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# A characterization of Cesàro average utility\*

Marcus Pivato<sup>†</sup>

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## Abstract

Let  $\mathcal{X}$  be a connected metric space, and let  $\geq$  be a weak order defined on a suitable subset of  $\mathcal{X}^{\mathbb{N}}$ . We characterize when  $\geq$  has a *Cesàro average utility representation*. This means that there is a continuous real-valued function  $u$  on  $\mathcal{X}$  such that, for all sequences  $\mathbf{x} = (x_n)_{n=1}^{\infty}$  and  $\mathbf{y} = (y_n)_{n=1}^{\infty}$  in the domain of  $\geq$ , we have  $\mathbf{x} \geq \mathbf{y}$  if and only if the limit as  $N \rightarrow \infty$  of the average value of  $u(x_1), \dots, u(x_N)$  is higher than limit as  $N \rightarrow \infty$  of the average value of  $u(y_1), \dots, u(y_N)$ . This has applications to decision theory, game theory, and intergenerational social choice.

**JEL class:** D81, D63, D71.

## 1 Introduction

Certain problems in normative economics involve preferences over countably infinite sequences  $(x_1, x_2, x_3, \dots)$  of outcomes. For example, such preferences arise in infinite-horizon intertemporal choice, in infinitely repeated games, and in decisions under uncertainty with a countable state space. Given a utility function  $u : \mathcal{X} \rightarrow \mathbb{R}$ , one natural way to evaluate an infinite sequence is via the *Cesàro average utility*  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N u(x_n)$ , assuming this limit

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<sup>†</sup>CY Cergy Paris Université, CNRS, THEMA, F-95000 Cergy, France, marcus-pivato@cyu.fr

is well-defined. In models of intertemporal choice, Cesàro average utility describes preferences with *infinite patience*. This paper axiomatically characterizes preferences admitting such a representation. Preferences with similar representations have previously been studied by [Lauwers \(1998\)](#), [Marinacci \(1998\)](#), [Rébillé \(2007\)](#), [Khan and Stinchcombe \(2018\)](#) and [Jonsson and Voorneveld \(2018\)](#). But to our knowledge, this is the first characterization of Cesàro average utilities.

The remainder of the paper is organized as follows. Section 2 explains the formal framework and the four axioms, and states the main result. Section 3 reviews prior literature. All proofs are in the Appendix.

## 2 Model and main result

Let  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Let  $\mathcal{X}$  be a connected metric space, and let  $\mathcal{X}^{\mathbb{N}}$  denote the set of all sequences  $\mathbf{x} = (x_n)_{n=1}^{\infty}$ , where  $x_n \in \mathcal{X}$  for all  $n \in \mathbb{N}$ . For any  $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$ , we will say that  $\mathbf{x}$  is *totally bounded* if, for any  $\epsilon > 0$ , the set  $\{x_n\}_{n=1}^{\infty}$  can be covered with a finite number of  $\epsilon$ -radius balls in  $\mathcal{X}$ . Let  $\ell^{\infty}(\mathbb{N}, \mathcal{X}) := \{\mathbf{x} \in \mathcal{X}^{\mathbb{N}}; \mathbf{x} \text{ is totally bounded}\}$ . For example, if  $\mathcal{X} = \mathbb{R}^N$ , then this is equivalent to requiring  $\mathbf{x}$  to be a *bounded* sequence —i.e. requiring  $\sup_{n \in \mathbb{N}} |x_n| < \infty$ , so this definition reduces to the standard definition of  $\ell^{\infty}(\mathbb{N}, \mathbb{R}^N)$ . But in a general metric space, total boundedness is more restrictive than mere boundedness. Define the *supremum metric*  $d_{\infty}$  on  $\ell^{\infty}(\mathbb{N}, \mathcal{X})$  in the standard way:

$$d_{\infty}(\mathbf{x}, \mathbf{y}) \quad := \quad \sup_{n \in \mathbb{N}} d(x_n, y_n).$$

If  $\mathcal{X} = \mathbb{R}^N$ , then this is just the standard supremum norm metric on  $\ell^{\infty}(\mathbb{N}, \mathbb{R}^N)$ .

**Regular sequences.** Let  $\mathfrak{D}(\mathcal{X})$  be the set of all sets of the form  $\mathcal{O} \cap \mathcal{C}$ , where  $\mathcal{O} \subseteq \mathcal{X}$  is open and  $\mathcal{C} \subseteq \mathcal{X}$  is closed. In particular,  $\mathfrak{D}(\mathcal{X})$  contains all open and closed subsets of  $\mathcal{X}$ . (If  $\mathcal{X}$  is locally compact, then  $\mathfrak{D}(\mathcal{X}) = \{\text{locally compact subsets of } \mathcal{X}\}$ . But we will not use this fact.) A sequence  $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$  is *regular* if, for any subset  $\mathcal{O} \in \mathfrak{D}(\mathcal{X})$ , the limit

$$\lim_{N \rightarrow \infty} \frac{\#\{n \in \{1, \dots, N\}; x_n \in \mathcal{O}\}}{N} \tag{1}$$

exists.<sup>1</sup> Here are three examples of regular sequences.

- Any periodic sequence is regular.
- Suppose  $\mathcal{X} = \mathbb{R}$ . Let  $a, b \in \mathbb{R}$  be constants and let  $x_n = \sin(an + b)$  for all  $n \in \mathbb{N}$ ; then  $(x_n)_{n=1}^\infty$  is regular. So is any linear combination of such sequences.
- Let  $\mathcal{Y} \subseteq \mathcal{X}$  be a finite subset, let  $\rho$  be a probability measure on  $\mathcal{Y}$ , and let  $\mathbf{y} = (y_0, y_1, y_2, \dots)$  be an infinite sequence of independent random variables drawn from  $\rho$ . Then  $\mathbf{y}$  is regular with probability 1. The same thing is true if  $\mathbf{y}$  is a sequence of exchangeable random variables, or if  $\mathbf{y}$  is generated by a stationary Markov process, or indeed, *any* stationary stochastic process on  $\mathcal{Y}$  (see Lemma A.1 in the Appendix).

Let  $\text{Reg}(\mathcal{X})$  denote the set of all regular sequences in  $\ell^\infty(\mathbb{N}, \mathcal{X})$ .

**Semiregular sequences.** A sequence  $\mathbf{x} \in \mathcal{X}^\mathbb{N}$  is *semiregular* if either (1)  $\mathbf{x}$  is regular, or (2) there is a compact subset  $\mathcal{K} \subseteq \mathcal{X}$  and a sequence  $(\mathbf{x}^j)_{j=1}^\infty$  of elements of  $\text{Reg}(\mathcal{K})$  such that  $\lim_{j \rightarrow \infty} d_\infty(\mathbf{x}^j, \mathbf{x}) = 0$ . Let  $\overline{\text{Reg}}(\mathcal{X})$  denote the set of all semiregular sequences in  $\mathcal{X}^\mathbb{N}$ . (It is easily shown that  $\overline{\text{Reg}}(\mathcal{X}) \subseteq \ell^\infty(\mathbb{N}, \mathcal{X})$ .) For example, suppose  $\mathcal{X}$  is *ball-compact*, meaning that every closed ball in  $\mathcal{X}$  is compact. (This holds if  $\mathcal{X}$  itself is compact, but it also holds if  $\mathcal{X}$  is  $\mathbb{R}^N$  or any closed subset of  $\mathbb{R}^N$ .) Then  $\mathbf{x}$  is semiregular if and only if it is a  $d_\infty$ -limit of elements of  $\text{Reg}(\mathcal{X})$ . Thus, for ball-compact spaces,  $\overline{\text{Reg}}(\mathcal{X})$  is the closure of  $\text{Reg}(\mathcal{X})$  in the  $d_\infty$ -metric. But the definition works for any metric space.

Some semiregular sequences are not regular. For example, if  $\mathbf{x} = (x_n)_{n=1}^\infty$  is a convergent sequence in  $\mathcal{X}$ , then in general,  $\mathbf{x}$  is *not* regular. But if  $\mathcal{X}$  is locally compact, then  $\mathbf{x}$  is semiregular (see Lemma A.2 in the Appendix).

**Cesàro averages.** For any sequence  $\mathbf{r} = (r_1, r_2, \dots) \in \mathbb{R}^\mathbb{N}$ , the *Cesàro average* is defined

$$\text{CesAve}_{n \in \mathbb{N}} r_n := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N r_n, \quad (2)$$

when this limit exists. In this case, one says that  $\mathbf{r}$  is *Cesàro summable*.

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<sup>1</sup>In the case  $\mathcal{X} = \mathbb{R}$ , these are similar to what Khan and Stinchcombe (2020, §3.7) call *ergodic* sequences.

