Précise Asymptotics for the Complete Moment Convergence : The General Case

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Pub. Inst. Stat. Univ. Paris
LIV, fasc. 1-2, 2010, 7 à 22

Precise Asymptotics for the Complete Moment Convergence : The General Case

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Abstract: Let $(X,X_i,i\geq 1)$ be a sequence of i.i.d. random variables. Let f be a differentiable function, $0<f(x)\uparrow\infty$ and g be a one to one function, $0<g(x)\uparrow\infty$ with g^{-1} differentiable and regularly varying at infinity with index $\lambda\geq 0$. We achieve a general law of precise asymptotics of complete moment convergence. We give, in an adequate way, a generalization and extension of Lin and Liu's results [13]. Under optimal moment conditions, we give, for all $p\geq 0$, a precise asymptotics of: $1/g^{-1}(\frac{1}{\varepsilon})\sum_{n\geq n_0} f'(n)\ \mathbf{E}\Big|\frac{S_n}{\sqrt{n}}\Big|^p \mathbf{I}\{|S_n|\geq \varepsilon g(f(n))\}\}$.

Keywords: Complete moment convergence, Independent random variables, Precise asymptotics

2000 MR Subject Classification: 60F15, 60G50

1 Introduction:

Throughout this paper, let $(X, X_i, i \geq 1)$ be a sequence of i.i.d. random variables and $S_n = \sum_{k=1}^{n} X_k$. For all increasing positive functions K and positive functions H, we associate the functional defined for all $\varepsilon > 0$, by

$$\nu(\varepsilon) = \sum_{n\geq 1} H(n)\ \mathbf{I}\big(|S_n| \geq \varepsilon\sqrt{n}K(n)\big)\ , \tag{1.1}$$

where $\mathbf{I}$ is the indicator function. The complete convergence of the partial sums of r.v. studies under certain conditions on the moments, the expectation convergence of $\nu(\varepsilon)$, i.e.:

$$\mathbf{E}\nu(\varepsilon) = \sum_{n\geq 1} H(n)\ \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n}K(n)\right\} < \infty. \tag{1.2}$$

It was Hsu and Robbins [12] who introduced this notion. They proved that (1.2) occurs for all $\varepsilon > 0$ if, and only if $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$ when $H(n) = 1$ and $K(n) = \sqrt{n}$, $n \geq 1$. Baum and Katz [1] have generalized this result for the case $H(n) = n^{r-p}$, $K(n) = n^{1/p-1/2}$with $n \geq 1$, $1 \leq p < r$ and $r \geq 2$. They proved that (1.2) occurs for all $\epsilon > 0$ if, and only if $\mathbf{E}X = 0$ and $\mathbf{E}|X|^{rp} < \infty$. Davis [6] has extended this result to the case when H is non polynomial, $H(n) = \ln(n)/n$ and $K(n) = \sqrt{\ln n}$, $n \geq 1$. In the last few years, many authors have given precise asymptotics of $\mathbf{E}\nu(\varepsilon)$ when $\varepsilon \downarrow 0$ (cf [4], [8], [9], [14]).

For all $n \geq n_0$ fixed, put $G(n) = H(n)/\sqrt{n}K(n)$ and define, for all $\varepsilon > 0$ and $p > 0$:

$$L(\varepsilon, p) = p\int_{\varepsilon}^{\infty} t^{p-1}\mathbf{E}\nu(t)dt = \sum_{n\geq n_0} G(n)\ \mathbf{E}\big[|S_n|^p - (\varepsilon\sqrt{n}K(n))^p\big]_+. \tag{1.3}$$

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Chow [5] has introduced the complete moment convergence by studying $L(\varepsilon,1)$ with $\sqrt{n}K(n) = n^{\alpha}$, $\alpha > 1/2$ and $G(n) = n^{\alpha(q-1)-2}$, $q \geq 1$. Lin and Liu [13] gave a precise asymptotics for $\mathbf{L}(\varepsilon,p)$, $0 \leq p < 2$, when $\varepsilon \downarrow 0$ in the case $G(n) = 1/n^2$ and $K(n) = \sqrt{n}$. For $p = 2$, they examined $\mathbf{L}(\varepsilon,2)$ when $\varepsilon \downarrow 0$ in the case $G(n) = \ln^{\delta-1}(n)/n^2, 0 < \delta \leq 1$ (resp. $G(n) = 1/n^2$), $n \geq 1$ and $K(n) = \sqrt{\ln n}$ (resp. $K(n) = \sqrt{n}$), $n \geq 1$. Their results are the following:

Theorem A (Lin, Liu, [13]): *Let $0 \leq p < 2$ The moment conditions*

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \tag{1.4}$$

are equivalent to

$$\lim_{\varepsilon \to 0} \varepsilon^{2-p} \sum_{n \geq 1} \frac{1}{n^p}\, \mathbf{E}|S_n|^p \mathbf{I}(|S_n| \geq \varepsilon n) = \frac{2\sigma^2}{2-p}. \tag{1.5}$$

Theorem B (Lin, Liu, [13]): *The moment conditions*

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}X^2 \ln(|X| \vee e) < \infty \tag{1.6}$$

are equivalent to

$$\lim_{\varepsilon \to 0} \frac{1}{\ln\left(\frac{1}{\varepsilon}\right)} \sum_{n \geq 1} \frac{1}{n^2}\, \mathbf{E}|S_n|^2 \mathbf{I}(|S_n| \geq \varepsilon n) = 2\sigma^2, \tag{1.7}$$

and, for $0 < \delta \leq 1$,

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 \text{ and } \mathbf{E}X^2 \ln^{\delta}(|X| \vee e) < \infty \tag{1.8}$$

are equivalent to

$$\lim_{\varepsilon \to 0} \varepsilon^{2\delta} \sum_{n \geq 1} \frac{(\ln n)^{\delta-1}}{n^2}\, \mathbf{E}|S_n|^2 \mathbf{I}(|S_n| \geq \varepsilon \sqrt{n \ln n}) = \frac{\sigma^{2\delta+2}}{\delta} \mathbf{E}|N(0,1)|^{2\delta+2}. \tag{1.9}$$

In this work, we extend and generalize theorems A and B by giving precise asymptotics, for a large class of functions G and H of $\mathbf{L}(\varepsilon,p)$, $0 \leq p < \infty$, when the random variable X satisfies certain moment conditions.

During the writing of the paper, I learned about the paper written by Y. Zhang, X. Yang and Z. Dong ([7]). The authors of that paper dealt with sufficient conditions to get the precise asymptotics of $\mathbf{L}(\varepsilon,2)$ and in two particular cases of the normalizer's term. Our results, as seen in the sequel, generalize theirs ([7], Theorem 2.1). We note though, for $p > 2$, the hypothesis of aforementioned paper [7] is never valid.

Throughout, $\log_0(x) = x$, $\log_1(x) = \log(x) = \ln(x \vee e)$, and for $n \geq 2$, $\log_n(x) = \log_{n-1}(\log(x))$, $N = N(0,1)$ is the standard normal r.v. Let $[x]$ denote the largest integer $\leq x$, $n_0 \geq 0$ a fixed integer and C denotes positive constants, possibly varying from place to place.

2 Main results :

In the following, f and g are two increasing positive functions defined on $\mathbf{R}^+$such that:
(a) $\lim\limits_{x \to \infty} f(x) = \infty$ and f is differentiable.
(b) $\lim\limits_{x \to \infty} g(x) = \infty$, g is one to one function with differentiable inverse function g^{-1}which is regularly varying, with index $\lambda \geq 0$.

