HAL open science

Cluster algebras associated with open Richardson varieties: an algorithm to compute initial seeds

Etienne Menard

▶ To cite this version:

Etienne Menard. Cluster algebras associated with open Richardson varieties: an algorithm to compute initial seeds. 2022. hal-03638968

HAL Id: hal-03638968
https://hal.science/hal-03638968v1

Preprint submitted on 12 Apr 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Cluster algebras associated with open Richardson varieties: an algorithm to compute initial seeds

Etienne Ménard

March 29, 2022

Abstract

We present a new algorithm to compute initial seeds for cluster structures on categories associated with coordinate rings of open Richardson varieties. This allows us to explicitely determine seeds first considered in [Lec16].

1 Introduction

Follwing [Lec16], we first recall the context of this paper.

Let G be a simple simply connected algebraic $\mathbb{C}$-group. We assume G to be simply-laced and fix H a maximal torus in G, B a Borel subgroup of G containing H, and we denote by B^- the Borel subgroup opposite to B with respect to H. Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group with length function $w \mapsto \ell(w)$ and longest element $w_0 \in W$, of length $r = \ell(w_0)$. We denote by $S = \{s_1, \dots, s_n\}$ its subset of standard Coxeter generators. Reduced representatives of $w \in W$ will be denoted by $\overline{w} = [i_{\ell(w)}, \dots, i_1]$ corresponding to products of $\ell(w)$ simple reflections $s_{i_{\ell(w)}} \cdots, s_{i_1}$. Our convention for the Dynkin diagram of G is:

$$A_n: \ 1 - 2 \cdots \cdots n-1 - n \ , n \geq 1$$

$$D_n: \ \begin{array}{ccccccc} & & & & & & n-1 \\ 1 - 2 & \cdots & \cdots & \cdots & n-2 & \diagup & \\ & & & & & \diagdown & n \end{array} \ , n \geq 4$$

$$E_n: \ \begin{array}{ccccccc} 1 - 3 - 4 - 5 - 6 - 7 - 8 \\ \quad\quad | \\ \quad\quad 2 \end{array} \ , n = 6, 7, 8$$

Figure 1: Dynkin diagrams

We consider the flag variety $X = B^- \backslash G$ and we denote by $\pi : G \to X$ the natural projection $\pi(g) := B^- g$. The Bruhat decomposition (resp. Birkhoff decomposition):

$$G = \bigsqcup_{w \in W} B^- w B^-, \quad (\text{resp. } G = \bigsqcup_{w \in W} B^- w B,)$$

projects to the Schubert decomposition (resp. *opposite* Schubert decomposition)

$$X = \bigsqcup_{w \in W} C_w, \quad (\text{resp. } X = \bigsqcup_{w \in W} C^w,)$$

where $C_w = \pi(B^- w B^-)$ is the *Schubert cell* attached to w, isomorphic to $\mathbb{C}^{\ell(w)}$ and $C^w = \pi(B^- w B)$ is the *opposite Schubert cell* attached to w, isomorphic to $\mathbb{C}^{\ell(w_0) - \ell(w)}$.

The intersection

$$\mathcal{R}_{v,w} := C^v \cap C_w$$

is called an *open Richardson variety*. In [KL80, Deo85] it is shown that $\mathcal{R}_{v,w}$ is non-empty if and only $v \leq w$ for the Bruhat order of W and that it is a smooth irreducible locally closed subset of C_w of dimension $\ell(w) - \ell(v)$. We get a finer stratification

$$X = \bigsqcup_{v \leq w} \mathcal{R}_{v,w}$$

though the strata $\mathcal{R}_{v,w}$ are no longer isomorphic to affine spaces (in general).

In [Lus98] Lusztig used this stratification for studying the nonnegative part $X_{\geq 0}$ of the flag variety X (more generally a partial flag variety). In [Lec16], Leclerc showed that there exists a cluster subalgebra $\widetilde{R}_{v,w}$ contained in the coordinate ring $\mathbb{C}[\mathcal{R}_{v,w}]$ and proposed the following conjecture:

Conjecture 1.1 (Leclerc's conjecture)**.** *For any strata $\mathcal{R}_{v,w}$ we have $\mathbb{C}[\mathcal{R}_{v,w}] = \widetilde{R}_{v,w}$.*

In order to help to solve this conjecture, we want to work with explicit seeds for $\widetilde{R}_{v,w}$. Some seeds are defined (but not explicitly computed) in [Lec16] using a categorification that we now introduce.

In [GLS11], Geiss, Leclerc and Schröer introduced a category called $\mathcal{C}_w$ whose definition will be recalled below. Independently, Buan, Iyama, Reiten and Scott introduced the same category in [BIRS09]. The latter also defined the cluster structure on 2-Calabi-Yau categories: a categorical analog of the cluster algebra structure with a notion of seed and its mutations.

In [GLS11] the authors build explicitely a seed for the cluster structure on $\mathcal{C}_w$. Via the cluster character φ that they defined in [GLS06, Section 9] they found a seed for the cluster algebra on $\mathbb{C}[C_w]$ as the image of a seed M of $\mathcal{C}_w$ via the cluster character.

We will here use the same strategy to find a seed for $\widetilde{R}_{v,w}$ based on a category $\mathcal{C}_{v,w}$, defined below. In order to do so, we need a seed for the cluster structure on $\mathcal{C}_{v,w}$ and we will compute this seed thanks to a mutation algorithm that we will design and prove. This algorithm takes as entry data a seed for the cluster structure on $\mathcal{C}_w$ computed thanks to a reduced representative $\overline{w}$ of w, and the Weyl group element v.

In this paper, relying on the existence of a cluster structure for $\mathcal{C}_{v,w}$, proved in [Lec16], we will define and prove an algorithm giving an explicit way to find a seed for this cluster structure (in the sense of [BIRS09]). This paper summarize the results of the author's thesis [Mén21a] written during his PhD internship at Université de Caen.

1.1 Organisation

In the second section we will define the categories $\mathcal{C}_w$, $\mathcal{C}^v$, and $\mathcal{C}_{v,w}$, their properties, and we will recall how to compute an initial seed $V_{\overline{w}}$ for $\mathcal{C}_w$. In the third section we will introduce some combinatorial notation that we will use for our algorithm. In the fourth section we will study the quiver associated to the initial seed $V_{\overline{w}}$, its combinatorial definition, and some properties of his graph. In the fifth section we will consider a way to represent the elements of $\mathcal{C}_w$ thanks to an integer vector called Δ-vector. We will study its properties, how it can be combinatorially defined for $V_{\overline{w}}$ and why it plays a key role in our algorithm proof.

We will then define our algorithm in the sixth section and give two equivalent formulations. In the following section we will prove it and end this article with some results obtained thanks to this algorithm.

In appendix we put a discussion about the different configurations of the quiver and their evolution thanks to mutation. We end this article with a step-by-step example of running the algorithm with given data.

2 Categorification of cluster algebras

In this section we will recall some results from Buan, Iyama, Reiten and Scott [BIRS09], from Geiss, Leclerc and Schröer [GLS11] and from Leclerc [Lec16] about the categories $\mathcal{C}_w, \mathcal{C}^v$ and $\mathcal{C}_{v,w}$.

2.1 The preprojective algebra

Given a Dynkin diagram of type Δ, we construct the preprojective algebra of type Δ. To do so we take any orientation on the Dynkin diagram to have a quiver $Q = (Q_0, Q_1)$, double each arrow (for each arrow $\alpha \in Q_1$ we now have α and $\alpha^* \in Q_1^*$, forming a 2-cycle), and obtain the quiver $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^*)$.

We then take the $\mathbb{C}$-path algebra $\mathbb{C}\overline{Q}$ of $\overline{Q}$ and quotient it by the two-sided ideal generated by:

$$c = \sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha.$$

We then obtain the preprojective algebra Λ of type Δ:

$$\Lambda = \mathbb{C}\overline{Q}/(c)$$

The reader will be able to find more information about this construction in [GLS08, Section 3].

In the following we will denote by $\mathbf{mod}(\Lambda)$ the category of finitely generated Λ-modules. The simple 1-dimensional Λ-modules in correspondence with the vertices of the Dynkin diagram will be denoted by S_i $(i \in I)$ and their respective injective envelopes by Q_i $(i \in I)$. We will denote by P_i $(i \in I)$ the projective cover of S_i.

We will now introduce the categories $\mathcal{C}_w$, $\mathcal{C}^v$ and $\mathcal{C}_{v,w}$ and their links with the cluster algebra structures on the rings $\mathbb{C}[C_w]$, $\mathbb{C}[C^v]$ and $\mathbb{C}[\mathcal{R}_{v,w}]$.

2.2 Cluster structure on subcategories of $\mathbf{mod}(\Lambda)$

In the following, $\mathcal{C}$ will be an extension closed subcategory of $\mathbf{mod}(\Lambda)$.

The cluster structure will be introduced by first defining a seed (a module and its quiver) and then the notion of mutation of cluster seeds. A seed for this structure is a module with some properties whose definitions are recalled now, adapting [BIRS09] to the specific context of subcategories of $\mathbf{mod}(\Lambda)$.

Definition 2.1. For a Λ-module M, $\mathrm{add}(M)$ is the full subcategory of $\mathbf{mod}(\Lambda)$ consisting of all modules isomorphic to direct summands of finite direct sums of copies of M.

M is *rigid* if $\mathrm{Ext}^1_\Lambda(M, M) = 0$.

It is *$\mathcal{C}$-maximal rigid* if for any $T' \in \mathcal{C}$ such that $M \oplus T'$ is rigid, then $T' \in \mathrm{add}(M)$.

M is said to be *cluster-tilting* if we have the following equivalence:

$$(X \in \mathcal{C} \text{ and } \mathrm{Ext}^1_\Lambda(M, X) = 0) \Leftrightarrow (X \in \mathrm{add}(M))$$

M is *basic* if its indecomposable direct summands are pairwise non-isomorphic.

Then to any basic $\mathcal{C}$-cluster-tilting module M we can associate a seed in the following way:

Definition 2.2 (Seed for a cluster structure)**.** Let $M = \bigoplus_{i=1}^{r} M_i$ be the indecomposable direct summands decomposition of M. Then the collection $\{M_1, \dots, M_r\}$ is a *cluster* whose projective direct summands are coefficients and the others are cluster variables.

To M one can associate a quiver Γ_M by taking the Gabriel quiver of the algebra $\mathrm{End}_\Lambda(M)$ associating the vertices $\mathrm{Hom}_\Lambda(M, M_i)$ of the latest to the vertex labelled by M_i in Γ_M and considering quivers to be defined up to arrows between two vertices corresponding to coefficients.

Remark 2.3. *Here we decide not to use the quiver of* $\mathrm{End}^{\mathrm{op}}(M)$ *contrary to many articles without any loss of generality.*

To define mutation we have to consider short exact sequences:

Theorem 2.4 ([BIRS09])**.** *For a seed $M \in \mathcal{C}$, M_i a cluster variable and the quiver $\Gamma_M = (Q_0, Q_1, s, t)$ of M, there exists a unique (up to isomorphism) indecomposable module M_i^* such that the two following exchange sequences exist:*

$$M_i \to \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha) = M_i}} M_{t(\alpha)} \to M_i^*, \quad M_i^* \to \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = M_i}} M_{s(\alpha)} \to M_i$$

Then $M^* = \bigoplus_{\substack{k=1 \\ k \neq i}}^{r} M_k \oplus M_i^*$ is another cluster-tilting module called the mutation of M in direction i, denoted by $\mu_i(M)$. The mutation of the quiver $\mu_i(\Gamma_M)$ is the classical mutation of quivers.

2.3 The category $\mathcal{C}_{v,w}$

Given a Coxeter generator s_i $(i \in I)$, as in [Lec16, Section 3.2], we define the endofunctor $\mathcal{E}_i = \mathcal{E}_{s_i}$ on the objects of $\mathbf{mod}(\Lambda)$ as the kernel of the surjection $X \to S_i^{\oplus m_i(X)}$ where $m_i(X)$ is the multiplicity of S_i in the head of X.

Dually we define the endofunctor $\mathcal{E}_i^\dagger$ on the objects of $\mathbf{mod}(\Lambda)$ as the cokernel of the injection $S_i^{\oplus m_i^\dagger(X)} \to X$ where $m_i^\dagger(X)$ is the multiplicity of S_i in the socle of X.

$\mathcal{E}_i$ acts on X by removing the S_i-isotypical part of its head, $\mathcal{E}_i^\dagger$ by removing the S_i-isotypical part of its socle. As the $\mathcal{E}_i$ and $\mathcal{E}_i^\dagger$ verify the braid relations of W, we can define unambiguously $\mathcal{E}_w$ on any $w \in W$ by taking any reduced representative and composing the corresponding endofunctors.

Given these functors we can now define the following categories which will be our main interest:

Definition 2.5. For $w \in W$ let $u := w^{-1}w_0$, $I_w := \mathcal{E}_u\left(\bigoplus_{i\in I} Q_i\right)$ and $J_w := \mathcal{E}^{\dagger}_{w^{-1}}\left(\bigoplus_{i\in I} Q_i\right)$.

We define:

$$\mathcal{C}_w := \mathrm{Fac}(I_w) = \mathcal{E}_u(\mathbf{mod}(\Lambda)), \quad \mathcal{C}^v := \mathrm{Sub}(J_v) = \mathcal{E}^{\dagger}_{v^{-1}}(\mathbf{mod}(\Lambda))$$

As explained in [Lec16] and [GLS11], these two categories give categorical models of the strata $\mathcal{R}_{e,w}$ ($\mathcal{R}_{w,w_0}$ respectively) of the flag variety X.

Example 2.6. Let W be of type A_3 and $w = s_1s_2s_3s_1$ and $v = s_2s_1$. To give a straightforward visual representation of the endofunctors' action we will represent the modules by their socle decomposition.

The preprojective algebra of type A_3 has 3 maximal indecomposable projective-injective modules:

$$Q_1 = \begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & \end{array}, \quad Q_2 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array}, \quad Q_3 = \begin{array}{ccc} 1 & & \\ & 2 & \\ & & 3 \end{array}$$

We compute $u = w^{-1}w_0 = s_2s_1$ and $v^{-1} = s_1s_2$. We have:

$$\mathcal{E}_1(Q_1) = Q_1, \quad \mathcal{E}_1(Q_2) = Q_2, \quad \mathcal{E}_1(Q_3) = \begin{array}{cc} 2 & \\ & 3 \end{array}$$

$$\mathcal{E}_u(Q_1) = Q_1 = \begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & \end{array}, \quad \mathcal{E}_u(Q_2) = \begin{array}{cc} 1 & \\ & 2 \end{array}\begin{array}{c} \\ 3 \end{array}, \quad \mathcal{E}_u(Q_3) = \begin{array}{c} 3 \end{array} = S_3$$

Similarly we have

$$\mathcal{E}^{\dagger}_{v^{-1}}(Q_1) = \begin{array}{c} 3 \end{array} = S_3, \quad \mathcal{E}^{\dagger}_{v^{-1}}(Q_2) = \begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array}, \quad \mathcal{E}^{\dagger}_{v^{-1}}(Q_3) = Q_3 = \begin{array}{ccc} 1 & & \\ & 2 & \\ & & 3 \end{array}$$

So

$$I_w = \begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & \end{array} \oplus \begin{array}{cc} 1 & \\ & 2 \end{array}\begin{array}{c} \\ 3 \end{array} \oplus S_3, \quad J_v = S_3 \oplus \begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array} \oplus \begin{array}{ccc} 1 & & \\ & 2 & \\ & & 3 \end{array}$$

and eventually the categories $\mathcal{C}_w$ and $\mathcal{C}^v$ have the following sets of indecomposable modules:

$$\mathrm{Ind}(\mathcal{C}_w) = \left\{ S_3, S_1, \begin{array}{cc} 1 & \\ & 2 \end{array}\begin{array}{c} \\ 3 \end{array}, \begin{array}{cc} & \\ 2 & \end{array}\begin{array}{c} \\ 3 \end{array}, \begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & \end{array} \right\}, \quad \mathrm{Ind}(\mathcal{C}^v) = \left\{ S_3, S_1, \begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array}, \begin{array}{cc} 2 & \\ & 3 \end{array}, \begin{array}{ccc} 1 & & \\ & 2 & \\ & & 3 \end{array} \right\}$$

One can easily verify that these categories are closed under extensions, and closed either under factors (for $\mathcal{C}_w$), or under submodules (for $\mathcal{C}^v$).

Definition 2.7 ([Lec16])**.** Given $v \leq w \in W$ we define the category:

$$\mathcal{C}_{v,w} = \mathcal{C}^v \cap \mathcal{C}_w$$

In [Lec16, Corollary 3.11], Leclerc showed that this category has a cluster structure but did not give an explicit seed of it.

Given a $\mathcal{C}_w$-cluster tilting object T, and denoting $t_v(T)$ the maximal submodule of T in $\mathcal{C}_v$, he showed that $T/t_v(T)$ is a $\mathcal{C}_{v,w}$-cluster-tilting object but it is neither basic in general nor do we know its quiver.

We recall some definitions, as in Section 2.2, $\mathcal{C}$ is an extension-closed subcategory of $\mathbf{mod}(\Lambda)$.

Definition 2.8. Let T be a $\mathcal{C}$-module, $\Sigma(T)$ is the number of isoclasses of indecomposable direct summands of T.

A module T having the maximal number $\Sigma(T)$ in $\mathcal{C}$ is said to be *$\mathcal{C}$-complete.*

We then have the following equivalence:

Theorem 2.9 ([GLS06, Theorem 2.2.],[GLS11, Theorem 2.9.])**.** *For a rigid Λ-module T the following are equivalent:*

- *T is $\mathcal{C}_w$-maximal rigid;*
- *T is $\mathcal{C}_w$-complete rigid;*

- *T is $\mathcal{C}_w$-cluster-tilting.*

The same holds for $\mathcal{C}^v$ by duality.

It is then possible to decide if a rigid module is cluster-tilting by counting its isoclasses of indecomposable summands and we have in addition the following information about the number of isoclasses:

Theorem 2.10 ([Lec16, Proposition 4.3])**.** *A $\mathcal{C}_{v,w}$-maximal rigid basic module M has $\Sigma(M) = \ell(w) - \ell(v)$ indecomposable summands.*

2.4 The initial module $V_{\overline{w}}$

In order to get some examples of Leclerc's conjecture, we need to have an explicit initial seed for the cluster structure on $\mathcal{C}_{v,w}$. We will not compute it directly as the authors did for $\mathcal{C}_w$ in [GLS11, Section 2.4] but we will in fact start from such a seed for $\mathcal{C}_w$. In this section we will recall the construction of this seed thanks to a reduced representative $\overline{w}$ of $w \in W$.

Definition 2.11 (Definition of soc notation)**.** For a Λ-module X and a simple module S_j, let $\mathrm{soc}_{(j)}(X) := \mathrm{soc}_{S_j}(X)$ be the S_j-isotypical part of the socle of X.

For a sequence $[j_1, \dots, j_t]$ with $1 \leq j_p \leq n$ for any p, there is a unique chain of submodules of X

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t \subseteq X$$

such that $X_p/X_{p-1} = \mathrm{soc}_{(j_p)}(X/X_{p-1})$. Then we define $\mathrm{soc}_{(j_1,\dots,j_t)}(X) := X_t$.

Example 2.12. Let W be of type A_4 and $X = Q_3$, we have:

$$Q_3 = \begin{array}{ccccccc} & & & 2 & & & \\ & & \diagup & & \diagdown & & \\ & 1 & & & & 3 & \\ & & \diagdown & & \diagup & & \\ & & & 2 & & & \\ & & \diagdown & & \diagup & & \\ & & & & 3 & & \\ & & & & & & 4 \end{array}.$$

Then $\mathrm{soc}_{(3)}(Q_3) = S_3$, $\mathrm{soc}_{(j)}(Q_3) = 0$ for all $j \neq 3$. We have the following chain of submodules of Q_3:

$$0 \subseteq S_3 \subseteq S_3 \subseteq \begin{array}{ccc} 2 & & \\ & \diagdown & \\ & & 3 \end{array} \subseteq \begin{array}{ccc} 2 & & \\ & \diagdown & \\ & & 3 \end{array} \subseteq \begin{array}{ccccc} 2 & & & & \\ & \diagdown & & & \\ & & 3 & \diagup & 4 \end{array} \subseteq \begin{array}{ccccccc} & & & 2 & & & \\ & & \diagup & & \diagdown & & \\ & 1 & & & & 3 & \\ & & \diagdown & & \diagup & & \\ & & & 2 & & & \\ & & \diagdown & & \diagup & & \\ & & & & 3 & & \\ & & & & & & 4 \end{array}$$

allowing us to define $\mathrm{soc}_{(3,1,2,3,4)}(Q_3) = \begin{array}{ccccc} 2 & & & & \\ & \diagdown & & & \\ & & 3 & \diagup & 4 \end{array}$.

Definition 2.13 (Modules $V_{k,\overline{w}}$)**.** Given a reduced representative $\overline{w} = [i_{\ell(w)}, \dots, i_1]$ of $w \in W$ and $1 \leq k \leq \ell(w)$, let

$$V_{k,\overline{w}} := \mathrm{soc}_{(i_k,\dots,i_1)}(Q_{i_k}).$$

Remark 2.14. *Note that defining $V_{k,\overline{w}}$ requires to read $\overline{w}$ from left to right even if it is indexed from right to left.*

Example 2.15. For W of type A_4 let $\overline{w} = [4, 2, 3, 1, 2, 3, 4]$ then, thanks to Example 2.12, we have

$$V_{5,\overline{w}} = \begin{array}{ccccc} 2 & & & & \\ & \diagdown & & & \\ & & 3 & \diagup & 4 \end{array}.$$

These $V_{k,\overline{w}}$ then gives what will be the starting point of our algorithm:

Definition 2.16 (Module $V_{\overline{w}}$)**.** Under the same assumptions as before, let:

$$V_{\overline{w}} = \bigoplus_{i=1}^{\ell(w)} V_{i,\overline{w}}.$$

Example 2.17. For $\overline{w} = [4, 2, 3, 1, 2, 3, 4]$ as in Example 2.15, we have (summands ordered by ascending index):

$$V_{\overline{w}} = S_4 \oplus \begin{array}{ccccc} & & & & 4 \\ & & & \diagup & \\ & & 3 & & \\ & \diagup & & & \\ 3 & & & & \end{array} \oplus \begin{array}{ccccc} & & & & 4 \\ & & & \diagup & \\ & & 3 & & \\ & \diagup & & & \\ 2 & & & & \end{array} \oplus \begin{array}{ccccc} & & & & 4 \\ & & & \diagup & \\ & & 3 & & \\ & \diagup & & & \\ 2 & & & & \\ & \diagup & & & \\ 1 & & & & \end{array} \oplus \begin{array}{ccccc} 2 & & & & \\ & \diagdown & & & \\ & & 3 & \diagup & 4 \end{array} \oplus \begin{array}{ccccccc} & & & 2 & & & \\ & & \diagup & & \diagdown & & \\ & 1 & & & & 3 & \\ & & \diagdown & & \diagup & & \\ & & & 2 & & & \\ & & \diagdown & & \diagup & & \\ & & & & 3 & & \\ & & & & & & 4 \end{array} \oplus \begin{array}{ccccc} 2 & & & & \\ & \diagdown & & & \\ & & 3 & & \\ & & & \diagdown & \\ & & & & 4 \end{array}.$$

$V_{\overline{w}}$ is enough to define the whole category $\mathcal{C}_w$ which has the following properties:

Theorem 2.18 ([GLS11, Section 2.4])**.** *$V_{\overline{w}}$ is a $\mathcal{C}_w$-cluster-tilting module, $\mathcal{C}_w = \mathrm{Fac}(V_{\overline{w}})$, and in $\mathcal{C}_w$ a module M is a cluster-tilting module iff it is $\mathcal{C}_w$-maximal rigid iff $\Sigma(M) = \ell(w)$.*

3 Combinatorial notations

In the following we will have to study in detail reduced representatives of Weyl group elements and their indices.

Definition 3.1. To a product $s_{i_k} \cdots s_{i_1}$ of simple reflections representing the Weyl group element w, we associate the integer tuple $\overline{w} = [i_k, \ldots, i_1]$. Given a Weyl group element w there exists a minimal number of factors among products representing this element. This number is called the length of w, denoted $\ell(w)$. A representative having as few factors as the length is called *reduced*. We will always suppose the representatives to be reduced.

Given a reduced representative $\overline{w} = [i_{\ell(w)}, \ldots, i_1]$ of $w \in W$ and $1 \leq k \leq \ell(w)$ we will define the following notions:

- the *color* of the index k is the numerical value of the integer i_k,
- the *successor*, k^+, of k is the smallest index strictly greater than k such that k and k^+ have the same color. If k is the greatest index of its color, then for any j such that $i_j = i_k$ (i.e. of the same color) we note $j_{\max} := k$ and $k^+ = \ell(w) + 1$,
- the *predecessor*, k^-, of k is the greatest index strictly smaller than k such that k and k^- have the same color. If k is the smallest index of its color, then for any j of the same color as k, we note $j_{\min} := k$ and $k^- = 0$.

Two colors are said to be adjacent if the associated vertex of the Dynkin diagram are linked by (at least) one edge.

Example 3.2. For instance, if we consider W of type A_5 and $w = s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1$, one reduced representative is $\overline{w} = [2, 1, 3, 4, 2, 1, 3, 5, 2, 4, 3, 2, 1] =: [i_{13}, i_{12}, i_{11}, i_{10}, i_9, i_8, i_7, i_6, i_5, i_4, i_3, i_2, i_1]$

The color of the fifth letter i_5 is the integer 2, the color of the tenth is $i_{10} = 4$.

The successor of the seventh letter is the eleventh: $7^+ = 11$ as 7 and 11 are of color 3 and there is no letter of this colour after the seventh and before the eleventh; its predecessor is $7^- = 3$. In the same sense, we have $8^+ = 12$, $8^- = 1$. In order to illustrate the boundary definitions: $6^+ = 14 = \ell(w) + 1$ and $6^- = 0$ as the sixth letter is the only one of its color (5).

4 Structure of $\Gamma_{V_{\overline{w}}}$

4.1 Definition

As explained in Definition 2.2, the quiver attached to a maximal-rigid basic module $M \in \mathbf{mod}(\Lambda)$ is the quiver of its endomorphism algebra $\mathrm{End}_\Lambda(M)$. This quiver has its vertices labelled by the $\mathrm{End}_\Lambda(M)$-modules $\mathrm{Hom}_\Lambda(M, M_i)$ where $M = \bigoplus M_i$ is a decomposition of M into indecomposable direct summands, and the arrows correspond to irreducible morphisms.

In general it is difficult to determine such a quiver, but thanks to [GLS11, Section 2.4] we have a combinatorial description of the quiver $\Gamma_{V_{\overline{w}}}$ (or $\Gamma_{\overline{w}}$). Note that this is a special case of the quivers associated in [BFZ05] to double Bruhat cells.

The vertices are labelled by the direct summands V_k, $1 \leq k \leq \ell(w)$. In order to fix a position we will put all vertices on a $n \times \ell(w)$ grid, the vertex labelled by V_k being on the kth column (numbered from right to left) and on the i_k-th line (numbered from up to bottom).

We will then add arrows by splitting them into two sets:

- the horizontal arrows will go from k to k^+ for any $1 \leq k \leq \ell(w)$ whenever possible,
- the ordinary arrows will go from k to j if $i_j \neq i_k$ and the inequality $j^+ \geq k^+ > j > k$ is verified. Then there are $-a_{i_j, i_k}$ arrows $V_{i_j} \to V_{i_k}$ where $A = (a_{i,j})$ is the Cartan matrix associated to W.

