and solve (31) with $\bar{\tau} > 0$ instead of τ_M , *i.e.*:

$$\begin{bmatrix} \mathcal{H}e \left\{ P \mathcal{A}(\Psi) - \mathcal{Y}^\top C \right\} + \bar{\tau} \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} < 0, \quad (49)$$

Hence, by choosing

$$\tau_M = \min(\bar{\tau}, \tau_1, \tau_2) \quad (50)$$

we guarantee exponential convergence of the observer because LMI (31) still feasible for any $\tau_M \leq \bar{\tau}$. Furthermore, inequality (49) can return a $\bar{\tau}$ value very close to zero, which is not suitable because the objective is to get a value of $\bar{\tau}$ large enough to force τ_M to take either the value of τ_1 or that of τ_2 . To this end, a solution consists in fixing $\bar{\tau}$ in (49). Indeed, we cannot maximise $\bar{\tau}$ because it is obvious to show that if (49) is feasible for $\bar{\tau} = 1$, then it still remains feasible for any $\bar{\tau} > 0$. A change of variable to eliminate $\bar{\tau}$ from (49) cannot be performed since μ_1 is used to calculate τ_1 and τ_2 . The procedure is summarized in Algorithm 1, which will be implemented in Section 5 to show the performances of the proposed methodology, compared to those of the standard high-gain observer.

Algorithm 1: A numerical design procedure

Step 1. Choose a small $\epsilon > 0$ for the gridding, take

$\varrho = \frac{1}{2}$, appropriate values for $\lambda > 0, \mu_1 > 0$, and a sufficiently high value $v_{gain} > 0$ and go to **Step 2**;

Step 2. While $\varrho + \epsilon < 1$, take $\varrho := \varrho + \epsilon$ and go to **Step 3**;

Step 3. Solve LMI (49) with respect to $\lambda, P > 0, \varrho > 0$, for a given $\bar{\tau} > 0$.

Step 4. Take $\sigma = \frac{\varrho}{(1-\varrho)}, K = P^{-1}\mathcal{Y}^\top$ and

compute

- $\theta = \max\left(\sigma, \sqrt[1+j_s]{\frac{2k_{j_s}\lambda_{\max}(P)}{\lambda}}\right)$;

- $L = T(\theta)K$;

if $v_{gain} > \|L\|$ then

 put $v_{gain} := \|L\|$ and go to **Step 2**

else

 return

- $\tau_1 = \sqrt{\frac{\left[\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)\varpi(\mu_1, \mu_2, \mu_3)}};$

- $\tau_2 = \frac{1}{\mu_1\left(1 + \frac{1}{\mu_2}\right)\theta^2 + \frac{\left[\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)}};$

- $\tau_M = \min(\bar{\tau}, \tau_1, \tau_2)$.

Remark 19 *There are several methods applicable for the same class of systems studied in this paper that avoid bounds on the delay by either using chain of observers (Germani et al., 2002), (Cacace et al., 2014) or by using predictors (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015). These papers proposed effective methods based on elegant mathematical arguments overcoming the problem of presence of arbitrarily long delay in the output. What we propose in this paper can be viewed as an alternative method, which improves existing results in the literature. Anywhere the standard high-gain observer is used for systems with delayed out-*

puts, the proposed methodology can be applied to improve the results while ensuring exponential convergence for large values of the delay. The choice of Lyapunov based-functional can lead to delay-independent stability conditions. For instance, the chain of observers (Germani et al., 2002), (Cacace et al., 2014) and observer-predictor in (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015) can be effective for the compensation of the delay. However, the determination of an implementable form for the observer-predictor feedback gains over the past time interval can be challenging. Furthermore, the construction of Lyapunov-Krasovskii functional for exponential stability analysis of the observer error dynamics under the observer-predictor scheme is difficult to carry due to the output delayed measurement state. It is worth mentioning that the practical implementation of the high-gain predictor-observer requires the future values of the measurement state, which can significantly increase the computation. On the other hand, the proposed observer in this paper is simple to implement on real-world models without any computational complexity. The observer design parameters are also easy to compute.

Remark 20 *The methodology established in the paper may open the door to further contributions and new ideas to solve other control problems, namely output feedback stabilization; reference trajectory tracking; self-synchronization in networks of multi-agent systems. More importantly, the proposed methodology can be used as a design tool in several alternative approaches, like those using a chain of observers (Germani et al., 2002), (Ahmed-Ali et al., 2009), (Cacace et al., 2014), or those using prediction part (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015).*

5 Numerical Comparisons

To show the effectiveness of the proposed methodology, we present in this section two numerical examples. We will provide some comparisons between the standard high-gain observer, the high-gain observer method presented in (Van Assche et al., 2011) and the proposed HG/LMI based observer. The simulations will be carried out by using MATLAB.