(c) For all $a > 0$, there exists $C > 0$ such that: $\lim_{x\to\infty} \frac{g(f(ax))}{\sqrt{x}} \leq C$.

Let h a function defined, for all $n \geq n_0$, by : $h(\sqrt{n}g(f(n))) = n$, extended to $\mathbf{R}^+$ by linear interpolating. The function h is strictly increasing, bijective and satisfying $h(\sqrt{n}) \leq n \leq h(n)$ for n large enough.

We will prove the following theorems: Let $g(x) = x^{\frac{1}{\lambda}}$, $x > 0$,

Theorem 1: *Let $\lambda \geq 2 - p$, $0 \leq p < 2$. The following propositions are equivalent:*

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}X^2 (f \circ h)^{1-\frac{2-p}{\lambda}}(|X|) < \infty \tag{2.1}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{\lambda} \sum_{n\geq n_0} f'(n)\, \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}(|S_n| \geq \varepsilon\sqrt{n}\, f^{1/\lambda}(n)) = \sigma^{\lambda+p}\mathbf{E}|N|^{\lambda+p}. \tag{2.2}$$

Remark 2.0 For x large enough, $f \circ h(x) = x^{\lambda}/h^{\frac{\lambda}{2}}(x)$. The condition (2.1) is equivalent to

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}\left\{|X|^{\lambda+p}/h^{\frac{\lambda+p}{2}}(|X|)\right\} < \infty.$$

Corollary 1: *For $0 \leq p < 2$, we have :*

$$\mathbf{E}X = 0 \text{ and } \mathbf{E}X^2 = \sigma^2 < \infty \Longleftrightarrow \lim_{\varepsilon\to 0} \varepsilon^{2-p} \sum_{n\geq n_0} f'(n)\, \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}(|S_n| \geq \varepsilon\sqrt{n}\, f^{\frac{1}{2-p}}(n)) = \sigma^2.$$

This generalizes Theorem A, which one can find by taking $f(n) = n^{1-p/2}$, $n \geq 1$. If $p = 0$, we obtain a result which is similar to that of Heyde [11]:

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \Longleftrightarrow \lim_{\varepsilon\to 0} \varepsilon^{\lambda} \sum_{n\geq n_0} f'(n)\, \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n} f^{\frac{1}{\lambda}}(n)\right\} = \sigma^{\lambda}\, \mathbf{E}|N|^{\lambda} \quad (\lambda \geq 2).$$

Let $s > 0$ such that $1/s > p \geq 0$. In Theorem 1, for $\lambda = 1/s - p \geq 2$ and let f_1 a function defined, for all $n \geq n_0$, by : $f(n) = f_1^{1-ps}(n)$, we obtain the result of [7] with the best moment conditions.

Let f and g satisfying conditions (a), (b), and (c). We have:

Theorem 2: *For $p \geq 2$ and $\lambda \geq 0$, the following propositions are equivalent:*

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}|X|^p\, f \circ h(|X|) < \infty \tag{2.1}'$$

$$\lim_{\varepsilon\to 0} \frac{1}{g^{-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n\geq n_0} \frac{f'(n)}{n^{p/2}}\, \mathbf{E}\left[|S_n|^p - (\varepsilon\sqrt{n}\, g \circ f(n))^p\right]_+ = \frac{p\,\sigma^{\lambda+p}}{\lambda+p}\mathbf{E}|N|^{\lambda+p}. \tag{2.2}'$$

Remark 2.1 The condition (b) above implies the existence of a slowly varying function at infinity, L, such that: $g^{-1}(x) = x^{\lambda}L(x)(x \geq 0)$. The condition (c) provides the connection between f and g and is in fact equivalent, for x large enough, to $f(x) \leq Cg^{-1}(\sqrt{x}) = Cx^{\lambda/2}L(\sqrt{x})$.

Remark 2.2 For $b > 0$ such that $f(x)\log^{-b}(x)(x > 0)$ is decreasing, the moment condition $\mathbf{E}X^2(f \circ h)^{1-\frac{2-p}{\lambda}}(|X|) < \infty$ (resp. $\mathbf{E}|X|^p\, f \circ h(|X|) < \infty$) is equivalent to $\mathbf{E}X^2\, f^{1-\frac{2-p}{\lambda}}(|X|) < \infty$ (resp. $\mathbf{E}|X|^p\, f(|X|) < \infty$).

Remark 2.3 For x large enough, $f \circ h(x) = g^{-1}(x/\sqrt{h(x)}) \leq g^{-1}(x)$ and $x\dfrac{(g \circ f)'(x)}{(g \circ f)(x)} \leq 1$.

Remark 2.4 One has $\lim_{\varepsilon\to 0} f\circ h(|X|/\varepsilon)/g^{-1}(1/\varepsilon) \le 1$, if $\lambda = 0$ and the limit equals 0, if $\lambda > 0$.
The following propositions will allow us to deduce the results of Lin and Liu [13] as particular cases of ours:

case 1: Let $e_1(x) = e^x$, and for $k \ge 2$, $e_k(x) = e_{k-1}(e^x)$. We assume that for $n \ge n_0$, $f(n) \le C\log_k(\sqrt{n})$. The condition (c) is then satisfied for $g(n) = e_k(n)$ and we have the following proposition:

Proposition 1: *Let $p \ge 2$. For any $k \ge 1$, the moment conditions :*

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}|X|^p f(|X|) < \infty,$$

are equivalent to

$$\lim_{\varepsilon\to 0} \frac{1}{\log_k\left(\frac{1}{\varepsilon}\right)} \sum_{n\ge n_0} f'(n)\ \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\{|S_n| \ge \varepsilon\sqrt{n}e_k(f(n))\}\} = \sigma^p\mathbf{E}|N|^p. \tag{2.3}$$

Observe that if $f(n) = \log_k\sqrt{n}$ or $\log_{k-1}\sqrt{\log n}$ then $(2.1)'$ is equivalent to

$$\mathbf{E}X = 0,\ \mathbf{E}X^2 = \sigma^2 < \infty \text{ and } \mathbf{E}|X|^p \log_k|X| < \infty, \tag{2.4}$$

it follows the corollaries:

Corollary 2: *For $p \ge 2$, we have the following equivalencies:*

$$(2.4) \Longleftrightarrow \lim_{\varepsilon\to 0} \frac{1}{\log_k\left(\frac{1}{\varepsilon}\right)} \sum_{n\ge n_0} \frac{1}{n\prod_{j=1}^{k-1}\log_j(\sqrt{n})}\ \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\{|S_n| \ge \varepsilon n\} = 2\sigma^p\mathbf{E}|N|^p, \quad k \ge 1,$$

$$\Longleftrightarrow \lim_{\varepsilon\to 0} \frac{1}{\log_{k-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n\ge n_0} \frac{\prod_{j=0}^{k-2}\log_j^{-1}(\sqrt{\log n})}{n\sqrt{\log n}}\ \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\{|S_n| \ge \varepsilon\sqrt{n\log n}\} = 2\sigma^p\mathbf{E}|N|^p,\ k \ge 2.$$

For $p = 2$, the first equivalence generalizes the first result of Theorem B and extends it to the case $p > 2$, and the second spreads the equivalence $(1.8) \Longleftrightarrow (1.9)$ to the case $\delta = 0$ and $p \ge 2$. In that case, the change of the normalizing term seems to be necessary and we have, for $p = 2$:
$\lim_{\varepsilon\to 0} \frac{1}{\log\left(\frac{1}{\varepsilon}\right)} \sum_{n\ge n_0} \frac{1}{n^2\log(n)}\ \mathbf{E}|S_n|^2\mathbf{I}\{|S_n| \ge \varepsilon\sqrt{n\log n}\} = \sigma^2$ if and only if, $\mathbf{E}X = 0$, $\mathbf{E}X^2 = \sigma^2$ and $\mathbf{E}X^2\log\log|X| < \infty$.
Let $k = 1$ and f_1 a function defined, for all $n \ge n_1$, by : $f_1(n) = e^{2f(n)}$. Proposition 1 gives, for $p = 2$, the result of [7], with necessary and sufficient conditions:

Corollary 3: *The moment conditions:* $\mathbf{E}X = 0$, $\mathbf{E}X^2 = \sigma^2$ *and* $\mathbf{E}X^2\log|X| < \infty$ *occur if and only if,*

$$\lim_{\varepsilon\to 0} \frac{1}{\log\left(\frac{1}{\varepsilon}\right)} \sum_{n\ge 1} \frac{f_1'(n)}{f_1(n)}\ \mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^2 \mathbf{I}\{|S_n| \ge \varepsilon\sqrt{nf_1(n)}\} = 2\sigma^2.$$

case 2: We define for $n \ge 1$, $g(n) = n^{1/\lambda}$. Condition (c) imposes that, for $n \ge n_0$, $f(n) \le Cn^{\lambda/2}$. In this case we have the following result:

Proposition 2: *The moment conditions* (2.1)′ *are equivalent to*

$$\lim_{\varepsilon\to 0}\varepsilon^{\lambda}\sum_{n\geq n_0} f'(n)\,\mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\left\{|S_n|\geq \varepsilon\sqrt{n}f^{\frac{1}{\lambda}}(n)\right\} = \sigma^{\lambda+p}\mathbf{E}|N|^{\lambda+p},\quad p\geq 2. \tag{2.5}$$

:t us point out that Proposition 2 is a natural extension of Theorem 1 to the case $p\geq 2$:
Corollary 4: *For $\lambda>0$, $p\geq 0$ and $\lambda+p\geq 2$, the following propositions are equivalent:*

$$\mathbf{E}X=0,\ \mathbf{E}X^2=\sigma^2<\infty \text{ and } \mathbf{E}|X|^{\lambda+p}/h^{\frac{\lambda}{2}}(|X|)<\infty.$$

$$\lim_{\varepsilon\to 0}\varepsilon^{\lambda}\sum_{n\geq n_0} f'(n)\,\mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\left\{|S_n|\geq \varepsilon\sqrt{n}f^{\frac{1}{\lambda}}(n)\right\} = \sigma^{\lambda+p}\mathbf{E}|N|^{\lambda+p}.$$

:t $k\geq 0$. For $\lambda=2\delta>0$, we put for any $n\geq 1$, $f(n)=\log_k^{\delta} n$. Then (2.1)′ becomes

$$\mathbf{E}X=0,\ \mathbf{E}X^2=\sigma^2 \text{ and } \mathbf{E}X^2\log_k^{\delta}|X|<\infty. \tag{2.6}$$

om this follows a generalization and an extension of the equivalence (1.8) $\Longleftrightarrow$ (1.9); the condi-
ɔn $\delta\leq 1$ is not necessary:

Corollary 5: *The moment conditions* (2.6)) *are equivalent to*

$$\lim_{\varepsilon\to 0}\varepsilon^{2\delta}\sum_{n\geq 1}\frac{\log_k^{\delta-1}(n)}{n\prod_{j=1}^{k-1}\log_j n}\,\mathbf{E}\left|\frac{S_n}{\sqrt{n}}\right|^p \mathbf{I}\left\{|S_n|\geq \varepsilon\sqrt{n\log_k n}\right\} = \frac{\sigma^{2\delta+p}}{\delta}\mathbf{E}|N|^{2\delta+p},\ \delta>0,\ p\geq 2. \tag{2.7}$$

Proof of Theorem 1 and Theorem 2 : We only discuss the case $p>0$, because for $p=0$,

$$\mathbf{E}|S_n|^p\mathbf{I}\big(|S_n|\geq\varepsilon\sqrt{n}g(f(n))\big)=\mathbf{P}\big(|S_n|\geq\varepsilon\sqrt{n}g(f(n))\big).$$

ithout loss of generality, we assume $\sigma^2=1$. We have:

$$\begin{aligned}&\mathbf{E}|S_n|^p\mathbf{I}\big(|S_n|\geq\varepsilon\sqrt{n}g(f(n))\big)\\ &=\varepsilon^p n^{\frac{p}{2}}g^p(f(n))\mathbf{P}\left\{|S_n|\geq\varepsilon\sqrt{n}g(f(n))\right\}+\int_{(\varepsilon\sqrt{n}g(f(n)))^p}^{\infty}\mathbf{P}\left\{|S_n|^p>t\right\}dt.\end{aligned}$$

ıen $\mathbf{E}\big[|S_n|^p-(\varepsilon\sqrt{n}g(f(n)))^p\big]_+=\int_{(\varepsilon\sqrt{n}g(f(n)))^p}^{\infty}\mathbf{P}\left\{|S_n|^p>t\right\}dt.$
ıt

$$\begin{aligned}J(\varepsilon)&:=\sum_{n\geq n_0}\frac{f'(n)}{n^{\frac{p}{2}}}\,\mathbf{E}\big[|S_n|^p-(\varepsilon\sqrt{n}g(f(n)))^p\big]_+\\ &=\sum_{n\geq n_0}\frac{f'(n)}{n^{\frac{p}{2}}}\int_{(\varepsilon\sqrt{n}g(f(n)))^p}^{\infty}\mathbf{P}\left\{|S_n|^p>t\right\}dt \qquad (3.0)\\ &=\sum_{n\geq n_0}f'(n)\int_{\varepsilon g(f(n))}^{\infty}pt^{p-1}\mathbf{P}\left\{|S_n|>t\sqrt{n}\right\}dt.\end{aligned}$$

A- Direct part of Theorems 1 and 2: We have to prove that if (2.1) (or (2.1)′) occurs,
en:

$$\lim_{\varepsilon\to 0}\frac{1}{g^{-1}\left(\frac{1}{\varepsilon}\right)}J(\varepsilon)=\frac{p}{p+\lambda}\mathbf{E}|N|^{\lambda+p}. \tag{3.1}$$

Assume then for $0<p<2$ (resp. $p\geq 2$), that (2.1) (resp. (2.1)') is satisfied. We prove equation (3.1) in two steps. At first, we establish it for the normal case, then we extend it to the general case. The following propositions will help us to reach the conclusion.

Proposition 3: *We have:*

$$\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\sum_{n\geq n_0} f'(n)\int_{\varepsilon g(f(n))}^{\infty} pt^{p-1}\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}dt=\frac{p}{p+\lambda}\mathbf{E}|\mathbf{N}|^{p+\lambda}. \tag{3.2}$$

Proof of proposition 3 : We have:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\sum_{n\geq n_0} f'(n)\int_{\varepsilon g(f(n))}^{\infty} pt^{p-1}\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}dt\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\int_{n_0}^{\infty} f'(x)\int_{\varepsilon g(f(x))}^{\infty} pt^{p-1}\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}dt\,dx\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\int_{\varepsilon g(f(n_0))}^{\infty} d(g^{-1}(z\varepsilon^{-1}))\int_{z}^{\infty} pt^{p-1}\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}dt\\
&\quad=\int_0^{\infty} pt^{p+\lambda-1}\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}dt\quad\text{(because } g^{-1}\text{ is regularly varying)}\\
&\quad=\frac{p}{p+\lambda}\mathbf{E}|\mathbf{N}|^{p+\lambda}.
\end{aligned}$$

Let $M>1$ and, for all $\varepsilon>0$, put $d(\varepsilon)=f^{-1}(g^{-1}((M/\varepsilon)))$. Then $\lim_{\varepsilon\to 0} d(\varepsilon)=\infty$.