Remark 4.1. *As we are in simply-laced Dynkin cases, we have one arrow $V_{i,j} \to V_{i,k}$ if $s_{i_k} s_{i_j} s_{i_k} = s_{i_j} s_{i_k} s_{i_j}$ and the inequality is verified and no arrow if $s_{i_j} s_{i_k} = s_{i_k} s_{i_j}$.*

Example 4.2. Let us consider the element w of the Weyl group W of type A_4 and the representative $\overline{w} = [3, 4, 2, 3, 1, 2, 4, 1]$. Then we have (indecomposable direct summands ordered by ascending index):

$$V_{\overline{w}} = 1 \oplus 4 \oplus \begin{tikzcd} 1 \arrow[dr] & \\ & 2 \end{tikzcd} \oplus \begin{tikzcd} & 2 \arrow[dl] \\ 1 & \end{tikzcd} \oplus \begin{tikzcd} 1 \arrow[dr] & & \\ & 2 \arrow[dr] & \\ & & 4 \\ & 3 \arrow[ur] & \end{tikzcd} \oplus \begin{tikzcd} & 2 \arrow[dr] & & 4 \arrow[dl] \\ 1 \arrow[ur] \arrow[dr] & & 3 & \\ & 2 \arrow[ur] & & \end{tikzcd} \oplus \begin{tikzcd} 1 \arrow[dr] & & \\ & 2 \arrow[dr] & \\ & & 3 \\ & & & 4 \end{tikzcd} \oplus \begin{tikzcd} & 2 \arrow[dr] & & \\ 1 \arrow[ur] \arrow[dr] & & 3 \arrow[dl] \arrow[dr] & \\ & 2 \arrow[ur] \arrow[dr] & & 4 \\ & & 3 \arrow[ur] & \end{tikzcd} .$$

If we compute the quiver of the endomorphism algebra $\mathrm{End}_\Lambda(V_{\overline{w}})$ we get:

$$\begin{tikzcd}
 & & \mathrm{Hom}(V_4, V_{\overline{w}}) \arrow[dr] & & \mathrm{Hom}(V_1, V_{\overline{w}}) \arrow[ll] \\
 & \mathrm{Hom}(V_6, V_{\overline{w}}) \arrow[ur] \arrow[dr] & & \mathrm{Hom}(V_3, V_{\overline{w}}) \arrow[ur] \arrow[ll] & \\
\mathrm{Hom}(V_8, V_{\overline{w}}) \arrow[ur] \arrow[dr] & & \mathrm{Hom}(V_5, V_{\overline{w}}) \arrow[ur] \arrow[ll] \arrow[drr] & & \\
 & \mathrm{Hom}(V_7, V_{\overline{w}}) \arrow[ur] & & & \mathrm{Hom}(V_2, \overline{w}) \arrow[lll]
\end{tikzcd}$$

and thus, the quiver $\Gamma_{\overline{w}}$ is Figure 2.

$$\begin{tikzcd}
 & & V_4 \arrow[dr] & & V_1 \arrow[ll] \\
 & V_6 \arrow[ur] \arrow[dr] & & V_3 \arrow[ur] \arrow[ll] & \\
V_8 \arrow[ur] \arrow[dr] & & V_5 \arrow[ur] \arrow[ll] \arrow[drr] & & \\
 & V_7 \arrow[ur] & & & V_2 \arrow[lll]
\end{tikzcd}$$

Figure 2: Quiver $\Gamma_{\overline{w}}$

If we follow the steps described above we first draw a 4×8 table in which we put the vertices according to their index and color:

8	7	6	5	4	3	2	1	
				V_4			V_1	1
		V_6			V_3			2
V_8			V_5					3
	V_7					V_2		4

,

we add the horizontal arrows (between two successive vertices on the same line going from right to left) and, eventually, we add the ordinary arrows and we get Figure 2. For instance we have the arrow $V_5 \to V_2$ as $i_5 = 3 \neq 4 = i_2$, $s_4 s_3 s_4 = s_3 s_4 s_3$ (so 3 and 4 are linked in the Dynkin diagram by one edge) and $5^+ = 8 \geq 2^+ = 7 > 5 > 2$.

There is no arrow $V_6 \to V_1$ as, even if $i_6 \neq i_1$ and these colors are related by a braid move, the inequality is not verified:

$$6^+ = 9 \geq 2^+ = 5 \not> 6 > 2$$

There is no arrow $V_2 \to V_1$ as i_2 and i_1 are two colors not linked in the Dynkin diagram and thus generate no arrow, even if the inequality $2^+ = 7 \geq 1^+ = 4 > 2 > 1$ is verified.

Definition 4.3 (Colored lines)**.** Thanks to this way of displaying the quiver, we can now associate a color to each line, the color of the i_k-th line being i_k.

4.2 Saw teeth structure

We now want to exhibit a remarkable structure on this quiver that we will need later for the proof of our algorithm. We need first to introduce another notion:

Definition 4.4 (Bicolor subquiver)**.** Given a quiver Γ having a structure of colored lines $c_1, \ldots, c_n$, the bicolor subquiver of color (c_j, c_k) is the subquiver of Γ obtained by taking:

- all vertices of color c_j or c_k
- all arrows between vertices both of color c_j
- all arrows between a vertex of color c_j and one of color c_k (and conversely)

Remark 4.5. *The definition is not symmetric: c_j and c_k do not play the same role.*

Example 4.6. Taking again Figure 2, we can extract the $(2, 3)$-bicolor subquiver and get Figure 3 a). We can also look at the $(3, 2)$-bicolor subquiver of Figure 3 b) and check that they are different.

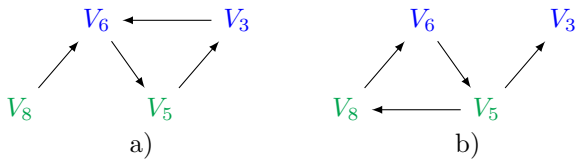


Figure 3: a) $(2,3)$-bicolor subquiver of $\Gamma_{\overline{w}}$, b) $(3,2)$-bicolor subquiver of $\Gamma_{\overline{w}}$

Definition 4.7 (Saw teeth)**.** Let Γ be a bicolor subquiver of colors (i_k, i_j). We call a saw tooth of Γ, a cycle of Γ arrows having the following elements:

- an arrow from a vertex of color i_k (let say R_k) to a vertex of color i_j (let say R_j)
- an arrow from R_j to a vertex of color i_k and of index stricly less than k, let say $R_{k^{\alpha-}}$, $\alpha > 0$
- a sequence of arrows between vertices of color i_k: $R_k \leftarrow R_{k^-} \leftarrow \cdots \leftarrow R_{k^{(\alpha-1)-}} \leftarrow R_{k^{\alpha-}}$.

We then say that $R_{k^{\alpha-}}$ is the right end of the saw tooth, R_k its left end and R_j its summit.

We call successive sequence of saw teeth of length m a collection of m teeth that we can index by $1, \ldots, m$ such that the left end of the tooth p is the right end of the tooth $p+1$ $(1 \leq p \leq m-1)$. We call right end of the sequence the right end of the tooth 1 and left end of the sequence the left end of the tooth m.

Definition 4.8 (Saw teeth structure)**.** A bicolor subquiver of colors (i_k, i_j) is said to have a saw teeth structure if it has the following structure (by ascending order of indices on vertices on the i_k-th line):

1. a sequence (possibly empty) of vertices of color i_k where successive vertices are linked by a horizontal arrow
$$R_{(k_{\min})^{\alpha+}} \leftarrow R_{(k_{\min})^{(\alpha-1)+}} \leftarrow \cdots \leftarrow R_{(k_{\min})^{+}} \leftarrow R_{k_{\min}}$$
where α is the length of this sequence
2. a unique arrow, called initial barb, $R_{(k_{\min})^{\alpha+}} \rightarrow R_{(j_{\min})^{\beta+}}$ (optional)
3. a sequence of m saw teeth of right end $R_{(k_{\min})^{\alpha+}}$ and of left end $R_{(k_{\max})^{\gamma-}}$ with $\gamma \geq \alpha$ (equality if empty)
4. a unique arrow, called final barb, $R_{(j_{\min})^{\delta+}} \rightarrow R_{(k_{\max})^{\gamma-}}$ with $\delta > \beta$ (optional)
5. a sequence (possibly empty) of vertices of color i_k where successive vertices are linked by a horizontal arrow
$$R_{k_{\max}} \leftarrow R_{(k_{\max})^{-}} \leftarrow \cdots \leftarrow R_{(k_{\max})^{(\gamma-1)-}} \leftarrow R_{(k_{\max})^{\gamma-}}$$

and some vertices of color i_j not linked to any vertex of color i_k by the arrows of the bicolor subquiver called isolated vertices.

A saw teeth structure is said to be pure if there is no initial barb.

A quiver Γ having a structure of colored lines is said to have a saw teeth structure if, any bicolor subquiver of Γ has a saw teeth structure.

Example 4.9. We will consider the quiver given by the representative

$$\overline{w} = [4, 3, 5, 2, 3, 4, 1, 2, 3, 5, 3, 4, 1, 2, 3, 1, 2]$$

of $w \in W$ of type D_5 (see Figure 1 p. 1 for the Dynkin diagram associated with this type). We have the quiver of Figure 4.

This quiver has a saw teeth structure. For instance if we check the $(3,2)$ bicolor subquiver we get Figure 5 where the initial and final sequence are empty, the initial barb is dashed and the teeth sequence is of length three, with the tooth $V_3 \rightarrow V_7 \rightarrow V_9 \rightarrow V_4 \rightarrow V_3$, the tooth $V_{13} \rightarrow V_{10} \rightarrow V_9 \rightarrow V_{13}$ and the tooth $V_{16} \rightarrow V_{14} \rightarrow V_{13} \rightarrow V_{16}$ which are consecutives. Here there is neither final barb nor isolated vertex.

Due to the presence of the initial barb, this subquiver is not of pure saw teeth structure.

The reader will be able to see a non-empty initial sequence by looking at the $(3,5)$ bicolor subquiver, both an initial and a final barb (as well as an isolated vertex) by looking at the $(5,3)$ bicolor subquiver.

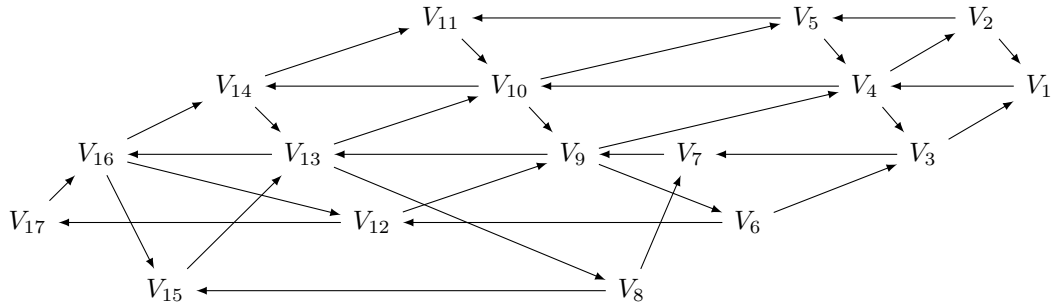


Figure 4: $\Gamma_{\overline{w}}$

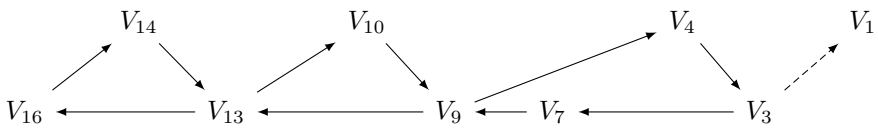
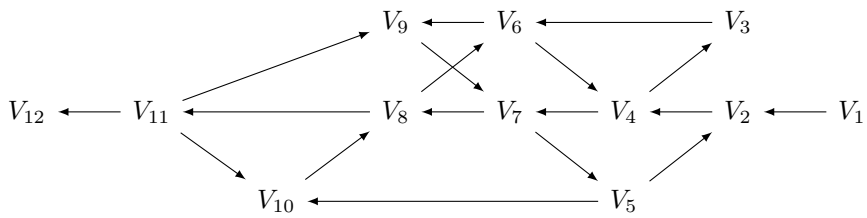


Figure 5: $(3, 2)$ bicolor subquiver

Counter-example 4.10. Let suppose we have the following quiver:



where the first and second lines and second and third lines are of adjacent colors and first and third are not adjacent. Then the $(1, 2)$ (Figure 6) and the $(2, 3)$ bicolor subquivers (Figure 7) (among others) do not have a saw teeth structure.

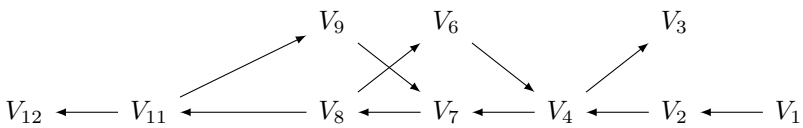


Figure 6: $(1, 2)$-bicolor subquiver

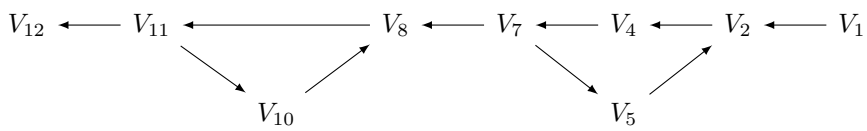


Figure 7: $(2, 3)$-bicolor subquiver

Indeed, on the $(2, 1)$ bicolor subquiver, there are some ordinary arrows after the final barb $V_8 \to V_6$, while on the $(1, 2)$ bicolor subquiver, the tooth $V_8 \to V_6 \to V_4 \to \cdots$ ends after the tooth $V_{11} \to V_9 \to V_7$ has begun. In the $(2, 3)$ bicolor subquiver, the left end of the first tooth (V_7) is not the right end of the second tooth (V_8). In these two latter cases the teeth are not consecutives.

Proposition 4.11. *If a bicolor subquiver (i_k, i_j) has a saw teeth structure then this structure is preserved by removing the first vertex of any line.*

Remark 4.12. *This does not hold in general for the pure saw teeth structure.*

Proof. We will proceed by case exhaustion. Let us call $R_{j_{\text{mini}}}$ the vertex of line i_j of minimal index and $R_{k_{\text{mini}}}$ the one of line i_k.

If we remove $R_{j_{\text{mini}}}$ we have 4 possible cases:

1. $R_{j_{\text{mini}}}$ is an isolated vertex: the structure is preserved
2. $R_{j_{\text{mini}}}$ is the target of an initial barb: the initial barb is removed and now the subquiver starts with a sequence of teeth, a final barb or a sequence of vertices. The structure is preserved.
3. $R_{j_{\text{mini}}}$ is the summit of the first tooth: removing it removes the first tooth and now the line starts by a one vertex more sequence of vertices. There cannot have an initial barb here by minimality of $R_{j_{\text{mini}}}$. The structure is preserved.
4. $R_{j_{\text{mini}}}$ is the source of the final barb. In that case it is the only ordinary arrow of the toothless bicolor subquiver and thus after removal, the bicolor subquiver consists in a set of isolated vertices on line i_j and a sequence of vertices on line i_k. The structure is preserved.

So, in any case, removing $R_{j_{\text{mini}}}$ will preserve the struture.

Now we study the effect of removing $R_{k_{\text{mini}}}$. We have once again 2 possible cases:

1. $R_{k_{\text{mini}}}$ is the first vertex of a vertices sequence linked by consecutives horizontal arrows. Then the sequence is of length one less (or disappeared). The structure is preserved.
2. $R_{k_{\text{mini}}}$ is the source of an initial barb or the right end of the teeth sequence or the target of the final barb (the last two possibilities being incompatible). Then the initial barb disappears (provided it existed) and the line starts by a new initial barb (resulting in the subsequent deletion of the right ordinary arrow of the first tooth provided it existed), or a sequence of horizontal arrows in a subquiver now with no ordinary arrows (initial and final barb being the only ones and both deleted by the vertex deletion). The structure is preserved.

Because of the possible apparition of initial barb due to the deletion of the right side of the initial teeth, even if there was no initial barb before the deletion in the last case, we cannot ensure that the pure saw teeth structure is preserved.

□

4.3 Saw teeth structure in $\mathcal{C}_w$ initial seeds

The point of introducing the previous notions was the following proposition that will be of use for the proof of the algorithm.

Proposition 4.13. *For any representative $\overline{w}$ of an element $w \in W$, $\Gamma_{\overline{w}}$ has a saw teeth structure.*

Proof. The proof will rely on the combinatorial description of the quiver, given in Section 4.1.

We need to look at any bicolor subquiver of $\Gamma_{\overline{w}}$ to prove the proposition. First, if the two chosen colors are not adjacent (in the sense of Definition 3.1) then the structure is trivial: there is no ordinary arrow in the subquiver thus all vertices of the first color form an initial sequence, there is not tooth, no barb and all vertices of the second color are isolated vertices. The saw teeth structure is verified.

Let us then consider a bicolor subquiver of adjacent colors. As for the description of the structure, we will here follow the first line (of color i_k and of vertices $k_{\min}, \dots, k_{\max}$ to fix the notation) by ascending order of indices.

We start by considering the special case $k_{\max} < j_{\min}$ and $j_{\min} \neq j_{\max}$, or $k_{\min} > j_{\max}$ and $k_{\min} \neq k_{\max}$. Then there is no ordinary arrows and we are again in the trivial case. Apart from this case, while $k^+ < j_{\min}$ there is no ordinary arrow allowed between k and $j_{\min}$ and we are in the initial sequence.

If we have $k_{\min} > j_{\min}$ then there exists an integer γ such that $(j_{\min})^{(\gamma+1)+} > k_{\min} > (j_{\min})^{\gamma+}$. Then there is a maximal index k (possibly greater than $k_{\min}$) such that $k^+ \geq (j_{\min})^{(\gamma+1)+} > k > (j_{\min})^{\gamma+}$. Then there is an ordinary arrow $k \to (j_{\min})^{\gamma+}$ and the vertices of color i_j and of index $< (j_{\min})^{\gamma+}$ are isolated vertices. This is the initial barb.

If we have $k_{\min} > j_{\min}$ (so no initial barb) there is a minimal k such that $k^+ > j_{\min} > k$. Let us call j the greatest index such that $k^+ > j > k$ (and thus $j^+ \geq k^+$). If we have an initial barb, let k be the source of the initial barb and j the successor of its target.

($*$) In both cases we then have $j^+ \geq k^+ > j > k$ and then there is an arrow $V_j \to V_k$. If $j = j_{\max}$ then there is no more ordinary arrow beyond this point and this arrow is a final barb and all following vertices of line i_k will form the final sequence. In the opposite case, this arrow will be the right side of the first tooth of the teeth sequence.

In the case where this is not the final barb, there exists a $\gamma > 0$ such that $k^{(\gamma+1)+} \geq j^+ > k^{\gamma+} > j$ and then we have an ordinary arrow $k^{\gamma+} \to j$ closing the teeth $k^{\gamma+} \to j \to k \to k^+ \to \cdots \to k^{\gamma+}$.

Given these k and j either there is a $\zeta > 0$ such that $j^{(\zeta+1)+} \geq k^{(\zeta+1)+} > j^{\zeta+} > k^{\gamma+}$ or not. If it is not true, then we have considered all ordinary arrows and vertices of color i_j and the last remaining i_k vertices form the final sequence. If such a ζ exist, we start again the reasoning at the point $(*)$ for the formation of the next tooth or the final barb.

Eventually we end with a possibly empty final sequence and then our subquiver has a saw teeth structure. □

5 Δ-vectors

Δ-vectors are a notion coming from Ringel's work on quasi-hereditary algebras ([Rin91]) which plays a crucial role in our paper. We present it here in the narrower context of our category $\mathcal{C}_w$.

5.1 Definition

In the quiver of the algebra $\mathrm{End}_\Lambda(\Gamma_{\overline{w}})$, the horizontal arrows correspond to irreducible morphisms which are in fact inclusions. These inclusions still hold for our $\Gamma_{\overline{w}}$ in the sense that $V_k \subset V_{k^+}$ for all $1 \leq k \leq \ell(w)$ such that $1 \leq k^+ \leq \ell(w)$.

This allows us to define a new collection of modules, related to an initial seed $V_{\overline{w}}$.

Definition 5.1 (Modules $M_{k,\overline{w}}$)**.** Given a module from an initial seed $V_{\overline{w}}$, we define:

$$M_{k,\overline{w}} := \begin{cases} V_k & \text{if } k = k_{\min} \\ V_k/V_{k^-} & \text{otherwise} \end{cases}, \quad 1 \leq k \leq \ell(w), \quad \text{and } M_{\overline{w}} = \bigoplus_{k=1}^{\ell(w)} M_{k,\overline{w}}.$$

Following [GLS11, Section 10], we denote $\mathcal{C}_{M_{\overline{w}}}$ the category of all modules having a stratification by the modules $(M_{k,\overline{w}})_k$ in ascending order.

Lemma 5.2 ([GLS11, Lemma 10.2])**.** *We have $\mathcal{C}_w = \mathcal{C}_{M_{\overline{w}}}$. Every module of $\mathcal{C}_w$ admits a stratification by the family of modules $(M_{k,\overline{w}})_k$ in ascending order.*

Definition 5.3. Let $B_{\overline{w}} := \mathrm{End}_\Lambda(V_{\overline{w}})$ and define the standard modules of $B_{\overline{w}}$ by:

$$\Delta_k := \mathrm{Hom}(V_{\overline{w}}, M_k), 1 \leq k \leq \ell(w)$$

The following results will allow us to introduce Δ-vectors.

Proposition 5.4 ([GLS11, Lemmas 10.3 & 10.4])**.** *The dimension vectors $\dim_{B_{\overline{w}}}(\Delta_k)$, $1 \leq k \leq r$ are linearly independent and we have*

$$\dim_{B_{\overline{w}}}(\mathrm{Hom}_\Lambda(V_{\overline{w}}, V_k)) = \dim_{B_{\overline{w}}}(\Delta_k) + \dim_{B_{\overline{w}}}(\Delta_{k^-}) + \cdots + \dim_{B_{\overline{w}}}(\Delta_{k_{\min}})$$

Proposition 5.5 ([GLS11, Proposition 10.5])**.** *Let $X \in \mathcal{C}_w$ and $\underline{a} = (a_1, \ldots, a_{\ell(w)})$ be. The following assumptions are equivalent:*

1. *$X \in \mathcal{C}_{M_{\overline{w}}}$ with the sequence $\underline{a}$ as multiplicity of the stratas;*
2. *there exists a short exact sequence: $0 \to \bigoplus_{k=1}^{\ell(w)} V_{k^-}^{a_k} \to \bigoplus_{k=1}^{r} V_k^{a_k} \to X \to 0$;*
3. *$\dim_{B_{\overline{w}}}(\mathrm{Hom}_\Lambda(V_{\overline{w}}, X)) = \dim_{B_{\overline{w}}}(\mathrm{Hom}_\Lambda(V_{\overline{w}}, M_{\overline{w}}(X))) = \sum_{k=1}^{r} a_k \dim_{B_{\overline{w}}}(\Delta_k)$ where*

$$M_{\overline{w}}(X) := M_1^{a_1} \oplus \cdots \oplus M_{\ell(w)}^{a_{\ell(w)}}.$$

Proposition 5.6 ([GLS11, Corollary 12.3])**.** *Let X and Y be indecomposable rigid modules in $\mathcal{C}_w$. If $\dim_{B_{\overline{w}}}(\mathrm{Hom}_\Lambda(V_{\overline{w}}, X)) = \dim_{B_{\overline{w}}}(\mathrm{Hom}_\Lambda(V_{\overline{w}}, Y))$ then $X \cong Y$.*

Thus, thanks to these propositions we can unambiguously determine an indecomposable rigid Λ-module $X \in \mathcal{C}_w$ by the $\dim_{B_{\overline{w}}}$-vector of $\mathrm{Hom}_\Lambda(V_{\overline{w}}, X)$ or equivalently by the corresponding vector $(a_1, \ldots, a_{\ell(w)})$, which gives rise to the following definition.

Definition 5.7. The previous sequence $\underline{a} = (a_1, \ldots, a_{\ell(w)})$ is called the $\Delta_{\overline{w}}$-vector of X, written $\Delta_{\overline{w}}(X) = (a_1, \ldots, a_{\ell(w)})$. We will write $(\Delta_{\overline{w}}(X))_i = \Delta_{\overline{w},i}(X)$ for the ith coordinate of the $\Delta_{\overline{w}}$ vector of X.

5.2 Mutation of Δ-vectors

In [GLS11] the authors give an explicit way to compute Δ-vectors after a mutation. We recall here the formula.

Definition 5.8. Given the family of modules $(M_{k,\overline{w}})_{k=1}^{\ell(w)}$ we define the vector

$$d_\Delta := (\dim(\Delta_1), \dots, \dim(\Delta_{\ell(w)})).$$

where $\dim(\Delta_s) = \sum_{k=1}^{\ell(w)} \dim \operatorname{Hom}_\Lambda(V_k, M_s)$.

Proposition 5.9 ([GLS11, Proposition 12.6])**.** *In a seed (R, Γ), the Δ-vector of R_k^*, image of R_k by μ_k $(1 \leq k \leq \ell(w))$ is:*

$$\Delta_{\overline{w}}(R_k^*) := \begin{cases} -\Delta_{\overline{w}}(R_k) + \sum_{(R_i \to R_k) \in \Gamma} \Delta_{\overline{w}}(R_i) & \text{if } \sum_{(R_i \to R_k) \in \Gamma} \Delta_{\overline{w}}(R_i) \cdot d_\Delta > \sum_{(R_k \to R_j) \in \Gamma} \Delta_{\overline{w}}(R_k) \cdot d_\Delta \\ -\Delta_{\overline{w}}(R_k) + \sum_{(R_k \to R_j) \in \Gamma} \Delta_{\overline{w}}(R_j) & \text{otherwise} \end{cases}$$

where $\Delta_{\overline{w}}(M) \cdot d_\Delta = \sum_{i=1}^{\ell(w)} \Delta_{\overline{w},i}(M) d_{\Delta,i}$ is the usual inner product.

Example 5.10. Examples of such computation are given in Example B.1.

5.3 Δ-vectors related to different representatives

In our definition of Δ-vector, we have to choose a particular reduced representative, $\overline{w}$, defining our initial seed $V_{\overline{w}}$ and the family of modules $(M_{\overline{w},k})_k$. But given another representative $\overline{w}_2$ of the same Weyl group element w we can define another family of module $(M_{\overline{w}_2,k})_k$ defining another notion of $\Delta_{\overline{w}_2}$-vector.

Clearly we have:

Proposition 5.11. *We have $\Delta_{\overline{w}}(V_{\overline{w},k}) = \sum_{j=1}^{k} \delta_{i_k,i_j} e_j$ where e_j is the j-th vector of the canonical basis.*

Proof. By definition of the modules $(M_{\overline{w},k})_k$. □

However it is far more difficult to compute the $\Delta_{\overline{w}_1}$-vector of the indecomposable summands of the initial seed $V_{\overline{w}_2}$ when $\overline{w}_2 \neq \overline{w}_1$.

There exists an algorithm to obtain it, based on the fact that if we know a rewriting path from $\overline{w}_2$ to $\overline{w}$, we can decompose this rewriting path in 2 and 3-moves (commutation of letters or braid moves) thanks to Matsumoto's lemma [Mat64]. Then [BKT14, Propositions 5.26 & 5.27] gives us a way to compute changes in Δ-vectors and [SSVZ00, Theorem 3.5] changes in the quiver. It is the idea used in the algorithmic implementation [Mén21b] of this article's algorithm.

The drawback of this method is that we do not have a closed formula for the output, other than making the computation. However, we can obtain a closed formula for some particular Δ-coordinates using another method relying on Mirković-Vilonen polytopes that we now introduce.

5.4 Relation with Mirković-Vilonen polytopes

In [Kas95, Section 8], Kashiwara introduces the crystal $B(\infty)$ as the crystal base of the negative part of the quantized universal envelopping algebra of a Lie algebra $\mathfrak{g}$, $U_q^-(\mathfrak{g})$. Lusztig then gives a more geometric description of $B(\infty)$, linking it with representations of preprojective algebras in [Lus90b].

More precisely, if we take a quiver $Q = (Q_0, Q_1)$, being an orientation of a Dynkin quiver associated to the root system Φ of the Lie algebra $\mathfrak{g}$, and a dimension vector $\nu \in \mathbb{N}Q_0$, any Λ-module can be seen as equivalent to the data of its linear maps. This data can been seen as points of a variety called Lusztig nilpotent variety and denoted by $\Lambda(\nu)$. We denote by $\mathfrak{B}(\nu)$ the set of irreducible components of $\Lambda(\nu)$ and:

$$\mathfrak{B} := \bigsqcup_{\nu \in \mathbb{N}Q_0} \mathfrak{B}(\nu)$$

Theorem 5.12 ([KS97, Theorem 5.3.2] & [Lus90a, Theorem 1.8])**.** *$\mathfrak{B}$ has a crystal structure and there exists a unique crystal isomorphism:*

$$\Phi : \begin{array}{ccc} B(\infty) & \to & \mathfrak{B} \\ b & \mapsto & \Lambda_b \end{array}$$

Any irreducible component of $\mathfrak{B}$ can also be associated with a formal sum of positive roots of Φ given an indexation of elements of $B(\infty)$ by Poincaré-Birkhoff-Witt bases.