**Cesàro average utility.** Let  $\geq$  be a *weak order* (i.e. a complete, transitive binary relation) on  $\overline{\text{Reg}}(\mathcal{X})$ . Let us say that  $\geq$  has a *Cesàro average utility representation* if there is a function  $u : \mathcal{X} \rightarrow \mathbb{R}$  such that, for all  $\mathbf{x}, \mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ ,

$$\mathbf{x} \geq \mathbf{y} \iff \text{CesAve}_{n \in \mathbb{N}} u(x_n) \geq \text{CesAve}_{n \in \mathbb{N}} u(y_n). \quad (3)$$

This paper axiomatically characterizes weak orders on  $\overline{\text{Reg}}(\mathcal{X})$  that admit such a representation. Such weak orders have a variety of applications in economics. We will now describe four: intertemporal decisions, infinitely repeated games, intertemporal social choice, and decisions under uncertainty.

**Intertemporal choice.** Suppose  $\mathbb{N}$  represents an infinite sequence of future times, and  $\mathcal{X}$  is the set of outcomes that could occur at each time. So an element of  $\overline{\text{Reg}}(\mathcal{X})$  represents an infinite stream of outcomes. Thus, a weak order on  $\overline{\text{Reg}}(\mathcal{X})$  represents the *intertemporal preferences* of an agent. The Cesàro average utility representation (3) is something like the limit of the discounted utility sum  $\sum_{n=1}^{\infty} \delta^n u(x_n)$  as  $\delta \nearrow 1$ . (See equation (12) in Section 3.) It thus describes an agent with infinite patience. If an intertemporal preference has a representation (3), then it satisfies a strong version of [Koopmans's \(1960\) Stationarity axiom](#);<sup>2</sup> for all sequences  $(x_0, x_1, x_2, \dots)$  and  $(y_0, y_1, y_2, \dots)$ , we have

$$(x_0, x_1, x_2, \dots) \geq (y_0, y_1, y_2, \dots) \iff (x_1, x_2, \dots) \geq (y_1, y_2, \dots). \quad (4)$$

**Infinitely repeated games.** Closely related to intertemporal choice is the theory of *repeated games*. To obtain equilibria, it is common to assume that the game will be repeated an *infinite* number of times. So it is necessary to endow players with preferences over the resulting infinite-length payoff streams. For this purpose, the Cesàro average utility (3) is quite convenient. It describes the preferences of infinitely patient players, and appears in several important folk theorems; see e.g. [Osborne and Rubinstein \(1994, §8.6\)](#) or [Maschler et al. \(2013, §13.5\)](#).

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<sup>2</sup>Koopman's original *Stationarity* axiom requires (4) only when  $x_0 = y_0$ .

**Intertemporal social choice.** Suppose the elements of  $\mathbb{N}$  represent all the people who will exist in the future, ordered chronologically by their moment of birth, and  $\mathcal{X}$  is the set of possible life outcomes for each future person. Thus, each element of  $\overline{\text{Reg}}(\mathcal{X})$  represents an intertemporal social outcome, and a weak order on  $\overline{\text{Reg}}(\mathcal{X})$  is an intertemporal social welfare order. In this interpretation, the Cesàro average (3) is a utilitarian SWF.

**Uncertainty and insufficient reason.** Let  $\mathcal{S}$  be a countably infinite space of possible states of nature with a “natural” enumeration  $\mathcal{S} = \{s_1, s_2, \dots\}$ . Let  $\mathcal{X}$  be a topological space of outcomes. An *act* is a function  $\alpha : \mathcal{S} \rightarrow \mathcal{X}$ . Since the foundational work of Savage (1954), a weak order over acts has been used as a model of rational decision-making under uncertainty. If we identify  $\mathcal{S}$  with  $\mathbb{N}$  in the obvious way (using the enumeration given), then the act  $\alpha$  can be identified with an element of  $\mathcal{X}^{\mathbb{N}}$ . In this context, the Cesàro average utility representation (3) can be interpreted as an *expected utility* representation, where  $u$  is the agent’s utility function, and her probabilistic beliefs correspond roughly to density of subsets of  $\mathbb{N}$ .<sup>3</sup> For any subset  $\mathcal{B} \subseteq \mathbb{N}$ , its *density* is defined

$$\delta(\mathcal{B}) := \lim_{N \rightarrow \infty} \frac{|\mathcal{B} \cap \{1, \dots, N\}|}{N}, \quad (5)$$

whenever this limit exists. We say that the agent’s beliefs correspond “roughly” to  $\delta$  because the limit (5) does not exist for all subsets of  $\mathbb{N}$ , and the sets for which it *does* exist do not form a Boolean algebra. Nevertheless,  $\delta$  is finitely additive on the sets for which it is well-defined: if  $\mathcal{A}$  and  $\mathcal{B}$  are disjoint subsets of  $\mathbb{N}$  such that  $\delta(\mathcal{A})$  and  $\delta(\mathcal{B})$  are both well-defined, then it is easily verified that  $\delta(\mathcal{A} \sqcup \mathcal{B}) = \delta(\mathcal{A}) + \delta(\mathcal{B})$ . Thus  $\delta$  is something like a “uniformly distributed probability measure” on the state space  $\mathcal{S}$ .<sup>4</sup> Such

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<sup>3</sup>Given the Savage framework, it is tempting to describe this as *subjective* expected utility. However, as noted by a referee, the Cesàro average (2) and density (5) are entirely *objective*. (But see footnote 4.)

<sup>4</sup>In fact,  $\delta$  can be extended to a finitely additive measure on the entire power set  $\wp(\mathbb{N})$ . Such extensions are called a *density measures* (Sleziak and Ziman, 2008; Letavaj et al., 2015). Likewise, the Cesàro averaging operator can be extended to a continuous, positive linear functional on  $\ell^\infty(\mathbb{N}, \mathbb{R})$  (Blümlinger and Obata, 1991, Theorem 3). But these extensions are far from unique. Thus, notwithstanding footnote 3, some “subjectivity” would creep into the expected utility representation, if one tried to extend it beyond  $\overline{\text{Reg}}(\mathcal{X})$ .

a uniform distribution arises as an expression of Laplace’s (1829) *Principle of Insufficient Reason*, which Keynes (1921) called the *Principle of Indifference*: if all elements of  $\mathcal{S}$  look indistinguishable *a priori* to the decision-maker, then she has no reason to assign greater probability to some of them than she does to others.

**The Lévy group.** A *Lévy permutation*<sup>5</sup> is a bijection  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$\lim_{N \rightarrow \infty} \frac{\#\{n \in \{1, \dots, N\}; \gamma(n) > N\}}{N} = 0. \quad (6)$$

For example, if  $\gamma$  only moves a finite subset of  $\mathbb{N}$  and fixes all other elements, then it is a Lévy permutation. More generally, if  $\gamma$  only moves a zero-density subset of  $\mathbb{N}$  (e.g. the set of all perfect squares) and fixes all other elements, then it is a Lévy permutation. For another example, consider this permutation that swaps the even and odd elements of  $\mathbb{N}$ :

$$\gamma(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd;} \\ n - 1 & \text{if } n \text{ is even.} \end{cases} \quad (7)$$

It is easily verified that  $\gamma$  is a Lévy permutation. The set  $\Gamma$  of all Lévy permutations forms a group under composition, called the *Lévy group*. If  $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$ , then define  $\gamma(\mathbf{x}) := (y_n)_{n=1}^{\infty} \in \mathcal{X}^{\mathbb{N}}$  by setting  $y_n := x_{\gamma(n)}$  for all  $n \in \mathbb{N}$ . Clearly,  $\gamma(\mathbf{x}) \in \ell^{\infty}(\mathbb{N}, \mathcal{X})$  if and only if  $\mathbf{x} \in \ell^{\infty}(\mathbb{N}, \mathcal{X})$ . Furthermore, if  $\gamma \in \Gamma$ , then  $\gamma(\mathbf{x}) \in \text{Reg}(\mathcal{X})$  if and only if  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ .

**Axioms.** Let  $\geq$  be a weak order on  $\overline{\text{Reg}}(\mathcal{X})$ . Define the weak order  $\geq_*$  on  $\mathcal{X}$  by restricting  $\geq$  to constant sequences in the obvious way. Let  $>$  be the antisymmetric part of  $\geq$ . Let  $>_*$  be the antisymmetric part of  $\geq_*$ . We will say that  $\geq$  is *nondegenerate* if  $>_*$  is nontrivial.<sup>6</sup> For any  $\mathbf{x} = (x_n)_{n=1}^{\infty} \in \mathcal{X}^{\mathbb{N}}$  and any  $\mathcal{L} \subseteq \mathbb{N}$ , define  $\mathbf{x}_{\mathcal{L}} := (x_{\ell})_{\ell \in \mathcal{L}}$ , an element of  $\mathcal{X}^{\mathcal{L}}$ . A

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<sup>5</sup>These permutations were introduced by Lévy (1951, Part III). Papers on infinite-population social welfare such as Lauwers (1998) call them *bounded* permutations (a slightly misleading term, because the set  $\{n \in \mathbb{N}; \gamma(n) \neq n\}$  is may be infinite, and the distance  $|\gamma(n) - n|$  is may be unbounded as  $n \rightarrow \infty$ ).