Proposition 4: *We have:*

$$\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\sum_{n\geq n_0} f'(n)\int_{\varepsilon g(f(n))}^{\infty} pt^{p-1}\left|\mathbf{P}\left\{|S_n|\geq t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}\right|dt=0. \tag{3.3}$$

The proof of this proposition will obviously follow from the following Lemmas 1 and 3:

Lemma 1: *We have:*

$$\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\sum_{n=n_0}^{[d(\varepsilon)]} f'(n)\int_{\varepsilon g(f(n))}^{\infty} pt^{p-1}\left|\mathbf{P}\left\{|S_n|\geq t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}\right|dt=0. \tag{3.4}$$

Proof of Lemma 1:
Put $\Delta_n=\sup_{n\geq 0}\left|\mathbf{P}\left\{|S_n|\geq t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}\right|$. Then, by taking $t=(z+\varepsilon)g(f(n))$, we have:

$$\begin{aligned}
&\int_{\varepsilon g(f(n))}^{\infty} pt^{p-1}\left|\mathbf{P}\left\{|S_n|\geq t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\geq t\right\}\right|dt\\
&=g^p(f(n))\int_0^{\infty} p(z+\varepsilon)^{p-1}\left|\mathbf{P}\left\{|S_n|\geq (z+\varepsilon)\sqrt{n}g(f(n))\right\}-\mathbf{P}\left\{|\mathbf{N}|\geq (z+\varepsilon)g(f(n))\right\}\right|dz\\
&\leq T_{n1}+T_{n2}+T_{n3}
\end{aligned}$$

where

$$T_{n1}=g^p(f(n))\int_0^{1/g(f(n))\Delta_n^{1/2p}} p(z+\varepsilon)^{p-1}\left|\mathbf{P}\left\{|S_n|\geq (z+\varepsilon)\sqrt{n}g(f(n))\right\}-\psi\big((z+\varepsilon)g(f(n))\big)\right|dz,$$

$$T_{n2}=g^p(f(n))\int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty} p(z+\varepsilon)^{p-1}\mathbf{P}\left\{|S_n|\geq (z+\varepsilon)\sqrt{n}g(f(n))\right\}dz\quad\text{and}$$

$$T_{n3}=g^p(f(n))\int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty} p(z+\varepsilon)^{p-1}\psi\big((z+\varepsilon)g(f(n))\big)\,dz.$$

This gives

$$\begin{aligned} T_{n1} &\le g^p(f(n)) \int_0^{1/g(f(n))\Delta_n^{1/2p}} p(z+\varepsilon)^{p-1}\Delta_n \, dz \\ &\le g^p(f(n))\Delta_n \left(\frac{1}{g(f(n))\Delta_n^{1/2p}} + \varepsilon \right)^p \\ &\le \left(\Delta_n^{1/2p} + M\Delta_n^{1/p}\right)^p. \end{aligned}$$

The last inequality is due to the fact that, if $n \le d(\varepsilon)$ then $\varepsilon g(f(n)) \le M$.
For T_{n3}, by Markov's inequality, we have:

$$\begin{aligned} T_{n3} &\le \frac{p}{g^2(f(n))} \int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty} \frac{dz}{(z+\varepsilon)^3} \\ &\le \frac{p}{2g^2(f(n))} (1/g(f(n))\Delta_n^{1/2p} + \varepsilon)^{-2} \\ &\le \frac{p\Delta_n^{1/p}}{2(1+\varepsilon g(f(n))\Delta_n^{1/2p})^2} \le \frac{p}{2}\Delta_n^{1/p}. \end{aligned}$$

By the CLT, and for any continues function H, we have: $\lim_{n\to\infty} H(\Delta_n) = 0$. This implies

$$\lim_{\varepsilon\to 0} \frac{1}{f(d(\varepsilon))} \sum_{n=n_0}^{[d(\varepsilon)]} f'(n)H(\Delta_n) = \lim_{\varepsilon\to 0} \frac{1}{g^{-1}(M/\varepsilon))} \sum_{n=n_0}^{[d(\varepsilon)]} f'(n)H(\Delta_n) = 0.$$

Since g^{-1} is regularly varying with index $\lambda \ge 0$, we have:

$$\lim_{\varepsilon\to 0} \frac{1}{g^{-1}(\varepsilon^{-1})} \sum_{n=n_0}^{[d(\varepsilon)]} f'(n)H(\Delta_n) = M^\lambda \lim_{\varepsilon\to 0} \frac{1}{g^{-1}(M/\varepsilon))} \sum_{n=n_0}^{[d(\varepsilon)]} f'(n)H(\Delta_n) = 0. \tag{3.5}$$

Then $\lim_{\varepsilon\to 0} \frac{1}{g^{-1}(\varepsilon^{-1})} T_{n1} = \lim_{\varepsilon\to 0} \frac{1}{g^{-1}(\varepsilon^{-1})} T_{n3} = 0.$

To get (3.4), it remains to prove that $\lim_{\varepsilon\to 0} \frac{1}{g^{-1}(\varepsilon^{-1})} \sum_{n=n_0}^{[d(\varepsilon)]} f'(n)T_{n2} = 0$. For this, we will use the following lemma due to Spătaru [14]:

Lemma 2: *Let* $1 < \gamma \le 2$. *Suppose that* $\mathbf{E}|X|^\gamma < \infty$. *For all* $x, y > 0$, *we have:*

$$\mathbf{P}(|S_n| \ge x) \le n\mathbf{P}(|X| \ge y) + 2n^{x/y} \left[\frac{e\mathbf{E}|X|^\gamma}{xy^{\gamma-1}} \right]^{x/y}.$$

Then, we have : $T_{n2} \le T'_{n2} + T''_{n2}$, where

$$T''_{n2} \le C\, g^p(f(n)) \int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty} \frac{p(z+\varepsilon)^{p-1}}{(z+\varepsilon)^{2p} g^{2p}(f(n))}\, dz \le C\,[g(f(n))]^{-p}[g(f(n))\Delta_n^{1/2p}]^p \le C\sqrt{\Delta_n},$$

and

$$T'_{n2} = pn\, g^p(f(n)) \int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty} (z+\varepsilon)^{p-1} \mathbf{P}\left\{ p|X| \ge (z+\varepsilon)\sqrt{n} g(f(n)) \right\} dz.$$

According to (3.5), $\lim\limits_{\varepsilon\to 0}\dfrac{1}{g^{-1}(\varepsilon^{-1})}T''_{n2}=0$. On the other hand,

$$
\begin{aligned}
T'_{n2} &\le pn\,g^p(f(n))\mathbf{E}\int_{1/g(f(n))\Delta_n^{1/2p}}^{\infty}(z+\varepsilon)^{p-1}\mathbf{I}\left\{p|X|\ge(z+\varepsilon)\sqrt{n}g(f(n))\right\}dz\\
&\le n\,g^p(f(n))\mathbf{E}\left(\mathbf{I}\left\{\frac{1}{g(f(n))\Delta_n^{1/2p}}\le\frac{p|X|}{\sqrt{n}g(f(n))}\right\}\int_{1/g(f(n))\Delta_n^{1/2p}}^{\frac{p|X|}{\sqrt{n}g(f(n))}}d(z+\varepsilon)^p\right)\\
&\le n\,g^p(f(n))\mathbf{E}\left[\left(\frac{p|X|}{\sqrt{n}g(f(n))}+\varepsilon\right)^p-\left(\frac{1}{g(f(n))\Delta_n^{1/2p}}+\varepsilon\right)^p\right]\mathbf{I}\left\{\sqrt{n}\le p|X|\Delta_n^{1/2p}\right\}\\
&\le n^{1-p/2}\mathbf{E}\left(p|X|+\varepsilon\sqrt{n}g(f(n))\right)^p\mathbf{I}\left\{\sqrt{n}\le p|X|\Delta_n^{1/2p}\right\}\\
&\le n^{1-p/2}\mathbf{E}\left(p|X|+M\sqrt{n}\right)^p\mathbf{I}\left\{\sqrt{n}\le 2p|X|\right\}\\
&\le Cn^{1-p/2}\,\mathbf{E}\,|X|^p\,\mathbf{I}\left\{\sqrt{n}\le 2p|X|\right\}\le\mathbf{E}\,|X|^2\,\mathbf{I}\left\{\sqrt{n}\le 2p|X|\right\},
\end{aligned}
$$