On the other side, Kamnitzer in [Kam07] associated to each element of $B(\infty)$ a Mirković-Vilonen polytope following an idea of Anderson-Mirković. Thus we have two bijections linking $B(\infty)$ with irreducible components of the varieties of quiver representations on one hand and with Mirković-Vilonen polytopes on the other hand. In [BK12], Baumann and Kamnitzer study directly the connection between Mirković-Vilonen polytopes and Luzstig's nilpotent varieties. More precisely they introduce the Harder-Naramsimhan polytopes as:

Definition 5.13 ([BKT14])**.** The Harder-Naramsimhan polytope $\mathrm{Pol}(T)$ of a Λ-module T is the convex hull of the dimension vectors of all submodules of T in $\mathbb{R}I$ (where I is the set of vertices in the Dynkin diagram). We denote it $\mathrm{Pol}(T)$.

and they prove:

Theorem 5.14 ([BKT14, Section 1.3])**.** *For each $b \in B(\infty)$ the map $T \mapsto \mathrm{Pol}(T)$ takes a generic value $\Psi(b)$ on the irreducible component Λ_b defining the map $\Psi : B(\infty) \to \mathcal{MV}(\infty)$. Moreover the map $\Psi \circ \Phi^{-1} : \mathfrak{B} \to \mathcal{MV}(\infty)$ is a crystal isomorphism.*

where $\mathcal{MV}(\infty)$ is the set of Mirković-Vilonen polytopes starting at the origin of the underlying space.

Then, the crystal isomorphism $\Psi \circ \Phi^{-1} : \mathfrak{B} \to \mathcal{MV}(\infty)$ associate to any irreductible component of the Lusztig variety a Mirković-Vilonen polytope.

Finally, in [NS09, Theorem 4.15], Naito and Sagaki give a combinatorial description of some Mirković-Vilonen polytopes corresponding to extremal vectors of irreducible representations of $\mathfrak{g}$. Given an irreducible representation $L(\lambda)$ with crystal basis $B(\lambda)$ seen as a subset of $B(\infty)$, the extremal weight vectors are the elements of the orbit of the highest weight $W \cdot \lambda$ by the Weyl group. To the datum (λ, w) we can associate an extremal weight vector corresponding to a Mirković-Vilonen polytope that Naito and Sagaki compute explicitly. To the same datum, Geiss, Leclerc and Schröer associate an indecomposable summand $V_k \in V_{\overline{w}}$ where λ is the fundamental weight corresponding to the socle of V_k and $w = s_{i_k} \cdots s_{i_1}$ by [GLS11, Proposition 9.1]. Examples and concrete computations can be found in [Mén21a, Sections 2.2 & 2.9]

We will now give an explicit interpretation of this reasoning restricting the combinatorial description of [NS09, Theorem 4.15] to only two paths along edges of the Mirković-Vilonen polytope.

5.5 Explicitation of the isomorphism

In the following we will not only work with two reduced representatives of the same Weyl group element but with two reduced representatives of distincts elements. In order to do so we will in fact see any of these elements as "left parts" of reduced representatives of w_0.

Definition 5.15. Given a representative $\overline{w} = [i_{\ell(w)}, \dots, i_1]$ of $w \in W$, we call *left-completion of* $\overline{w}$ in w_0 a reduced representative $\dot{w} = [j_r, \dots, j_1]$ of w_0 such that $i_k = j_k$ for all $1 \leq k \leq \ell(w)$.

Remark 5.16. *This completion exists due to the lattice structure of W for the weak left Bruhat order.*

The completion $\dot{w}$ is not unique in general, and we just pick one of them and fix it.

Definition 5.17. Given $\overline{w}$ and $\dot{w} = [i_r, \dots, i_1]$, let $1 \leq k \leq \ell(w)$. We denote by $u_k \in W$ the element defined by:

$$u_k = \begin{cases} s_{i_r} \cdots s_{i_{k+1}} & \text{if } 1 \leq k \leq \ell(w) \\ e & k = r \end{cases}$$

Given $\dot{w}_0 = [j_r, \dots, j_1]$ another representative of w_0, we define $[j_{q_{r-k}}, \dots, j_{q_1}]$ to be the leftmost representative of u_k in $\dot{w}_0$ and we write

$$[j_{r_k}, \dots, j_{r_1}] = [j_r, \dots, j_1] \setminus [j_{q_{r-k}}, \dots, j_{q_1}]$$

for the set of letters of $\dot{w}_0$ not taking part in the writing of the leftmost representative of u_k in $\dot{w}_0$.

Example 5.18. Let W be of type A_3, $\dot{w} = [2, 1, 2, 3, 2, 1]$, $\dot{w}_0 = [1, 2, 3, 1, 2, 1]$ and $k = 4$. Then $u_4 = s_2 s_1$, the leftmost representative of u_4 in $\dot{w}_0$ uses the letters of indices $q_1 = 3$ and $q_2 = 5$. Then $r_1 = 1, r_2 = 2, r_3 = 4$ and $r_4 = 6$.

Proposition 5.19 ([Hum90, Section 1.7])**.** *Let $\overline{w_0} = [i_r, \dots, i_1]$ be a reduced representative of $w_0 \in W$. Then, given the simple roots $(\alpha_i)_{i\in I}$ and the simple reflections $(s_j)_{j\in I}$ in the root system Φ associated to W, the roots of the sequence (β_i):*

$$\beta_i = \begin{cases} \beta_1 = \alpha_{i_1} \\ \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad 2 \le k \le r \end{cases}$$

are pairwise distincts and form the set of all positive roots of Φ

Let ϖ_i be the ith fundamental weight associated to the coroot α_i^* of the underlying root system Φ of W.

Definition 5.20 ([NS09, Section 4.1])**.** With the previous notations, we define a sequence of weights associated to a representative $\dot{w}_0$ of w_0 $(\xi_i^{\dot{w}_0})_i$ in the following way. Let $\xi_0^{\dot{w}_0} = \varpi_{i_k}$ and $r_0 = 0$. We have

$$\begin{cases} \xi_{r_c}^{\dot{w}_0} := s_{\beta_{r_c}^{\dot{w}_0}} s_{\beta_{r_{c-1}}^{\dot{w}_0}} \cdots s_{\beta_{r_1}^{\dot{w}_0}}(\varpi_{i_k}) & 1 \le c \le k \\ \xi_{q_d}^{\dot{w}_0} := \xi_{r_c}^{\dot{w}_0} & \text{if } r_{c+1} > q_d > r_c \end{cases}$$

We can now express the Δ-coordinates of any direct summand $V_{k,\overline{w}}$ of $V_{\overline{w}}$, seen as a direct summand of the module $V_{\dot{w}} \supset V_{\overline{w}}$.

Proposition 5.21 ([NS09, Lemma 4.1.3][BKT14, Example 5.14 & Proposition 5.24])**.** *With the previous notations we denote $n_i^{\dot{w}_0}$, $1 \le i \le \ell(w)$ the coefficient:*

$$\xi_{i-1}^{\dot{w}_0} - \xi_i^{\dot{w}_0} = n_i^{\dot{w}_0} \beta_i^{\dot{w}_0}$$

and we have $\Delta_{\dot{w}_0,i}(V_{\overline{w},k}) = n_i^{\dot{w}_0}$.

Example 5.22. Given W of type A_3, we will look at two representatives of w_0: $\dot{w} = [2,1,2,3,2,1]$, $\dot{w}_0 = [1,2,3,1,2,1]$. We have the quivers given at Figure 8 with the modules given by the socle decomposition of Table 1.

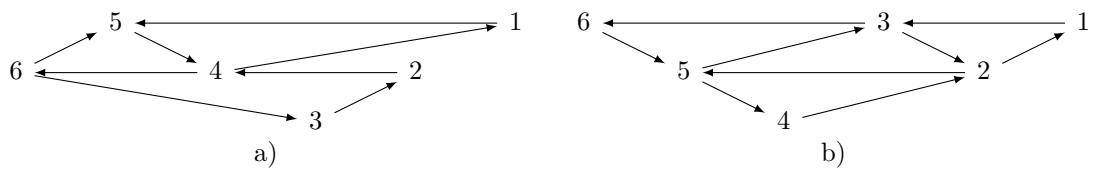


Figure 8: a) $\Gamma_{\dot{w}}$, b) $\Gamma_{\dot{w}_0}$

k	$V_{\dot{w},k}$	$V_{\dot{w}_0,k}$	k	$V_{\dot{w},k}$	$V_{\dot{w}_0,k}$
1	1	1	4	1 2 3	1 2 3
2	1 2	1 2	5	1 2 3	1 2 3 2
3	1 2 3	1 2	6	1 2 3 2	1 2 3

Table 1: Socle decomposition of indecomposable summands of $V_{\dot{w}}$ and $V_{\dot{w}_0}$

Thanks to Proposition 5.11 we have the Δ vectors of left hand side of Table 2 (seen as a linear combination of the canonical basis $(e_i)_i$ or $(f_j)_j$ of $\mathbb{Z}^r$).

Using the socle decomposition of M-modules given by red parts of summands of Table 1 we can find the $\Delta_{\dot{w}_0}$-vectors of the indecomposable direct summands of $V_{\dot{w}}$ and, conversely, the $\Delta_{\dot{w}}$-vectors of the indecomposable direct summands of $V_{\dot{w}_0}$.

k	$\Delta_{\dot{w}}(V_{\dot{w},k})$	$\Delta_{\dot{w}_0}(V_{\dot{w}_0,k})$	$\Delta_{\dot{w}_0}(V_{\dot{w},k})$	$\Delta_{\dot{w}}(V_{\dot{w}_0,k})$
1	e_1	f_1	f_1	e_1
2	e_2	f_2	f_2	e_2
3	e_3	f_1+f_3	f_4	e_1+e_6
4	e_2+e_4	f_4	f_2+f_6	e_3
5	e_1+e_5	f_2+f_5	$f_1+f_3+f_6$	$e_2+e_4+e_6$
6	$e_2+e_4+e_6$	$f_1+f_3+f_6$	f_2+f_5	e_1+e_5

Table 2: Δ-vectors directly computable (first two) and computed by hand (last two)

We will now illustrate Proposition 5.21 by computing $\Delta_{\dot{w}}(V_{\dot{w}_0,5})$. Given $\dot{w}_0$, we have $u_5 = s_1$ then the leftmost representative of u_5 in $\dot{w}$ is given by the letter of index $q_1 = 5$. We then have $(r_i) = (1,2,3,4,6)$. We can compute the roots $\beta_i^{\dot{w}}$:

$$\beta_1 = \alpha_1, \quad \beta_2 = s_1(\alpha_2) = \alpha_1+\alpha_2, \quad \beta_3 = s_1(s_2(\alpha_3)) = \alpha_1+\alpha_2+\alpha_3$$

$$\beta_4 = s_1s_2s_3(\alpha_2) = \alpha_3, \quad \beta_5 = s_1s_2s_3s_2(\alpha_1) = \alpha_2+\alpha_3, \quad \beta_6 = s_1s_2s_3s_2s_1(\alpha_2) = \alpha_2$$

We can now compute the $(\xi_i^{\dot{w}_0})_i$ sequence. $\xi_0 = \varpi_{i_5} = \varpi_2$. We have:

$$\begin{array}{lllllllllll}
\xi_1 & := & s_{\alpha_1}(\varpi_2) & = & \varpi_2 & n_1\beta_1 & = & \xi_0-\xi_1 & = & 0\alpha_1 \\
\xi_2 & := & s_{\alpha_1+\alpha_2}(\xi_1) & = & \varpi_2-\alpha_1-\alpha_2 & n_2\beta_2 & = & \xi_1-\xi_2 & = & 1(\alpha_1+\alpha_2) \\
\xi_3 & := & s_{\alpha_1+\alpha_2+\alpha_3}(\xi_2) & = & \varpi_2-\alpha_1-\alpha_2 & n_3\beta_3 & = & \xi_2-\xi_3 & = & 0(\alpha_1+\alpha_2+\alpha_3) \\
\xi_4 & := & s_{\alpha_3}(\xi_3) & = & \varpi_2-\alpha_1-\alpha_2-\alpha_3 & n_4\beta_4 & = & \xi_3-\xi_4 & = & 1\alpha_3 \\
\xi_5 & := & \xi_4 & = & \varpi_2-\alpha_1-\alpha_2-\alpha_3 & n_5\beta_5 & = & \xi_4-\xi_5 & = & 0(\alpha_2+\alpha_3) \\
\xi_6 & := & s_{\alpha_2}(\xi_5) & = & \varpi_2-\alpha_1-2\alpha_2-\alpha_3 & n_6\beta_6 & = & \xi_5-\xi_6 & = & 1\alpha_2
\end{array}$$

and finally we get that $\Delta_{\dot{w}}(V_{\dot{w}_0,5}) = (0,1,0,1,0,1)$, as expected.

5.6 Δ-vectors of $V_{\overline{w}}$

We just saw how to compute $\Delta_{\dot{v}}(V_{\overline{w}})$. Even if it is sufficient to use it algorithmically and to do some computation, we will now show a combinatorial description of a part of these coordinates.

5.6.1 Specific notations

We will first introduce some specific notations in order to give a more concise formulation of the main result.

Definition 5.23. Given a reduced representative $\overline{w} = [i_{\ell(w)}, \dots, i_1]$ of $w \in W$ and $\overline{v} = [i_{p_{\ell(v)}}, \dots, i_{p_1}]$ the rightmost representative of $v \leq w$ in $\overline{w}$, $1 \leq k \leq \ell(w)$ and $1 \leq m \leq \ell(v)$ we define

$$f_{\min}(k) := \min(\{1 \leq j \leq \ell(v) \mid i_{p_j} = i_k\} \cup \{0\}),$$

$$f(k) := \max(\{1 \leq j \leq \ell(v) \mid p_j \leq k \text{ and } i_{p_j} = i_k\} \cup \{0\}),$$

$$m^{\oplus} := \min(\{1 \leq j \leq \ell(v) \mid p_j > p_m \text{ et } i_{p_j} = i_{p_m}\} \cup \{\ell(v)+1\}),$$

$$\alpha(k,m) = \#\{1 \leq j \leq m \mid i_{p_j} = i_k\}, \gamma_m := \alpha(p_m, m) \quad \text{and}$$

$$\beta_m = \#\{1 \leq j \leq p_m \mid \forall 1 \leq k \leq m, p_k \neq j \text{ and } i_j = i_{p_m}\}$$

We fix the convention $\alpha(k,0) = 0$.

Here we deal with two systems of indices. $1 \leq k \leq \ell(w)$ is the usual indexation (from right to left) of the letters of $\overline{w}$. $1 \leq m \leq \ell(v)$ is the indexation of the letters of $\overline{v}$. As $\overline{v}$ is the rightmost subword of $\overline{w}$, any $1 \leq m \leq \ell(v)$ correspond to an index $1 \leq k \leq \ell(w)$, namely $m \rightsquigarrow p_m$. In the following we will speak about $\overline{w}$-indices for $1 \leq k \leq \ell(w)$ and $\overline{v}$-indices for $1 \leq m \leq \ell(v)$.

$f_{\min}(k)$ is the lowest $\overline{v}$-index of color i_k. $f(k)$ is the highest $\overline{v}$-index of the letters of color i_k in the subword of $\overline{w}$ consisting of all the letter on the right of the k-th (k-th included).

$m^{\oplus}$ is the $\overline{v}$-successor of the letter of $\overline{v}$-index m.

$\alpha(k,m)$ is the number of $\overline{v}$-indices of color i_k lower or equal to m.

β_m is the number of $\overline{w}$-indices of color i_k lower than p_m which have no $\overline{v}$-index. It counts the number of letter of $\overline{w}$ on the right of the p_m-th, of color i_{p_m}, not taking part in the writing of $\overline{v}$.

Example 5.24. Let $W = D_5$ of Dynkin diagram $\begin{array}{ccccc} & & & & 4 \\ 1 & - & 2 & - & 3 \\ & & & & 5 \end{array}$. We have

$$\overline{w} = [2, \underline{3}, \underline{4}, \underline{1}, \underline{2}, \underline{3}, \underline{5}, 1, 2, \underline{3}, \underline{4}, 1, 2, \underline{3}, 1, \underline{2}, 1]$$

where underlined letters give $\overline{v}$.

The notations previously introduced give us Table 3.

k	m	i_k	$f_{\min}(k)$	$f(k)$	$m^{\oplus}$	β_m	γ_m	k	m	i_k	$f_{\min}(k)$	$f(k)$	$m^{\oplus}$	β_m	γ_m
1		1	8	0				10		1	8	0			
2	1	2	1	1	7	0	1	11	5	5	5	5	18	0	1
3		1	8	0				12	6	3	2	6	10	0	3
4	2	3	2	2	4	0	1	13	7	2	1	7	18	2	2
5		2	1	1				14	8	1	8	8	18	4	1
6		1	8	0				15	9	4	3	9	18	0	2
7	3	4	3	3	9	0	1	16	10	3	2	10	18	0	4
8	4	3	2	4	6	0	2	17		2	1	7			
9		2	1	1											

Table 3: Example of the introduced notations

5.6.2 Combinatorial description

In the following we will consider $\overline{w} = [i_{\ell(w)}, \dots, i_1]$ a reduced representative of $w \in W$ and $\overline{v} = [i_{p_{\ell(v)}}, \dots, i_{p_1}]$ the rightmost subword of $\overline{w}$ representing $v \leq w$ for the Bruhat order in W. We denote $\dot{w}$ the completion of $\overline{w}$ in a representative of w_0 and $\dot{v}$ the one of $\overline{v}$.

By construction we have the following result:

Proposition 5.25. *The module $V_{\overline{w}}$ is a submodule of $V_{\dot{w}}$ obtained by taking the $\ell(w)$ first summands in the indexation given by Definition 2.13.*

The quiver $\Gamma_{\overline{w}}$ is a subquiver of the quiver $\Gamma_{\dot{w}}$ obtained by taking the $\ell(w)$ first vertices and the arrows between them.

In the following we will identify $V_{k,\overline{w}}$ and $V_{k,\dot{w}}$ for $1 \leq k \leq \ell(w)$ and $V_{m,\overline{v}}$ and $V_{m,\dot{v}}$ for $1 \leq m \leq \ell(v)$.

Thanks to Proposition 5.11 we already know a formulation for $\Delta_{\overline{w}}(V_{k,\overline{w}})$. We will now establish a combinatorical description of the $\ell(v)$ first coordinates of $\Delta_{\dot{v}}(V_{k,\overline{w}})$.

Notation 5.26. With the above notations, we will denote $\widetilde{\Delta}_{\dot{v}}(X) := (\Delta_{\dot{v},i}(X))_{i=1}^{\ell(v)}$ the $\ell(v)$ first $\Delta_{\dot{v}}$-coordinates of a module X.

In the following we will use the notations introduced in Section 5.5. Precisely we will focus on the $\Delta_{\dot{v}}$-vector of $V_{k,\overline{w}}$ seen as $V_{k,\dot{w}}$ (thus $1 \leq k \leq \ell(w)$). As $V_{k,\overline{w}}$ is only defined by the k first letters of $\dot{w} = [i_r, \dots, \underbrace{i_{\ell(w)}, \dots, i_1}_{\overline{w}}]$ we define

$$\overline{w}_k = [i_k, \dots, i_1], \quad w_k = s_{i_k} \cdots s_{i_1}, \quad \overline{u}_k = [i_r, \dots, i_{k+1}]$$

and as $u_k = s_{i_r} \cdots s_{i_{k+1}} = w_0 w_k^{-1}$ we take $[j_{q_{r-k}}, \dots, j_{q_1}]$ as the leftmost representative of u_k in $\dot{v} = [j_r, \dots, j_1]$ (where $r = \ell(w_0)$). We denote $1 \leq r_1 < \cdots < r_k \leq r$ the elements of $\{1, \dots, r\} \setminus \{q_1, \dots, q_{r-k}\}$.

Proposition 5.27 (Relation between indices of representatives of u_k)**.** *With the above notations*

1. *$q_1 > \ell(v)$ iff $p_{\ell(v)} < k+1$*
2. *there exists an integer $0 \leq t \leq \ell(v) - 1$ such that $q_1 = \ell(v) - t$ iff $p_{\ell(v)-t} \geq k+1$ and $p_{\ell(v)-t-1} \leq k$.*

Proof. We begin by proving the first part of the statement. Suppose that we have $p_{\ell(v)} \leq k$. Then $\overline{v} = [i_{p_{\ell(v)}}, \dots, i_{p_1}]$ is a subword of $[i_k, \dots, i_1]$, representative of w_k and we have $v \leq u_k^{-1} w_0 = w_k$.

Conversely if $v \leq w_k$, there is at least one subword of $[i_k, \dots, i_1]$ representing v. The rightmost subword representing v is the smallest for the lexicographic order among any representative of v in $\overline{w}_k$

and thus a subword of w_k (see [Mén21a, Lemma 2.1.38]). In particular we have $p_{\ell(w)} \leq k$. We then have shown that $p_{\ell(v)} \leq k \Leftrightarrow v \leq u_a^{-1} w_0$.

We now establish the equivalence $v \leq u_a^{-1} w_0 \Leftrightarrow q_1 > \ell(v)$. Suppose that $q_1 > \ell(v)$ then $[j_{q_{r-k}}, \dots, j_{q_1}]$ is a subword of $[j_r, \dots, j_{\ell(v)+1}]$. We thus have $u_k \leq w_0 v^{-1}$ which is equivalent to $u_k^{-1} \leq v w_0$ (as $w_0^{-1} = w_0$ and by inversion of the words) and finaly $u_k^{-1} w_0 \geq v$ (by multiplication by w_0).

Conversely, suppose that $u_k \leq w_0 v^{-1}$ then $[j_r, \dots, j_{\ell(v)+1}]$ is a representative of $w_0 v^{-1}$ and there exists at least one representative of u_k in this representative. As we are looking at the left part of $\dot{v}$, the leftmost representative of u_k in $\dot{v}$ is among them and eventually $[j_{q_{r-k}}, \dots, j_{q_1}]$ is a subword of $[j_r, \dots, j_{\ell(v)+1}]$. Then it is now obvious that $q_1 \geq \ell(v) + 1$.

We now prove the second statement by introducing again a third statement and proceed via double equivalence.

Let t be such that $q_1 = \ell(v) - t$, we want to show that it is equivalent to

$$(u \leq w_0 s_{j_1} \cdots s_{j_{\ell(v)-t-1}}) \quad \text{and} \quad (u \not\leq w_0 s_{j_1} \cdots s_{j_{\ell(v)-t}}) \tag{1}$$

As $q_1 = \ell(v) - t$, $[j_{q_{r-k}}, \dots, j_{q_1}]$ is a subword of $[j_r, \dots, j_{\ell(v)-t}]$ and we have $u_k \leq w_0 s_{j_1} \cdots s_{j_{\ell(v)-t-1}}$.

On the other hand, as $[j_{q_{r-k}}, \dots, j_{q_1}]$ is the leftmost representative of u_k in $\dot{v}$, there is no representative of u_k $[j_{n_{r-k}}, \dots j_{n_1}]$ such that $n_1 > q_1$. Then u_k has no representative in $[j_r, \dots, j_{q_1+1}]$ and we have $u_k \not\leq w_0 s_{j_1} \dots s_{j_{\ell(v)-t}}$.

Conversely, by hypothesis, we know that there is a representative of u_k among the subwords of $[j_r, \dots, j_{\ell(v)-t}]$ and none among the subwords of $[j_r, \dots, j_{\ell(v)-t+1}]$. In particular the leftmost representative has as a first letter $j_{\ell(v)-t-1}$ and $\ell(v) - t = q_1$.

We now show the other equivalence. If $p_{\ell(v)-t} \geq k+1$ and $p_{\ell(v)-t-1} \leq k$, $s_{i_{p_{\ell(v)-t-1}}} \cdots s_{i_{p_1}}$ has at least one representative among the subwords of $[i_k, \dots, i_1]$. One of them is $[i_{p_{\ell(v)-t-1}}, \dots, i_{p_1}]$ itself and we have $s_{i_{p_{\ell(v)-t-1}}} \cdots s_{i_{p_1}} \leq u_k^{-1} w_0$. Conversely, by right minimality, $s_{i_{p_{\ell(v)-t}}} \cdots s_{i_{p_1}}$ has no representative among the subwords of $[i_k, \dots, i_1]$ and $s_{i_{p_{\ell(v)-t}}} \cdots s_{i_{p_1}} \not\leq u_k^{-1} w_0$ and we get the wanted conjunction of inequalities by inverting and multiplication by w_0.

On the other hand, if we have the two above inequalities there is no representative of $s_{i_{p_{\ell(v)-t}}} \cdots s_{i_{p_1}}$ in $\overline{u}_k$ and there is at least one of $s_{i_{p_{\ell(v)-t-1}}} \cdots s_{i_{p_1}}$. By minimality of the rightmost representative of u_k we have $p_{\ell(v)-t} > k$ and $p_{\ell(v)-t-1} \leq k$.

The inequalities used in this equivalence are the same as the one of Equation 1 as, by definition, $(j_i)_{1 \leq i \leq \ell(v)} = (i_{p_m})_{1 \leq m \leq \ell(v)}$. □

Proposition 5.28. *For $m < q_1$, $\dot{x} = [l_m, \dots, l_1]$ a reduced representative of $\dot{w}$ (possibly equal to $\dot{w}$) we have $\xi_m^{\dot{x}} = s_{l_1} \cdots s_{l_m}(\varpi_{i_k})$ and*

$$\Delta_{\dot{x},m}(V_k) = \begin{cases} 1 & \text{if } l_m = i_k \\ 0 & \text{otherwise} \end{cases}$$

where ξ is the sequence defined in Definition 5.20 defining the $\Delta_{\dot{x}}$-coordinates of $V_{\overline{w},k}$.

Proof. We have $\beta_m^{\dot{x}} = s_{l_1} \cdots s_{l_{m-1}}(\alpha_{l_m})$ and so $s_{\beta_m^{\dot{x}}} = (s_{l_1} \cdots s_{l_m-1}) s_{l_m} (s_{l_1} \cdots s_{l_m-1})$ then

$$s_{\beta_m^{\dot{x}}} \cdots s_{\beta_1^{\dot{v}}} = s_{l_1} \cdots s_{l_m}.$$

As we suppose that $m < q_1$, we have $\xi_m^{\dot{x}} = s_{\beta_m^{\dot{x}}} \cdots s_{\beta_1^{\dot{x}}}(\varpi_{i_k})$ and then

$$\xi_{m-1}^{\dot{x}} - \xi_m^{\dot{x}} = s_{l_1} \cdots s_{l_{m-1}}(\varpi_{i_k} - s_{l_m}(\varpi_{i_k})).$$

We have $s_{l_m}(\varpi_{i_k}) = \varpi_{i_k}$ if $i_k \neq j_m$ and $\varpi_{i_k} - \alpha_{l_m}$ if $i_k = l_m$. In this last case $\xi_{m-1}^{\dot{x}} - \xi_m^{\dot{x}} = \beta_m^{\dot{x}}$ and then:

$$n_m^{\dot{x}} = \begin{cases} 1 & \text{if } l_m = i_k \\ 0 & \text{otherwise} \end{cases}$$

□

Lemma 5.29. *If there is $0 \leq t \leq \ell(v) - 1$ such that $q_1 = \ell(v) - t$, we have*

$$[q_{t+1}, \dots, q_1] = [\ell(v), \dots, \ell(v) - t].$$

Proof. We prove this lemma for a given $\dot{w}$ and $\dot{v}$, we will proceed by decreasing induction on the value of k. The sequences $(q_i)_{1 \leq i \leq r-k}$ depending on the value of k we add an index indicating to which value of k they refer to.

If $k > p_{\ell(v)} - 1$, we have $q_{1,k} \geq \ell(v)$ by Proposition 5.27 and thus we are not in the hypothesis of the theorem.

If $k = p_{\ell(v)} - 1$ then $p_{\ell(v)} = k + 1$ and as $p_{\ell(v)-1} \leq p_{\ell(v)} - 1$, $p_{\ell(v)-1} \leq k$. By the same proposition $q_{1,k} = \ell(v)$ and so $[q_{1,k}] = [\ell(v)]$ and the lemma is verified.

Suppose that $1 < k \leq p_{\ell(v)}$, that there exists $0 \leq t_k \leq \ell(v) - 1$ such that $q_{1,k} = \ell(v) - t_k$ and assume that $[q_{t_k+1,k}, \ldots, q_{1,k}] = [\ell(v), \ldots, \ell(v) - t_k]$. We want to show that there exists $0 \leq t_{k-1} \leq \ell(v) - 1$ such that $q_{1,k-1} = \ell(v) - t_{k-1}$ and such that $[q_{t_{k-1}+1,k-1}, \ldots, q_{1,k-1}] = [\ell(v), \ldots, \ell(v) - t_{k-1}]$.

First we show that t_{k-1} exists. As $q_{1,k} = \ell(v) - t_k$, we have $p_{\ell(v)-t_k} \geq k+1$ and $p_{\ell(v)-t_k-1} \leq k$ which can be rewritten as $p_{\ell(v)-t_k} \geq (k-1) + 2$ and $p_{\ell(v)-t_k-1} \leq (k-1) + 1$.

We have two possibilities: either $p_{\ell(v)-t_k-1} = (k-1) + 1 = k$ or $p_{\ell(v)-t_k-1} \leq k - 1$.