5.1 Example 1

The aim of this example is to compare the proposed approach to the standard high-gain design and the high-gain observer proposed in (Van Assche et al., 2011). We consider the following fifth order nonlinear system with only a single nonlinearity in the last component including a delay at output measurement:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)) \\ y(t) = Cx(t - \tau(t)), \end{cases} \quad (51)$$

Table 1

Comparison between the proposed HG/LMI observer (HG/LMI), the standard (SHG) high-gain observer and the high-gain observer (VVA) proposed in (Van Assche *et al.*, 2011) for different values of k_f .

Methods	k_f	j_s	σ	θ	τ_M	$\ L\ $
VVA	0.1	0	/	3407	3.3760×10^{-11}	4.5906×10^{17}
SHG		0	/	6.3638	2.3661×10^{-06}	15875
HG/LMI		1	2.5088	2.5088	8.2499 $\times 10^{-05}$	269.5916
		2	1.8571	1.8571	1.6505×10^{-04}	89.7547
VVA	1	0	/	34070	3.0255×10^{-12}	4.5906×10^{22}
SHG		0	/	42.3560	1.5305×10^{-07}	5.2321×10^{08}
HG/LMI		1	7	7	2.3810×10^{-05}	84446
		2	4.1282	4.1282	3.6851×10^{-05}	9330.5
VVA	5	0	/	170350	3.9542×10^{-13}	1.4346×10^{26}
SHG		0	/	186.6738	9.0240×10^{-09}	1.8313×10^{12}
HG/LMI		1	17.1818	17.1818	3.8231×10^{-06}	1.9828×10^{07}
		2	10.7647	10.7647	4.1180×10^{-06}	3.3268×10^{06}

where

$$(A)_{i,j} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{if } j \neq i + 1. \end{cases}$$

$$B = [0 \ 0 \ 0 \ 0 \ 1]^\top \quad \text{and} \quad C = [1 \ 0 \ 0 \ 0 \ 0].$$

The nonlinearity is defined by

$$f(x) = \frac{k_f}{5} \sum_{i=1}^5 \sin(x_i). \quad (52)$$

In the sequel, we will provide comparisons between the standard high-gain observer, the observer by (Van Assche *et al.*, 2011) and the HG/LMI based design.

The comparison results are shown in Table 1, which illustrates how the values of the tuning parameter are decreased and the maximum bounds on the delay become larger. For $j_s = 2$, the value of the tuning parameter θ is significantly reduced to $\theta = 1.8571$, compared to $\theta = 3407$ obtained by (Van Assche *et al.*, 2011), for $k_f = 0.1$. It is also reduced from $\theta = 170350$ with (Van Assche *et al.*, 2011) to $\theta = 10.7647$ for $k_f = 5$. Meanwhile, in comparison to the standard high-gain observer, the design parameter θ is decreased from $\theta = 6.3638$ to $\theta = 1.8571$ for $k_f = 0.1$ and from $\theta = 186.6738$ to $\theta = 10.7647$ for $k_f = 5$. On the other hand, the maximum bound on the delay given by the observer in (Van Assche *et al.*, 2011) and the standard high-gain observer considerably increased from $\tau_M = 3.3760 \times 10^{-11}$ and $\tau_M = 2.3661 \times 10^{-6}$, respectively, to $\tau_M = 1.6505 \times 10^{-4}$ for $k_f = 0.1$ with HG/LMI approach. The previous values are decreased more by increasing the value of the compromise index j_s . We notice that the HG/LMI observer gain is considerably reduced compared to the standard high gain observer and the observer in (Van Assche

et al., 2011). For instance, for $k_f = 0.1$, the norms of the gains obtained by the standard high-gain observer and the observer in (Van Assche *et al.*, 2011) are significantly decreased from 15875 and 4.5906×10^{17} , respectively, to 89.7547 with the HG/LMI approach for $j_s = 2$.