Using Toeplitz' lemma [10], we obtain: $\lim\limits_{\varepsilon\to 0}\dfrac{1}{g^{-1}(\varepsilon^{-1})}\sum\limits_{n=n_0}^{[d(\varepsilon)]}f'(n)T'_{n2}=0$. Lemma 1 is then proved.

Lemma 3: *We have:*

$$
\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}\sum_{n\ge[d(\varepsilon)]+1}f'(n)\int_{\varepsilon g(f(n))}^{\infty}pt^{p-1}\left|\mathbf{P}\left\{|S_n|\ge t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\ge t\right\}\right|dt=0. \tag{3.6}
$$

Proof of Lemma 3: Let

$$
\begin{aligned}
T_4 &:= \sum_{n\ge[d(\varepsilon)]+1}f'(n)\int_{\varepsilon g(f(n))}^{\infty}pt^{p-1}\left|\mathbf{P}\left\{|S_n|\ge t\sqrt{n}\right\}-\mathbf{P}\left\{|\mathbf{N}|\ge t\right\}\right|dt\\
T'_4 &:= \sum_{n\ge[d(\varepsilon)]+1}f'(n)g^p(f(n))\int_0^{\infty}p(z+\varepsilon)^{p-1}\mathbf{P}\left\{|S_n|\ge(z+\varepsilon)\sqrt{n}g(f(n))\right\}dz \qquad \text{and}\\
T''_4 &:= \sum_{n\ge[d(\varepsilon)]+1}f'(n)g^p(f(n))\int_0^{\infty}p(z+\varepsilon)^{p-1}\mathbf{P}\left\{|\mathbf{N}|\ge(z+\varepsilon)g(f(n))\right\}dz.
\end{aligned}
$$

Taking $t=(z+\varepsilon)\sqrt{n}g(f(n))$, we obtain: $T_4\le T'_4+T''_4$.
Let $\theta>\lambda$ be fixed. Markov inequality gives:

$$
\begin{aligned}
T''_4 &\le C\sum_{n\ge[d(\varepsilon)]+1}f'(n)[g(f(n))]^{-\theta}\int_0^{\infty}\frac{dz}{(z+\varepsilon)^{\theta+1}}\\
&\le C\sum_{n\ge[d(\varepsilon)]+1}f'(n)[g(f(n))]^{-\theta}\varepsilon^{-\theta}\\
&\le C\varepsilon^{-\theta}\int_{d(\varepsilon)}^{\infty}f'(t)[g(f(t))]^{-\theta}dt\\
&\le C\varepsilon^{-\theta}\int_{M/\varepsilon}^{\infty}y^{-\theta}(g^{-1})'(y)dy, \quad \text{by putting} \quad y=g(f(t)).
\end{aligned}
$$

Since the function g is regularly varying at infinity with index $\lambda \geq 0$, there exists $C > 0$ such that ([2],Theorem 1.7.2.):

$$y^{-\theta}(g^{-1})'(y) \leq Cy^{-\theta-1}g^{-1}(y). \tag{3.7}$$

The previous inequality becomes

$$\begin{aligned} T_4'' &\leq C\varepsilon^{-\theta}\int_{M/\varepsilon}^{\infty} y^{-\theta-1}g^{-1}(y)dy \\ &\leq C\int_{M}^{\infty} y^{-\theta-1}g^{-1}(\frac{y}{\varepsilon})dy\,. \end{aligned}$$

By the monotonous convergence theorem we have:

$$\begin{aligned} \lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\frac{1}{g^{-1}(\varepsilon^{-1})}T_4'' &\leq C\lim_{M\longrightarrow\infty}\limsup_{\varepsilon\searrow 0}\frac{g^{-1}\left(\frac{M}{\varepsilon}\right)}{g^{-1}\left(\frac{1}{\varepsilon}\right)}M^{-\theta} \\ &\leq C\lim_{M\longrightarrow\infty} M^{\lambda-\theta} = 0 \qquad (\lambda<\theta). \end{aligned}$$

It remains to prove that $\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}T_4' = 0$. By Lemma 2, we have for $K > 0$:

$$\mathbf{P}\left\{|S_n| \geq (z+\varepsilon)\sqrt{n}g(f(n))\right\} \leq n\mathbf{P}\left\{|X| \geq \frac{(z+\varepsilon)\sqrt{n}g(f(n))}{K}\right\} + \frac{Cn^K}{((z+\varepsilon)\sqrt{n}g(f(n)))^{2K}}.$$

Then $T_4' \leq T_4^{(1)} + T_4^{(2)}$, with:

$$T_4^{(1)} = \sum_{n\geq[d(\varepsilon)]+1} nf'(n)g^p(f(n))\int_0^{\infty} p(z+\varepsilon)^{p-1}\mathbf{P}\left\{|X| \geq \frac{(z+\varepsilon)\sqrt{n}g(f(n))}{K}\right\}dz$$

and

$$T_4^{(2)} = C\sum_{n\geq[d(\varepsilon)]+1} f'(n)g^{p-2K}(f(n))\int_0^{\infty}\frac{dz}{(z+\varepsilon)^{2K-p+1}}.$$

To prove $\lim_{\varepsilon\to 0}\frac{1}{g^{-1}(\varepsilon^{-1})}T_4^{(1)} = 0$, we need the following lemma:

Lemma 4: *There exist C_1, $C_2 > 0$ such that:*

$$C_1k^2\,f'(k)g^p(f(k)) \leq \sum_{n=n_0}^{k} nf'(n)g^p(f(n)) \leq C_2k^2\,f'(k)g^p(f(k)).$$

Proof of Lemma 4: Conditions (b) and (c) (see Remark 2.1 also) imply that there exists $\beta > \lambda/2+1$ such that $x^{1-\beta}f'(x)g^p(f(x))$ is decreasing. Then we have :

$$\begin{aligned} \sum_{n=n_0}^{k} nf'(n)g^p(f(n)) &\geq k^{1-\beta}f'(k)g^p(f(k))\sum_{n=n_0}^{k} n^{\beta} \geq k^{1-\beta}f'(k)g^p(f(k))\frac{k^{1+\beta}}{n_0(\beta+1)} \\ &= C_1k^2f'(k)g^p(f(k)). \end{aligned}$$

There exists $0 < \gamma \leq 1/2$ such that $x^{1+\gamma}f'(x)g^p(f(x))$ is increasing and we have:

$$\begin{aligned} \sum_{n=n_0}^{k} nf'(n)g^p(f(n)) &\leq k^{1+\gamma}f'(k)g^p(f(k))\sum_{n=n_0}^{k} n^{-\gamma} \leq 2k^{1+\gamma}f'(k)g^p(f(k))k^{1-\gamma} \\ &= C_2k^2f'(k)g^p(f(k)). \end{aligned}$$