In the first case we then have $p_{\ell(v)-t_k-1} \geq (k-1)+1$ and $p_{\ell(v)-t_k-2} < (k-1)+1$ so $p_{\ell(v)-t_k-2} \leq k-1$ then, according to Proposition 5.27 we have $q_{1,k-1} = \ell(v) - t_k - 1$ and so $t_{k-1} = t_k + 1$

In the second case we have $p_{\ell(v)-t_k} \geq (k-1) + 2 \geq (k-1) + 1$ and $p_{\ell(v)-q_k-1} \leq k - 1$ and so $q_{1,k-1} = \ell(v) - t_k$, eventually $t_k = t_{k-1}$.

In both cases, such a t_{k-1} exists. We now show that any of the two cases imply the equality.

In the first case $t_{k-1} = t_k + 1$ and $q_{1,k-1} = \ell - t_{k-1}$. We first show that $[q_{r-(k-1),k-1}, \ldots, q_{2,k-1}]$ is a representative of u_k. By definition $u_{k-1} = s_{j_{q_{r-(k-1),k-1}}} \cdots s_{j_{q_{1,k-1}}} = s_{i_r} \cdots s_{i_{(k-1)+1}}$.

Then $u_{k-1}s_{i_{(k-1)+1}} = s_{i_r} \cdots s_{i_{k+1}} = u_k$. However, $i_{p_m} = j_m$ for any $1 \leq m \leq \ell(v)$ by definition of $\dot{v}$. As $q_{1,k-1} = \ell(v) - t_{k-1}$ we have $j_{q_{1,k-1}} = j_{\ell(v)-t_{k-1}} = i_{p_{\ell(v)-t_{k-1}}} = i_{p_{\ell(v)-t_k-1}}$ and, as we are in the first case, $p_{\ell(v)-t_k-1} = (k-1)+1 = k$, $j_{q_{1,k-1}} = i_k$. Then $u_k = u_{k-1}s_{i_k} = u_k s_{j_{q_{1,k-1}}} = s_{j_{q_{r-(k-1),k-1}}} \cdots s_{j_{q_{2,k-1}}}$ and $[q_{r-(k-1),k-1}, \ldots, q_{2,k-1}]$ is a representative of u_k in $\dot{v}$. As the leftmost representative of u_k in $\dot{v}$ is $[q_{r-k,k}, \ldots, q_{1,k}]$ for any $2 \leq \alpha \leq r - (k-1)$ we have $q_{\alpha-1,k} \geq q_{\alpha,k-1}$.

We will show that, in fact, $q_{\alpha-1,k} = q_{\alpha,k-1}$ for any $2 \leq \alpha \leq t_k + 2$.

As $q_{2,k-1} > q_{1,k-1}$ we have $\ell(v) - t_k = q_{1,k} \geq q_{2,k-1} > q_{1,k-1} = \ell(v) - t_{k-1} = \ell(v) - t_k - 1$ so $q_{1,k} = q_{2,k-1}$. As for any $2 \leq \alpha \leq t_k + 2$ we have $q_{\alpha-1,k} = \ell(v) - t_k + (\alpha - 2)$ by induction hypothesis, we can repeat this proof for any value of $\alpha \leq t_k + 2$ and get:

$$\forall 2 \leq \alpha \leq t_k + 2,\ q_{\alpha,k-1} = \ell(v) - t_{k-1} + (\alpha - 1) = \ell(v) - t_k + (\alpha - 2) = q_{\alpha-1,k}$$

and so $[q_{t_{k-1}+1,k-1}, \ldots, q_{2,k-1}] = [q_{t_k+1,k}, \ldots, q_{1,k}] = [\ell(v), \ldots, \ell(v) - t_{k-1} + 1]$.

As we have already shown that $q_{1,k-1} = \ell(v) - t_{k-1}$ we finally obtain

$$[q_{t_{k-1}+1,k-1}, \ldots, q_{1,k-1}] = [\ell(v), \ldots, \ell(v) - t_{k-1}]$$

In the second case, $\ell(v) - t_k = q_{1,k} = q_{1,k-1} = \ell(v) - t_{k-1}$. As previously, by definition, $u_{k-1} = u_k s_{i_k}$ and so $u_{k-1} = s_{j_{q_{r-k,k}}} \cdots s_{j_{q_{1,k}}} s_{i_k}$.

However $\ell(u_{k-1}) = r - k - 1$ and $\ell(u_k s_{j_{q_{1,k-1}}}) \leq r - (k-1) - 1$ as it admits $s_{j_{q_{r-(k-1),k-1}}} \cdots s_{j_{q_{2,k-1}}}$ as representative. We then have $\ell(u_{k-1}) > \ell(u_{k-1}s_{j_{q_{1,k-1}}})$.

According to the exchange relation there is $r - (k-1) \geq \alpha \geq q_{1,k-1}$ such that

$$u_k = u_{k-1}s_{i_k} = s_{j_{q_{r-(k-1),k-1}}} \cdots \hat{s}_{j_{q_{\alpha,k-1}}} \cdots s_{j_{q_{1,k-1}}}.$$

As $[j_{q_{r-k,k}}, \ldots, j_{1,k}]$ is the leftmost representative of u_k we have:

$$\forall 1 \leq \beta \leq \alpha,\ q_{\beta,k-1} \leq q_{\beta,k} \text{ and } \forall \alpha + 1 \leq \beta \leq r - (k-1),\ q_{\beta,k-1} \leq q_{\beta-1,k}$$

We will show by induction that for all $2 \leq \beta \leq \min(\alpha, t_{k-1} + 1)$, we have $q_{\beta,k-1} = q_{\beta,k}$.

By hypothesis, we have $q_{1,k-1} = q_{1,k} = \ell(v) - t_{k-1}$ and so for $\beta = 2$ we have

$$\ell(v) - t_{k-1} = q_{1,k-1} < q_{2,k-1} \leq q_{2,k} = \ell(v) - t_k + 1 = \ell(v) - t_{k-1} + 1$$

and thus $q_{2,k-1} = \ell - t_{k-1} + 1 = q_{2,k}$. We then have the initialization.

By induction, we suppose that we have $q_{\beta-1,k-1} = \ell(v) - t_{k-1} + \beta - 2 = q_{\beta-1,k}$. Then we have: $\ell(v) - t_{k-1} + \beta - 2 = q_{\beta-1,k-1} < q_{\beta,k-1}$ and, by minimality we have

$$q_{\beta,k-1} \leq q_{\beta,k} = \ell(v) - t_{k-1} + \beta - 1$$

and so we have the following sequence of inequalities.

$$\ell(v) - t_{k-1} + \beta - 2 = q_{\beta-1,k-1} < q_{\beta,k-1} \leq q_{\beta,k} = \ell(v) - t_{k-1} + \beta - 1$$

and so $q_{\beta,k-1} = q_{\beta,k}$.

We know want to determine relative positions of α and $t_{k-1}+1$. Indeed, if $\alpha > t_{k-1}+1$ we can deduce the lemma from the above induction.

By contradiction, let us suppose $\min(\alpha, t_{k-1}+1) = \alpha \leq t_{k-1}+1$ then we have:

$$\ell(v) - t_{k-1} + \alpha - 2 = q_{\alpha-1,k-1} < q_{\alpha,k-1} < q_{\alpha+1,k-1} \leq q_{\alpha,k} = \ell(v) - t_{k-1} + \alpha - 1$$

which is absurd by a cardinality argument. Then $\alpha > t_{k-1}+1$ and $q_{\beta,k} = q_{\beta,k-1}$ for all $2 \leq \beta \leq t_{k-1}+1$ and as we previously show that $q_{1,k-1} = q_{1,k}$, we can eventually write:

$$[q_{t_{k-1}+1,k-1}, \ldots, q_{1,k-1}] = [\ell(v), \ldots, \ell(v) - t_{k-1}]$$

and we have the heredity and thus the lemma is proved in both cases. □

Theorem 5.30. *We have the following correspondance between Δ-vector coordinates:*

$$\Delta_{\dot{v},m}(V_k) = \Delta_{\dot{w},i_{p_m}}(V_k), \quad \forall 1 \leq m \leq \ell(v) \text{ and } 1 \leq k \leq \ell(w).$$

Proof. First, if $k = \ell(w_0)$ (and thus $w = w_0$), $u_k = e$ and we can always apply Proposition 5.28 and we have the desired equality.

In any other case we will prove the identities, for any coordinate of index $1 \leq m \leq \ell(v)$.

If $m < q_1$, thanks to Proposition 5.28 we take $\dot{x} = \dot{v}$ and then $\Delta_{\dot{v},m}(V_k) = \begin{cases} 1 & \text{if } j_m = i_k \\ 0 & \text{otherwise} \end{cases}$.

Now we will show we can use the same result but with $\dot{w} = \dot{x}$ and p_m instead of m to get

$$\Delta_{\dot{w},m}(V_k) = \begin{cases} 1 & \text{if } i_{p_m} = i_k \\ 0 & \text{otherwise} \end{cases}$$

As $m < q_1$ we have two possibilities. Either we have $m < m_1$ for any $1 \leq m \leq \ell(v)$ (first case of Proposition 5.27) or not. If we do, $p_{\ell(v)} \leq k$. As $p_m \leq p_{\ell(v)}$, q_1 is the first index of u_k in $\dot{w}$, $k+1$ by definition. We then have $p_m \leq q_1$ and we can apply Proposition 5.28.

In the other case, there exists $0 \leq t \leq \ell(v) - 1$ such that $q_1 = \ell(v) - t$. However, in any case we have $m < q_1$ so $p_m \leq p_{\ell(v)-t-1} \leq k$ and we can once again apply Proposition 5.28. We then have

$$\Delta_{\dot{w},p_m}(V_k) = \begin{cases} 1 & \text{if } i_{p_m} = i_k \\ 0 & \text{otherwise} \end{cases}$$

hence the result.

Now we consider the case where $m \geq q_1$. In this case, thanks to Proposition 5.27 we now that $m \geq \ell(v) - t$ and thus $p_m \geq p_{\ell(v)-t} \geq k+1$. By definition of u_k in $\dot{w}$, we know that $\xi^{\dot{w}}_{p_m} = \xi^{\dot{w}}_{p_m-1}$ and so $n^{\dot{w}}_{p_m} = 0$ and finally $\Delta_{\dot{w},p_m}(V_k) = 0$. We then have to prove that, in this case, $\Delta_{\dot{v},m}(V_k) = 0$.

We use Lemma 5.29. We have that $[\ell(v), \ldots, \ell(v) - t] = [q_{t+1}, \ldots, q_1]$ and thus, for all $\ell(v) \geq m \geq \ell(v) - t$, we have $\Delta_{\dot{v},m}(V_k) = 0$.

Finally, for all $1 \leq m \leq \ell(v)$ we have: $\Delta_{\dot{v},m}(V_k) = \Delta_{\dot{w},p_m}(V_k)$. □

Example 5.31. In order to illustrate the previous notions let us consider the following case. Here $W = D_6$

We consider $\overline{w} = [5,3,4,2,3,6,4,2,1,5,2,3,2,4,3,6,4,5,3,4,1,3]$ and complete it in:

$$\dot{w} = [2,3,4,6,1,2,3,4,\underline{5},3,4,2,3,6,\underline{4},2,\underline{1,5,2,3,2,4,3,6,4,5},3,\underline{4},1,\underline{3}].$$

We chose $v = s_4s_5s_1s_2s_3s_4s_6s_2s_3s_4s_5s_4s_2s_3$ (so $\ell(v) = 14$) and we have

$$\dot{v} = [4,6,2,3,4,5,1,2,3,4,6,5,4,2,3,2,\underbrace{5,4,1,5,2,3,2,4,3,6,4,5,4,3}_{\overline{v}}].$$

The indices of $\overline{w}$ letters forming $\overline{v}$ are $(p_m)_{1\leq m\leq 14} = (1,3,5,6,7,8,9,10,11,12,13,14,16,22)$.

Let us take $k = 10$ we then have

$$\dot{w} = [\underbrace{2,3,4,6,1,2,3,4,5,3,4,2,3,6,4,2,1,5,2,3}_{\overline{u}_{10}},\underbrace{2,4,3,6,4,5,3,4,1,3}_{\overline{w}_{10}}].$$

and now we look for the leftmost representative of u_{10} in $\dot{v}$.

We find $[6,2,3,4,5,1,2,3,4,6,4,2,3,2,5,4,1,5,2,3]$ which uses the letters in positions

$$(q_i)_{1\leq i\leq \ell(w_0)-k} = (9,10,11,12,13,14,15,16,17,18,20,21,22,23,24,25,26,27,28,29).$$

In particular we then have $q_1 = 9$. $p_9 = 11 \geq k+1$ et $p_8 = 10 \leq k$, as in the second case of Proposition 5.27.

To sum up, we have

$$\dot{w} = [\overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{3}, \overline{4}, \overline{2}, \overline{3}, \overline{6}, \overline{4}, \overline{2}, \overline{1}, \overline{5}, \overline{2}, \overline{3}, 2, 4, 3, 6, 4, 5, 3, 4, 1, 3]$$

$$\dot{w} = [2, 3, 4, 6, 1, 2, 3, 4, \underline{5}, 3, 4, 2, 3, 6, \underline{4}, 2, \underline{1}, \underline{5}, \underline{2}, \underline{3}, \underline{2}, \underline{4}, \underline{3}, \underline{6}, \underline{4}, \underline{5}, 3, \underline{4}, 1, \underline{3}]$$

$$\dot{v} = [4, \overline{6}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, 5, \overline{4}, \overline{2}, \overline{3}, \overline{2}, \overline{5}, \overline{4}, \overline{1}, \overline{5}, \overline{2}, \overline{3}, 2, 4, 3, 6, 4, 5, 4, 3]$$

$$\dot{v} = [4, 6, 2, 3, 4, 5, 1, 2, 3, 4, 6, 5, 4, 2, 3, 2, \underline{5}, \underline{4}, \underline{1}, \underline{5}, \underline{2}, \underline{3}, \underline{2}, \underline{4}, \underline{3}, \underline{6}, \underline{4}, \underline{5}, \underline{4}, \underline{3}]$$

where the overlined red letters represent positions of the letters of the leftmost representative of u_{10} in $\dot{w}$ and $\dot{v}$ and the underlined blue ones, the one of v. We can see Lemma 5.29 in the fact that, for indices $9 = q_1 \leq m \leq \ell(v) = 14$, all letters are (the right) part of the representative of u_{10}.

We now place ourselves in a close case. We keep $\dot{w}$, and take $v = s_4 s_5 s_1 s_2 s_3 s_4 s_6 s_2 s_3 s_4 s_5 s_4 s_2 s_3$. We have $\overline{v} = [4, 1, 2, 3, 2, 4, 3, 6, 4, 5, 4, 3]$, so, in particular, $p_{12} = 15$ and

$$\dot{v} = [4, 6, 2, 3, 4, 5, 1, 2, 3, 4, 6, 2, 3, 4, 5, 4, 3, 2, 4, 1, 2, 3, 2, 4, 3, 6, 4, 5, 4, 3].$$

Let $k = 16$ be. We have $p_{12} = 15 < 17 = k + 1$. Thus we have

$$\dot{w} = [\overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{3}, \overline{4}, \overline{2}, \overline{3}, \overline{6}, 4, 2, 1, 5, 2, 3, 2, 4, 3, 6, 4, 5, 3, 4, 1, 3]$$

$$\dot{w} = [2, 3, 4, 6, 1, 2, 3, 4, 5, 3, 4, 2, 3, 6, 4, \underline{2}, \underline{1}, 5, \underline{2}, \underline{3}, \underline{2}, \underline{4}, \underline{3}, \underline{6}, \underline{4}, \underline{5}, 3, \underline{4}, 1, \underline{3}]$$

$$\dot{v} = [4, \overline{6}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{2}, 3, \overline{4}, \overline{5}, 4, \overline{3}, 2, 4, 1, 2, 3, 2, 4, 3, 6, 4, 5, 4, 3]$$

$$\dot{v} = [4, 6, 2, 3, 4, 5, 1, 2, 3, 4, 6, 2, 3, 4, 5, 4, 3, 2, \underline{4}, \underline{1}, \underline{2}, \underline{3}, \underline{2}, \underline{4}, \underline{3}, \underline{6}, \underline{4}, \underline{5}, \underline{4}, \underline{3}]$$

The representative of u_{16} is $\overline{u}_{16} = [2, 3, 4, 6, 1, 2, 3, 4, 5, 3, 4, 2, 3, 6]$. The leftmost representative of u_{16} in $\dot{v}$ is then: $[4, 6, 2, 3, 4, 5, 1, 2, 3, 4, 6, 2, 4, 5, 3]$. In particular $q_1 = 14 > 12 = \ell(v)$, we are in the first case of Proposition 5.27.

The previous results give us now a combinatorial description of the first coordinates of the $\Delta_{\dot{v}}$-vectors of indecomposable summands of $V_{\overline{w}}$.

Theorem 5.32 (Structure of initial $\Delta_{\dot{v}}$-vectors)**.** *The $\ell(v)$-first coordinates of $\Delta_{\dot{v}}$-vectors of direct indecomposable summands of $V_{\overline{w}}$ are given by:*

$$\widetilde{\Delta}_{\dot{v}}(V_k) = \sum_{j=1}^{f(k)} \delta_{i_{p_j}, i_k} f_j \quad 1 \leq k \leq \ell(w)$$

where $(f_j)_{1 \leq j \leq \ell(v)}$ is the canonical basis.

We can also say that the indices of coordinates equal to 1 of $\Delta_{\dot{v}}(V_k)$ are the integers of the set

$$\{1 \leq j \leq \ell(v) \mid f_{\min}(k) \leq j \leq f(k) \mid i_{p_j} = i_k\},$$

the other $\ell(v)$ first being zero.

Proof. By definition, $f(k) = \max\{1 \leq j \leq \ell(v) \mid p_j \leq k \text{ et } i_{p_j} = i_k\}$. We will use Theorem 5.30 and look at any index $1 \leq m \leq \ell(v)$ of $\Delta_{\dot{v}}$.

First suppose that we have m such that $p_m \leq k$. For these coordinates, thanks to Proposition 5.27, we have that $m < q_1$ and so we can directly use Proposition 5.21.

In the other case, by Lemma 5.29 and the proof of Theorem 5.30, we now that these coordinates are zero and here they are not part of the sum.

For the second formulation, it is equivalent as $\{1 \leq j < f_{\min}(k) \mid i_{p_j} i_k\}$ is empty by minimality of $f_{\min}(k)$. □

5.7 Criterion for being in $\mathcal{C}^v$

We now want to use Δ-vector to be able to tell if a module is in $\mathcal{C}^v$ or in $\mathcal{C}_w$.

Proposition 5.33. *Let $\overline{w}$ be a reduced representative of $w \in W$ and $\dot{w}$ a reduced representative of w_0 having $\overline{w}$ as a right factor. A module $M \in \mathbf{mod}(\Lambda)$ is in $\mathcal{C}_w$ iff its $\Delta_{\dot{w}}$-vector has zero coordinates for indices $k \geq \ell(w)$.*

Proof. Thanks to Lemma 5.2, any module of $\mathcal{C}_w$ admits a stratification by modules $(M_{i,\dot{w}})_{i=1}^{\ell(w)}$ and conversely, any module admitting such a stratification lies in the category. □

In order to prove an analoguous result for $\mathcal{C}^v$, we first need some preparatory lemmas.

Lemma 5.34. *Let $M \in \mathbf{mod}(\Lambda)$ rigid, $\overline{v}$ a representative of v. The maximal submodule of M in $\mathcal{C}_v$, denoted $t_v(M)$, has as $\Delta_{\dot{v}}$-vector the one coming from the $\ell(v)$ first coordinates of $\Delta_{\dot{v}}(M)$.*

Proof. Let $N \subset M$ of $\Delta_{\dot{v}}$-vector given by the $\ell(v)$ first coordinates of M. As M is rigid, so is N and it is completely determined by its $\Delta_{\dot{v}}$-vector. We will show that it is the maximal submodule of M being in $\mathcal{C}_v$.

$N \subset M \in \mathcal{C}_v$. By maximality, we have $N \subset t_v(M)$ and then $M/N \supseteq M/t_v(M)$. By definition of $\Delta_{\dot{v}}$-vectors, any M submodule containing N has as $\ell(v)$ first coordinates the one of N. So if there exists $t_v(M) \supsetneq N$ in $\mathcal{C}_v$, then $t_v(M)$ has at least one coordinate in its $\Delta_{\dot{v}}$-vector nonzero and of index more than $\ell(v)$. Thanks to Proposition 5.33, such a module does not belong to $\mathcal{C}_v$ which is absurd. So $t_v(M) = N$. □

Proposition 5.35 (Criterion to be in $\mathcal{C}^v$)**.** *Let $v \in W$ and M be a rigid indecomposable Λ-module then $M \in \mathcal{C}^v$ iff $\widetilde{\Delta}_{\dot{v}}(M) = 0$.*

Proof. Thanks to [Lec16, Proposition 3.12], if M is a rigid indecomposable module of $\mathcal{C}_w$ then $M/t_v(M)$ is a rigid indecomposable module of $\mathcal{C}_{v,w}$. Then $M/t_v(M) \in \mathcal{C}^v$.

Thus $M = M/t_v(M)$ or, equivalently, $t_v(M) = 0$ iff $M \in \mathcal{C}^v$. According to the previous lemma, $t_v(M) = 0$ iff $\widetilde{\Delta}_{\dot{v}}(M) = 0$. □

6 Seed computation algorithm

Thanks to Section 4, we have an explicit way to build a seed for the cluster structure on $\mathcal{C}_w$ and thanks to Proposition 5.35 we have a way to check if the underlying module of the seed is in $\mathcal{C}^v$. We must now combine both in order to get a $\mathcal{C}_{v,w}$-rigid maximal basic module and its quiver. Doing so we will then have an initial seed for the cluster structure on $\mathcal{C}_{v,w}$.

6.1 Concept

As it would be very difficult to build the quiver of a seed G of $\mathcal{C}_{v,w}$ by looking at the endormorphism algebra of the module, we want to use the fact that we already know the quiver of $V_{\overline{w}}$ and we can compute, following the mutation, the quiver of any seed mutation-equivalent to G. Our idea is then to apply a well-chosen sequence of mutation to $G_{\overline{w}} = (V_{\overline{w}}, \Gamma_{\overline{w}})$ in order to get a seed $G' = (V', \Gamma')$ such that a subseed is a seed for the cluster structure on $\mathcal{C}_{v,w}$. That is to say that we want V' to be a $\mathcal{C}_{v,w}$-rigid maximal basic module and Γ' the quiver of its endomorphism algebra.

The rigidity of the seed will not be an issue as rigidity is preserved through mutation and cluster reduction as well as the basicity. Regarding the $\mathcal{C}_{v,w}$-maximal rigidity we already saw in Theorem 2.18 that to check it, it is enough to check that the module has $\ell(w) - \ell(v)$ indecomposable summands. Thus the cluster reduction will have to remove $\ell(v)$ summands to the seed.

Eventually the main issue in this method will be to ensure that the obtained module does lie in $\mathcal{C}_{v,w}$. We will use the criterion of Proposition 5.35 and reach it thanks to a very specific choice of mutations. We will now introduce two different formulations of the sequence of mutations to perform, each one coming from a different idea. We will show in Section 8.1 that these two formulations are in fact equivalent.

6.2 First formulation

The idea of this formulation, due to Jan Schröer, is to mimic the sequence of mutation in [GLS11, Section 13]. In this sequence, the authors look at the modules V_k of $V_{\overline{w}}$ which have the $\Delta_{\dot{w}}$-vectors as described in Proposition 5.11, making them interval modules. The sequence of mutation aims to "shift" these $\Delta_{\dot{w}}$-coordinates in the following sense: initial V_k modules have nonzero (and equal to 1) coordinates for the indices $\leq k$ of letters in $\overline{w}$ of color i_k; final T_k modules then have nonzero (equal to 1) $\Delta_{\dot{w}}$-coordinates for the last k indices of color i_k.

Here we want to do the same but with $\Delta_{\dot{v}}$-coordinates of modules originally equal to $V_{\overline{w}}$, shifting the coordinates far enough from the start, such that first $\ell(v)$ coordinates are all zero and thus the corresponding modules are in $\mathcal{C}^v$. As we are now dealing with $\Delta_{\dot{v}}$ coordinates of $V_{\overline{w}}$ the sequence of mutation will be slightly different and we will only find the same sequence if $\overline{v}$ is a right factor of $\overline{w}$.

The advantage of this formulation is that we can directly read the sequence of mutation to perform from the positions of the letters of $\overline{v}$ in $\overline{w}$. This formulation will not be very easy to use in a proof but we will see in the next section a more helpful one.

We will express the mutation sequence using the combinatorial numbers defined in Definition 5.23

Definition 6.1. Given a reduced representative $\overline{w}$ of $w \in W$ and the rightmost representative $\overline{v}$ of $v \leq w$ in $\overline{w}$ we define for any letter $1 \leq m \leq \ell(v)$ of $\overline{v}$ the following sequence of mutations:

$$\tilde{\mu}_m := \begin{cases} \mu_{(k_{\max})^{\gamma_m -}} \circ \mu_{(k_{\max})^{(\gamma_m+1)-}} \circ \cdots \circ \mu_{(k_{\min})^{(\beta_m+1)+}} \circ \mu_{(k_{\min})^{\beta_m +}} & \text{if } (k_{\max})^{\gamma_m -} \geq (k_{\min})^{\beta +} \\ \text{id} & \text{otherwise} \end{cases}$$

where $k = p_m$. Then we combine all the $\tilde{\mu}_m$ to form the sequence: $\widetilde{M} = \tilde{\mu}_{\ell(v)} \circ \cdots \circ \tilde{\mu}_1$

A way to rephrase this algorithm is to say that given the index $1 \leq m \leq \ell(v)$, the sequence $\widetilde{\mu}_m$ consists in mutating all the summands whose color is i_{p_m} by ascending order of index (following the indexation given in Definition 2.13) apart from a number of initial summands (depending on the number of letters of color i_{p_m} among the letters of $\overline{w}$ of index $< p_m$ not used in the writing of $\overline{v}$) and a number of final summands (depending on the number of times we have already mutated this line).

Example 6.2. Here W is of type A_5 with $\overline{w} = [1, 3, 2, 4, 3, \underline{2}, \underline{4}, \underline{5}, 4, \underline{3}, 2, \underline{1}, \underline{2}]$ and $v = s_2 s_4 s_5 s_3 s_1 s_2$.

The integers p_m, β and γ and the colors i_{p_m} are written for any $1 \leq m \leq 6 = \ell(v)$ in Table 4.

m	1	2	3	4	5	6
p_m	1	2	4	6	7	8
i_{p_m}	2	1	3	5	4	2
β_m	0	0	0	0	1	1
γ_m	1	1	1	1	1	2

Table 4: Values of p_m, i_{p_m}, β_m and γ_m

We then have $\widetilde{\mu}_1 = \mu_8 \circ \mu_3 \circ \mu_1$, $\widetilde{\mu}_2 = \mu_2$, $\widetilde{\mu}_3 = \mu_9 \circ \mu_4$, $\widetilde{\mu}_4 = id$, $\widetilde{\mu}_5 = \mu_7$, $\widetilde{\mu}_6 = \mu_3$. Note that whenever $\beta_m = 0$ we get the same sequence of mutations as in [GLS11, Section 13].

6.3 Recursive formulation

We have another formulation of the algorithm, allowing us to see it as a kind of greedy algorithm. The idea is to first cancel the first $\Delta_{\dot{v}}$ coordinates of (almost) all the summands, then the second coordinates etc. until the $\ell(v)$-th. Then, removing all the summands which still have nonzero coordinates among the $\ell(v)$ first, we will obtain a module in $\mathcal{C}^v$, thus in $\mathcal{C}_{v,w}$.

As this algorithm is defined by induction we need first to introduce some notations before giving its formulation.

Notation 6.3. In the following we will define a sequence of sequences of mutations $(\hat{\mu}_m)_{m=1}^{\ell(v)}$. In order to keep track of the successive modules we will call $V_{\overline{w}} = R_0$ (respectively $V_{k,\overline{w}} = R_{k,0}$ for $1 \leq k \leq \ell(w_0)$) and for $0 \leq m \leq \ell(v) - 1$ $R_{m+1} = \hat{\mu}_{m+1}(R_m)$ (respectively $R_{k,m+1}$ is the k-th summand of R_{m+1}).

Definition 6.4 (Index set $A_m(R)$)**.** Let $1 \leq m \leq \ell(v)$ and $R_{m-1} = \hat{\mu}_{m-1} \circ \cdots \circ \hat{\mu}_1(R_0)$ with summands indexed as they were in $V_{\overline{w}} = R_0$. Let $b_m = ((p_m)_{\max})^{\gamma_m -}$.