<sup>6</sup>In Savage’s (1954) axiomatization, nondegeneracy is Axiom P5. If  $>$  is trivial, then  $\geq$  is the *totally indifferent* relation; this has a trivial Cesàro average utility representation (3) in which  $u$  is any constant function. So we can assume  $>$  is nontrivial without any loss of generality. Nondegeneracy implies that  $>$  is nontrivial. Under mild assumptions, the converse is also true; see footnote 10.

subset  $\mathcal{M} \subseteq \mathbb{N}$  is *non-null* (for  $\geq$ ) if there exist  $\mathbf{x}, \mathbf{y} \in \text{Reg}(\mathcal{X})$  such that  $\mathbf{x}_{\mathbb{N} \setminus \mathcal{M}} = \mathbf{y}_{\mathbb{N} \setminus \mathcal{M}}$  but  $\mathbf{x} > \mathbf{y}$ . Otherwise  $\mathcal{M}$  is *null* (for  $\geq$ ).<sup>7</sup> We will need  $\geq$  to satisfy four axioms.

- A1. (*Monotonicity*) Let  $\mathcal{M} \subseteq \mathbb{N}$  be non-null, and let  $\mathbf{x}, \mathbf{y} \in \text{Reg}(\mathcal{X})$ . Suppose  $\mathbf{x}_{\mathbb{N} \setminus \mathcal{M}} = \mathbf{y}_{\mathbb{N} \setminus \mathcal{M}}$ , and there exist  $x, y \in \mathcal{X}$  such that  $x_m = x$  and  $y_m = y$  for all  $m \in \mathcal{M}$ . If  $x >_* y$ , then  $\mathbf{x} > \mathbf{y}$ .
- A2. (*Continuity*) Let  $\mathbf{x} \in \overline{\text{Reg}}(\mathcal{X})$ . Let  $\{\mathbf{x}^k\}_{k=1}^\infty$  be a sequence in  $\overline{\text{Reg}}(\mathcal{X})$  with  $\lim_{k \rightarrow \infty} d_\infty(\mathbf{x}^k, \mathbf{x}) = 0$ . If  $\mathbf{x}^k \geq \mathbf{y}$  for all  $k \in \mathbb{N}$ , then  $\mathbf{x} \geq \mathbf{y}$ . If  $\mathbf{x}^k \leq \mathbf{y}$  for all  $k \in \mathbb{N}$ , then  $\mathbf{x} \leq \mathbf{y}$ .
- A3. (*Separability*) Let  $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \text{Reg}(\mathcal{X})$ . Let  $\mathcal{J} \subseteq \mathbb{N}$  and  $\mathcal{K} = \mathbb{N} \setminus \mathcal{J}$ . Suppose  $\mathbf{x}_{\mathcal{J}} = \mathbf{x}'_{\mathcal{J}}$  and  $\mathbf{y}_{\mathcal{J}} = \mathbf{y}'_{\mathcal{J}}$ , while  $\mathbf{x}_{\mathcal{K}} = \mathbf{y}_{\mathcal{K}}$  and  $\mathbf{x}'_{\mathcal{K}} = \mathbf{y}'_{\mathcal{K}}$ . Then  $\mathbf{x} \geq \mathbf{y}$  if and only if  $\mathbf{x}' \geq \mathbf{y}'$ .
- A4. ( $\Gamma$ -*invariance*) For all  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ , and all  $\gamma \in \Gamma$ ,  $\mathbf{x} \approx \gamma(\mathbf{x})$ .

Note that Axioms A1, A3, and A4 apply only to  $\text{Reg}(\mathcal{X})$ . Axiom A2 is the only one that also extends to  $\overline{\text{Reg}}(\mathcal{X})$ . Axioms A1 - A3 are standard and ubiquitous in social choice and decision theory. It remains to explain Axiom A4. In an application to intertemporal choice or infinitely-repeated games, A4 describes *infinite patience*: the agent is indifferent between the payoff stream  $\mathbf{x}$  and its permutation  $\gamma(\mathbf{x})$ . This means that she assigns equal importance to any two moments in time. More generally, she assigns equal importance to any two *sequences* of future times, as long as one time-sequence is the image of the other under an element of  $\Gamma$ . For example, the sequence of *even* times and the sequence of *odd* times are treated equally, via the permutation defined in formula (7).<sup>8</sup>

In an application to intertemporal social choice (where  $\mathbb{N}$  indexes future people), A4 encodes *impartiality*. It says that any two future people are treated equally; furthermore, any two *populations* are treated equally, if one is the image of the other under an element of  $\Gamma$ . Lauwers (1998) introduced A4 under the name *Bounded anonymity*. Impartiality axioms

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<sup>7</sup>Here we have defined “null” with reference to  $\geq$ , as in Savage’s P3. But we shall later see that in our framework, nullity does not depend on  $\geq$ : a set is null for  $\geq$  if and only if it has zero density (Lemma A.7).

<sup>8</sup>In fact, we shall later see that Cesàro average utility is invariant under any permutation of  $\mathbb{N}$  that preserves the asymptotic frequency (5) of time sequences, encoding an even stronger notion of infinite patience. But this stronger invariance property is not required for our axiomatization.



of this nature are prevalent in intertemporal social choice theory; [Khan and Stinchcombe \(2018\)](#) discuss the relevant ethical considerations, and [Asheim \(2010\)](#) provides a good review of the literature.

In an application to decisions under uncertainty (where  $\mathbb{N}$  represents a space of states of nature), [A4](#) encodes *insufficient reason*. It says that all states appear identical, *a priori*, so they should have equal weight in the decision; furthermore, any two *events* (i.e. subsets of  $\mathbb{N}$ ) should have equal weight, if one is the image of the other under an element of  $\Gamma$ .

We will also need  $\geq_*$  to satisfy the following richness condition:

(B) (*Boundedness*) For any countable, totally bounded  $\mathcal{Y} \subseteq \mathcal{X}$ , there exist  $x, z \in \mathcal{X}$  such that  $x \leq_* y \leq_* z$  for all  $y \in \mathcal{Y}$ .

This is a mild condition that is usually satisfied automatically, as shown by the next result.

**Proposition.** *If  $(\mathcal{X}, d)$  is a complete metric space, then [A2](#) implies (B).*

Here is the main result of this paper.

**Theorem.** *Let  $\geq$  be a nondegenerate weak order on  $\overline{\text{Reg}}(\mathcal{X})$  satisfying condition (B). Then  $\geq$  satisfies Axioms [A1](#) - [A4](#) if and only if there is a continuous function  $u : \mathcal{X} \rightarrow \mathbb{R}$  such that  $\geq$  has a Cesàro average utility representation (3). In this representation,  $u$  is unique up to positive affine transformations.*

**Proof sketch.** The first part of the proof roughly follows the proof strategy of [Wakker and Zank \(1999\)](#) to derive what they call an “additive representation” for  $\geq$  from Axioms [A1](#) - [A3](#). Then Axiom [A4](#) and a theorem of [Obata \(1988a\)](#) are used to show that this additive representation takes the form (3). See the Appendix for details.  $\square$

The Cesàro average utility representation (3) does not satisfy the Weak Pareto axiom.<sup>9</sup> But it does satisfy two weaker conditions. First, it clearly satisfies:

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<sup>9</sup>Weak Pareto says: If  $x_n \succ_* y_n$  for all  $n \in \mathbb{N}$ , then  $\mathbf{x} \succ \mathbf{y}$ . This “Pareto” terminology is appropriate for applications to intertemporal social choice. In applications to individual intertemporal choice or choice under uncertainty, the formally identical axiom is called *Dominance*. We will use the term “Pareto” for brevity, and let the reader make the obvious translations.

Loose Pareto. For all  $\mathbf{x} \in \overline{\text{Reg}}(\mathcal{X})$ , if  $x_n \geq_* y_n$  for all  $n \in \mathbb{N}$ , then  $\mathbf{x} \geq \mathbf{y}$ .<sup>10</sup>

Recall that the *lower density* of a subset  $\mathcal{M} \subseteq \mathbb{N}$  is defined

$$\underline{\delta}(\mathcal{M}) := \liminf_{N \rightarrow \infty} \frac{|\mathcal{M} \cap \{1, \dots, N\}|}{N}.$$

If a weak order has a Cesàro average utility representation (3), then it is easy to verify that it satisfies the following Pareto condition:

Weak asymptotic Pareto.<sup>11</sup> For all  $\mathbf{x}, \mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$  and null  $\mathcal{M} \subset \mathbb{N}$ , if  $x_n \geq_* y_n$  for all  $n \in \mathbb{N} \setminus \mathcal{M}$ , and there is  $\epsilon > 0$  such that  $\underline{\delta} \{n \in \mathbb{N}; u(x_n) \geq u(y_n) + \epsilon\} > 0$ , then  $\mathbf{x} > \mathbf{y}$ .