Therefore, Lemma 4 is proved.
Let $\phi(x) = x^{2-p/2} f'(x)$. We have:

$$T_4^{(1)} = p \sum_{n \geq [d(\varepsilon)]+1} n f'(n) g^p(f(n)) \int_0^\infty (z+\varepsilon)^{p-1} \mathbf{P}\left\{ |X| \geq \frac{(z+\varepsilon)\sqrt{n} g(f(n))}{K} \right\} dz$$

$$\leq C\mathbf{E} \sum_{n \geq [d(\varepsilon)]+1} n f'(n) g^p(f(n)) \int_\varepsilon^\infty z^{p-1} \mathbf{I}\left\{ \sqrt{n} g(f(n)) \leq \frac{K|X|}{z} \right\} \mathbf{I}\left\{ \sqrt{d(\varepsilon)} g(f(d(\varepsilon))) \leq \frac{K|X|}{z} \right\} dz$$

$$\leq C\mathbf{E} \int_\varepsilon^\infty z^{p-1} \sum_{n \geq [d(\varepsilon)]+1} n f'(n) g^p(f(n)) \sum_{l \geq n} \mathbf{I}\left\{ l \leq h\left(\frac{K|X|}{z}\right) < l+1 \right\} \mathbf{I}\left\{ z \leq \frac{K|X|}{h^{-1}(d(\varepsilon))} \right\} dz.$$

But

$$\sum_{n \geq [d(\varepsilon)]+1} n f'(n) g^p(f(n)) \sum_{l \geq n} \mathbf{I}\left\{ l \leq h\left(\frac{K|X|}{z}\right) < l+1 \right\}$$

$$= \sum_{l \geq [d(\varepsilon)]+1} \mathbf{I}\left\{ l \leq h\left(\frac{K|X|}{z}\right) < l+1 \right\} \sum_{n=[d(\varepsilon)]+1}^{l} n f'(n) g^p(f(n))$$

$$\leq C \sum_{l \geq [d(\varepsilon)]+1} l^2 f'(l) g^p(f(l)) \mathbf{I}\left\{ l \leq h\left(\frac{K|X|}{z}\right) < l+1 \right\}$$

$$\leq C \left(\frac{K|X|}{z}\right)^p \phi\left(h\left(\frac{K|X|}{z}\right)\right).$$

It follows that

$$\begin{aligned} T_4^{(1)} &\leq C\mathbf{E} \int_\varepsilon^{K|X|/h^{-1}(d(\varepsilon))} \frac{K^p |X|^p}{z} \phi\left(h\left(\frac{K|X|}{z}\right)\right) \mathbf{I}\left\{ \varepsilon \leq \frac{K|X|}{h^{-1}(d(\varepsilon))} \right\} dz \\ &\leq C\mathbf{E} \int_\varepsilon^{K|X|/h^{-1}(d(\varepsilon))} \frac{K^p |X|^p}{z} \phi\left(h\left(\frac{K|X|}{z}\right)\right) \mathbf{I}\left\{ K|X| \geq M\sqrt{d(\varepsilon)} \right\} dz, \end{aligned} \tag{3.8}$$

since $\varepsilon h^{-1}(d(\varepsilon)) = M\sqrt{d(\varepsilon)}$. Putting $y = h(\frac{K|X|}{z})$, inequality (3.8) gives, for all $\varepsilon > 0$:

$$\begin{aligned} T_4^{(1)} &\leq CK^p \mathbf{E}|X|^p \int_{d(\varepsilon)}^{h(K|X|/\varepsilon)} \frac{\phi(y)}{y} \mathbf{I}\left\{ K|X| \geq M\sqrt{d(\varepsilon)} \right\} dy \\ &\leq CK^p \mathbf{E}|X|^p \int_{d(\varepsilon)}^{h(K|X|/\varepsilon)} y^{1-p/2} f'(y) \mathbf{I}\left\{ K|X| \geq M\sqrt{d(\varepsilon)} \right\} dy. \end{aligned} \tag{3.9}$$

Case : $\mathbf{p \geq 2}$. According to (2.1)′, the inequality (3.9) gives for all $\varepsilon > 0$:

$$\begin{aligned} T_4^{(1)} &\leq CK^p (d(\varepsilon))^{1-p/2} \mathbf{E}|X|^p f \circ h(K|X|/\varepsilon) \mathbf{I}\left\{ K|X| \geq M\sqrt{d(\varepsilon)} \right\} \\ &\leq C \frac{\varepsilon^{p-2}}{M^{p-2}} \mathbf{E}|X|^p f \circ h(K|X|/\varepsilon) \mathbf{I}\left\{ K|X| \geq M\sqrt{d(\varepsilon)} \right\} < \infty. \end{aligned}$$

As previously (Remark 2.4), this allows us to obtain $\lim_{\varepsilon \to 0} \frac{1}{g^{-1}(\frac{1}{\varepsilon})} T_4^{(1)} = 0$.

Case : p<2. In this case, for $x \geq 0$, we put $g(x) = x^{1/\lambda}$, $\lambda \geq 2-p$. By (2.1) and the definition of the function h, the inequality (3.9) gives, for all $\varepsilon > 0$:

$$T_4^{(1)} \leq C\mathbf{E}|X|^p \left[h^{1-p/2}\big(K|X|/\varepsilon\big)\,(f\circ h)\big(K|X|/\varepsilon\big)\right] \mathbf{I}\left\{K|X| \geq M\sqrt{d(\varepsilon)}\right\}$$

$$\leq C\varepsilon^{p-2}\mathbf{E}|X|^2 \frac{(f\circ h)\big(K|X|/\varepsilon\big)}{(g\circ f\circ h)^{2-p}\big(K|X|/\varepsilon\big)} \mathbf{I}\left\{K|X| \geq M\sqrt{d(\varepsilon)}\right\}$$

$$\leq C\varepsilon^{p-2}\mathbf{E}|X|^2 (f\circ h)^{1-\frac{2-p}{\lambda}}\big(K|X|/\varepsilon\big)\mathbf{I}\left\{K|X| \geq M\sqrt{d(\varepsilon)}\right\} < \infty.$$

By Remark 2.3:

$$\frac{1}{g^{-1}\big(\frac{1}{\varepsilon}\big)} T_4^{(1)} \leq \varepsilon^{\lambda+p-2}\mathbf{E}|X|^2 (f\circ h)^{1-\frac{2-p}{\lambda}}\big(K|X|/\varepsilon\big)\mathbf{I}\left\{K|X| \geq M\sqrt{d(\varepsilon)}\right\}$$

$$= \mathbf{E}|X|^{\lambda+p} h^{-(\frac{\lambda+p}{2}-1)}\big(K|X|/\varepsilon\big)\mathbf{I}\left\{K|X| \geq M\sqrt{d(\varepsilon)}\right\}.$$

we deduce: $\lim\limits_{\varepsilon\to 0} \dfrac{1}{g^{-1}\big(\frac{1}{\varepsilon}\big)} T_4^{(1)} = 0.$

On the other hand,

$$T_4^{(2)} = C \sum_{n\geq [d(\varepsilon)]+1} f'(n) g^{p-2K}(f(n)) \int_0^\infty \frac{dz}{(z+\varepsilon)^{2K-p+1}}$$

$$\leq C\varepsilon^{p-2K} \sum_{n\geq [d(\varepsilon)]+1} f'(n) g^{p-2K}(f(n))$$

$$\leq C\varepsilon^{p-2K} \int_{d(\varepsilon)}^\infty f'(y) g^{p-2K}(f(y))dy$$

$$= C\varepsilon^{-1} \int_M^\infty t^{p-2K}(g^{-1})'\big(\frac{t}{\varepsilon}\big)dt, \quad \text{by putting} \quad t = \varepsilon g(f(y))$$

$$\leq C \int_M^\infty t^{p-2K-1} g^{-1}\big(\frac{t}{\varepsilon}\big)dt, \quad \text{by (3.7)}.$$

The assumption on the function g^{-1} gives, for M large enough and for $t \geq M$: $g^{-1}\big(\frac{t}{\varepsilon}\big) \leq t^\lambda g^{-1}\big(\frac{1}{\varepsilon}\big)$. Then $1/g^{-1}\big(\frac{1}{\varepsilon}\big) T_4^{(2)} \leq CM^{p+\lambda-2K}$. Taking $K > (p+\lambda)/2$, we obtain $\lim\limits_{M\to\infty} \limsup\limits_{\varepsilon\searrow 0} 1/g^{-1}\big(\frac{1}{\varepsilon}\big) T_4^{(2)} = 0$, and Lemma 3 follows.