We define: $A_m(R_{m-1}) = \{1 \leq i \leq b_m \mid \Delta_{\dot{v},m}(R_{i,m-1}) \neq 0\}$. We order the elements of A_m by ascending order, denote $(A_m(R_{m-1}))_1$ the smallest one, the j-th smaller $(A_m(R_{m-1}))_j$ and $(A_m(R_{m-1}))_{\max}$ the largest.

We can now define the sequence of mutation $\hat{\mu}_m$ for any m by induction.

Definition 6.5. Let $m \geq 1$. Suppose we have already defined: $R_{m-1} = \hat{\mu}_{m-1} \circ \cdots \circ \hat{\mu}_1(R_0)$. We write $A_m = A_m(R_{m-1})$. Then

$$\hat{\mu}_m := \begin{cases} \mu_{A_{m,\max}} \circ \mu_{A_{m,\max-1}} \circ \cdots \circ \mu_{A_{m,1}} & \text{if } A_m \neq \varnothing \\ \text{id} & \text{otherwise} \end{cases}$$

We can summarize it by saying that we will mutate in increasing order all the summands of indices smaller than a given bound, which have a nonzero m-th coordinate in their $\Delta_{\dot{v}}$-vector.

Definition 6.6. We define $\mu_\bullet(V_{\overline{w}}) = S(\hat{\mu}_{\ell(v)} \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}}))$ where S is the deletion of all the direct summands $R_{k,\ell(v)}$ such that $k > (k_{\max})^{\alpha(k,\ell(v))-}$.

Example 6.7. Please see Example B.1 p. 40.

6.4 Main theorem

We will now state the main theorem of this article:

Theorem 6.8. *$\mu_\bullet(V_{\overline{w}})$ is a $\mathcal{C}_{v,w}$-rigid maximal basic module. The quiver of $\mu_\bullet(V_{\overline{w}})$ is obtained from $\hat{\mu}_{\ell(v)} \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$ by removing all the arrows adjacent to a vertex deleted by S and only these ones.*

The next section will be dedicated to the proof of this theorem.

7 Proof

As we said, the main issue here is to prove that $\mu_\bullet(V_{\overline{w}})$ is in $\mathcal{C}_{v,w}$. In order to prove it we will do an induction which will study some properties of a part of the successive seeds. In the next section we define this subseed.

We recall that $R_0 = V_{\overline{w}}$ and for $1 \leq m \leq \ell(v)$, $R_m = \hat{\mu}_m \circ \cdots \circ \hat{\mu}_1(R_0)$. Accordingly, $\Gamma_0 = \Gamma_{\overline{w}}$ and $\Gamma_m = \hat{\mu}_m \circ \cdots \circ \hat{\mu}_1(\Gamma_0)$.

7.1 Cut seed and evicted summands

Definition 7.1. Given a seed $G_m = \hat{\mu}_m \circ \cdots \circ \hat{\mu}_1((V_{\overline{w}}, \Gamma_{\overline{w}}))$, we define the cut seed $C(R_m)$ as the couple $(C(R_m), C(\Gamma_m))$ where:

- the module $C(R_m)$ is obtained from R_m by removing the direct indecomposable summands $R_{i,m}$ such that $\widetilde{\Delta}_{\dot{v}}(R_{i,m}) = 0$. We call them *evicted modules*. We also remove the direct indecomposable summands $R_{i,m}$ such that $i > (i_{\max})^{\alpha(i,m)-}$. We call them *deleted modules*.
- the quiver $C(\Gamma_{\overline{w}})$ obtained from $\Gamma_{\overline{w}}$ by removing all deleted or evicted vertices and the arrows iff they are adjacent to at least one of these vertices.

The other summands keep the same index as in R_m.

We can note that, for the final step $m = \ell(v)$, the deleted summands correspond to the summands erased by S. Provided our algorithm works as intended, at this point, all the other summands must be in $\mathcal{C}_{v,w}$ and thus evicted and so the final cut seed must be empty.

We now state some additional properties of the evicted summands. We will place ourselves under the following hypothesis using the notations introduced in Section 5.6.1. This hypothesis generalizes the description given in Theorem 5.32 and will be a part of the induction hypothesis for the proof of the algorithm.

Hypothesis 7.2. *The non-zero coordinates of $\widetilde{\Delta}_{\dot{v}}(R_k)$ are the elements of the subset:*

$$\{1 \leq j \leq \ell(v) \mid f_{\min}(k)^{\alpha(k,m)\oplus} \leq j \leq f(k^{\alpha(k,m)+}) \mid i_{p_j} = i_k\}$$

Example 7.3. Example B.1 is an example where this hypothesis is always verified.

Lemma 7.4. *If $i_k = i_{p_m}$ then $f_{\min}(k)^{\alpha(k,m-1)\oplus} = f(p_m) = m$*

Proof. $f_{\min}(k)^{\alpha(k,m-1)\oplus}$ is the $\alpha(k,m-1)$-th successor (among the letters of $\overline{v}$) of the first letter in $\overline{v}$ of color i_k. As the m-th letter of $\overline{v}$ is of color i_k, $f_{\min}(k) = f_{\min}(p_m)$ and $f_{\min}(p_m)^{\alpha(k,m-1)\oplus} = m = f(p_m)$ by definition. □

In order to have such a result for any index in $\overline{w}$, we introduce the following notation.

Definition 7.5. The application $\xi(k,m)$, $1 \leq k \leq \ell(w)$ and $0 \leq m \leq \ell(v)$ is defined by

$$\xi(k,m) = \begin{cases} \min\{p_m < j \leq \ell(w) \mid \exists 1 \leq q \leq \ell(v) \mid j = p_q \text{ et } i_j = i_k\} & \text{if non-empty} \\ \ell(w)+1 & \text{otherwise} \end{cases}$$

where $p_0 = 0$.

$\xi(k,m)$ is the $\overline{w}$-coordinate of the first letter of $\overline{v}$ of color i_k and of index strictly greater than p_m.

Proposition 7.6. $f(\xi(k,m)) = f_{\min}(k)^{\alpha(k,m)\oplus}$.

Proof. If $m = 0$, $\xi(k,0) = f_{\min}(k)$ by definition.

If $m \neq 0$ and $i_k = i_{p_m}$, $f(\xi(k,m)) = p_m^{\oplus} = f_{\min}(k)^{(\alpha(k,m-1)+1)\oplus}$ thanks to Lemma 7.4. As $\alpha(k,m) = \alpha(k,m-1)+1$ here as $i_k = i_{p_m}$ we deduce the result.

If $i_k \neq i_{p_m}$ we prove by induction on m. We already have the initialization when $m = 0$. Suppose that $f(\xi(k,m-1)) = f_{\min}(k)^{\alpha(k,m-1)\oplus}$. As in this case $\alpha(k,m-1) = \alpha(k,m)$, we have to show that $\xi(k,m-1) = \xi(k,m)$.

If these two numbers were different, there would be a letter of $\overline{v}$ of color i_k in the interval $[\![p_{m-1}+1, p_m - 1]\!]$. This is impossible, as $\overline{v}$ is the rightmost representative. Thus $\xi(k,m-1) = \xi(k,m)$ and the heredity follows. □

Corollary 7.7. *Let R_k a module verifying Hypothesis 7.2 then $R_k \in \mathcal{C}^v$ iff $k^{\alpha(k,m)+} < \xi(k,m)$.*

Proof. In this case, as we have $f(k^{\alpha(k,m)+}) < f(\xi(k,m)) = f_{\min}(k)^{\alpha(k,m)\oplus}$, the set of $\widetilde{\Delta}_{\dot{v}}$-coordinates not equal to 0 is empty.

The inequality is strict. Even if f is not strictly increasing, $\xi(k,m) = p_j$ (for a $1 \leq j \leq \ell(v)$) is the $\overline{w}$-index of a letter of $\overline{v}$ thus, as $k^{\alpha(k,m)+} < \xi(k,m)$, $f(k^{\alpha(k,m)+})$ cannot be equal to $f(\xi(k,m)) = j$. □

Corollary 7.8 (Initial evictions)**.** *The indecomposable summands V_k of $V_{\overline{w}}$ such that $f(k) < f_{\min}(k)$ are in $\mathcal{C}^v$ and thus are evicted summands.*

Proof. By using Theorem 5.32 we are in Hypothesis 7.2 for $m = 0$ and thus the Corollary 7.7 holds. □

Corollary 7.9. *If R_k and R_{k^-} are direct indecomposable summands of R verifying Hypothesis 7.2 and $R_k \in \mathcal{C}^v$ then $R_{k^-} \in \mathcal{C}^v$.*

Proof. $(k^-)^{\alpha(k,m)+} < k^{\alpha(k,m)+} < \xi(k,m) = \xi(k^-,m)$. By Corollary 7.7 $R_k^- \in \mathcal{C}^v$. □

7.2 Induction theorem

The goal of the greedy algorithm is to annihilate all $\Delta_{\dot{v}}$-coordinates of index $\leq \ell(v)$. This will translate on the cut seed as a greedy induction where the goal is to remove all summands. In order for the heredity to be fulfilled we will have to introduce some technical conditions.

Theorem 7.10 (Induction Theorem)**.** *With the previously defined notations, let $1 \leq k \leq \ell(w)$ and $0 \leq m \leq \ell(v)$, $G_m = C(R_m)$ and $\widetilde{\Gamma}_m = C(\Gamma_m)$. We have that:*

1. *any summand $G_{k,m}$ has $\Delta_{\dot{v},j}(G_{k,m}) = 0$ for any $1 \leq j \leq m$.*
2. *For any vertex $G_{k,m}$ such that $G_{k^+,m}$ is a vertex of $\widetilde{\Gamma}_m$, there is an arrow $G_{k^+,m} \leftarrow G_{k,m}$ in $\widetilde{\Gamma}_m$. Such arrows are called horizontal arrows.*
3. *The other arrows of $\widetilde{\Gamma}_m$ are between vertices of adjacent colors . The vertices are called ordinary arrows.*
4. *For any couple of adjacent colors (i_k, i_j), the bicolor subquiver of $\widetilde{\Gamma}_m$ according to (i_k, i_j) has a saw-teeth configuration. In particular for a given vertex k and a given adjacent color to i_k there is at most two ordinary arrows between k and vertices of color i_k.*
5. *for any color i_j adjacent to $i_{p_{m+1}}$, the bicolor subquiver of $\widetilde{\Gamma}_m$ according to $(i_{p_{m+1}}, i_j)$ has a pure saw teeth configuration.*

6. *the coordinate* $1 \leq j \leq \ell(v)$ *of* $\widetilde{\Delta}_{\dot{v}}(G_{k,m})$ *is:*
   - 1 *if* $i_{p_j} = i_k$ *and* j *satisfies* $f_{\min}(k)^{\alpha(k,m)\oplus} \leq j \leq f(k^{\alpha(k,m)+})$
   - 0 *in any other case.*

   If $m = \ell(v)$ *the point 5 will be considered as trivially true.*

Notation 7.11. We recall that the notion of horizontal/ordinary arrow is introduced in Section 4.1, as well as the notion of bicolor subquiver and (pure) saw teeth configuration, the notion of adjacent colors in Definition 3.1 and the notations of point 6 in Definition 5.23.

Remark 7.12. *Note that point 6 is the same hypothesis as Hypothesis 7.2.*

7.3 Initialization

For $m = 0$ we have $R_0 = V_{\overline{w}}$.

We first need to look at vertices evicted or deleted in the cut seed. There is no deleted summands as $i_{\max}^{\alpha(i,m)-} = i_{\max}$ and, by definition, there exists no index i such that $i > i_{\max}$.

The point 1 is an empty condition.

For the points 2 and 3 we remove only arrows adjacent to vertices also removed from Γ_0 which satisfy the condition by definition in Section 4.1. Then $\widetilde{\Gamma}$ verifies also the condition.

For the point 4, thanks to Corollary 7.8 we know that, on the one hand, thanks to Proposition 4.13 the quiver has a saw teeth structure. On the other hand, $\widetilde{\Gamma}_0$ contains only vertices of index k such that $f(k) \geq f_{\min}(k)$. In particular, $\widetilde{\Gamma}_0$ is a subquiver of the quiver $\Gamma_{\overline{w}_{\geq p_1}}$ where $\overline{w} = [i_{\ell(w)}, \cdots, i_{p_1+1}, i_{p_1}]$. The quiver $\Gamma_{\overline{w}_{\geq p_1}}$ is obtained by removing consecutive initial vertices of Γ_0. By Proposition 4.11, the saw teeth structure is preserved by such a removal of initial vertices. Now, in order to get $\widetilde{\Gamma}_0$ from $\Gamma_{\overline{w}_{\geq p_1}}$ we need to remove vertices on lines other than i_{p_1}. However, thanks to Corollary 7.9 we remove consecutively the first vertices of given lines. Then, using once again Proposition 4.13 we have that the saw teeth structure is preserved.

We will now prove that $V_{\overline{w}}$ verifies the point 5 by contradiction.

Proof. First, note that, due to the rightmostness of $\overline{v}$, $(p_1)_{\min} = p_1$. Thus any vertex of color i_{p_1} is in $\widetilde{\Gamma}_0$. Now suppose that for one line of color i_j adjacent to the line of color i_{p_1} the ordinary arrow of minimal target index is an arrow $V_{(p_1)^{\gamma+}} \to V_j$ for $\gamma \geq 0$ and $1 \leq j \leq \ell(w_0)$ of color i_j. We will show that there exists an arrow $V_j \to V_{(p_1)^{\zeta+}}$ where $\zeta < \gamma$. This implies that the inequality: $p_1^{(\gamma+1)+} \geq j^+ > p_1^{\gamma+} > j$ holds and we want to show that there exists $\zeta < \gamma$ such that $j^+ \geq p_1^{(\zeta+1)+} > j > p_1^{\zeta+}$.

We have $\gamma > 0$: if $\gamma = 0$ then we have j such that $j < p_1$ which is impossible as the quiver is a subquiver of $\Gamma_{\overline{w}_{\geq p_1}}$. So $p_1^{\gamma+} > j > 1$ and there existe $\zeta < \gamma$ such that $p_1^{(\zeta+1)+} > j > p_1^{\zeta+}$. As, in addition we have $j^+ > p_1^{\gamma+}$ and $p_1^{\gamma+} > 1^{\zeta+}$, eventually we have: $j^+ \geq p_1^{(\zeta+1)+} > j > p_1^{\zeta+}$ and, as j and $p_1^{\zeta+}$ are vertices of $\widetilde{\Gamma}_0$ subquiver of $\Gamma_{\overline{w}}$ there is an arrow $V_j \to V_{p_1^{\zeta+}}$ and so the arrow $V_j \to V_{p_1^{\gamma+}}$ was not the one of minimal target index. Then the ordinary arrow of minimal target index between both lines has as target a vertex of color i_{p_1}. So the bicolor subquiver has a pure saw teeth structure. □

For point 6, the considered modules are a subset of the indecomposable summands of $V_{\overline{w}}$. Thanks to Theorem 5.32 we know that each of these summands has a $\widetilde{\Delta}$-vector as described.

Then we have proved that R_0 satisfies all the properties of Theorem 7.10.

7.4 Induction lemma

We now want to prove that, given the seed $(G_m, \widetilde{\Gamma}_m)$ satisfying all properties of Theorem 7.10, the next seed $(G_{m+1}, \widetilde{\Gamma}_{m+1})$ satisfies all properties of Theorem 7.10 for any $0 \leq m \leq \ell(v)$.

We know how to get a cut seed from (R_m, Γ_m) and how to mutate it in (R_{m+1}, Γ_{m+1}) via $\hat{\mu}_m$ but obtaining $(G_{m+1}, \widetilde{\Gamma}_{m+1})$ from $(G_m, \widetilde{\Gamma}_m)$ is more subtle and will be proven by another induction. We can summarize the situation by Figure 9.

Notation 7.13. In this section we will consider the seed

$$(R, \Gamma) = \mu_{j^{\gamma+}} \circ \cdots \circ \cdots \mu_j \circ \hat{\mu}_m \circ \cdots \circ \hat{\mu}_1(R_0, \Gamma_0) = \mu_{j^{\gamma+}} \circ \cdots \circ \mu_j(R_m, \Gamma_m)$$

where $\hat{\mu}_{m+1} = \mu_{j^{\zeta+}} \circ \cdots \circ \mu_{j^{\gamma+}} \circ \cdots \circ \mu_j$ and $\zeta \geq \gamma \geq 0$.

We denote by $R_{i,m+1}$ the i-th summand of R if $i \in \{j, j^+, \ldots, j^{\gamma+}\}$ and $R_{i,m}$ otherwise.

We define the cut seed $C(R, \Gamma) =: (G, \widetilde{\Gamma})$ as the seed where

$$\begin{tikzcd}
(R_m, \Gamma_m) \arrow[r, "\hat{\mu}_{m+1}"] \arrow[d, "C"] & (R_{m+1}, \Gamma_{m+1}) \arrow[d, "C"] \\
(G_m, \widetilde{\Gamma}_m) \arrow[r, dashed, "?"] & (G_{m+1}, \widetilde{\Gamma}_{m+1})
\end{tikzcd}$$

Figure 9: Cut seeds and mutated seeds

- the module G is obtained from R by removing, for $p \in \{m, m+1\}$, the direct indecomposable summands $G_{k,p}$ such that $\Delta_{\dot{v}}(R_{k,p}) = 0$ which we call evicted and the direct indecomposable summands $G_{k,p}$ such that $k > (k_{\max})^{\alpha(k,p)-}$ which we call deleted.
- the quiver $\widetilde{\Gamma}$ is obtained from Γ by removing all vertices corresponding to evicted or deleted summand and arrows adjacents to these vertices.

We will now prove by induction the following lemma

Lemma 7.14 (Induction lemma)**.** *In the cut seed $(G, \widetilde{\Gamma})$:*

1. *The summands $G_{k,m}$, provided $i_k \neq i_j$, and the summands $G_{j,m+1}, \dots, G_{j^{\gamma+},m+1}$ have all their $\Delta_{\dot{v}}$-coordinates of index $\leq m+1$ equal to 0.*
2. *The horizontal arrows are as in Theorem 7.10, apart from the horizontal arrow $G_{j^{(\gamma+1)+},m} \to G_{j^{\gamma+},m+1}$ if $(G, \widetilde{\Gamma}) \neq (G_m, \widetilde{\Gamma}_m)$.*
3. *All non-horizontal arrows are between lines with adjacent colors.*
4. *For any couple of color (i_k, i_l) with $i_k, i_l \neq i_j$, the bicolor subquiver (i_k, i_l) has a saw teeth configuration. The bicolor subquivers (i_j, i_l) with i_l adjacent to i_j, where we remove all vertices of color i_j and of index $> j^{\gamma+}$, have a saw teeth configuration.*
5. *The bicolor subquivers (i_j, i_l) where we remove the vertices of color i_j and of index $< j^{\gamma+}$ have a pure saw teeth structure.*
6. *$\widetilde{\Delta}_{\dot{v}}(G_{k,p})$ has all coordinates equal to 0 except from the one of color i_j and of index in the interval:*

$$f_{\min}(k)^{\alpha(k,p)\oplus} \leq l \leq f(k^{\alpha(k,p)+})$$

which are equal to 1.

Example 7.15. There are examples of these properties in Example B.1, with the notation $V_k^* = R_{k,1}$ and $R_{k,m}^* = R_{k,m+1}$. See especially Figure 13 & 14.

We will first prove the initialization of Lemma 7.14, i.e., the case where $(G, \widetilde{\Gamma}) = (G_m, \widetilde{\Gamma}_m)$. It consists in translating the points of Theorem 7.10 into the points of Lemma 7.14.

The point 1 of Lemma 7.14 comes from the points 1 and 6 of Theorem 7.10. The first point ensures that all coordinates of index $\leq m$ are equal to 0. As the $m+1$ summand is of color $i_{p_{m+1}} = i_j$, for summands of color different from i_j, the $m+1$th coordinate is equal to zero thanks to point 6 of the Theorem. The list of summands $G_{j,m+1}, \dots, G_{j^{\gamma+},m+1}$ is empty. The point is proved.

The points 2 and 3 of the Lemma are equivalent to the points 2 and 3 of the Theorem.

The point 4 of the Lemma consists in looking only at bicolor subquivers of colors (i_k, i_l) with $i_k, i_l \neq i_j$, a subset of the bicolor subquivers known to have a saw teeth structure thanks to the point 4 of the Theorem.

The points 5 and 6 of the lemma are exactly the points 5 and 6 of the Theorem.

Then the seed $(G_m, \widetilde{\Gamma}_m)$ satisfies all conditions of Lemma 7.14.

7.5 Heredity of the induction lemma

We will now establish the heredity of the hypotheses of Lemma 7.14.

We will deduce point 1 of the lemma from point 6.

For the point 2, we have two cases. If the vertex we are mutating, $G_{j^{\gamma+},m}$, is the first of its line, it has only one horizontal arrow $G_{j^{(\gamma+1)+},m} \leftarrow G_{j^{\gamma+},m}$. As, from point 5 we know that there is no ordinary

arrow in $\widetilde{\Gamma}$ having $G_{j^{\gamma+},m}$ as source, we know that there will not be any disturbance of other horizontal arrow on other lines.

Then the only horizontal arrow to be changed will be the one having $G_{j^{\gamma+},m}$ as source which will become $G_{j^{(\gamma+1)+},m} \to G_{j^{\gamma+},m+1}$, which is precisely the only derogation we allowed, now considering $G_{j^{(\gamma+1)+},m}$. Now, in order to get the next $\widetilde{\Gamma}$, either we evict $G_{j^{\gamma+},m+1}$ and we are once again in the initial case with no derogation, or not and we still validate the hypothesis.

If we are not in an initial case then we have $G_{j^{(\gamma+1)+},m} \leftarrow G_{j^{\gamma+},m} \to G_{j^{(\gamma-1)+},m+1}$ before mutation. As before, thanks to point 5, we know that the only horizontal arrow to be changed will be the one adjacent to $G_{j^{\gamma+},m}$ and after mutation we have $G_{j^{(\gamma+1)+},m} \to G_{j^{\gamma+},m+1} \leftarrow G_{j^{(\gamma-1)+},m+1}$ so we still validate the hypothesis when looking at $G_{j^{(\gamma+1)+},m}$. We cannot have eviction here thanks to Corollary 7.9.

The points 3, 4 and 5 will result from an enumeration of cases, which we postpone to Section A in the appendix.

Indeed, we initially verify the condition of the case enumeration and then with the evictions and mutation we stay in one of the studied case.

As all the bicolor subquivers of first color i_j are in pure saw teeth structure, all the ordinary arrow adjacent to a vertex to mutate will have it as target and thus it is impossible, in the cut quiver, to create arrow between two lines of non-adjacent colors.

By the study of configurations, looking only at what happens at the already mutated vertices, we see that we keep the saw teeth structure. By the same point, when only looking at the vertices still to mutate, we see that we still are in a pure saw teeth structure.

We now need to compute the Δ-vectors after mutation. Using Proposition 5.9 we have that Δ-vectors can be computed by exactly one of the two computations, one corresponding to taking into account neighbours of G_j via arrow of source G_j, the other one via arrow of target G_j.

Here we will only focus on the $\widetilde{\Delta}$-vector, only the $\ell(v)$ first coordinates being relevant to our study and then we allow ourself to only consider non-evicted vertices, the evicted one having zero $\widetilde{\Delta}_{\dot{v}}$-vectors.

We have a more precise criterion to know which summands to take into account on the computation:

Proposition 7.16. *The neighbouring of G_j in $\widetilde{\Gamma}$ just before mutating is composed of the ordinary arrow having G_j as target, the horizontal arrows having G_j as source, and no other arrows.*

It follows from the study of all cases. Thus the computation of the $\widetilde{\Delta}_{\dot{v}}$-vectors amounts to taking into account only the ordinary arrows or only the horizontal arrows.

We now should use the computational criterion of [GLS11] to know which one to use for the computation but we will show, by contradiction, that in fact we can only consider the horizontal arrows.

Lemma 7.17 (Evolution of $\Delta_{\dot{v}}$-vectors coordinates by mutation)**.** *In the cut quiver $\widetilde{\Gamma}$, if the vertex G_j we mutate has a configuration described by points 4 and 5 of the lemma 7.14, then*

$$\widetilde{\Delta}_{\dot{v}}(G_{j,m+1} = \widetilde{\Delta}_{\dot{v}}(G_{j^+,m}) + \widetilde{\Delta}_{\dot{v}}(G_{j^-,m+1}) - \widetilde{\Delta}_{\dot{v}}(G_{j,m})$$

where we write $\Delta_{\dot{v}}(G_{j^-,m+1}) = 0$ if there is no summand/vertex of index j^- in the cut module/quiver.

Proof. First we will show that we have to consider summands of the same color as they are linked to $G_{j,m}$ by horizontal arrows.

If we look at ordinary arrows, we will have the computation

$$\sum_{G_{i,m} \to G_{j,m}} \widetilde{\Delta}_{\dot{v}}(G_{i,m}) - \widetilde{\Delta}_{\dot{v}}(G_{j,m})$$

and if we look at the $m+1$st coordinate, then $\Delta_{\dot{v},m+1}(G_{i,m}) = 0$ for any G_i adjacent to G_j via an ordinary arrow and $\Delta_{\dot{v},m+1}(G_{j,m}) = 1$ by the induction hypothesis 6 of Lemma 7.14. Thus we would have $\Delta_{\dot{v},m+1}(G_{j,m+1}) = 0 - 1 < 0$. This is absurd by the definition of Δ-vectors and then we have to consider horizontal arrows.

Then there are only two neighbours of $G_{j,m}$: $G_{j^-,m+1}$ (if it exists) and $G_{j^+,m}$ (which always exists). Thus the computation is the one given in the lemma. □

Now we suppose that, according to point 6 of Lemma 7.14, we have that the non-zero and equal to 1 coordinates of $\widetilde{\Delta}_{\dot{v}}(G_{k,p})$ are the one of index $1 \leq l \leq \ell(v)$ having the color i_k and being in the interval:

$$f_{\min}(k)^{\alpha(k,p)\oplus} \leq l \leq f(k^{\alpha(k,p)+})$$

In particular we have:

$$f_{\min}(j)^{(\alpha(j,m)+1)\oplus} \leq l \leq f(j^{\alpha(j,m)+}) \quad \text{for } G_{j^-,m+1}$$

$$f_{\min}(j)^{\alpha(j,m)\oplus} \leq l \leq f(j^{\alpha(j,m)+}) \quad \text{for } G_{j,m}$$

$$f_{\min}(j)^{\alpha(j,m)\oplus} \leq l \leq f(j^{(\alpha(j^+,m)+1)+}) \quad \text{for } G_{j^+,m}$$

using the facts that $f_{\min}(j) = f_{\min}(j^+) = f_{\min}(j^-)$, that $\alpha(j,m) = \alpha(j^+,m) = \alpha(j^-,m)$ and that $\alpha(j,m)+1 = \alpha(j,m+1)$.

We want to show that

$$f_{\min}(j)^{\alpha(j,m+1)\oplus} \leq l \leq f(j^{\alpha(j,m+1)+}) \quad \text{for } G_{j,m+1}$$

Then we can cut these intervals in three intervals:

$$f_{\min}(j)^{\alpha(j,m)\oplus} \leq l < f_{\min}(j)^{(\alpha(j,m)+1)\oplus}$$

$$f_{\min}(j)^{(\alpha(j,m)+1)\oplus} \leq l \leq f(j^{\alpha(j,m)+})$$

$$f(j^{\alpha(j,m)+}) < l \leq f(j^{(\alpha(j,m)+1)+})$$

One these intervals, we have respectively :

$$(\Delta_{\dot{v},l}(G_{j^-,m+1}), \Delta_{\dot{v},l}(G_{j,m}), \Delta_{\dot{v},l}(G_{j^+,m})) = (0,1,1), \text{on the first},$$

$$(\Delta_{\dot{v},l}(G_{j^-,m+1}), \Delta_{\dot{v},l}(G_{j,m}), \Delta_{\dot{v},l}(G_{j^+,m})) = (1,1,1), \text{on the second},$$

$$(\Delta_{\dot{v},l}(G_{j^-,m+1}), \Delta_{\dot{v},l}(G_{j,m}), \Delta_{\dot{v},l}(G_{j^+,m})) = (0,0,1), \text{on the third}.$$

So we have as coordinate for $\widetilde{\Delta}_{\dot{v},l}(G_{j,m+1})$, on the first interval $0-1+1=0$, on the second: $1-1+1=1$, on the third $0-0+1=1$.

Finally, in $\widetilde{\Delta}_{\dot{v}}(G_{j,m+1})$, the non-empty coordinates are equal to 1 and are the ones of index l of color i_j and in the interval $f_{\min}(j)^{\alpha(j,m+1)\oplus} \leq l \leq f(j^{\alpha(j,m+1)+})$ whence the heredity of point 6 of Lemma 7.14 and the point 1 coming from point 6.

Then we have proved that Lemma 7.14 holds by induction.