Under Axioms A1 and A4, a subset  $\mathcal{M} \subset \mathbb{N}$  is null if  $\delta(\mathcal{M}) = 0$  (see Lemma A.7). Thus, Weak asymptotic Pareto applies in particular if  $\delta(\mathcal{M}) = 0$  and  $x_n \geq_* y_n$  for all  $n \in \mathbb{N} \setminus \mathcal{M}$ .

A weak order  $\geq$  with representation (3) does not satisfy Full anonymity (i.e. invariance under *all* permutations of  $\mathbb{N}$ ). But as already noted, Axiom A4 itself is an anonymity condition. In fact,  $\geq$  satisfies a stronger anonymity property. To explain this, recall the density  $\delta$  defined in formula (5). Let  $\mathfrak{B}$  be the family of all subsets of  $\mathbb{N}$  with a well-defined density. A bijection  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  is *density preserving* if  $\pi(\mathfrak{B}) = \mathfrak{B}$ , and for all  $\mathcal{B} \in \mathfrak{B}$  we have  $\delta[\pi(\mathcal{B})] = \delta(\mathcal{B})$ . It is easily verified that  $\geq$  satisfies the following axiom:

Density anonymity. If  $\mathbf{x}, \mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ , and there is a density-preserving permutation  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\mathbf{y} = \pi(\mathbf{x})$ , then  $\mathbf{x} \approx \mathbf{y}$ .

The Lévy group is a proper subgroup of the group of density-preserving permutations (Obata, 1988b, Propositions 3.1 and 4.1). So Density anonymity is stronger than A4.

The representation (3) might be criticized for failing to satisfy Weak Pareto or Full anonymity. But in infinite-population social choice, there is a stark tradeoff between Pareto

<sup>10</sup>This is often called *Monotonicity*. The term *loose Pareto* is due to Lauwers and Vallentyne (2004). As noted in footnote 6, nontriviality of  $>_*$  implies the nontriviality of  $>$ . If we add Loose Pareto to our axioms, then the converse is also true —i.e. the nontrivialities of  $>$  and  $>_*$  are logically equivalent. But it seems logically simpler to just assume the nontriviality of  $>_*$  directly.

<sup>11</sup>Van Liedekerke and Lauwers (1997) refer to this as *Infinite sensitivity*, while Khan and Stinchcombe (2018, Definition 2.2) call it *Pareto responsiveness*.

and anonymity principles. **Weak Pareto** is logically incompatible with **Full anonymity**, and indeed is barely even compatible with **Finite anonymity** (i.e. invariance under *finite* permutations, the weakest possible anonymity axiom); this has been shown in increasing generality by [Diamond \(1965, §5\)](#), [Fleurbaey and Michel \(2003, Theorem 2\)](#), [Zame \(2007\)](#), [Dubey \(2011\)](#), and [Dubey and Mitra \(2011\)](#). In fact, [Petri \(2019\)](#) has recently shown that even **Weak asymptotic Pareto** is barely compatible with **Finite anonymity**. See [Petri \(2019\)](#) for a good review of the relevant literature.

In light of these incompatibility results, the relevant question is not *whether* one should compromise on Pareto or anonymity principles, but rather, *how much* one should compromise on each. Some authors (e.g. [Jonsson and Voorneveld 2018](#)) are willing not only to fall back to **Finite anonymity**, but also to sacrifice completeness in order to satisfy **Strong Pareto**. At the opposite extreme, [Asheim et al. \(2021a,b\)](#) maintain **Full Anonymity** by sacrificing Pareto and ignoring all but the least fortunate members of an infinite population. A third group (e.g. [Van Liedekerke and Lauwers 1997](#), [Khan and Stinchcombe 2018](#)) adopt an intermediate position, combining **Weak asymptotic Pareto** with infinite anonymity principles; this paper follows this third approach.

### 3 Prior literature

[Lauwers \(1998\)](#), [Marinacci \(1998\)](#), [Rébillé \(2007\)](#), [Khan and Stinchcombe \(2018\)](#) and [Jonsson and Voorneveld \(2018\)](#) have studied Cesàro average utility representations or variants. [Lauwers \(1998, Proposition 3\)](#) shows that if a linear function  $F : \ell^\infty(\mathbb{N}, \mathbb{R}) \rightarrow \mathbb{R}$  is  $d_\infty$ -continuous and invariant under Lévy permutations, it is a *medial limit* —i.e. it satisfies<sup>12</sup>

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_t \leq F(\mathbf{x}) \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_t \quad \text{for all } \mathbf{x} \in \mathcal{X}^{\mathbb{N}}.$$

[Marinacci \(1998, Theorem 5\)](#) characterizes weak orders on  $\mathcal{X}^{\mathbb{N}}$  represented by value func-

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<sup>12</sup>Lauwers refers to Lévy permutations as *bounded permutations*. His Proposition 3 follows from Theorem 2 of [Blümlinger and Obata \(1991\)](#). Earlier, [Van Liedekerke and Lauwers \(1997\)](#) had proposed medial limits as suitable SWFs for infinite populations, but without any axiomatic characterization.

tions of the form

$$V_1(\mathbf{x}) := \lim_{T \rightarrow \infty} \left[ \inf_{n \in \mathbb{N}} \left( \frac{1}{T} \sum_{t=n}^{T+n} u(x_t) \right) \right], \quad \text{for all } \mathbf{x} \in \mathcal{X}^{\mathbb{N}}, \quad (8)$$

where  $u : \mathcal{X} \rightarrow \mathbb{R}$  is some utility function. Meanwhile, [Marinacci's](#) Theorem 14 characterizes weak orders on  $\mathcal{X}^{\mathbb{N}}$  represented by a *Polya index* of the form

$$V_2(\mathbf{x}) := \lim_{\epsilon \rightarrow 0} \left( \lim_{T \rightarrow \infty} \frac{1}{\epsilon T} \sum_{t=(1-\epsilon)T}^T u(x_t) \right) \quad \text{for all } \mathbf{x} \in \mathcal{X}^{\mathbb{N}}. \quad (9)$$

These representations extend the Cesàro average utility representation (3) because if  $\text{CesAve}_{n \in \mathbb{N}} u(x_n)$  exists, then  $V_2(\mathbf{x}) = \text{CesAve}_{n \in \mathbb{N}} u(x_n)$ , and if a uniform convergence condition is satisfied, then also  $V_1(\mathbf{x}) = \text{CesAve}_{n \in \mathbb{N}} u(x_n)$ . For both representations, [Marinacci](#) assumes that  $\mathcal{X}$  is the space of all finite-support probability distributions over some underlying set  $\mathcal{C}$  of consequences, and assumes that  $\geq$  satisfies the six axioms of [Gilboa and Schmeidler \(1989\)](#); he then enriches these with additional axioms of time-shift invariance and permutation invariance to achieve his characterizations.

[Rébillé \(2007, Theorems 2 and 3\)](#) characterizes weak orders on  $\ell^\infty(\mathbb{N}, \mathbb{R})$  that can be represented by *Banach limits* and *Banach-Mazur limits*. Like the present paper, his axioms include versions of monotonicity, continuity, separability, and permutation-invariance. He also requires invariance under addition of any element of  $\ell^\infty(\mathbb{N}, \mathbb{R})$ ; this yields a representation in terms of an integral relative to a purely finitely additive measure on  $\mathbb{N}$ .

[Khan and Stinchcombe \(2018\)](#) study intergenerational social welfare orders on  $\ell^\infty(\mathbb{N}, \mathbb{R}_+)$  that satisfy the *first-order distribution overtaking criterion*. That is: for any  $\mathbf{x}, \mathbf{y} \in \ell^\infty(\mathbb{N}, \mathbb{R}_+)$ , we have  $\mathbf{x} \geq \mathbf{y}$  whenever

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T (f(x_t) - f(y_t)) > 0, \quad (10)$$

for all differentiable, increasing functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f'(r) > \epsilon$  for all  $r \in \mathbb{R}$  (for some  $\epsilon > 0$ ). This criterion alone does not select a unique social welfare order; [Khan and Stinchcombe](#) study the family of SWOs satisfying this criteria and other normative axioms.

Jonsson and Voorneveld (2018, Theorem 1) axiomatically characterize the *limit-discounted utilitarian* preorder  $\succeq_{\text{LDU}}$  on  $\ell^\infty(\mathbb{N}, \mathbb{R})$ , which is defined as follows: for any  $\mathbf{x}, \mathbf{y} \in \ell^\infty(\mathbb{N}, \mathbb{R})$ ,

$$\mathbf{x} \succeq_{\text{LDU}} \mathbf{y} \iff \liminf_{\delta \nearrow 1} \sum_{t=1}^{\infty} \delta^t (x_t - y_t) \geq 0. \quad (11)$$

(This preorder is incomplete: it may be the case that neither  $\mathbf{x} \succeq \mathbf{y}$  nor  $\mathbf{y} \succeq \mathbf{x}$ .) Their characterization involves three axioms: Strong Pareto, invariance under addition of a constant, and the *Compensation Principle*, which states that if  $\mathbf{x} = (x_1, x_2, \dots)$  has Cesàro average  $\bar{x}$ , then  $\mathbf{x} \approx (\bar{x}, x_1, x_2, x_3, \dots)$  (i.e. delaying an infinite utility stream by one period can be “compensated” by supplying the Cesàro average utility during the first period).