Combining (3.2) and (3.3), the direct part of Theorem 2 is then proven. To get the direct part of Theorem 1, it remains to prove :

$$\lim_{\varepsilon\to 0} \varepsilon^{\lambda+p} \sum_{n\geq n_0} f'(n) f^{\frac{p}{\lambda}}(n)\, \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n} f^{\frac{1}{\lambda}}(n)\right\} = \frac{\lambda\sigma^{\lambda+p}}{\lambda+p}\mathbf{E}|N|^{\lambda+p}.$$

But,

$$\lim_{\varepsilon\to 0} \varepsilon^{\lambda+p} \sum_{n\geq n_0} f'(n) f^{\frac{p}{\lambda}}(n)\, \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n} g(f(n))\right\}$$

$$= \frac{\lambda}{\lambda+p} \lim_{\varepsilon\to 0} \varepsilon^{\lambda+p} \sum_{n\geq n_0} \big(f^{\frac{\lambda+p}{\lambda}}\big)'(n)\, \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n}\big(f^{\frac{\lambda+p}{\lambda}}(n)\big)^{\frac{1}{\lambda+p}}\right\} \tag{3.10}$$

$$= \frac{\lambda\sigma^{\lambda+p}}{\lambda+p}\mathbf{E}|N|^{\lambda+p}.$$

The last limit is deduced from ([4],Theorem 1), and we are done.

B- Proof of the inverse part of Theorems 1 and 2: We assume then (2.2) is satisfied. Let us first prove that $\mathbf{E}|X|^2 < \infty$. Let $(Y, Y_i, i \geq 1)$ be a symmetrized sequence of $(X, X_i, i \geq 1)$ and put $S_n(Y) = \sum_{k=1}^n Y_k$. We have:

$$\sum_{n \geq n_0} f'(n) \int_{\varepsilon g(f(n))}^{\infty} pt^{p-1} \mathbf{P}\left\{|S_n(Y)| > t\sqrt{n}\right\} dt \leq 2J(\varepsilon)$$

where $J(\varepsilon)$ is defined in (3.0). This implies, by (3.1):

$$\limsup_{\varepsilon \to 0} \frac{1}{g^{-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n \geq n_0} f'(n) \mathbf{E}\left[|S_n(Y)|^p - (\varepsilon\sqrt{n} g(f(n)))^p\right]_+ \leq \frac{2p\sigma_X^{\lambda+p}}{p+\lambda} \mathbf{E}|N|^{\lambda+p}. \tag{3.11}$$

For $M > 0$, we define $Z = Y\mathbf{I}\{|Y| \leq M\}$, $Z_n = Y_n \mathbf{I}\{|Y_n| \leq M\}$ and $S_n(Z) = \sum_{k=1}^n Z_k$. By (3.1) and the triangular inequality we have:

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \frac{1}{g^{-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n \geq n_0} f'(n) \mathbf{E}\left[|S_n(Z)|^p - (\varepsilon\sqrt{n} g(f(n)))^p\right]_+ \\ &\qquad \leq \limsup_{\varepsilon \to 0} \frac{2}{g^{-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n \geq n_0} f'(n) \mathbf{E}\left[|S_n(Y)|^p - (\varepsilon\sqrt{n} g(f(n)))^p\right]_+ \\ &\qquad \leq \frac{4p\sigma_X^{\lambda+p}}{p+\lambda} \mathbf{E}|N|^{\lambda+p}. \end{aligned} \tag{3.12}$$

By a similar way to the direct part of the proof, for $|Z| \leq M$, we have:

$$\limsup_{\varepsilon \to 0} \frac{1}{g^{-1}\left(\frac{1}{\varepsilon}\right)} \sum_{n \geq n_0} f'(n) \mathbf{E}\left[|S_n(Z)|^p - (\varepsilon\sqrt{n} g(f(n)))^p\right]_+ = \frac{p\left[\mathbf{Var}(Z)\right]^{\lambda+p}}{p+\lambda} \mathbf{E}|N|^{\lambda+p}.$$

It follows that $\mathbf{Var}(Z) \leq 4\sigma_X^{\lambda+p}$. As Y is symmetric, taking M to infinity, we get: $\mathbf{E}Y^2 \leq 4\sigma_X^{\lambda+2}$; and then, $\mathbf{E}X^2 < \infty$.

Suppose X is not centred. By the strong law of large numbers, almost surely $\lim_{n\to\infty} \frac{S_n}{n} = \mathbf{E}X = \mu \neq 0$. This implies that $\lim_{n\to\infty} \mathbf{P}\left\{\frac{|S_n|}{n} > \frac{\mu}{2}\right\} = 1$. But, according to (c), for all $\delta > 0$, there exists an integer n_0 such that: for all $n \geq n_0$

$$\mathbf{P}\left\{\frac{|S_n|}{n} > \frac{\mu}{2}\right\} = \mathbf{P}\left\{\frac{|S_n|}{\sqrt{n} g(f(n))} > \frac{\mu\sqrt{n}}{2g(f(n))}\right\} \leq \mathbf{P}\left\{\frac{|S_n|}{\sqrt{n} g(f(n))} > \frac{\mu\delta}{2}\right\}.$$

Then, for all $0 < \varepsilon < \mu\delta/2$, $\lim_{n\to\infty} \mathbf{P}\left\{|S_n| > \varepsilon\sqrt{n} g(f(n)) > \varepsilon\right\} = 1$.

But then the series $J(\varepsilon) = p \sum_{n \geq n_0} f'(n) \int_{\varepsilon g(f(n))}^{\infty} t^{p-1} \mathbf{P}\left\{|S_n| > t\sqrt{n}\right\} dt$ would be divergent which contradicts (2.2). Therefore, $\mathbf{E}X = 0$.