7.6 Toward heredity proof of the induction theorem

As Lemma 7.14 is proved, when taking the result for the seed $(R,\Gamma) = \hat{\mu}_{m+1} \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}}, \Gamma_{\overline{w}})$ we have:

1. The summands $G_{k,m+1}$ belonging to $C(R_{m+1})$ have all their coordinates of $\Delta_{\dot{v}}$-vector of index $\leq m+1$ equal to 0.
2. In the cut quiver $\widetilde{\Gamma}_{m+1}$ (obtained from Γ by removing the last vertex of line i_j and the evicted vertices), all the horizontal arrows are conform to 2 of Theorem 7.10.
3. All ordinary arrows are between lines of adjacent colors in $\widetilde{\Gamma}_{m+1}$.
4. The quiver $\widetilde{\Gamma}_{m+1}$ has a saw teeth structure.
5. This condition is empty.
6. The description of $\widetilde{\Delta}_{\dot{v}}$-coordinates of the summands is conform to point 6 of Theorem 7.10.

Then, up to point 5 we have the heredity of the hypothesis. It remains to show that the next line to mutate will have a pure saw teeth structure.

To show that final point we need a new lemma, introduced and proved in the next section.

7.7 Heredity: inclusion lemma

Our goal here is to show that in the seed $\widetilde{\Gamma}_{m+1}$ the line of color $i_{p_{m+1}}$ has a pure saw teeth structure for any bicolor subquiver having $i_{p_{m+1}}$ as first color. In order to do so, we will use (and prove) the fact that the cut seed $\widetilde{\Gamma}_{m+1}$ has the same quiver as the left (maximal indices) part of the quiver of Γ_0.

Definition 7.18 (Graph morphism)**.** Let $\Gamma = (\Gamma_0, \Gamma_1)$ and $\Gamma' = (\Gamma'_0, \Gamma'_1)$ be two graphs. A map $f : \Gamma_0 \to \Gamma'_0$ is a graph morphism if, for any vertex $\alpha_{j_s,j_t} \in \Gamma_1$ of source the vertex $V_{j_s} \in \Gamma_0$ and of target V_{j_t}, the image vertex $f(\alpha_{j_s,j_t})$ of source $f(Vj_s)$ and of target $f(V_{j_t})$ belongs to Γ'_1. If f is injective, we say that Γ is included in Γ'.

Lemma 7.19 (Inclusion lemma)**.** *Let $0 \le m \le \ell(v)$, $\Gamma_{W_m} = (W_{i,m}, \beta_{j_s,j_t,m})$ and $\widetilde{\Gamma}_m = (G_{i,m}, \alpha_{j_s,j_t,m})$ be the quivers of the modules (respectively) $V_{\overline{w}_{\ge p_{m+1}}}$ and $C(\hat{\mu}_m \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}}))$,using the convention that $s(\gamma_{j_s,j_t,m}) = j_s$ and $t(\gamma_{j_s,j_t,m}) = j_t$ for any vertex.*

We define the family of maps $(\rho_m)_{m=0}^{\ell(v)}$ by: $\rho_m : \begin{cases} (G_{i,m}, \alpha_{j_s,j_t,m}) & \to & (W_{i,m+1}, \beta_{j_s,j_t,m+1}) \\ G_{i,m} & \mapsto & W_{i^{\alpha(i,m)+},m+1} \end{cases}$.

These maps are injective graph morphisms and we have a partial inverse:

$$\rho^*_m : \begin{cases} (W_{i,m+1}, \beta_{j_s,j_t,m+1}) & \to & (G_{i_m}, \alpha_{j_s,j_t,m}) \\ W_{i,m+1} & \mapsto & \begin{cases} G_{i^{\alpha(i,m)-},m} & \text{if} & i^{\alpha(i,m)-} \ge \xi(i,m) \\ \notin D_{\rho^*_m} & \text{otherwise} \end{cases} \end{cases}$$

Proof. We need two more functions (generally not graph morphisms) to prove the result by induction.

We define $\iota_m : \Gamma_{W_m} \to \Gamma_{W_{m+1}}$ by the following identifications:

$$\iota_m : \begin{cases} W_{j,m} & \mapsto & \begin{cases} W_{j^+,m+1} & \text{if} & i_j = i_{p_m} \text{ and } (p_m)_{\max} \ge j^+ \ge p_{m+1} \\ W_{j,m+1} & \text{otherwise if} & i_j \ne i_{p_m} \text{ and } j \ge p_{m+1} \\ \notin D_{\iota_m} & \text{otherwise,} \end{cases} \\ \beta_{j_s,j_t,m} & \mapsto & \begin{cases} \notin D_{\iota_m} & \text{if} & W_{j_s,m} \notin D_{\iota_m} \text{ or } W_{j_t} \notin D_{\iota_m} \\ \beta_{j_s^+,j_t^+,m+1} & \text{otherwise if} & i_{j_s} = i_{j_t} = i_{p_m} \\ \beta_{j_s,j_t,m+1} & \text{otherwise,} \end{cases} \end{cases}$$

where D_{ι_m} is the domain of ι_m.

We define $\overline{\mu}_m : \widetilde{\Gamma}_{m-1} \to \widetilde{\Gamma}_m$ as:

$$\overline{\mu}_m : \begin{cases} G_{j,m-1} & \mapsto & \begin{cases} G_{j,m} & \text{if} & i_j \ne i_{p_m} \text{ or } j_{\max} > j^{\alpha(j,m)+} \ge \xi(j,m) \\ \notin D_{\overline{\mu}_m} & \text{otherwise,} \end{cases} \\ \alpha_{j_s,j_t,m-1} & \mapsto & \begin{cases} \alpha_{j_s,j_t,m} & \text{if} & i_{j_s} \ne i_{p_m} \ne i_{j_t} \text{ or } i_{j_s} = i_{p_m} = i_{j_t} \\ \alpha_{j_s^-,j_t,m} & \text{if} & i_{j_s} = i_{p_m} \ne i_{j_t} \text{ and } G_{j_s^-,m-1}, G_{j_t,m-1} \in D_{\overline{\mu}_m} \\ \alpha_{j_s,j_t^-,m} & \text{if} & i_{j_s} \ne i_{p_m} = i_{j_t} \text{ and } G_{j_s,m-1}, G_{j_t^-,m-1} \in D_{\overline{\mu}_m} \\ \notin D_{\overline{\mu}_m} & \text{otherwise} \end{cases} \end{cases}$$

and for any arrow $\alpha_{j_s,j_t,m-1}$ such that there is no arrow $\alpha_{j_t^{\gamma+},j_s,m-1}$ with $\gamma > 0$ and verifying that $j_t^{\alpha(j_t,m-1)+} \ne (j_t)_{\max}$ and $i_{j_t} = i_{p_m}$ (when we are on a final barb shifted), we add an arrow $\alpha_{j_{t_2},j_s,m}$ where $j_{t_2} = (j_t)_{\max}^{\alpha(j_t,m)-}$.

First we show that ι_m is well-defined, and that $\iota_m(\Gamma_{W_m}) = \Gamma_{W_{m+1}}$. Then we will show that $\overline{\mu}_m$ is well-defined, that $\overline{\mu}_m(\widetilde{\Gamma}_{m-1}) = \widetilde{\Gamma}_m$ and that ρ_m is a graph morphism.

Let $0 \le m < \ell(v)$. By definition, $\Gamma_{W_{m+1}}$ is the quiver obtained from Γ_{W_m} by removing all summands of index $p_{m+1} > j \ge p_m$ and the adjacent arrows.

The vertices having no image by ι_m are $W_{(p_m)_{\max},m}$, the vertices $W_{j,m}$ such that $i_j \ne i_{p_m}$ and $p_{m+1} > j > p_m$, and the vertices $W_{j,m}$ such that $i_j = i_{p_m}$ and $p_{m+1} > j^+ > p_m$.

Thus among the vertices having an image via ι_m, we have a bijection $W_{j,m} \leftrightarrow W_{j,m+1}$ if $i_j \ne i_{p_m}$ and $W_{j,m} \leftrightarrow W_{j^+,m}$ if $i_j = i_{p_m}$.

In addition the vertices of $\Gamma_{W_{m+1}}$ are the same of those of Γ_{W_m} except the ones adjacent to a vertex of index $p_{m+1} > j \ge p_m$. Among the preserved arrows, for the arrows $\alpha_{j_s,j_t,m}$ such that at least one of the extremity is not of color i_{p_m} we have a bijection $\alpha_{j_s,j_t,m} \leftrightarrow \alpha_{j_s,j_t,m+1}$. If both extremities are of color i_{p_m} then the arrow is an horizontal arrow $\alpha_{k,k^+,m}$ and we have the bijection $\alpha_{k,k^+,m} \leftrightarrow \alpha_{k^+,k^{2+},m+1}$. The arrow $\alpha_{k,k^{2+},m+1}$ belongs to $\Gamma_{W_{m+1}}$ because we suppose that $k^+ \ge p_{m+1}$ and $k^+ \ne k_{\max}$ so $W_{k^{2+},m+1} \in \Gamma_{W_{m+1}}$.

Thus we have a bijection between $\Gamma_{W_{m+1}}$ and $\iota_m(\Gamma_{W_m})$ for $0 \leq m < \ell(v)$.

Now we consider $\widetilde{\Gamma}_{m+1}$. We will show that $\overline{\mu}_m(\widetilde{\Gamma}_{m+1})$. To fix notations we will suppose that $\hat{\mu}_m = \mu_{k^{\gamma+}} \circ \cdots \circ \mu_k$.

Then $\overline{\mu}_m$ amounts exactly to the modifications described by Proposition A.3 and the cut.

Thus ι_m and $\overline{\mu}_m$ are well-defined and the images are the one expected.

We will now show that ρ_m is a morphism of graphs for any $0 \leq m \leq \ell(v)$ by induction.

For ρ_0, $\widetilde{\Gamma}_0$ is the subquiver of $\Gamma_{\overline{w}}$ containing all vertices $R_{j,0}$ such that $j \geq \xi(j,0)$ (by Corollary 7.8), Γ_{W_1} is the subquiver of $\Gamma_{\overline{w}}$ containing all summands such that $j \geq p_1$ and for $1 \leq j \leq \ell(w)$ we have $\xi(j,0) \geq p_1$. $\widetilde{\Gamma}_0$ is then a subquiver of Γ_{W_1}. As for all j we have $\alpha(j,0) = 0$, there is no vertex shift nor arrows shift thus ρ_0 is well-defined and is just an inclusion. The partial inverse exists and is trivial.

Now we will show the heredity. Suppose that ρ_{m-1} is a morphism of graphs. We will show that ρ_m is also a morphism of graphs. To do so, we will show that we have $\rho_m \circ \overline{\mu}_m = \iota_m \circ \rho_{m-1}$ on their definition domain, having the commutative diagram:

$$\begin{tikzcd}
\widetilde{\Gamma}_{m-1} \arrow[r, "\rho_{m-1}"] \arrow[d, "\overline{\mu}_m"'] & \Gamma_{W_m} \arrow[d, "\iota_m"] \\
\widetilde{\Gamma}_m \arrow[r, "\rho_m"'] & \Gamma_{W_{m+1}}
\end{tikzcd}$$

We first show the equality on vertices. We have four cases (provided the previous one are not verified) for the index of a vertex $G_{j,m-1}$:

1. either $j^{\alpha(j,m-1)+} = (p_m)_{\max}$
2. or $j^{\alpha(j,m)+} < p_{m+1}$
3. or $\xi(j,m) > j^{\alpha(j,m)+} \geq p_{m+1}$
4. or $j^{\alpha(j,m)+} \geq \xi(j,m)$

In the first case $G_{j,m-1} \notin D_{\overline{\mu}_m}$. On the other side $\rho_{m-1}(G_{j,m-1}) = W_{j^{\alpha(j,m-1)+},m}$ and $W_{j^{\alpha(j,m-1)+},m} \notin D_{\iota_m}$ as $j^{\alpha(j,m-1)+} = (p_m)_{\max}$ and thus $(j^{\alpha(j,m-1)+})^+ > (p_m)_{\max}$.

In the second case, if $i_j = i_{p_m}$, then $G_{j,m-1} \notin D_{\overline{\mu}_m}$ as $\xi(i,m) \geq p_{m+1}$ for any $1 \leq i \leq \ell(w)$. On the other side we have $\rho_{m-1}(G_{j,m-1}) = W_{j^{\alpha(j,m-1)+},m} \notin D_{\iota_m}$. If $i_j \neq i_{p_m}$ then $j^{\alpha(j,m)+} < p_{m+1} \leq \xi(j,m) = \xi(j,m-1)$ and thus this vertex is not in $\widetilde{\Gamma}_{m-1}$.

In the third case, it depends on the color. If $i_j \neq i_{p_m}$ then $\xi(j,m) = \xi(j,m-1)$ and thus $G_{j,m-1}$ does not exist. Then $i_j = i_{p_m}$ and as $\xi(j,m) = p_{m+1}$, the interval is empty.

In the fourth case we have $\overline{\mu}_m(G_{j,m-1}) = G_{j,m}$ and $\rho_m(G_{j,m}) = W_{j^{\alpha(j,m)+},m+1}$. On the other side, we have $\rho_{m-1}(G_{j,m-1}) = W_{j^{\alpha(j,m-1)+},m}$ and then $\iota_m(W_{j^{\alpha(j,m-1)+},m}) =_{j^{\alpha(j,m)+},m+1}$ using the fact that $\alpha(j,m-1) = \alpha(j,m)$ if $i_j \neq i_{p_m}$ and $\alpha(j,m-1)+1 = \alpha(j,m)$ if $i_j = i_{p_m}$.

Thus we do have $\rho_m \circ \overline{\mu}_m = \iota_m \circ \rho_{m-1}$ for any vertex of $\widetilde{\Gamma}_m$.

We will now prove the equality for all arrows. We have four cases for $\alpha_{j_s,j_t,m-1}$

1. $G_{j_s,m-1}$ or $G_{j_t,m-1}$ is not in $D_{\rho_m \circ \overline{\mu}_m} = D_{\iota_m \circ \rho_{m-1}}$,
2. $i_{j_s} \neq i_{p_m} \neq i_{j_t}$,
3. $i_{j_s} = i_{p_m} = i_{j_t}$,
4. $i_{j_s} = i_{p_m}$ or $i_{j_t} = i_{p_m}$ but not both.

In the first case, the arrow will not be in $D_{\overline{\mu}_m}$ neither in $D_{\iota_m \circ \rho_{m-1}}$.

In the second case $\overline{\mu}_m(\alpha_{j_s,j_t,m-1}) = \alpha_{j_s,j_t,m}$ and $\rho_m(\alpha_{j_s,j_t,m}) = \beta_{j_s^{\alpha(j_s,m)+},j_t^{\alpha(j_t,m)+},m+1}$. On the other hand we have $\rho_{m-1}(\alpha_{j_s,j_t,m-1}) = \beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m}$ and

$$\iota_m(\beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m}) = \beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m+1}.$$

As $\alpha(j_t,m-1) = \alpha(j_t,m)$ and $\alpha(j_s,m-1) = \alpha(j_s,m)$ we have the equality.

In the third case, we have $\overline{\mu}_m(\alpha_{j_s,j_t,m-1}) = \alpha_{j_s,j_t,m}$ and then $\rho_m(\alpha_{j_s,j_t,m}) = \beta_{j_s^{\alpha(j_s,m)+},j_t^{\alpha(j_t,m)+},m+1}$. On the other hand, we have $\rho_{m-1}(\alpha_{j_s,j_t,m-1}) = \beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m}$ and

$$\iota_m(\beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m}) = \beta_{j_s^{(\alpha(j_s,m-1)+1)+},j_t^{(\alpha(j_t,m-1)+1)+},m}$$

and, as $\alpha(j_s,m)=\alpha(j_t,m)=\alpha(j_s,m-1)+1=\alpha(j_t,m-1)+1$ we have the equality.

The last case is symmetrical, and we will suppose that $i_{j_s}=i_{p_m}$. Thus $\overline{\mu}_m(\alpha_{j_s,j_t,m-1})=\alpha_{j_s^-,j_t,m}$ and we have $\rho_m(\alpha_{j_s^-,j_t,m})=\beta_{j_s^{\alpha(j_s,m)-)+},j_t^{\alpha(j_t,m)+},m+1}$. On the other side we have $\rho_{m-1}(\alpha_{j_s,j_t,m-1})=\beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m}$ and $\iota_m(\beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m})=\beta_{j_s^{\alpha(j_s,m-1)+},j_t^{\alpha(j_t,m-1)+},m+1}$. As $\alpha(j_s,m)=\alpha(j_s,m-1)+1$ and $\alpha(j_t,m)=\alpha(j_t,m-1)$ we have the equality.

Finally in all cases we have the equality, and the diagram commutes. Thus for any vertex or arrow in $\widetilde{\Gamma}_{m-1}$ being in the domain, we have $\rho_m\circ\overline{\mu}_m=\iota_m\circ\rho_{m-1}$.

As ι_m is clearly invertible for vertices of index $\geq p_{m+1}$ and for adjacent arrows, we can define $\rho_m^*=\overline{\mu}_m\circ\rho_{m-1}^*\circ\iota_m^{-1}$ on any vertex of $\Gamma_{W_{m+1}}$ of index $\geq p_{m+1}$.

To end our proof we need to show that all arrows created between $\widetilde{\Gamma}_{m-1}$ and $\widetilde{\Gamma}_m$ (by tooth shift of final barbs) have an image through ρ_m. In that case we have an arrow $\alpha_{((p_m)_{\max})^{\alpha(p_m,m)-},j,m}$ where i_j is adjacent to i_{p_m}. In addition we know that in $\widetilde{\Gamma}_{m-1}$ the last ordinary arrow between i_j and i_{p_m} was the arrow $\alpha_{j,k,m-1}$ where $i_k=i_{p_m}$ and $k\leq((p_m)_{\max})^{\alpha(p_m,m)-}$. As ρ_{m-1} is a graph morphism by hypothesis, there exists an arrow $\beta_{j^{\alpha(j,m-1)+},k^{\alpha(k,m-1)+},m}$ in Γ_{W_m}. As Γ_{W_m} is a subquiver of $V_{\overline{w}}$, its arrows satisfies the index condition of Section 4 and then:

$$j^{(\alpha(j,m-1)+1)+}\geq k^{(\alpha(k,m-1)+1)+}>j^{\alpha(j,m-1)+}>k^{\alpha(k,m-1)+} \tag{2}$$

which can be rewritten, using the facts that $\alpha(j,m)=\alpha(j,m-1)$ and $\alpha(k,m)=\alpha(k,m-1)+1$:

$$j^{(\alpha(j,m)+1)+}\geq k^{\alpha(k,m)+}>j^{\alpha(j,m)+}>k^{\alpha(k,m-1)+} \tag{3}$$

To show that ρ_m is a graph morphism, we must show that the image $\rho_m(\alpha_{((p_m)_{\max})^{\alpha(p_m,m)-},j,m})$ is an arrow of $\Gamma_{W_{m+1}}$. In that case the image arrow is:

$$\beta_{(p_m)_{\max},j^{\alpha(j,m)+},m+1}.$$

This arrow exists iff the following inequality is verified:

$$\ell(w)+1\geq j^{(\alpha(j,m)+1)+}>(p_m)_{\max}>j^{\alpha(j,m)+}$$

As we have $((p_m)_{\max})^{\alpha(p_m,m)-}>k$, $k^{\alpha(k,m)+}>j^{\alpha(j,m)+}$ and $\alpha(k,m)=\alpha(p_m,m)$, we obtain

$$(p_m)_{\max}>j^{\alpha(j,m)+}.$$

We always have $\ell(w)+1\geq j^{(\alpha(j,m)+1)+}$ with equality iff $j^{\alpha(j,m)+}=j_{\max}$. In that case it is obvious that $j^{(\alpha(j,m)+1)+}>(p_m)_{\max}$.

Suppose that we have $(p_m)_{\max}>j^{(\alpha(j,m)+1)+}$. Then there exists $\delta>0$ such that:

$$\ell(w)+1\geq j^{(\delta+\alpha(j,m)+1)+}>(p_m)_{\max}>j^{(\delta+\alpha(j,m))+}$$

and thus we have the arrow $\beta_{(p_m)_{\max},j^{(\delta+\alpha(j,m))+},m+1}\in\Gamma_{W_{m+1}}$ and then $\beta_{(p_m)_{\max},j^{(\delta+\alpha(j,m))+},m+1}\in\Gamma_{W_m}$. Then, by ρ_{m-1}^* we have that $\alpha_{((p_m)_{\max})^{\alpha(k,m-1)-},j^{\delta+},m-1}\in\widetilde{\Gamma}_{m-1}$. That is impossible by hypothesis on the final barb.

Thus $j^{(\alpha(j,m)+1)+}>(p_m)_{\max}$ and ρ_m is a graph morphism.

□

Lemma 7.20. *In the cut seed $(G_m,\widetilde{\Gamma}_m)$, $0\leq m<\ell(v)$ the line of color $i_{p_{m+1}}$ has a pure saw teeth structure.*

Proof. We consider the quiver $\Gamma_{W_{m+1}}$. Thanks to Section 7.3 we know it has a saw teeth structure and a pure saw teeth structure for the line $i_{p_{m+1}}$. Then we know that $\widetilde{\Gamma}_m=\rho_m^*(\Gamma_{W_{m+1}})$ and so we get $\widetilde{\Gamma}_m$ from $\Gamma_{W_{m+1}}$ by removing all vertices $G_{k,m}$ such that $\xi(k,m)>k\geq p_{m+1}$ and for $i_k=i_{p_m}$ this interval is empty. Thus we remove initial vertices of some line but not of i_{p_m} and, using the same reasonning as Section 7.3, we have the result. □

7.8 Conclusion of the induction

Lemma 7.20 gives us point 5 of the induction hypothesis of Theorem 7.10 and thus the induction is proved.

If we look at point 6 of Theorem 7.10 when $m=\ell(v)$ we have that $f_{\min}(k)^{\alpha(k,m)\oplus}=\ell(v)+1$ and then the interval is empty and in fact $(G_{\ell(v)},\widetilde{\Gamma}_{\ell(v)})$ is the empty seed.

Then $\mu_\bullet(V_{\overline{w}})$ is the sum of all evicted summands and the quiver $\mu_\bullet(\Gamma_{\overline{w}})$ is the quiver obtained by applying the mutations to $\Gamma_{\overline{w}}$ and removing all deleted vertices. We will now show that $\mu_\bullet(V_{\overline{w}})$ is a $\mathcal{C}_{v,w}$-rigid maximal basic module and its Gabriel quiver is the quiver $\mu_\bullet(\Gamma_{\overline{w}})$.

7.9 Conclusion

Proposition 7.21. *The module $\mu_\bullet(V_{\overline{w}})$ is rigid.*

Proof. $V_{\overline{w}}$ is rigid as it is the module of a seed for the cluster structure of $\mathcal{C}_w$. Moreover, rigidity is preserved through mutation. Then $\hat{\mu}_{\ell(v)} \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$ is rigid. In addition, if the module M is rigid, then any $N \subset M$ is also rigid. Then $\mu_\bullet(V_{\overline{w}}) \subset \hat{\mu}_{\ell(v)} \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$ is rigid. □

Proposition 7.22. *The module $\mu_\bullet(V_{\overline{w}})$ is basic.*

Proof. $V_{\overline{w}}$ was basic. Mutations preserves basicity and so does removing direct summands thus $\mu_\bullet(V_{\overline{w}})$ is basic. □

Proposition 7.23. *The module $\mu_\bullet(V_{\overline{w}})$ is in the category $\mathcal{C}_{v,w}$.*

Proof. We will show first that $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}_w$ and then that $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}^v$.

$V_{\overline{w}} \in \mathcal{C}_w$. $\mathcal{C}_w$ is stable by extensions so $\hat{\mu}_{\ell(v)} \circ \cdots \circ \hat{\mu}_1 \in \mathcal{C}_w$. In addition $\mathcal{C}_w$ is stable by taking direct summands and direct sums so $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}_w$.

To show that $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}^v$ we use Proposition 5.35. We know that every direct summands of $\mu_\bullet(V_{\overline{w}})$ is an evicted module of the cut module $G_{\ell(v)}$ so, if M is a direct summand of $\mu_\bullet(V_{\overline{w}})$, $\widetilde{\Delta}_{\dot{v}}(M) = 0$ and, by Proposition 5.35, $M \in \mathcal{C}^v$. As $\mathcal{C}_v$ is stable by taking direct summands and direct sums, $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}^v$.

Then $\mu_\bullet(V_{\overline{w}}) \in \mathcal{C}_{v,w}$. □

Proposition 7.24. *The module $\mu_\bullet(V_{\overline{w}})$ is $\mathcal{C}_{v,w}$-maximal rigid.*

Proof. We know from Theorem 2.18 that a basic rigid module of $\mathcal{C}_{v,w}$ is maximal if it has $\ell(w) - \ell(v)$ direct summands. We know that $V_{\overline{w}}$ had $\ell(w)$ direct summands and that we removed all summands whose indices are in the set $\bigcup_{k \in I}\{1 \leq j \leq \ell(w) \mid i_j = i_k \mid j > (j_{\max})^{\alpha(k,\ell(v))-}\}$ or equivalently, as seen with the cut seed: to remove one summand for each sequence of mutation $\hat{\mu}_m$. As we perform $\ell(v)$ such sequencess, we remove $\ell(v)$ summands and the final total number of summands is $\ell(w) - \ell(v)$ and so $\mu_\bullet(V_{\overline{w}})$ is $\mathcal{C}_{v,w}$-rigid maximal. □

Proposition 7.25. *The $\mathcal{C}_{v,w}$-projectives direct indecomposable summands of $\mu_\bullet(V_{\overline{w}})$ are the summands corresponding to vertices of $\Gamma_{R_{\ell(v)}}$ adjacent to summands deleted by S and to the vertices of $S(\Gamma_{\ell(v)})$ adjacent to no other vertex of $S(\Gamma_{\ell(v)})$.*

Proof. A module which is not in any of the two cases is in $\mathcal{C}_{v,w}$ and only adjacent to direct summands of $\mathcal{C}_{v,w}$. So, as $\mathcal{C}_{v,w}$ is extension-closed, mutating this summands will still give a $\mathcal{C}_{v,w}$-module. If the summand R_k to be mutated is adjacent to a summand of $\mathcal{C}_w \setminus \mathcal{C}_{v,w}$ then the mutated summand will be in $\mathcal{C}_w \setminus \mathcal{C}_{v,w}$ so it is non $\mathcal{C}_{v,w}$-mutable and so it has to be $\mathcal{C}_{v,w}$-projective.

If the summand is adjacent to no other vertex, then it is non-mutable and thus $\mathcal{C}_{v,w}$-projective. □

Proposition 7.26. *The quiver $\mu_\bullet(\Gamma_{\overline{w}})$ obtained from $\Gamma_{\overline{w}}$ by mutations and vertex removal is the quiver of $\mu_\bullet(V_{\overline{w}})$*

Proof. By Proposition 7.25 the deleted vertices are adjacent to only $\mathcal{C}_{v,w}$-frozen vertices and then deleting these summands does not change the quiver of the neighbours of mutable vertices.

The quiver obtained is thus the quiver of a $\mathcal{C}_{v,w}$-seed. □

This finishes the proof of Theorem 6.8.

8 Corollaries

We now state some consequences of the existence of this algorithm.

8.1 Mutation sequence equivalence

In Sections 6.2 & 6.3 we saw two different formulations for the sequence of mutations, respectively $\tilde{\mu}_m$ and $\hat{\mu}_m$ for $1 \leq m \leq \ell(v)$. We now show that these two formulations are in fact equivalent.

Proposition 8.1. *For any $1 \leq m \leq \ell(v)$ we have $\tilde{\mu}_m = \hat{\mu}_m$.*

Proof. Using the results of Theorem 7.10, we can describe the vertices/summands mutated by $\hat{\mu}_m$. We know that the summands $G_{k,m-1}$ of the cut seed having $\widetilde{\Delta}_{\dot{v},m}(G_{k,m-1}) \neq 0$ are the one mutated by $\hat{\mu}_m$. It amounts to the summands of color i_{p_m} with index $f_{\min}(k)^{\alpha(k,m-1)\oplus} \leq m \leq f(k^{\alpha(k,m-1)+})$ and such that $k < (k_{\max})^{\alpha(k,m)-}$.

The smallest index to be mutated is k such that $m = f(k^{\alpha(k,m-1)+}) = f_{\min}(k)^{\alpha(k,m-1)\oplus}$ and the bigest is $(k_{\max})^{\alpha(k,m)-}$.

On the other side, following the $\tilde{\mu}_m$ sequence, we mutate the indices $((p_m)_{\min})^{\beta_m+}, \ldots, ((p_m)_{\max})^{\gamma_m-}$.

We recall that $\gamma_m = \alpha(p_m, m)$ so $\widetilde{\mu}_m$ mutates all indices $((p_m)_{\min})^{\beta_m+}, \ldots, ((p_m)_{\max})^{\alpha(p_m,m)-}$.