Table 2 provides percentage of reduction/increment of the design parameter and the maximum bound of the delay, according to the following formulas:

$$\Delta_{\theta,i} = \frac{\theta_i - \theta_{HG/LMI}}{\theta_i} \%, \quad (53)$$

$$\Delta_{\tau,i} = \frac{\tau_{M,HG/LMI} - \tau_{M,i}}{\tau_{M,HG/LMI}} \%, \quad (54)$$

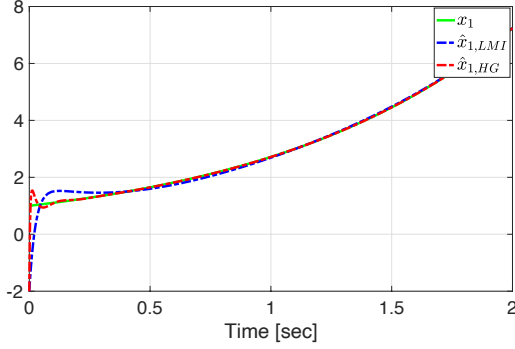
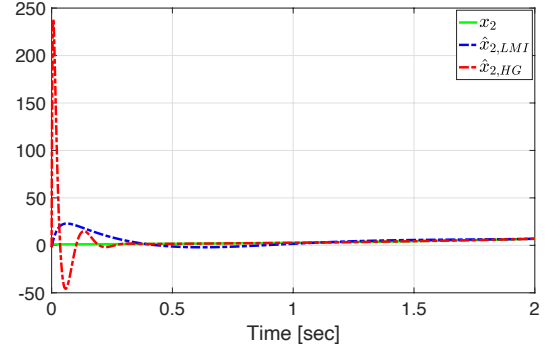
where $\theta_{M,i}$, τ_i , $\theta_{HG/LMI}$, and $\tau_{M,HG/LMI}$ stand for the design parameter and the maximum bound of the delay of the observer in (Van Assche *et al.*, 2011), the standard high-gain observer, and the HG/LMI observer, respectively. The index i refers to VVA or SHG. We notice that the percentage of reduction of the design parameter is up to 90.2536% for $j_s = 2$ and $k_f = 1$ in comparison to the initial value given by the standard high-gain observer. It is also up to 99.999879% in comparison to the initial value provided by the observer in (Van Assche *et al.*, 2011). Meanwhile, the improvement of the maximum bound of the delay is greater than 99.5846% and 99.999991% for $j_s = 2$ and $k_f = 1$, compared to both the standard high-gain observer and the observer by (Van Assche *et al.*, 2011), respectively.

Simulations have been carried out with $k_f = 1$, and comparisons between the HG/LMI based observer and the standard high-gain observer are provided. The simulation with the approach proposed in (Van Assche *et al.*, 2011) cannot occur due to the high value of the

Table 2

Percentage of reduction/increment for different values of $j_s = 0, 1, 2$.

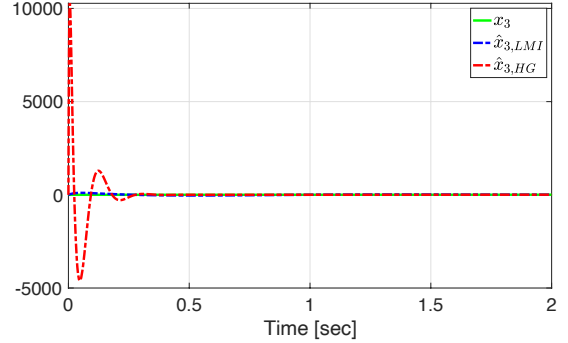
Methods	k_f	j_s	θ	$\Delta_{\theta,SHG}$	$\Delta_{\theta,VVA}$	τ_M	$\Delta_{\tau,SHG}$	$\Delta_{\tau,VVA}$
VVA		0	3407	/	0	3.3760×10^{-11}	/	0
SHG		0	6.3638	0	/	2.3661×10^{-06}	0	/
HG/LMI	0.1	1	2.5088	60.5770	99.9264	8.2499×10^{-05}	97.1319	99.999959
		2	1.8571	70.8178	99.9455	1.6505×10^{-04}	98.5664	99.999979
VVA		0	34070	/	0	3.0255×10^{-12}	/	0
SHG		0	42.3560	0	/	1.5305×10^{-07}	0	/
HG/LMI	1	1	6	85.8344	99.9824	2.3810×10^{-05}	99.3572	99.999987
		2	4.1282	90.2536	99.9879	3.6851×10^{-05}	99.5846	99.999991

Fig. 1. Behaviour of x_1 and its estimates for $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$.Fig. 2. Behaviour of x_2 and its estimates for $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$.

gain and numerical instabilities.