The relation  $\succeq_{\text{LDU}}$  defined by formula (11) is related to the Cesàro average utility representation (3) because for any  $\mathbf{x} \in \ell^\infty(\mathbb{N}, \mathbb{R})$ , a classical theorem of Frobenius (1880) states that

$$\lim_{\delta \nearrow 1} (1 - \delta) \sum_{t=1}^{\infty} \delta^t x_t = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_t, \quad (12)$$

when these limits exist. Furthermore, the right-hand limit exists if and only if the left-hand limit exists. Thus, if  $\mathbf{x}, \mathbf{y} \in \ell^\infty(\mathbb{N}, \mathbb{R})$  are Cesàro-summable sequences, and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_t > \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T y_t,$$

then it follows from (12) that

$$\lim_{\delta \nearrow 1} (1 - \delta) \sum_{t=1}^{\infty} \delta^t (x_t - y_t) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T (x_t - y_t) > 0, \quad \text{hence} \quad \lim_{\delta \nearrow 1} \sum_{t=1}^{\infty} \delta^t (x_t - y_t) > 0,$$

and hence  $\mathbf{x} \succ_{\text{LDU}} \mathbf{y}$ .<sup>13</sup>

Finally, Blackorby et al. (2005, Theorem 6.14, §6.8), Gravel et al. (2011, Theorem 2) and Kothiyal et al. (2014, Theorem 7) axiomatically characterize average utilitarianism. But these papers work with *finite* populations, so they are not closely related to this paper.

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<sup>13</sup>However, it is possible that  $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T x_t = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T y_t$  while  $\lim_{\delta \nearrow 1} \sum_{t=1}^{\infty} \delta^t (x_t - y_t) < 0$ , so that  $\mathbf{x} \prec_{\text{LDU}} \mathbf{y}$ . For example, this can occur if  $x_t = y_t$  for all but finitely many  $t \in \mathbb{N}$ . In particular, Cesàro average utility does not satisfy Strong Pareto, whereas  $\succeq_{\text{LDU}}$  does.

## A Appendix: Proofs

First we will prove some claims made early in Section 2 about regular and semiregular sequences. Let  $\mathcal{Y}$  be a finite set with the discrete topology. Endow  $\mathcal{Y}^{\mathbb{N}}$  with the product topology and the resulting Borel sigma algebra. Let  $\Delta(\mathcal{Y}^{\mathbb{N}})$  be the set of Borel probability measures on  $\mathcal{Y}^{\mathbb{N}}$ ; an element of  $\Delta(\mathcal{Y}^{\mathbb{N}})$  is called a *stochastic process* on  $\mathcal{Y}$ . If  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  is any function, and  $\mathbf{y} = (y_n)_{n=1}^{\infty} \in \mathcal{Y}^{\mathbb{N}}$ , then we define  $\phi_*(\mathbf{y}) := (y'_n)_{n=1}^{\infty}$ , where  $y'_n := y_{\phi(n)}$  for all  $n \in \mathbb{N}$ . This defines a continuous (hence, Borel-measurable) function  $\phi_* : \mathcal{Y}^{\mathbb{N}} \rightarrow \mathcal{Y}^{\mathbb{N}}$ . A measure  $\mu \in \Delta(\mathcal{Y}^{\mathbb{N}})$  is  *$\phi$ -invariant* if  $\mu[\phi_*^{-1}(\mathcal{B})] = \mu[\mathcal{B}]$  for all Borel subsets  $\mathcal{B} \subseteq \mathcal{Y}^{\mathbb{N}}$ .

In particular, the *shift map*  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  is defined by  $\phi(n) := n + 1$  for all  $n \in \mathbb{N}$ ; a  $\phi$ -invariant measure in  $\Delta(\mathcal{Y}^{\mathbb{N}})$  is called a *stationary stochastic process*. For example, any stationary  $\mathcal{Y}$ -valued Markov chain defines a stationary stochastic process. A *finite permutation* of  $\mathbb{N}$  is a bijection  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  that fixes all but a finite subset of  $\mathbb{N}$ . If  $\mu \in \Delta(\mathcal{Y}^{\mathbb{N}})$  is invariant under all finite permutations, then it is called an *exchangeable* stochastic process. For example, if  $\rho \in \Delta(\mathcal{Y})$  and  $\mu = \bigotimes_{n=1}^{\infty} \rho$  (describing an infinite sequence of independent  $\rho$ -random variables), then  $\mu$  is exchangeable. Any exchangeable process is stationary. The next result says that if  $\mathbf{y} = (y_0, y_1, y_2, \dots)$  is a random sequence generated by any stationary stochastic process on  $\mathcal{Y}$ , then  $\mathbf{y}$  is regular, almost-surely.

**Lemma A.1** *If  $\mathcal{Y}$  is a finite set, and  $\mu \in \Delta(\mathcal{Y}^{\mathbb{N}})$  is a stationary stochastic process, then  $\mu[\text{Reg}(\mathcal{Y})] = 1$ .*

*Proof:* Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be the shift map. Then  $(\mathcal{Y}^{\mathbb{N}}, \mu, \phi_*)$  is a measure-preserving dynamical system. For any  $\mathcal{O} \subseteq \mathcal{Y}$ , define  $f_{\mathcal{O}} : \mathcal{Y}^{\mathbb{N}} \rightarrow \{0, 1\}$  by setting  $f_{\mathcal{O}}(\mathbf{y}) := 1$  if  $y_1 \in \mathcal{O}$ , and  $f_{\mathcal{O}}(\mathbf{y}) := 0$  if  $y_1 \notin \mathcal{O}$ . Then for any  $\mathbf{y} \in \mathcal{Y}^{\mathbb{N}}$ , the expression (1) becomes

$$\lim_{N \rightarrow \infty} \frac{\#\{n \in \{1, \dots, N\}; y_n \in \mathcal{O}\}}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_{\mathcal{O}} \circ \phi_*^n(\mathbf{y}). \quad (\text{A1})$$

The right hand side of (A1) is the *ergodic average* of  $f_{\mathcal{O}}$  with respect to  $\phi_*$ . Let  $\mathcal{R}_{\mathcal{O}} := \{\mathbf{y} \in \mathcal{Y}^{\mathbb{N}}; \text{the limit (A1) exists}\}$ . Then  $\mu[\mathcal{R}_{\mathcal{O}}] = 1$ , by the Birkhoff Ergodic Theorem; see e.g. Theorem 2.3 on p.30 of Petersen (1989) or Theorem 372J on p.190 of Fremlin

(2002). By definition,  $\text{Reg}(\mathcal{Y}) := \bigcap_{\mathcal{O} \in \mathfrak{D}(\mathcal{Y})} \mathcal{R}_{\mathcal{O}}$ . Thus,  $\text{Reg}(\mathcal{Y})$  is the intersection of a finite number of subsets of measure 1 (because  $\mathcal{Y}$  is finite), so  $\mu[\text{Reg}(\mathcal{Y})] = 1$ .  $\square$

In the special case when  $\mu$  is a product measure (describing an infinite sequence of i.i.d. random variables), the almost-sure convergence of (A1) follows from the Law of Large Numbers. In the special case when  $\mu$  is an exchangeable process, it follows from de Finetti's Theorem (Fremlin, 2006a, Theorem 459C, p.564). The finiteness of  $\mathcal{Y}$  is not required to invoke these results: for any *particular*  $\mathcal{O} \in \mathfrak{D}(\mathcal{Y})$ , we have  $\mu[\mathcal{R}_{\mathcal{O}}] = 1$ . But  $\text{Reg}(\mathcal{Y}) = \bigcap_{\mathcal{O} \in \mathfrak{D}(\mathcal{Y})} \mathcal{R}_{\mathcal{O}}$ , and if  $\mathcal{Y}$  was infinite, then  $\mathfrak{D}(\mathcal{Y})$  would be uncountably infinite, so this intersection would not generally have positive measure.

**Lemma A.2** *Let  $x_* \in \mathcal{X}$ , and let  $\mathbf{x} = (x_n)_{n=1}^{\infty}$  be a sequence in  $\mathcal{X}$  converging to  $x_*$ .*

(a) *Suppose that  $x_n \neq x_m$  whenever  $n \neq m$ . Then  $\mathbf{x}$  is not regular.*

(b) *If  $\mathcal{X}$  is locally compact, then  $\mathbf{x}$  is semiregular.*

*Proof:* (a) Let  $\mathcal{M} \subseteq \mathbb{N}$  be arbitrary. The set  $\mathcal{O} := \{x_*\} \cup \{x_m\}_{m \in \mathcal{M}}$  is closed, hence an element of  $\mathfrak{D}(\mathcal{X})$ . For this set, the limit (1) is  $\delta(\mathcal{M})$ , which does not exist unless  $\mathcal{M} \in \mathfrak{B}$ . Thus,  $\mathbf{x}$  is not regular.