So we have to show that $k = (k_{\max})^{\alpha(k,m)-} \Leftrightarrow k = ((p_m)_{\max})^{\alpha(p_m,m)-}$ and

$$m = f(k^{\alpha(k,m-1)+}) \Leftrightarrow k = ((p_m)_{\min})^{\beta_m+}.$$

The first equivalence is direct as $(p_m)_{\max} = k_{\max}$ and $\alpha(k,m) = \alpha(p_m,m)$ as $i_k = i_{p_m}$.

For the second equivalence, by definition of f, the smallest k such that $f(k^{\alpha(k,m-1)+}) = m$ verifies $k^{\alpha(k,m-1)+} = p_m$.

We can now rewrite the equivalence as $p_m = k^{\alpha(k,m-1)+} \Leftrightarrow ((p_m)_{\min})^{\beta_m+} = k$.

Then, applying $(\gamma_m - 1)+ = (\alpha(k,m) - 1)+ = \alpha(k, m-1)+$ to the right side we obtain:

$$((p_m)_{\min})^{(\beta_m+\gamma_m-1)+} = k^{\alpha(k,m-1)+}$$

However we have by definition $\gamma_m + \beta_m = \#\{1 \leq k \leq p_m \mid i_k = i_{p_m}\}$ that is, the number of letters of color i_{p_m} and of index $\leq p_m$ in $\overline{w}$. So, the $\beta_m + \gamma_m - 1$-th successor of $(p_m)_{\min}$ is p_m and so we have $p_m = k^{\alpha(k,m-1)+}$ (see Section 7.1). Thus we have proved the equivalence. □

8.2 Red-green sequences

In this section we examine the mutation sequence according to the theory of Keller's red-green sequences. We recall some definitions based on [KD19].

Given a quiver Q, we define the framed quiver $\widehat{Q}$ from Q by adding a vertex i' for any vertex $i \in Q_0$ and an arrow $i' \to i$. In this context the vertices i' are called frozen.

Let Γ be a quiver obtained by $\widehat{Q}$ by a sequence of mutations. A vertex i is said to be green with respect to $\widehat{Q}$ if it is the source of no arrow $i \to j'$ where j is a frozen vertex. Conversely a vertex i is said to be red if it is the target of no arrow $j' \to i$.

Remark 8.2. *Here we use the opposite conventions of [KD19] as we are working with the quiver of* $\mathrm{End}_\Lambda(V_{\overline{w}})$.

We have the following Theorem:

Theorem 8.3 ([KD19, Theorem 1.7.])**.** *With respect to any quiver of the mutation class of* Γ*, every non-frozen vertex of* Γ *is either green or red.*

A sequence of vertices $(k_1, \ldots, k_N)$ of an initial quiver Q is said to be green if for any $1 \leq i \leq N$, the vertex k_i is green with respect to Q in the mutated quiver $\mu_{k_{i-1}} \circ \cdots \circ \mu_{k_1}(\widehat{Q})$.

A sequence is maximal green with respect to Γ if it is green with respect to Γ and if all the non-frozen vertices of $\mu_{k_N} \circ \cdots \mu_{k_1}(\widehat{Q})$ are red with respect to Γ.

A sequence is reddening with respect to Γ if all non-frozen vertices of $\mu_{k_N} \circ \cdots \mu_{k_1}(\widehat{Q})$ are red with respect to Γ (the sequence does not have to be green).

We have the following conjecture and proposition:

Conjecture 8.4. *The sequence of vertices of* $\Gamma_{\overline{w}}$ *mutated by* $\mu_\bullet$ *is a green sequence with respect to* $\Gamma_{\overline{w}}$.

Proposition 8.5. *The sequence of mutated vertices of* $\Gamma_{\dot{w}}$ *by* $\mu_\bullet$ *is a green sequence with respect to* $\Gamma_{\dot{v}}$.

Proof. The quiver $\Gamma_{\dot{w}}$ is in the mutation class of $\Gamma_{\dot{v}}$. Indeed, thanks to Matsumoto's lemma (see Section 5.3) we have a sequence of 2 and 3-moves linking $\dot{v}$ and $\dot{w}$. Moreover 2-moves translate in a reindexation of the vertices and 3-moves as a mutation and a reindexation of vertices.

Let $F_{\dot{v}}$ denote the functor $\mathrm{Hom}_\Lambda(V_{\dot{v}}, -)$. According to the proof of the algorithm (and particularly the proof of Lemma 7.17), in any of our mutation to compute $\widetilde{\Delta}_{\dot{v}}$-vectors of mutated summands, we only consider short exact sequences where the summand to mutate is the kernel. Thanks to [GLS11, Proposition 12.4.], there is an equivalence between the $F_{\dot{v}}$-exactness of a short exact sequence and a condition on dimension vectors of the summands which is equivalent to taking the sequence where R_k is the kernel. According to [BK20] this is equivalent to the fact that the mutation is green with respect to $V_{\dot{v}}$.

Thus the mutations of $\mu_\bullet$ are green mutations with respect to $V_{\dot{v}}$. □

Remark 8.6. *According to [Mén21a, Exemple 5.2.7.] the mutation sequence of $\mu_\bullet$ can be neither maximal green nor reddening.*

9 Aknowledgements

The author wants to thank Bernard Leclerc for his supervision, and Pierre-Guy Plamondon for fruitful conversations on implementing the computation, leading to study the Δ-vectors. He also wants to thank Bernhard Keller for the discussion about red/green sequences and the rewieving of the PhD thesis from which these results are taken.

A Configurations, inheritance and teeth shift principle

In this section, we translate [Mén21a, Sections 4.10 & 4.11].

Consider the seed (R_m, Γ_m) with the notations of Theorem 7.10, verifying in particular the points 4 and 5.

We will consider the configurations in the cut seed $(G, \widetilde{\Gamma})$ (with the notations of Lemma 7.14) around the vertex that will be mutated in any bicolor subquiver (i_k, i_j).

Due to the definition of the sequence of mutations in Definition 6.5, and the fact we are working in the cut seed, the mutation sequence amounts to mutating all vertices of the line except the last which is deleted. There can be eviction, but, according to Corollary 7.9, the vertex G_{k^+} can only be evicted if G_k has been evicted. Conversely, if G_k is not evicted, no successor of G_k can be evicted. Thus we will only study evictions in initial cases here.

At any time we will take configurations of Tables 5 & 6, adapting if necessary the indexation of vertices. We will always suppose that $G_{k^{2+}}$ exists, on the contrary, G_{k^+} is the last vertex of its line so it will not be mutated and it will be erased when considered.

A.1 Initial configurations

If G_k is the first of its line (in $\widetilde{\Gamma}$) then, according to hypotheses 4 and 5 of Theorem 7.10, the only possible configurations are the one listed in Table 5. Due to point 5, the first vertex of the line can have at most one ordinary arrow, being its target.

Case α_0 if eviction If we are in case α_0 before mutation, after its mutation we evict G_k and now G_{k^+} is the first vertex of the line in the new quiver $\widetilde{\Gamma}$.If G_{k^+} had no ordinary arrow we are once again in a case α_0. If there is one, it has G_{k^+} as target, being $G_j \to G_{k^+}$ and then, we are in case α_1 if there is no arrow $G_{k^{2+}} \to G_j$ or in case α_2 if it exists.

Case α_0 without eviction If there is no eviction, we continue by now considering the vertex G_{k^+}. If G_{k^+} had no ordinary arrow, it still does not and we are in case β_0. As previously if it has an ordinary arrow, it has only one $G_j \to G_{k^+}$, being its target. Then, depending on the existence of an arrow $G_{k^+} \to G_j$ we are in case β_1 or β_2.

Case α_1 with eviction If G_k^* is evicted after mutation, G_k^+ becomes the first vertex of the line. However, we have an arrow $G_j \to G_k$ coming from the mutation at G_k^*. It is the only ordinary arrow adjacent to G_{k^+} because either the initial arrow $G_j \to G_k$ was the only ordinary between two lines (as a final barb) or it was the beginning of a tooth $G_{k^{\gamma+}} \to G_j \to G_k$ with $\gamma > 1$ as the tooth is long, the left arrow being preserved in spite of the eviction. Then we have only one ordinary arrow pointing at G_{k^+} and if we were on the beginning of tooth and $\gamma = 2$ the we are now in case α_2, otherwise in case α_1.

Case α_1 without eviction If we have now eviction, we still know there is only one ordinary arrow adjacent to G_{k^+}: $G_j \to G_{k^+}$ and there is one ordinary arrow $G_j \leftarrow G_k$. Thus, depending on the length of the tooth or, equivalently, the configuration around $G_{k^{2+}}$ we are either in case β_3 or β_4.

Case α_2 with eviction If we have eviction G_{k^+} becomes the first vertex of its line. In addition there is no arrow between G_{k^+} and G_j (the original $G_{k^+} \to G_j$ being deleted by the mutation at G_k). Then either there exists a $\gamma > 0$ such that $G_{j^{\gamma+}} \to G_{k^+}$ exists and then we are now in case α_1 or α_2 depending on what happens at $G_{k^{2+}}$, or not and there was only one ordinary arrow, the final barb and now we are on case α_0 then β_0 (when evictions stop) until the end of the line.

Case α_2 without eviction Without eviction then G_{k^+} is, after mutation of G_k, the source of two horizontal arrows and, potentially an arrow $G_{j^{\gamma+}} \to G_{k^+}$ where $\gamma > 0$. The ordinary arrow $G_j \to G_k$ is no longer our concern so, up to renaming $j^{\gamma+} \rightsquigarrow j$ and $k^+ \rightsquigarrow k$ we are in case β_0 if there was no ordinary arrow or β_1 or β_2 if there was one.

Finally, for any of the 3 initial possible configurations, after mutation, we are in only 8 possible different configurations for G_{k^+}.

Config.	α_0	α_1	α_2
Nickname	Initial simple	Initial long tooth	Initial short tooth
G	$\leftarrow G_{k+} \leftarrow G_k$	$\begin{tikzcd} G_{j+} & G_j \arrow[l, dashed] \arrow[dr] & \\ & \emptyset & \\ G_{k+} & & G_k \arrow[ll] \end{tikzcd}$	$\begin{tikzcd} G_{j+} & G_j \arrow[l, dashed] \arrow[dr] & \\ & & \\ G_{k+} \arrow[uur] & & G_k \arrow[ll] \end{tikzcd}$
$\mu_k(G)$	$\leftarrow G_{k+} \rightarrow G_k^*$	$\begin{tikzcd} G_{j+} & G_j \arrow[l, dashed] \arrow[dl] & \\ & & \\ G_{k+} \arrow[rr] & & G_k^* \arrow[uul] \end{tikzcd}$	$\begin{tikzcd} G_{j+} & G_j \arrow[l, dashed] & \\ & \emptyset & \\ G_{k+} \arrow[rr] & & G_k^* \arrow[uul] \end{tikzcd}$
Next if no eviction Next if eviction	$\beta_0, \beta_1, \beta_2$ $\alpha_0, \alpha_1, \alpha_2$	β_3, β_4 α_1, α_2	$\beta_0, \beta_1, \beta_2$ $\alpha_0, \alpha_1, \alpha_2$

Table 5: Initial possible configurations and evolutions

Config.	β_0	β_1	β_2	β_3	β_4
Nickname	Start and end	Entrance long tooth	Entance short tooth	Middle (long) tooth	End tooth
G	$G_{k+} \leftarrow G_k \rightarrow G_{k-}^*$	$\begin{tikzcd} & G_j \arrow[d] & \\ \emptyset & \emptyset & \\ G_{k+} & G_k \arrow[l] \arrow[r] & G_{k-}^* \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[d] & \\ & \emptyset & \\ G_{k+} & G_k \arrow[l] \arrow[r] \arrow[uul] & G_{k-}^* \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[d] & \\ \emptyset & & \\ G_{k+} & G_k \arrow[l] \arrow[r] & G_{k-}^* \arrow[uul] \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[d] & \\ & & \\ G_{k+} & G_k \arrow[l] \arrow[r] \arrow[uul] & G_{k-}^* \arrow[uul] \end{tikzcd}$
$\mu_k(G)$	$G_{k+} \rightarrow G_k^* \leftarrow G_{k-}^*$	$\begin{tikzcd} & G_j \arrow[dl] \arrow[d] \arrow[dr] & \\ G_{k+} & G_k^* \arrow[l] & G_{k-}^* \arrow[l] \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[d] \arrow[dr] & \\ \emptyset & & \\ G_{k+} & G_k^* \arrow[l] & G_{k-}^* \arrow[l] \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[dl] \arrow[d] & \\ & & \emptyset \\ G_{k+} & G_k^* \arrow[l] & G_{k-}^* \arrow[l] \end{tikzcd}$	$\begin{tikzcd} & G_j \arrow[d] & \\ \emptyset & & \emptyset \\ G_{k+} & G_k^* \arrow[l] & G_{k-}^* \arrow[l] \end{tikzcd}$
Next	$\beta_0, \beta_1, \beta_2$	β_3, β_4	$\beta_0, \beta_1, \beta_2$	β_3, β_4	$\beta_0, \beta_1, \beta_2$

Table 6: General configuration possible and evolutions

Example A.1. We illustrate all this reasoning by Figure 10.

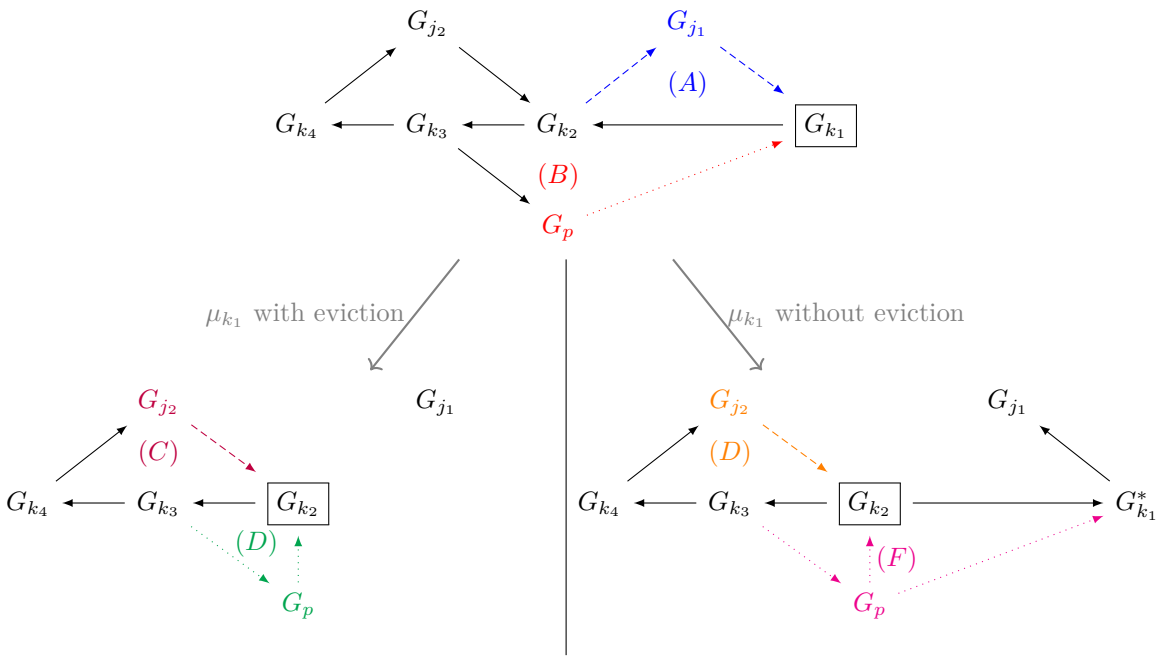


Figure 10: Evolutions of initial configurations

Initially we have a quiver whose vertices j_2 and j_1 are of color i_j, k_i of color i_k and p of color i_p.

The bicolor subquiver (i_k, i_j) is in configuration α_2 for G_{k_1} (arrows in blue dashes and horizontal arrows around (A)). The bicolor subquiver (i_k, i_p) is in configuration α_1 around G_{k_1} (dotted red arrows around (B) and horizontal arrows).

Then, after mutation a G_{k_1}, if it is evicted, we observe that the bicolor subquiver (i_k, i_j) is now in configuration α_1 around G_{k_2} (violet dashes, letter (C)). On the other side, the bicolor subquiver (i_k, i_p) is now in configuration α_2 (green dots, (D)). We remark that if the initial teeth was $G_{k_4} \to G_p \to G_{k_2}$, (D) would still be in configuration α_1.

If there is no eviction, then the bicolor subquiver (i_k, i_j) configuration around G_{k_2} is of type β_1 (orange dashes, (D)). If the teeth $G_{k_4} \to G_{j_2} \to G_{k_2}$ was shorter ($G_{k_3} \to G_{j_2} \to G_{k_2}$), we would have a configuration of type β_2.

The configuration of the bicolor subquiver (i_k, i_p) is of type β_4 (3 magenta dotted ordinary arrows, (F)). We remark, as before that if the tooth was longer and if we had originally $G_{k_4} \to G_p \to G_{k_1}$ we would be in configuration β_3.

A.2 General configurations

At this point we will now encounter at least 5 general configurations. We will show that under our hypothesis, there is no other configuration emerging. Recall that thanks to Corollary 7.9, as here G_{k^-} is not evicted, G_k cannot be evicted either.

Case β_0 After mutating G_k, G_{k^+} can have at most one ordinary arrow (whom it is the target). If there is not such an arrow we are again in a case β_0, if there is one, we are in case β_1 or β_2.

Case β_1 After mutation at G_k, there is a triangle $G_k^* \to G_j \to G_{k^+}$. Moreover, as in case α_1, there cannnot be an arrow $G_{j^{\gamma+}} \to G_{k^+}$ for any $\gamma > 0$. Thus we are in the case β_3 or β_4

Case β_2 In that case, as in α_2, we will potentially have an arrow $G_{j^{\gamma+}} \to G_{k^+}$ and, as there is no arrow $G_{k^+} \not\to G_j$, we can rename the vertices. If the arrow $G_{j^{\gamma+}} \to G_{k^+}$ does not exist, we are in case β_0 and that was the last tooth with no final barb and we will only have β_0 cases until the end. If this arrow exists, we are now in case β_1 or β_2.

Case β_3 As in β_1, we have only one ordinary arrow $G_j \to G_{k^+}$. Thus, depending on $G_{k^{2+}}$ we are in case β_3 or β_4

Case β_4 As in β_2 there is no connection between G_j and G_{k^+} and, if there is an ordinary arrow it is $G_{j^{\gamma+}} \to G_{k^+}$. If there is no such arrow we are in case β_0 until the end, if there is one we are in case β_1 or β_2.

Example A.2. We will illustrate this on Figure 11.

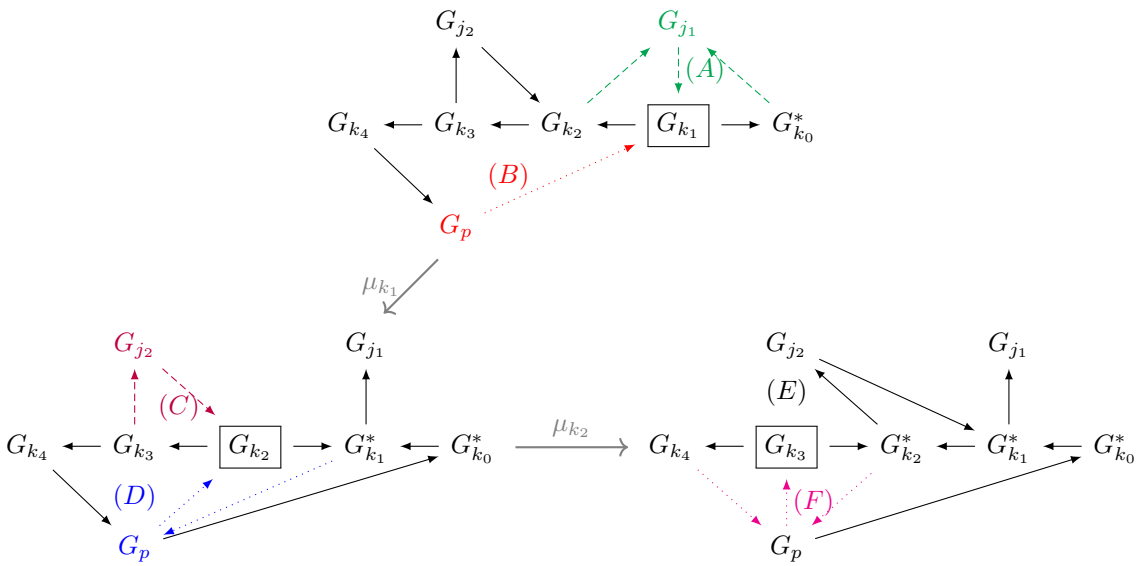


Figure 11: Evolutions of general configurations

Initially in the bicolor subquiver (i_k, i_j) we have a configuration β_4 around G_{k_1} (green dashes, (A)). In the bicolor subquiver (i_k, i_p) we have a configuration β_1 (red dots, (B)).

After mutation μ_{k_1}, in the subquiver (i_k, i_j), we have around G_{k_2} a configuration β_2 (violet dashed, (C)). In the subquiver(i_k, i_p), we have a configuration β_3 (blue dots, (D)).

After mutation μ_{k_2}, in the subquiver (i_k, i_j) we have a configuration β_0 around G_{k_3} ((E), no ordinary arrow). In the subquiver (i_k, i_p), we have a configuration β_4 once again (magenta dots).

A.3 Teeth shift

Using the previous knowledge of configuration, we will describe a more "macro" behaviour on the bicolor subquiver (i_k, i_j) after performing the whole sequence of mutation $\hat{\mu}$.

Proposition A.3 (Teeth shift)**.** *Suppose that a line i_k of the subquiver (i_k, i_j) is in pure saw teeth configuration. After mutating all the vertices of the line except the last one that we delete, we get a saw teeth structure. Precisely, any tooth $G_{k^{\alpha+}} \to G_j \to G_k$ becomes a tooth $G^*_{k^{(\alpha-1)+}} \to G_j \to G^*_{k^-}$ with the following derogation:*

- *if k is the first of its line, the tooth $G_{k^{\alpha+}} \to G_j \to G_k$ becomes the arrow $G^*_{k^{(\alpha-1)+}} \to G_j$.*
- *If the last ordinary arrow is in the orientation $G_j \to G_k$ and that k is not the last vertex of the line, then there appears a tooth $G^*_{k^{\alpha+}} \to G_j \to G^*_{k^-}$ (with the previous correction if k is the first of its line) where $G^*_{k^{\alpha+}}$ is the last vertex of the line after deletion of the previous one.*

Proof. We will rely on the previous study of all cases.

We will start the reasoning at the first teeth non evicted.

If the line starts with a sequence of vertices linked by horizontal arrows without any ordinary arrow (in the sense of Definition 4.7), we do the mutations until reaching the first tooth (there is no initial barb due to point 5 of Theorem 7.10). For the sequence of initial vertices, the arrow $G_{k^+} \leftarrow G_k$ is reversed by mutation at G_k but reversed back by mutation at G_{k^+} so, globally there is no change on it.

	Long tooth	Short tooth
Initial vertex	α_1	α_2
Non-initial vertex.	β_1	β_2

Table 7: Initial configurations

When reaching the right end of the teeth sequence, we have 4 possible cases. The vertex is either the initial one or not and the initial tooth has length one or more. We then have the configuration shown in Table 7.

We first consider short cases. If we are in case α_2, after mutation, we have an only arrow $G_k^* \to G_j$ as remnant of the initial tooth as instructed by the first derogation. Then there is no arrow between G_j and another vertex of the line of G_k and thus we will be in a case β for the next summuit, considering a new tooth of the sequence.

If we are in case β_2, we directly have the tooth switch: $G_{k^+} \to G_j \to G_k$ becoming $G_k \to G_j \to G_{k^-}$. By the same reasoning we will then be in case β_1 or β_2 for the next vertex, right end of the next tooth.

We now study the long cases. The case α_1 will be almost the same reasoning as for the case β_1 provided we consider that the arrow $G_j \to G_k$ is deleted as in case α_2. We will see this case through the β_1 case (but there α_1 leads to the apparition of an inital barb).

For the general case of a long tooth, we start with configuration β_1. The arrow $G_{k^{\gamma+}} \to G_j$ forming the left side of the tooth is not represented on the drawing, as it is too remote from G_k ($\gamma > 1$). Mutation of G_k gives us the arrow $G_j \to G_{k^-}^*$ (provided we are not looking at the α_1 case), the tilting of the ordinary arrow in $G_k^* \to G_j$ and the appearance of an arrow $G_j \to G_{k^+}$. The first ordinary arrow targeting G_{k^-} will now stay untouched by following mutations and will form the right end of the shifted tooth.

We now consider the next vertex, being in case β_3 if $\gamma > 2$ or β_4 if $\gamma = 2$.

If we are now in case β_3 around the new vertex G_{k^+}, by mutating it, the arrow $G_k \to G_j$ is deleted, the arrow $G_j \to G_{k^+}$ is tilted and there is a new arrow $G_j \to G_{k^{2+}}$. Then the following vertex will be in configuration β_3 if the left end of the initial tooth is not $G_{k^{3+}}$ and then we use the same reasoning as the current one.

On the contrary if we reach a configuration β_4 we will mutate $G_{k^{\gamma-1}+}$ and from the initially considered tooth, we have:

- the shifted arrow $G_j \to G_{k^-}^*$ (not existing if G_k was the initial vertex of the line)
- the arrow $G_{k^{(\gamma-2)+}} \to G_j$
- the arrow $G_j \to G_{k^{(\gamma-1)+}}$
- the arrow $G_{k^{\gamma+}} \to G_j$ which is the initial left side of the tooth

After mutation the first arrow in unchanged, the second one is deleted, the third is tilted in, $G_{k^{(\gamma-1)+}} \to G_j$, the left side of the new tooth and the fourth arrow is deleted. In total the initial tooth $G_{k^{\gamma+}} \to G_j \to G_k$ has been shifted in $G_{k^{(\gamma-1)+}} \to G_j \to G_{k^-}$.

After the tooth shift, the next vertex will be on configuration β_1 or β_2 (if there is another tooth or a final barb) or β_0 (if we are at the end of the tooth sequence). In that final case we use the same reasoning as the first β_0 until the end of the line.

If we have a tooth we repeat the reasoning as before.

If we have a final barb, then we will have a configuration β_1 followed by configurations β_3 but with no β_4 configuration to end the tooth. If $G_{k^{\zeta+}}$ is the last vertex of the line, the last mutated vertex will be $G_{k^{(\zeta-1)+}}$. After mutation we will have an arrow $G_{k^{(\zeta-1)+}} \to G_j$ closing the end having the shift of the final barb as right side and an arrow $G_j \to G_{k^{\zeta+}}$ which will disappear immediatly with the deletion of the vertex $G_{k^{\zeta+}}$. The new appeared left side of the tooth will be a minor inconvenience as we will never more mutate on $G_{k^{(\zeta-1)+}}$, the new last vertex of the line. □

Example A.4. We will take again Example A.2

We have initially the tooth $G_{k_3} \to G_{j_2} \to G_{k_2}$ shifted in $G_{k_2}^* \to G_{j_2} \to G_{k_1}^*$ after $\mu_{k_2} \circ \mu_{k_1}$, this is the non-initial short tooth case.

Let us look at the configuration before the mutation μ_{k_0}. It is:

$$\begin{tikzcd}
 & G_{j_2} \arrow[dr] & & G_{j_1} \arrow[dr] & \\
G_{k_4} \arrow[dr] & G_{k_3} \arrow[u] & G_{k_2} \arrow[ul] \arrow[ur] & (A) \; G_{k_1} \arrow[l] \arrow[r] & \boxed{G_{k_0}} \\
 & G_p \arrow[ur] & & &
\end{tikzcd}$$

Then we see a tooth (A) $G_{k_2} \to G_{j_1} \to G_{k_0}$ that, after $\mu_{k_1} \circ \mu_{k_0}$ reduces to the initial barb $G^*_{k_1} \to Gj_1$. That is the case of an initial long teeth.