By using MATLAB, the obtained value of the design parameter of the standard high-gain observer is $\theta = 42.3560$. By choosing the compromise index as $j_s = 2$, the obtained value of the tuning parameter by applying the HG/LMI approach is $\theta = 4.1282$, which is significantly reduced compared to that obtained by the standard high-gain based approach. In addition, the maximum bound on the delay obtained with the standard high-gain based approach is $\tau_M = 1.5310 \times 10^{-7}$, while with the HG/LMI observer we got a larger value, $\tau_M = 3.6851 \times 10^{-5}$. Denote by $\hat{x}_{LMI} = [\hat{x}_{1,LMI}, \hat{x}_{2,LMI}, \hat{x}_{3,LMI}, \hat{x}_{4,LMI}, \hat{x}_{5,LMI}]^T$ and $\hat{x}_{HG} = [\hat{x}_{1,HG}, \hat{x}_{2,HG}, \hat{x}_{3,HG}, \hat{x}_{4,HG}, \hat{x}_{5,HG}]^T$ the state estimates for the system (51) by using the observer design method proposed in the present paper and the standard high-gain observer, respectively. Let $\hat{x}_{LMI}(0) = [-1, -1, -1, -1, -1]^T$ and $\hat{x}_{HG}(0) = [-2, -2, -2, -2, -2]^T$. The simulation results are depicted in Figures 1-5, which provide the behaviors of x_i and its estimates $\hat{x}_{i,LMI}$, $\hat{x}_{i,HG}$, $i = 1, \dots, 5$, respectively. It is quite clear that both estimated states converge to the actual states, however, the proposed HG/LMI based observer considerably reduces the peaking phenomenon.

5.2 Example 2

Fig. 3. Behaviour of x_3 and its estimates for $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$.

To evaluate the performance and superiority of the proposed HG/LMI observer based design method, compared to the observer developed in (Van Assche *et al.*, 2011) for systems with multi-nonlinearties, we consider the example studied in (Van Assche *et al.*, 2011) with a slight modification on the last component to cope with the same class of system investigated in this paper. The system is described by the following equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - l_1 x_1(t) \\ \dot{x}_2(t) = c_1 c_2 \sin(x_1(t)) + c_1 c_3 \cos(x_2(t)) - c_1 c_4 u(t) \\ y(t) = x_1(t - \tau(t)), \end{cases} \quad (55)$$

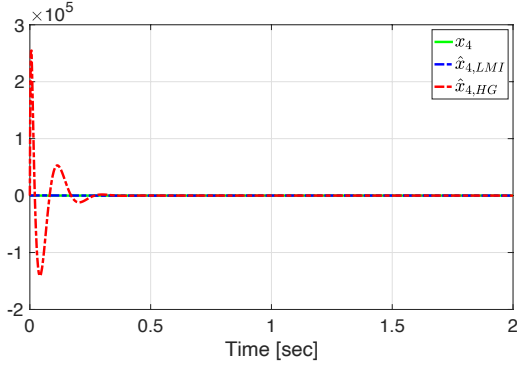


Fig. 4. Behaviour of x_4 and its estimates for $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$.

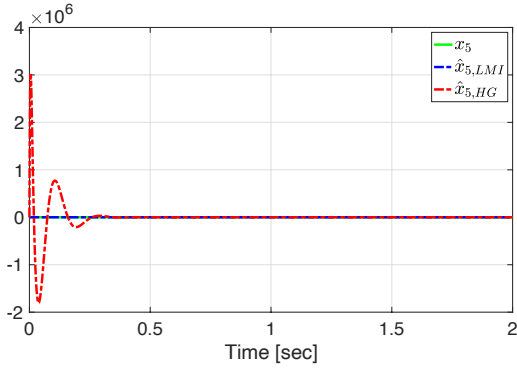


Fig. 5. Behaviour of x_5 and its estimates for $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$.

The values of the parameters are set to $c_1 = 1$, $c_2 = c_3 = 0.02$, $c_4 = 8$, $l_1 = 0.04$. The input function is $u(t) = \sin(0.35t)$. System (55) is in the canonical form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u) \\ y(t) = x_1(t - \tau(t)), \end{cases} \quad (56)$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f(x, u) = \begin{pmatrix} f_1(x, u) \\ f_2(x, u) \end{pmatrix}$$

where

$$\begin{cases} f_1(x, u) = -l_1 x_1 \\ f_2(x, u) = c_1 c_2 \sin(x_1) + c_1 c_3 \cos(x_2) - c_1 c_4 u(t). \end{cases}$$

The Lipschitz constants of the nonlinearities are $k_{f_1} = k_{f_2} = 0.04$. In (Van Assche *et al.*, 2011), the following high gain observer design was proposed:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}, u) - \theta \Delta^{-1} S^{-1} C^T (C\hat{x}(t - \tau(t)) - y) \\ y(t) = x_1(t - \tau(t)), \end{cases} \quad (57)$$

where

$$\Delta = \text{diag}\left(1, \frac{1}{\theta}\right), \theta > 1, \\ S = S^T > 0, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The matrix $S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is determined by solving the following equation

$$SA + A^T S - C^T C = -S. \quad (58)$$

Table 3 illustrates the results obtained by the approach in (Van Assche *et al.*, 2011) and the HG/LMI based observer design.