(b) Let  $\mathcal{K}_0$  be a compact neighbourhood of  $x_*$ . Find  $M \in \mathbb{N}$  such that  $x_m \in \mathcal{K}_0$  for all  $m \geq M$ . Let  $\mathcal{K} := \mathcal{K}_0 \cup \{x_1, \dots, x_M\}$ ; then  $\mathcal{K}$  is compact. For all  $j \in \mathbb{N}$ , let  $\mathbf{x}^j := (x_1, \dots, x_j, x_*, x_*, \dots)$ . Then  $\mathbf{x}^j \in \text{Reg}(\mathcal{K})$  (because  $\mathbf{x}^j$  takes only a finite number of distinct values, all in  $\mathcal{K}$ ), and  $\lim_{j \rightarrow \infty} d_{\infty}(\mathbf{x}^j, \mathbf{x}) = 0$  (because  $\lim_{j \rightarrow \infty} x_n = x_*$ ); thus  $\mathbf{x} \in \overline{\text{Reg}(\mathcal{X})}$ .  $\square$

*Proof of the Proposition.* It is easily deduced from Axiom A2 that  $\geq_*$  is a continuous order on  $\mathcal{X}$ —in other words, all upper and lower contour sets of  $\geq_*$  are closed subsets of  $\mathcal{X}$ . From this, it follows by standard arguments that any compact subset of  $\mathcal{X}$  contains at least one  $\geq_*$ -maximal element and at least one  $\geq_*$ -minimal element.

Now let  $\mathcal{Y} \subseteq \mathcal{X}$  be totally bounded. Let  $\overline{\mathcal{Y}}$  be its closure. Then  $\overline{\mathcal{Y}}$  is complete because  $\mathcal{X}$  is complete (Willard, 2004, Thm 24.10), and is totally bounded because  $\mathcal{Y}$  is totally bounded. Thus,  $\overline{\mathcal{Y}}$  is compact (Willard, 2004, Thm 39.9). Thus, by the remarks in the previous paragraph, there exist some  $x, z \in \overline{\mathcal{Y}}$  such that  $x \leq_* y \leq_* z$  for all  $y \in \overline{\mathcal{Y}}$ .  $\square$

The proof of the Theorem requires some machinery. For any subset  $\mathcal{B} \subseteq \mathbb{N}$ , recall that  $\delta(\mathcal{B})$  is defined by the limit (5), if this limit exists, and  $\mathfrak{B}$  is the collection of all subsets  $\mathcal{B} \subseteq \mathbb{N}$  for which  $\delta(\mathcal{B})$  is well-defined. This collection is closed under complementation (i.e.  $\mathcal{B} \in \mathfrak{B} \iff \mathcal{B}^c \in \mathfrak{B}$ ) and under finite disjoint unions (i.e. if  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$  and  $\mathcal{A} \cap \mathcal{B} = \emptyset$ , then  $\mathcal{A} \sqcup \mathcal{B} \in \mathfrak{B}$ ). But it is not a Boolean algebra, or even a  $\lambda$ -system.<sup>14</sup> However,  $\delta$  is “finitely additive”: for all disjoint  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$ , we have  $\delta(\mathcal{A} \sqcup \mathcal{B}) = \delta(\mathcal{A}) + \delta(\mathcal{B})$ . Note that a sequence  $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$  is regular in the sense of formula (1) if and only if  $\mathbf{x}$  is “measurable” with respect to  $\mathfrak{B}$ , in the sense that for any  $\mathcal{O} \in \mathfrak{D}(\mathcal{X})$ , the set  $\{n \in \mathbb{N}; x_n \in \mathcal{O}\}$  is in  $\mathfrak{B}$ .

A *partition* of  $\mathbb{N}$  is a finite collection of disjoint sets  $\mathcal{P}_1, \dots, \mathcal{P}_J \subseteq \mathbb{N}$  (called *cells*) such that  $\mathbb{N} = \mathcal{P}_1 \sqcup \dots \sqcup \mathcal{P}_J$ . If  $\mathcal{P}_1, \dots, \mathcal{P}_J \in \mathfrak{B}$ , then let us call this a  $\mathfrak{B}$ -*partition*. Let  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  and  $\mathfrak{Q} = \{\mathcal{Q}_1, \dots, \mathcal{Q}_K\}$  be two partitions. Say that  $\mathfrak{P}$  is a *refinement* of  $\mathfrak{Q}$  if, for all  $j \in \{1, \dots, J\}$  there exists  $k \in \{1, \dots, K\}$  such that  $\mathcal{P}_j \subseteq \mathcal{Q}_k$ . It follows that every cell of  $\mathfrak{Q}$  is a disjoint union of cells of  $\mathfrak{P}$ . Thus,

$$\text{If } \mathfrak{P} \text{ is a } \mathfrak{B}\text{-partition, and } \mathfrak{P} \text{ refines } \mathfrak{Q}, \text{ then } \mathfrak{Q} \text{ is also a } \mathfrak{B}\text{-partition.} \quad (\text{A2})$$

Any two partitions have a common refinement. But the common refinement of two  $\mathfrak{B}$ -partitions might not be a  $\mathfrak{B}$ -partition.

An element  $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$  is *simple* if it only takes finitely many values—in other words, there exist  $y_1, \dots, y_N \in \mathcal{X}$  and a partition  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  such that  $x_p = y_j$  for all  $p \in \mathcal{P}_j$ , for all  $j \in \{1, \dots, J\}$ . In this case, say  $\mathbf{x}$  is *subordinate* to  $\mathfrak{P}$ . If  $\mathbf{x}$  is simple, then  $\mathbf{x} \in \ell^\infty(\mathbb{N}, \mathcal{X})$ . If  $\mathfrak{P}$  is a  $\mathfrak{B}$ -partition, then  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ . Let  $\text{Smp}(\mathcal{X})$  be the set of simple elements in  $\ell^\infty(\mathbb{N}, \mathcal{X})$ . Let  $\text{SmpReg}(\mathcal{X})$  be the set of simple elements in  $\text{Reg}(\mathcal{X})$ .

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<sup>14</sup>A  $\lambda$ -*system* (or *Dynkin system*) is a collection of subsets that is closed under complementation and under *countable* disjoint unions (Aliprantis and Border, 2006, §4.3). But  $\mathbb{N}$  itself is countable, so the only  $\lambda$ -system on  $\mathbb{N}$  containing all singleton sets is the entire power set  $\wp(\mathbb{N})$ .



**Lemma A.3**  $\text{Smp}(\mathcal{X})$  is  $d_\infty$ -dense in  $\ell^\infty(\mathbb{N}, \mathcal{X})$ , and  $\text{SmpReg}(\mathcal{X})$  is  $d_\infty$ -dense in  $\overline{\text{Reg}}(\mathcal{X})$ .

*Proof:* Let  $\mathbf{x} \in \ell^\infty(\mathbb{N}, \mathcal{X})$ . Then the set  $\{x_n\}_{n \in \mathbb{N}}$  is totally bounded. So for any  $\epsilon > 0$ , there exists a finite subset  $\{y_1, \dots, y_J\} \subset \mathcal{X}$  such that  $\{x_n\}_{n \in \mathbb{N}} \subset \bigcup_{j=1}^J \mathcal{B}(y_j, \epsilon)$ . For all  $j \in \{1, \dots, J\}$ , let  $\mathcal{M}_j := \{n \in \mathbb{N}; d(x_n, y_j) < \epsilon\}$ . Then  $\mathbb{N} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_J$ . But these sets are not necessarily disjoint. Define  $\mathcal{A}_1 := \mathcal{M}_1$ ,  $\mathcal{A}_2 := \mathcal{M}_2 \setminus \mathcal{M}_1$ , and inductively,  $\mathcal{A}_j := \mathcal{M}_j \setminus (\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{j-1})$  for all  $j \in \{3, \dots, J\}$ . Then  $\mathcal{A}_1, \dots, \mathcal{A}_J$  are disjoint and  $\mathbb{N} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_J$ . Now define  $\mathbf{z} \in \text{Smp}(\mathcal{X})$  by setting  $z_a := y_j$  for all  $a \in \mathcal{A}_j$ , for all  $j \in \{1, \dots, J\}$ . Clearly,  $\mathbf{z}$  is simple, and  $d_\infty(\mathbf{z}, \mathbf{x}) < \epsilon$  by construction. This works for any  $\epsilon > 0$ . This shows that  $\text{Smp}(\mathcal{X})$  is  $d_\infty$ -dense in  $\ell^\infty(\mathbb{N}, \mathcal{X})$ .