To illustrate the second part of Proposition A.3, we consider the quiver:

$$\begin{tikzcd}
 & G_{j_2} \arrow[dr] & & G_{j_1} \arrow[dr] & & \\
G_{k_5} & G_{k_4} \arrow[l] & G_{k_3} \arrow[l] \arrow[ur] & & G_{k_2} \arrow[ll] & G_{k_1} \arrow[l]
\end{tikzcd}.$$

In that case, after $\mu_{k_4} \circ \cdots \circ \mu_{k_1}$ we have:

$$\begin{tikzcd}
 & G_{j_2} \arrow[dl] \arrow[drrr] & & G_{j_1} \arrow[drr] & & \\
G_{k_5} \arrow[r] & G^*_{k_4} & G^*_{k_3} \arrow[l] & & G^*_{k_2} \arrow[ll] \arrow[ul] & G^*_{k_1} \arrow[l]
\end{tikzcd},$$

which becomes, after deletion of G_{k_5}:

$$\begin{tikzcd}
G_{j_2} \arrow[drrr] & & G_{j_1} \arrow[drr] & & \\
G^*_{k_4} \arrow[u] & G^*_{k_3} \arrow[l] & & G^*_{k_2} \arrow[ll] \arrow[ul] & G^*_{k_1} \arrow[l]
\end{tikzcd}.$$

We have here the appearance of a tooth $G^*_{k_4} \to G_{j_2} \to G^*_{k_2}$ when we initially only have a final barb $G_{j_2} \to G_{k_2}$.

Nevertheless, we remark that it does not perturbate the saw teeth structure neither upcoming mutations, the vertex R_{k_4} will not be mutated anymore and then the arrow will not be taken into account in the computation of Δ-vectors in upcoming mutations.

B Examples

Example B.1. We take again the data of Example 6.2 p.22. Recall that we are in the case W of type A_5 with $\overline{w} = [1, 3, 2, 4, 3, \underline{2}, \underline{4}, \underline{5}, 4, \underline{3}, 2, \underline{1}, \underline{2}]$ and $v = s_2 s_4 s_5 s_3 s_1 s_2$. We represent the initial seed on Figure 12 with the quiver $\Gamma_{\overline{w}}$ labelled by socle decomposition of the direct summands of $V_{\overline{w}}$. On Example 6.2 we saw that we had the mutation sequences:

$$\tilde{\mu}_1 = \mu_8 \circ \mu_3 \circ \mu_1, \quad \tilde{\mu}_2 = \mu_2, \quad \tilde{\mu}_3 = \mu_9 \circ \mu_4, \quad \tilde{\mu}_4 = \mathrm{id}, \quad \tilde{\mu}_5 = \mu_7, \quad \tilde{\mu}_6 = \mu_3$$

We recall on Table 8 the modules defining the $\Delta_{\dot{v}}$-vectors. This gives the Δ-vectors of the Table 9. On this table, the color of the canonical vector f_m relates to the color i_{p_m}. The indices $m > 6$ are not colored as their refer to an arbitrary completion of $\overline{v}$ in $\dot{v}$.

In order to define the first sequence of mutation $\hat{\mu}_1(V_{\overline{w}})$ we determine the set:

$$A_1(V_{\overline{w}}) = \{1 \leq j \leq 8 \mid \Delta_{\dot{v},1}(V_{\overline{w}}) \neq 0\} = \{1, 3, 8\}$$

We then have $\hat{\mu}_1(V_{\overline{w}}) = \mu_8 \circ \mu_3 \circ \mu_1$ that we apply. Note that, as expected, we have $\hat{\mu}_1 = \widetilde{\mu}_1$ of Example 6.2.

We apply Proposition 5.9 to compute the $\Delta_{\dot{v}}$-vector of V_1^*. Here we do not do the auxilliary computation with d_Δ as, thanks to the proof of Lemma 7.17, only one of the two computations gives a vector of $\mathbb{N}^{\ell(w_0)}$.

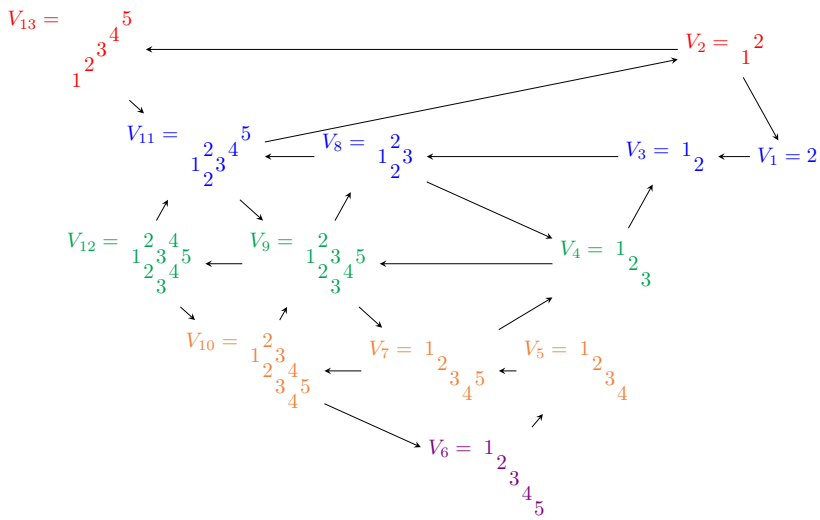


Figure 12: Seed $(V_{\overline{w}}, \Gamma_{\overline{w}})$

k	1	2	3	4	5
$M_{k,\dot{v}}$	2	1^{2}	2_{3}	5	$2_{3_{4}}^{5}$
k	6	7	8	9	10
$M_{k,\dot{v}}$	1^{2}_{3}	3	$1^{2}_{3_{4}}5$	3_{4}^{5}	4^{5}
k	11	12	13	14	15
$M_{k,\dot{v}}$	1	$2_{3_{4}}$	$1^{2}_{3_{4}}$	3_{4}	4

Table 8: Strata defining the $\Delta_{\dot{v}}$-stratification

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(V_k)$	f_1	f_2	$f_1 + f_{11}$	$f_3 + f_{11}$	f_{12}	$f_4 + f_{12}$	$f_5 + f_{11}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(V_k)$	$f_1 + f_6$	$f_3 + f_8$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 9: $\Delta_{\dot{v}}$-vectors of $V_{\overline{w}}$ direct indecomposable summands

We either have

$$\Delta_{\dot{v}}(V_1^*) \stackrel{?}{=} \Delta_{\dot{v}}(V_3) - \Delta_{\dot{v}}(V_1) = f_1 + f_{11} - f_1 = f_{11}$$

or

$$\Delta_{\dot{v}}(V_1^*) \stackrel{?}{=} \Delta_{\dot{v}}(V_2) - \Delta_{\dot{v}}(V_1) = f_2 - f_1.$$

Only the first computation is then possible and $\Delta_{\dot{v}}(V_1^*) = f_1$.

Note that these computations echoes the short exact sequences used to determine the mutated modules:

$$2 \to 1_{2} \to 1, \quad 1 \to {}_{1}^{2} \to 2. \tag{4}$$

and then the mutated module V_1^* is the simple module S_1. But only one short exact sequence translated into a valid computation regarding $\Delta_{\dot{v}}$-vectors: the first one.

After this mutation, we have the module of Figure 13 (modified arrows are in thick magenta).

We now compute $\Delta_{\dot{v}}(V_3^*)$ by the same method.

$$\Delta_{\dot{v}}(V_3^*) = \Delta_{\dot{v}}(V_1^*) + \Delta_{\dot{v}}(V_8) - \Delta_{\dot{v}}(V_3) = f_{11} + f_1 + f_6 - f_1 - f_{11} = f_6$$

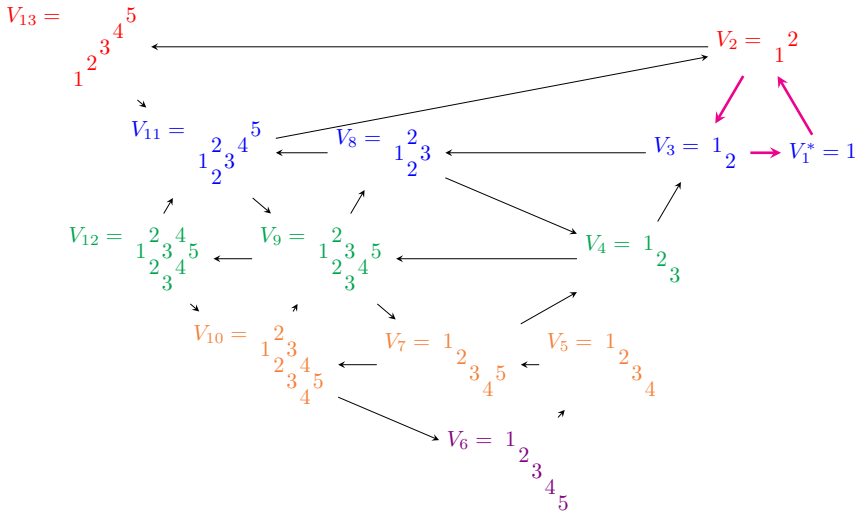


Figure 13: Module $\mu_1(V_{\overline{w}})$

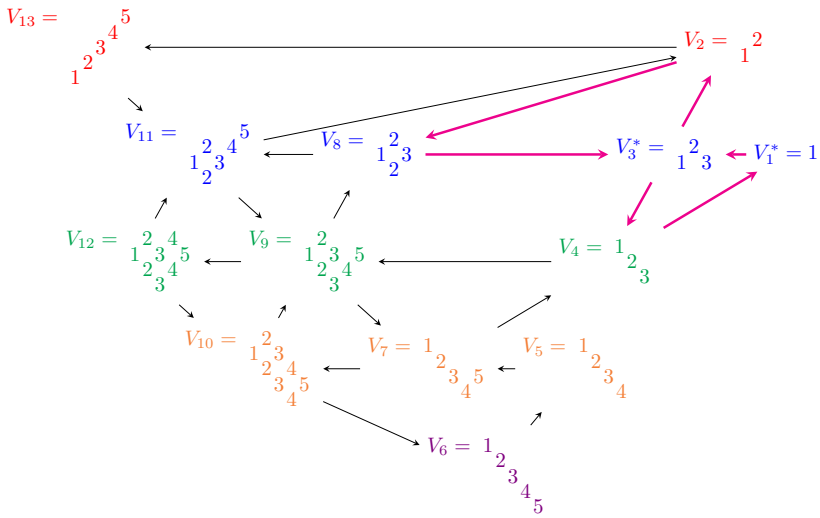


Figure 14: Module $\mu_3 \circ \mu_1(V_{\overline{w}})$

We have $V_3^* = {}_1{}^2{}_3$ and the module $\mu_3 \circ \mu_1(V_{\overline{w}})$ is the one of Figure 14.
And finally we get $\Delta_{\dot{v}}(V_8^*)$ by the computation:

$$\Delta_{\dot{v}}(V_8^*) = \Delta_{\dot{v}}(V_3^*) + \Delta_{\dot{v}}(V_{11}) - \Delta_{\dot{v}}(V_8) = f_6 + f_1 + f_6 + f_{10} - f_1 - f_6 = f_6 + f_{10}.$$

We then have as $\Delta_{\dot{v}}$-vectors for $\hat{\mu}_1(V_{\overline{w}})$ Table 10 and the module is on Figure 15.

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(R_{k,1})$	f_{11}	f_2	f_6	$f_3 + f_{11}$	f_{12}	$f_4 + f_{12}$	$f_5 + f_{11}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(R_{k,1})$	$f_6 + f_{10}$	$f_3 + f_8$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 10: $\Delta_{\dot{v}}$-vectors of $\hat{\mu}_1(V_{\overline{w}})$

Having completed $\hat{\mu}_1$, we now rename the summands

$$V_1^* \rightsquigarrow R_{1,1},\ V_3^* \rightsquigarrow R_{3,1},\ V_8^* \rightsquigarrow R_{8,1},\ V_k \rightsquigarrow R_{k,1},\ k \notin \{1,3,8\}.$$

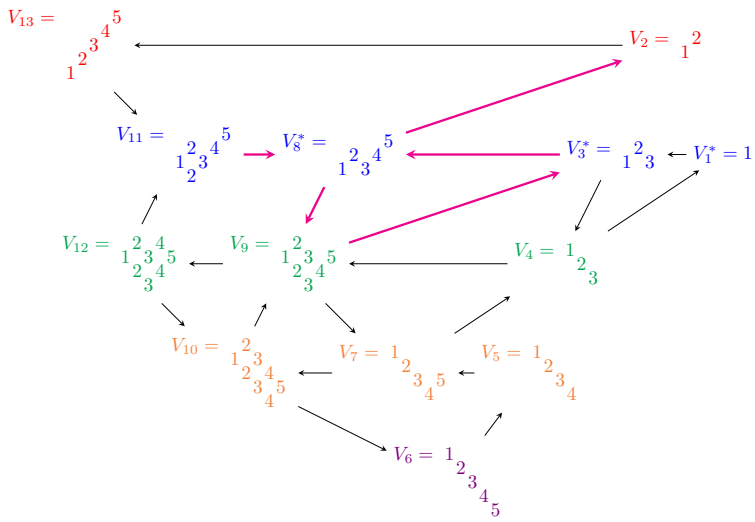


Figure 15: Module $\hat{\mu}_1(V_{\overline{w}})$

We now determine $A_2(\hat{\mu}_1(V_{\overline{w}}))$ et $\hat{\mu}_2$.
We have

$$A_2(R_1) = \{1 \leq i \leq 2 \mid \Delta_{\dot{v},2}(R_{i,1})\} = \{2\}$$

and so

$$\hat{\mu}_2 = \mu_2 = \widetilde{\mu}_2.$$

We now apply $\hat{\mu}_2 = \mu_2$ to $\hat{\mu}_1(V_{\overline{w}})$, and get Figure 16 and $\Delta_{\dot{v}}$-vectors of Table 11.

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(R_{k,2})$	f_{11}	$f_7 + f_{10}$	f_6	$f_3 + f_{11}$	f_{12}	$f_4 + f_{12}$	$f_5 + f_{11}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(R_{k,2})$	$f_6 + f_{10}$	$f_3 + f_8$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 11: $\Delta_{\dot{v}}$-vectors of $\hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}})$

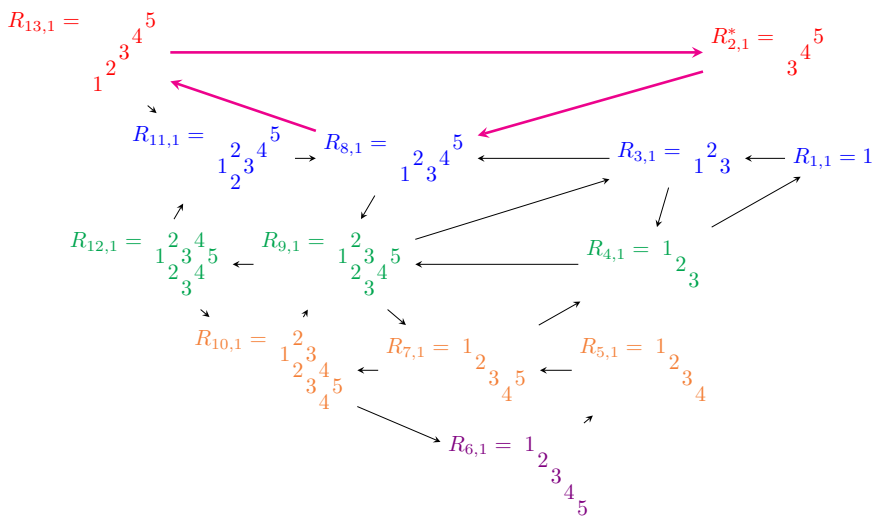


Figure 16: Module $\hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}})$

We again rename the indecomposables factors $R_{2,1}^* \rightsquigarrow R_{2,2}$ et $R_{k,1} \rightsquigarrow R_{k,2}$ pour $k \neq 2$.

We now compute:

$$A_3(R_2) = \{1 \leq i \leq 9 \mid \Delta_{\dot{v},3}(R_{i,3}) \neq 0\} = \{4, 9\}$$

and we have $\hat{\mu}_3 = \mu_9 \circ \mu_4$ and again $\hat{\mu}_3 = \widetilde{\mu}_3$.

We get the module R_3 represented on Figure 17 with $\Delta_{\dot{v}}$-vectors of Table 12.

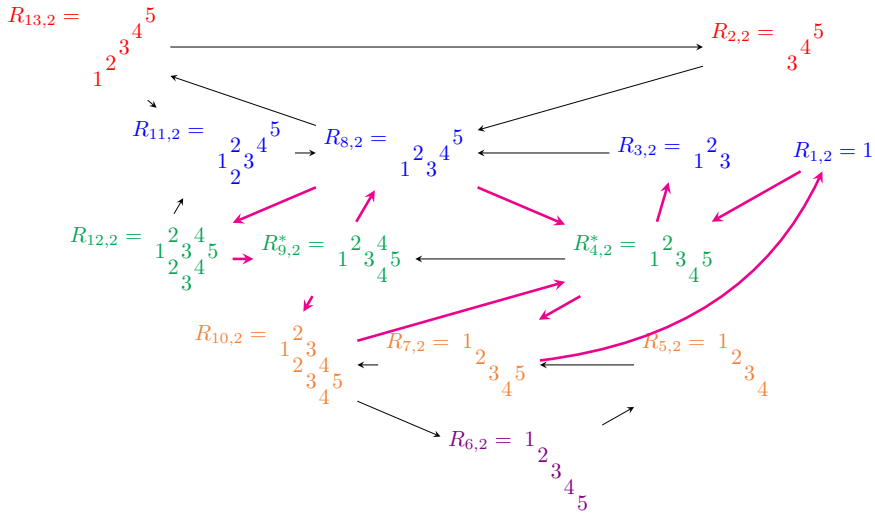


Figure 17: Module $\hat{\mu}_4 \circ \hat{\mu}_3 \circ \hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}}) = \hat{\mu}_3 \circ \hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}})$

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(R_{k,3})$	f_{11}	$f_7 + f_{10}$	f_6	f_8	f_{12}	$f_4 + f_{12}$	$f_5 + f_{11}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(R_{k,3})$	$f_6 + f_{10}$	$f_8 + f_{15}$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 12: $\Delta_{\dot{v}}$-vectors of $\hat{\mu}_3 \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$

We rename $R^*_{4,2} \rightsquigarrow R_{4,3}$, $R^*_{9,2} \rightsquigarrow R_{9,3}$ et $R_{k,2} \rightsquigarrow R_{k,3}$.

We compute

$$A_4(R_3) = \{1 \leq i \leq 0 | \Delta_{\dot{v},4}(R_{i,3}) \neq 0\} = \varnothing$$

and so $\hat{\mu}_4 = id = \widetilde{\mu}_4$.

As $\hat{\mu}_4 = id$,we rename $R_{k,3} \rightsquigarrow R_{k,4}$.

We compute

$$A_5(R_4) = \{1 \leq i \leq 7 | \Delta_{\dot{v},5}(R_{i,4}) \neq 0\} = \{7\}.$$

So $\hat{\mu}_5 = \mu_7 = \widetilde{\mu}_5$ and then $R_5 = \hat{\mu}_5 \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$ is the module on Figure 18 whose $\Delta_{\dot{v}}$-vectors are on Table 13.

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(R_{k,5})$	f_{11}	$f_7 + f_{10}$	f_6	f_8	f_{12}	$f_4 + f_{12}$	$f_{11} + f_{13}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(R_{k,5})$	$f_6 + f_{10}$	$f_8 + f_{15}$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 13: $\Delta_{\dot{v}}$-vectors of $\hat{\mu}_5 \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$

We rename $R^*_{7,4} = R_{7,5}$ et $R_{k,4} = R_{k,5}$ et $k \neq 7$.

$$A_6 = \{1 \leq i \leq 3 | \Delta_{\dot{v},6}(R_5) \neq 0\} = \{3\}$$

and so we have $\hat{\mu}_6 = \mu_3 = \widetilde{\mu}_6$.

That gives $\hat{\mu}_6 \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$ represented on Figure 19 whose $\Delta_{\dot{v}}$-vectors are on Table 14.

We rename $R^*_{3,5} \rightsquigarrow R_{3,6}$ and $R_{k,5} \rightsquigarrow R_{k,6}$ for $k \neq 3$.

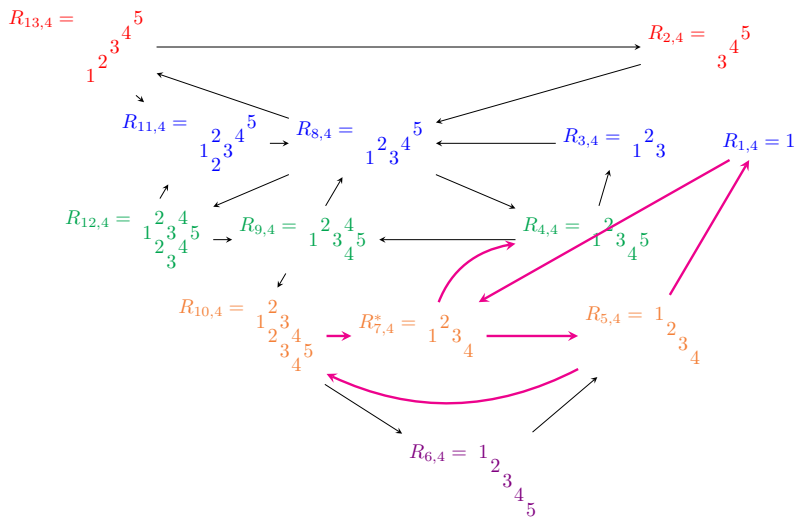


Figure 18: Module $\hat{\mu}_5 \circ \hat{\mu}_4 \circ \hat{\mu}_3 \circ \hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}})$

k	1	2	3	4	5	6	7
$\Delta_{\dot{v}}(R_{k,6})$	f_{11}	$f_7 + f_{10}$	f_{10}	f_8	f_{12}	$f_4 + f_{12}$	$f_{11} + f_{13}$
k	8	9	10	11	12	13	
$\Delta_{\dot{v}}(R_{k,6})$	$f_6 + f_{10}$	$f_8 + f_{15}$	$f_5 + f_{11} + f_{13}$	$f_1 + f_6 + f_{10}$	$f_3 + f_8 + f_{15}$	$f_2 + f_7 + f_{10}$	

Table 14: $\Delta_{\dot{v}}$-vectors of $\hat{\mu}_6 \circ \cdots \circ \hat{\mu}_1(V_{\overline{w}})$

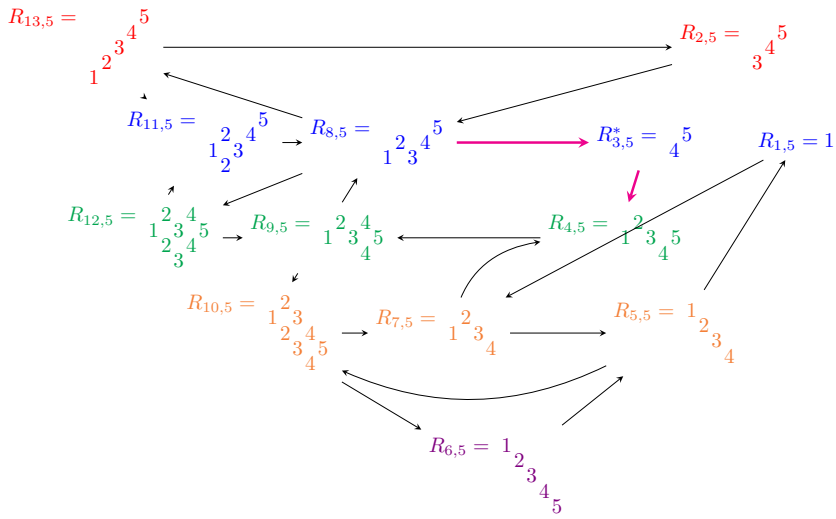


Figure 19: Module $\hat{\mu}_6 \circ \hat{\mu}_5 \circ \hat{\mu}_4 \circ \hat{\mu}_3 \circ \hat{\mu}_2 \circ \hat{\mu}_1(V_{\overline{w}})$

In order to get $\mu_\bullet$, we compute $(k_{\max})^{\alpha(k,\ell(v))-}$ for any color. For color 1, we have $(2_{\max})^{\alpha(2,6)-} = 13^- = 2$. For color 2, we have $(1_{\max})^{\alpha(1,6)-} = 11^{2-} = 3$. For color 3, we have $(4_{\max})^{\alpha(4,6)-} = 12^- = 9$. For color 4, we have $(5_{\max})^{\alpha(5,6)-} = 10^- = 7$. For color 5, we have $(6_{\max})^{\alpha(6,6)-} = 6^- = 0$.

Then $\mu_\bullet$ removes the indecomposable summands of indices $6, 8, 10, 11, 12, 13$ and we get module of Figure 20 (with $R_{2,6}$ a non-connected vertex).

Then, according to Theorem 6.8, $\mu^\bullet(V_{\overline{w}})$ is a $\mathcal{C}_{v,w}$-maximal rigid basic module, whose quiver is the one of Figure 20.

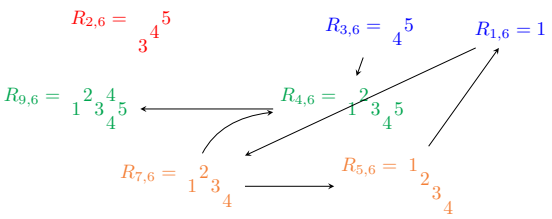


Figure 20: Module $\mu_\bullet(V_{\overline{w}})$

References

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras III: Upper bounds and double Bruhat cells. *Duke Mathematical Journal*, 126(1):1–52, January 2005.

[BIRS09] Aslak Bakke Buan, Osamu Iyama, Idun Reiten, and Jeanne Scott. Cluster structures for 2-Calabi–Yau categories and unipotent groups. *Compositio Mathematica*, 145(4):1035–1079, July 2009.

[BK12] Pierre Baumann and Joel Kamnitzer. Preprojective algebras and MV polytopes. *Representation Theory of the American Mathematical Society*, 16(5):152–188, 2012.

[BK20] Sarjick Bakshi and Bernhard Keller. Private communication, 2020.

[BKT14] Pierre Baumann, Joel Kamnitzer, and Peter Tingley. Affine Mirković-Vilonen polytopes. *Publications mathématiques de l'IHÉS*, 120(1):113–205, November 2014.

[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Inventiones mathematicae*, 79(3):499–511, October 1985.

[GLS06] Christof Geiss, Bernard Leclerc, and Jan Schröer. Rigid modules over preprojective algebras. *Inventiones mathematicae*, 165(3):589–632, September 2006.

[GLS08] Christof Geiss, Bernard Leclerc, and Jan Schröer. Preprojective algebras and cluster algebras. *arXiv:0804.3168 [math]*, pages 253–283, September 2008.

[GLS11] Christof Geiss, Bernard Leclerc, and Jan Schröer. Kac–Moody groups and cluster algebras. *Advances in Mathematics*, 228(1):329–433, September 2011.

[Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[Kam07] Joel Kamnitzer. The crystal structure on the set of Mirković–Vilonen polytopes. *Advances in Mathematics*, 215(1):66–93, October 2007.

[Kas95] Masaki Kashiwara. On crystal bases. In *Representations of Groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995.

[KD19] Bernhard Keller and Laurent Demonet. A survey on maximal green sequences. *arXiv:1904.09247 [math]*, October 2019.

[KL80] David Kazhdan and George Lusztig. Schubert varieties and Poincaré duality. In *Geometry of the Laplace Operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, volume 36 of *Proc. Sympos. Pure Math., XXXVI*, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.

[KS97] Masaki Kashiwara and Yoshihisa Saito. Geometric construction of crystal bases. *Duke Mathematical Journal*, 89(1):9–36, July 1997.

[Lec16] Bernard Leclerc. Cluster structures on strata of flag varieties. *Advances in Mathematics*, 300:190–228, September 2016.

[Lus90a] G. Lusztig. Canonical Bases Arising from Quantized Enveloping Algebras. II. *Progress of Theoretical Physics Supplement*, 102:175–201, March 1990.

[Lus90b] George Lusztig. Canonical bases arising from quantized enveloping algebras. *Journal of the American Mathematical Society*, 3(2):447–447, May 1990.

[Lus98] George Lusztig. Total positivity in partial flag manifolds. *Representation Theory. An Electronic Journal of the American Mathematical Society*, 2:70–78, 1998.

[Mat64] Hideya Matsumoto. Générateurs et relations des groupes de Weyl généralisés. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, 258(13):3419, 1964.

[Mén21a] Etienne Ménard. *Algèbres Amassées Associées Aux Variétés de Richardson Ouvertes : Un Algorithme de Calcul de Graines Initiales.* Theses, Normandie Université, March 2021.

[Mén21b] Etienne Ménard. Implementation of Ménard's algorithm. https://framagit.org/EtienneMenard/TheseSage, 2021.

[NS09] Satoshi Naito and Daisuke Sagaki. Mirković–Vilonen polytopes lying in a Demazure crystal and an opposite Demazure crystal. *Advances in Mathematics*, 221(6):1804–1842, August 2009.

[Rin91] Claus Michael Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Mathematische Zeitschrift*, 208(1):209–223, December 1991.

[SSVZ00] Boris Shapiro, Michael Shapiro, Alek Vainshtein, and Andrei Zelevinsky. Simply laced Coxeter groups and groups generated by symplectic transvections. *The Michigan Mathematical Journal*, 48(1):531–551, 2000.