Table 3

Percentage of reduction/increment of θ and τ_M , respectively.

Methods	θ	τ_M
VVA observer	1.55	0.01
HG/LMI observer	1.0202	0.0202
Relative error	34.18%	50.49%

We can see that the HG/LMI-based observer decreases the value of the tuning parameter θ by more than 34.18% compared to the one obtained by the observer proposed in (Van Assche *et al.*, 2011). In addition, the maximum bound on the delay is increased by 50.49%. It should be mentioned that the reduction/increment of the design parameter θ and the maximum bound on delay τ_M using the HG/LMI observer is obtained with $j_s = 1$.

6 Conclusion

In this paper, we considered the problem of observer design for a class of nonlinear systems with time-varying delayed output measurements. The delay is assumed to be time-varying. The objective was to develop a state observer with a small tuning parameter allowing a maximum bound of the delay as high as possible while ensuring exponential convergence. To this end, we extended the HG/LMI observer design introduced in (Zemouche *et al.*, 2019) for delayed output measurements, which led to a considerably higher allowable maximum bound on the delay compared to the standard high-gain methodology. The convergence analysis is established by using a Lyapunov-Krasovskii functional, depending on the tuning parameter of the observer, jointly with the Halanay inequality. On the other hand, the explicit relation between the tuning parameter of the observer and the maximum bound of the delay provided in this note, show analytically the superiority of the proposed method with respect to the standard high-gain observer design. Two examples are provided to illustrate the performance of

the proposed observer design procedure in comparison to the standard high-gain and to the high-gain observer proposed by (Van Assche *et al.*, 2011). Moreover, extensions to systems with nonlinear outputs and to systems with sampled measurements are established.

As a future work, we aim at improving the result by exploring new ideas on high-gain observers, namely the introduction of specific nonlinear transformations to decrease the value of the tuning parameter. We also plan to extend this result to systems with event-triggered measurements. This will allow the application of the observer to industrial real-world applications such as wastewater treatment plants and petrochemical systems.

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A Some Details on the Proof of Theorem 17

The proof is based on the same Lyapunov-Krasovskii functional as defined in (26). From Schur lemma (Boyd *et al.*, 1994), the inequality

$$\begin{aligned} \mathcal{H}e\left\{P\mathcal{A}(\Psi^\sigma) - \frac{\partial h}{\partial z}(z)\mathcal{Y}^\top C\right\} + \frac{1}{\mu_1}\mathcal{Y}^\top \mathcal{Y} \\ + \tau_M \mathcal{R}^\top \mathcal{R} \leq -\lambda I, \quad \forall \Psi^\sigma \in \mathcal{H}_{j_s}^\sigma. \end{aligned}$$

is satisfied if the following one hold:

$$\begin{bmatrix} \mathcal{H}e\left\{P\mathcal{A}(\Psi^\sigma) - \frac{\partial h}{\partial z}(z)\mathcal{Y}^\top C\right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0. \quad (\text{A.1})$$

Using the fact that

$$\delta \leq \frac{\partial h}{\partial z}(z) \leq 1, \quad \forall z \in \mathbb{R}, \quad (\text{A.2})$$

and $\Psi^\sigma \in \mathcal{H}_{j_s}^\sigma$, we deduce that (A.1) is satisfied if the following two of LMIs hold:

$$\begin{bmatrix} \mathcal{H}e\left\{P\mathcal{A}(\Psi^\sigma) - \mathcal{Y}^\top C\right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} \mathcal{H}e\left\{P\mathcal{A}(\Psi^\sigma) - \delta \mathcal{Y}^\top C\right\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ & \mathcal{Y} & -\mu_1 \end{bmatrix} \leq 0.$$

By following the steps of the proof of Theorem (13), we obtain $\frac{d}{dt}V(t) < 0, \forall x(t) \neq 0$ if θ and τ_M satisfy conditions of Lemma 4. Then the conditions on θ and τ_M are derived similarly as in the previous theorems. This ends the proof.