$\text{Reg}(\mathcal{X})$  is  $d_\infty$ -dense in  $\overline{\text{Reg}}(\mathcal{X})$ , so to prove that  $\text{SmpReg}(\mathcal{X})$  is  $d_\infty$ -dense in  $\overline{\text{Reg}}(\mathcal{X})$ , it suffices to show that  $\text{SmpReg}(\mathcal{X})$  is  $d_\infty$ -dense in  $\text{Reg}(\mathcal{X})$ . So, let  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ . For any  $\epsilon > 0$  and choice of  $\{y_1, \dots, y_J\} \subset \mathcal{X}$ , if one defines  $\mathcal{A}_1, \dots, \mathcal{A}_J$  as in the previous paragraph, then  $\mathcal{A}_1, \dots, \mathcal{A}_J \in \mathfrak{B}$ . To see this, note that  $\mathcal{A}_j := \{n \in \mathbb{N}; x_n \in \mathcal{Y}_j\}$  where  $\mathcal{Y}_j := \mathcal{B}(y_j, \epsilon) \setminus \bigcup_{k=1}^{j-1} \mathcal{B}(y_k, \epsilon)$  is the difference of two open sets. Thus,  $\mathcal{Y}_j \in \mathfrak{D}(\mathcal{X})$ , so  $\mathcal{A}_j \in \mathfrak{B}$  because  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ . Thus,  $\mathbf{z} \in \text{SmpReg}(\mathcal{X})$ . This shows that  $\text{SmpReg}(\mathcal{X})$  is  $d_\infty$ -dense in  $\text{Reg}(\mathcal{X})$ .  $\square$

**Lemma A.4** Let  $\mathbf{x} \in \text{SmpReg}(\mathcal{X})$ , and let  $y \in \mathcal{X}$ . If  $\mathbf{z} \in \mathcal{X}^\mathbb{N}$  is defined by setting  $z_n := \max_{\geq *} \{x_n, y\}$  for all  $n \in \mathbb{N}$ , then  $\mathbf{z} \in \text{SmpReg}(\mathcal{X})$ . Likewise, if  $\mathbf{w} \in \mathcal{X}^\mathbb{N}$  is defined by setting  $w_n := \min_{\geq *} \{x_n, y\}$  for all  $n \in \mathbb{N}$ , then  $\mathbf{w} \in \text{SmpReg}(\mathcal{X})$ .

*Proof:* Suppose  $\mathbf{x}$  is subordinate to the  $\mathfrak{B}$ -partition  $\{\mathcal{B}_1, \dots, \mathcal{B}_J\}$ . It is easily verified that  $\mathbf{z}$  and  $\mathbf{w}$  are simple, and are subordinate to partitions that are refined by  $\{\mathcal{B}_1, \dots, \mathcal{B}_J\}$  (that is: the cells of these partitions are unions of cells from  $\{\mathcal{B}_1, \dots, \mathcal{B}_J\}$ ). Thus, these partitions are also  $\mathfrak{B}$ -partitions, by statement (A2).  $\square$

The next lemma provides some useful properties about null subsets of  $\mathbb{N}$ .

**Lemma A.5**

(a) Let  $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$  be disjoint. Then ( $\mathcal{A}$  and  $\mathcal{B}$  are both null)  $\Leftrightarrow$  ( $\mathcal{A} \sqcup \mathcal{B}$  is null).

(b) Suppose  $\geq$  satisfies A4,  $\mathcal{A} \subset \mathbb{N}$  and  $\gamma \in \Gamma$ . Then ( $\mathcal{A}$  is null)  $\Leftrightarrow$  ( $\gamma(\mathcal{A})$  is null).

*Proof:* (a) “ $\Leftarrow$ ” It follows from the definition that any subset of a null set is null.

“ $\Rightarrow$ ” Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are null. Let  $\mathbf{x}, \mathbf{z} \in \text{Reg}(\mathcal{X})$ , and suppose  $\mathbf{x}_{\mathbb{N} \setminus \mathcal{A} \cup \mathcal{B}} = \mathbf{z}_{\mathbb{N} \setminus \mathcal{A} \cup \mathcal{B}}$ . Then there exists  $\mathbf{y} \in \text{Reg}(\mathcal{X})$  such that  $\mathbf{x}_{\mathbb{N} \setminus \mathcal{A}} = \mathbf{y}_{\mathbb{N} \setminus \mathcal{A}}$  and  $\mathbf{y}_{\mathbb{N} \setminus \mathcal{B}} = \mathbf{z}_{\mathbb{N} \setminus \mathcal{B}}$ . Thus,  $\mathbf{x} \approx \mathbf{y} \approx \mathbf{z}$ , because  $\mathcal{A}$  and  $\mathcal{B}$  are null. Thus,  $\mathbf{x} \approx \mathbf{z}$ , by transitivity. Thus,  $\mathcal{A} \sqcup \mathcal{B}$  is null.

(b) “ $\Rightarrow$ ” Suppose  $\mathcal{A}$  is null. Let  $\mathbf{x}, \mathbf{y} \in \text{Reg}(\mathcal{X})$ , and suppose  $\mathbf{x}_{\mathbb{N} \setminus \gamma(\mathcal{A})} = \mathbf{y}_{\mathbb{N} \setminus \gamma(\mathcal{A})}$ . We must show that  $\mathbf{x} \approx \mathbf{y}$ . Let  $\mathbf{x}' := \gamma(\mathbf{x})$  and  $\mathbf{y}' := \gamma(\mathbf{y})$ . Then  $\mathbf{x}'_{\mathbb{N} \setminus \mathcal{A}} = \mathbf{y}'_{\mathbb{N} \setminus \mathcal{A}}$ . Thus,  $\mathbf{x}' \approx \mathbf{y}'$ , because  $\mathcal{A}$  is null. But Axiom A4 says  $\mathbf{x} \approx \mathbf{x}'$  and  $\mathbf{y}' \approx \mathbf{y}$ . Thus,  $\mathbf{x} \approx \mathbf{y}$ . This holds for any  $\mathbf{x}, \mathbf{y} \in \text{Reg}(\mathcal{X})$  with  $\mathbf{x}_{\mathbb{N} \setminus \gamma(\mathcal{A})} = \mathbf{y}_{\mathbb{N} \setminus \gamma(\mathcal{A})}$ . Thus,  $\gamma(\mathcal{A})$  is null.

“ $\Leftarrow$ ”  $\gamma^{-1} \in \Gamma$ . So exchange the roles of  $\mathcal{A}$  and  $\gamma(\mathcal{A})$  and apply the proof of “ $\Rightarrow$ ”.  $\square$

Here are three results by [Obata \(1988a\)](#) that we will use repeatedly.

**Lemma A.6** (a) For any  $\mathcal{B} \in \mathfrak{B}$  with  $\delta(\mathcal{B}) = B > 0$  and any  $A \in (0, B)$ , there exists  $\mathcal{A} \in \mathfrak{B}$  with  $\mathcal{A} \subseteq \mathcal{B}$  and  $\delta(\mathcal{A}) = A$ .

(b) For any  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$  with  $0 < \delta(\mathcal{A}) = \delta(\mathcal{B}) < 1$ , there exists  $\gamma \in \Gamma$  such that  $\gamma(\mathcal{A}) = \mathcal{B}$ .

(c) Let  $\mu : \mathfrak{B} \rightarrow \mathbb{R}_+$  be a nonzero function that is finitely additive (i.e.  $\mu[\mathcal{A} \sqcup \mathcal{B}] = \mu[\mathcal{A}] + \mu[\mathcal{B}]$  for any disjoint  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$ ) and  $\Gamma$ -invariant (i.e.  $\mu[\gamma^{-1}(\mathcal{B})] = \mu[\mathcal{B}]$  for any  $\mathcal{B} \in \mathfrak{B}$  and  $\gamma \in \Gamma$ ). Then there is some  $r > 0$  such that  $\mu = r \delta$ .

*Proof:* See Proposition 1.3, Proposition 2.5 and Theorem 2 of [Obata \(1988a\)](#).<sup>15</sup>  $\square$

<sup>15</sup>Part (c) also follows indirectly from Theorem 2 of [Blümlinger and Obata \(1991\)](#), using the equivalence between finitely additive measures on  $\mathbb{N}$  and linear functionals on  $\ell^\infty$ . For other very similar results, see Theorem 1.12 of [van Douwen \(1992\)](#), Proposition 3 of [Lauwers \(1998\)](#), and Proposition 2.4 of [Sleziak and Ziman \(2008\)](#). Importantly, Obata’s Theorem 2 only requires  $\mu$  to be defined on  $\mathfrak{B}$  —in other words, it does not need to be a  $\Gamma$ -invariant finitely additive measure defined on a Boolean algebra.

**Lemma A.7** Assume [A1](#) and [A4](#). Let  $\mathcal{B} \in \mathfrak{B}$ . Then  $(\delta(\mathcal{B}) = 0) \Leftrightarrow (\mathcal{B} \text{ is null for } \geq)$ .