In the following, let $\varepsilon = 1$. We have:

Lemma 5: *There exists an integer n_0 such that for all $n \geq n_0$,*

$$n\mathbf{P}\{|X| \geq 4\sqrt{n}g(f(n))\} \leq 4\mathbf{P}\{|S_n| > \sqrt{n}g(f(n))\}.$$

Proof of Lemma 5: We'll adapt the proof of ([13], Lemma 2.3) to our case. Since $\mathbf{E}|X|^2 < \infty$, Markov's inequality gives $\lim_{n\to\infty} n\mathbf{P}(|X| \geq 4\sqrt{n}g(f(n))) = 0$. Then, for n large enough, using Levy's inequality and the fact that, for all $0 \leq t < \ln 2$, $t \leq 2(1 - e^{-t})$, we have:

$$\begin{aligned} n\mathbf{P}\{|X| \geq 4\sqrt{n}g(f(n))\} &\leq 2\left(1 - \exp\left(\sum_{k=1}^{n} \mathbf{P}\{|X| \geq 4\sqrt{n}g(f(n))\}\right)\right) \\ &\leq 2\mathbf{P}\left\{\max_{1\leq k\leq n} |S_k| \leq 2\sqrt{n}g(f(n))\right\} \\ &\leq 4\mathbf{P}\{|S_n| \leq 2\sqrt{n}g(f(n)) - \sigma\sqrt{2n}\} \\ &\leq 4\mathbf{P}\{|S_n| \leq \sqrt{n}g(f(n))\}. \end{aligned}$$

Therefore, Lemma 5 is proved.
This gives:

$$J(1) \geq \textstyle\sum_{n\geq n_0} nf'(n)g^p(f(n)) \int_1^\infty pz^{p-1}\mathbf{P}\left\{|X| \geq 4z\sqrt{n}g(f(n))\right\} dz.$$

Let ϕ and h be as defined in the direct part. According to (2.2)′ (or (2.2)), we have:

$$\begin{aligned} \infty &> \sum_{n\geq n_0} nf'(n)g^p(f(n)) \int_1^\infty pz^{p-1}\mathbf{P}\left\{|X| \geq 4z\sqrt{n}g(f(n))\right\} \\ &\geq C\mathbf{E} \sum_{n\geq n_0} nf'(n)g^p(f(n)) \int_1^\infty z^{p-1}\mathbf{I}\left\{n \leq h\left(\frac{|X|}{4z}\right)\right\} \mathbf{I}\left\{h^{-1}(n_0) \leq \frac{|X|}{4z}\right\} dz \\ &\geq C\mathbf{E} \int_1^\infty z^{p-1} \sum_{n\geq n_0} nf'(n)g^p(f(n)) \sum_{l\geq n} \mathbf{I}\left\{l \leq h\left(\frac{|X|}{4z}\right) < l+1\right\} \mathbf{I}\left\{z \leq \frac{|X|}{4\sqrt{h^{-1}(n_0)}}\right\} dz \\ &\geq C\mathbf{E} \int_1^{K|X|} z^{p-1} \sum_{l\geq n_0} \mathbf{I}\left\{l \leq h\left(\frac{|X|}{4z}\right) < l+1\right\} \sum_{n=n_0}^{l} nf'(n)g^p(f(n))\, dz. \end{aligned} \tag{3.13}$$

Using Lemma 4 and putting $y = h\big(|X|/4z\big)$ and $K = 1/4\,h^{-1}(n_0)$, (3.13) becomes

$$\begin{aligned} \infty &> C\mathbf{E} \int_1^{K|X|} z^{p-1} \sum_{l\geq n_0} l^2 f'(l)g^p(f(l))\mathbf{I}\left\{l \leq h\left(\frac{|X|}{4z}\right) < l+1\right\} dz \\ &\geq C\mathbf{E} \int_1^{K|X|} z^{p-1} \sum_{l\geq n_0} (h^{-1}(l))^p \phi(l)\mathbf{I}\left\{l \leq h\left(\frac{|X|}{4z}\right) < l+1\right\} dz \qquad (3.14) \\ &\geq C\mathbf{E} \int_1^{K|X|} \frac{K^p|X|^p}{z}\phi\left[h\left(\frac{|X|}{4z}\right)\right] dz = C\mathbf{E}|X|^p \int_{n_0}^{h(K|X|/4)} y^{1-p/2}f'(y)\, dy. \end{aligned}$$

Case : $\mathbf{p \geq 2}$. The inequality (3.14) gives for all $\eta > 0$:

$$\begin{aligned} \infty &> \mathbf{E}|X|^p h^{1-p/2}\left(\frac{K|X|}{4}\right)\left[(f\circ h)\left(\frac{K|X|}{4}\right) - f(n_0)\right]\mathbf{I}\{K|X| \leq \eta\} \\ &\geq \left[\frac{1}{h^{\frac{p}{2}-1}(\eta)}\mathbf{E}|X|^p(f\circ h)\left(\frac{K|X|}{4}\right) - f(n_0)(g\circ f\circ h)^{p-2}(\eta)\,\mathbf{E}|X|^2\right]\mathbf{I}\{K|X| \leq \eta\}. \end{aligned}$$

Since $\mathbf{E}X^2 < \infty$, for all $\eta > 0$, we have: $\mathbf{E}|X|^p (f \circ h)(|X|)\mathbf{I}\{K|X| \leq \eta\} < \infty$. Taking η to infinity, we obtain $\mathbf{E}|X|^p (f \circ h)(|X|) < \infty$.

Case : p<2. The inequality (3.14) gives for all $\eta > 0$:

$$\infty > \mathbf{E}|X|^p \left[h^{1-p/2}\left(\frac{K|X|}{4}\right)(f \circ h)\left(\frac{K|X|}{4}\right) - {n_0}^{1-\frac{p}{2}} f(n_0) - f(n_0) h^{1-p/2}\left(\frac{K|X|}{4}\right)\right]$$

$$\geq \mathbf{E}X^2 (f \circ h)^{1-\frac{2-p}{\lambda}}\left(\frac{K|X|}{4}\right) - {n_0}^{1-\frac{p}{2}} f(n_0)\mathbf{E}|X|^p - f(n_0)\mathbf{E}X^2.$$

Since $\mathbf{E}X^2 < \infty$, we get: $\mathbf{E}X^2 (f \circ h)^{1-\frac{2-p}{\lambda}}(|X|) < \infty$.

At last, following the direct part of the proof, we find $\mathbf{E}X^2 = \sigma^2$. This finishes the proof of Theorem 1 and Theorem 2.

4 Proofs of propositions 1 and 2: We have:
$\mathbf{E}|S_n|^p \mathbf{I}\big(|S_n| \geq \varepsilon\sqrt{n} g(f(n))\big) = \varepsilon^p n^{\frac{p}{2}} g^p(f(n))\mathbf{P}\{|S_n| \geq \varepsilon\sqrt{n} g(f(n))\} + \mathbf{E}\big[|S_n|^p - (\varepsilon\sqrt{n} g(f(n)))^p\big]_+$.
To get (2.3), it is sufficient to prove that, for $g(n) = e_k(n)$ with $k \geq 2$, we have:

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^p}{\log_k(1/\varepsilon)} \sum_{n \geq n_0} f'(n) g^p(f(n))\ \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n} g(f(n))\right\} = 0. \tag{4.1}$$

But,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^p}{\log_k(1/\varepsilon)} \sum_{n \geq n_0} f'(n) g^p(f(n))\ \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n} g(f(n))\right\}$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon^p}{\log_k(1/\varepsilon)} \sum_{n \geq n_0} f'(n) e_k^p(f)\ \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n}\, e_k(f)\right\}$$

$$\leq C \lim_{\varepsilon \to 0} \frac{\varepsilon^p}{\log_k(1/\varepsilon)} \sum_{n \geq n_0} \left(e_k^p(f)\right)' \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n}\left(e_k^p(f)\right)^{1/p}\right\}.$$

By using ([4], Theorem 1), we obtain: $\lim_{\varepsilon \to 0} \varepsilon^p \sum_{n \geq n_0} \left(e_k^p(f)\right)' \mathbf{P}\left\{|S_n| \geq \varepsilon\sqrt{n}\left(e_k^p(f)\right)^{1/p}\right\} < \infty$. So (4.1) follows.

Proposition 2 is deduced from (3.10).

Acknowledgment The author would like to thank the referee for many valuable comments.

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