*Proof:* “ $\implies$ ” Suppose we enumerate  $\mathcal{B}$  by writing  $\mathcal{B} = \{b_0 < b_1 < b_2 < b_3 < \dots\}$ . For each  $r \in \{0, 1, 2\}$ , define  $\mathcal{B}_r := \{b_n; n \equiv r \pmod{3}\}$ . Thus,  $\mathcal{B} = \mathcal{B}_0 \sqcup \mathcal{B}_1 \sqcup \mathcal{B}_2$ . By Lemma [A.5\(a\)](#), to show that  $\mathcal{B}$  is null for  $\geq$ , it suffices to show that  $\mathcal{B}_0$ ,  $\mathcal{B}_1$ , and  $\mathcal{B}_2$  are all null.

First we show that  $\mathcal{B}_1$  is null. For this, we will construct sequences  $\mathbf{x}, \mathbf{y} \in \text{Reg}(\mathcal{X})$  that are equal everywhere outside of  $\mathcal{B}_1$ , and take different constant values  $a$  and  $b$  inside  $\mathcal{B}_1$ , such that  $a >_* b$  but  $\mathbf{x} \approx \mathbf{y}$ ; this forces  $\mathcal{B}_1$  to be null, on pain of contradicting Axiom [A1](#).

For this task, we first define a permutation  $\sigma : \mathcal{B} \rightarrow \mathcal{B}$  such that  $\sigma(\mathcal{B}_0) = \mathcal{B}_0 \sqcup \mathcal{B}_1$ , while  $\sigma(\mathcal{B}_1 \sqcup \mathcal{B}_2) = \mathcal{B}_2$ . For all  $r \in \{0, 1, 2\}$  and  $q \in \mathbb{N}$  let  $b_q^{(r)} := b_{3q+r}$ . Thus,  $\mathcal{B}_r = \{b_0^{(r)} < b_1^{(r)} < b_2^{(r)} < b_3^{(r)} < \dots\}$  for each  $r \in \{0, 1, 2\}$ . Now define  $\sigma_0 : \mathcal{B}_0 \rightarrow \mathcal{B}_0 \sqcup \mathcal{B}_1$  by

$$\sigma_0(b_q^{(0)}) := \begin{cases} b_{q/2}^{(0)} & \text{if } q \text{ is even;} \\ b_{(q-1)/2}^{(1)} & \text{if } q \text{ is odd.} \end{cases}$$

Clearly,  $\sigma_0$  is a bijection from  $\mathcal{B}_0$  to  $\mathcal{B}_0 \sqcup \mathcal{B}_1$ . Meanwhile, define  $\sigma_1 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \rightarrow \mathcal{B}_2$  by

$$\sigma_1(b_q^{(r)}) := \begin{cases} b_{2q}^{(2)} & \text{if } r = 2; \\ b_{2q+1}^{(2)} & \text{if } r = 1. \end{cases}$$

Clearly,  $\sigma_1$  is a bijection from  $\mathcal{B}_1 \sqcup \mathcal{B}_2$  to  $\mathcal{B}_2$ . Finally, define  $\sigma = \sigma_0 \sqcup \sigma_1$ ; then  $\sigma : \mathcal{B} \rightarrow \mathcal{B}$  is a permutation.

Now define  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  by setting  $\gamma(b) := \sigma(b)$  for all  $b \in \mathcal{B}$ , while  $\gamma(n) := n$  for all  $n \in \mathcal{B}^c$ . Then  $\gamma \in \Gamma$  (because  $\delta(\mathcal{B}) = 0$ , so equation [\(6\)](#) is clearly satisfied). Thus, Axiom [A4](#) says  $\geq$  is invariant under  $\gamma$ .

Now let  $a, b, c \in \mathcal{X}$  with  $a >_* b$ . (These exist because  $\geq$  is nondegenerate.) Define  $\mathbf{x} \in \ell^\infty(\mathbb{N}, \mathcal{X})$  as follows: for all  $k \in \mathbb{N}$ ,

$$x_k := \begin{cases} a & \text{if } k \in \mathcal{B}_0 \sqcup \mathcal{B}_1; \\ b & \text{if } k \in \mathcal{B}_2; \\ c & \text{if } k \in \mathcal{B}^c. \end{cases}$$

Note that  $\mathbf{x} \in \text{Reg}(\mathcal{X})$ . (To see this, note that  $\mathcal{B}^c \in \mathfrak{B}$  because  $\delta(\mathcal{B}^c) = 1$ , while  $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}$ , because any subset of  $\mathcal{B}$  has density 0, hence is an element of  $\mathfrak{B}$ .) Let  $\mathbf{y} = \gamma(\mathbf{x})$ . Then  $\mathbf{y} \approx \mathbf{x}$  by Axiom A4, because  $\gamma \in \Gamma$ . It is easily verified that

$$y_k := \begin{cases} a & \text{if } k \in \mathcal{B}_0; \\ b & \text{if } k \in \mathcal{B}_1 \sqcup \mathcal{B}_2; \\ c & \text{if } k \in \mathcal{B}^c. \end{cases}$$

In other words,  $\mathbf{x}_{\mathcal{B}_1^c} = \mathbf{y}_{\mathcal{B}_1^c}$ , while  $x_k = a >_* b = y_k$  for all  $k \in \mathcal{B}_1$ . If  $\mathcal{B}_1$  was non-null for  $\geq$ , this would contradict Axiom A1. Thus,  $\mathcal{B}_1$  must be null. By a similar construction (exchanging the roles of  $\mathcal{B}_0$ ,  $\mathcal{B}_1$ , and  $\mathcal{B}_2$ ), one can show that  $\mathcal{B}_0$  and  $\mathcal{B}_2$  are also null. Thus,  $\mathcal{B}$  itself is null, by Lemma A.5(a).

“ $\Leftarrow$ ” We will prove the contrapositive: if  $\delta(\mathcal{B}) > 0$ , then  $\mathcal{B}$  is *not* null for  $\geq$ . Find  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \delta(\mathcal{B})$ . Lemma A.6(a) yields some  $\mathcal{A}_1 \in \mathfrak{B}$  with  $\mathcal{A}_1 \subseteq \mathcal{B}$  such that  $\delta(\mathcal{A}_1) = \frac{1}{N}$ . It suffices to prove that  $\mathcal{A}_1$  is non-null (because any superset of a non-null set is non-null, by the contrapositive of Lemma A.5(a)). Lemma A.6(a) yields  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$  with  $\delta(\mathcal{A}_2) = \frac{1}{N}$ , and then  $\mathcal{A}_3 \subseteq (\mathcal{A}_1 \sqcup \mathcal{A}_2)^c$  with  $\delta(\mathcal{A}_3) = \frac{1}{N}$ , and so on. Inductively, we obtain disjoint  $\mathcal{A}_1, \dots, \mathcal{A}_{N-1} \in \mathfrak{B}$  with  $\delta(\mathcal{A}_n) = \frac{1}{N}$  for all  $n \in \{1, \dots, N-1\}$ . Let  $\mathcal{A}_N := (\mathcal{A}_1 \sqcup \dots \sqcup \mathcal{A}_{N-1})^c$ . Then  $\mathcal{A}_N \in \mathfrak{B}$  and  $\delta(\mathcal{A}_N) = 1 - \delta(\mathcal{A}_1 \sqcup \dots \sqcup \mathcal{A}_{N-1}) = 1 - \frac{N-1}{N} = \frac{1}{N}$ .

Now suppose  $\mathcal{A}_1$  is null. For all  $n \in \{2, \dots, N\}$ , Lemma A.6(b) yields  $\gamma \in \Gamma$  such that  $\gamma(\mathcal{A}_1) = \mathcal{A}_n$ . Thus,  $\mathcal{A}_2, \dots, \mathcal{A}_N$  are also null, by Lemma A.5(b). Thus,  $\mathcal{A}_1 \sqcup \dots \sqcup \mathcal{A}_N$  is null, by Lemma A.5(a). But  $\mathcal{A}_1 \sqcup \dots \sqcup \mathcal{A}_N = \mathbb{N}$ , so this contradicts the assumed nontriviality of  $\geq$ . Thus,  $\mathcal{A}_1$  cannot be null.  $\square$

**Lemma A.8** *Suppose  $\geq$  satisfies A1 and A4. For any  $\mathcal{B} \in \mathfrak{B}$ , if  $\mathcal{B}^c$  is not null, then there is a  $\mathfrak{B}$ -partition  $\mathfrak{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$  of  $\mathbb{N}$  such that  $\mathcal{P}_1 = \mathcal{B}$  while  $\mathcal{P}_2$  and  $\mathcal{P}_3$  are both non-null.*

*Proof:*  $\mathcal{B}^c \in \mathfrak{B}$  because  $\mathcal{B} \in \mathfrak{B}$ . Also,  $\delta(\mathcal{B}^c) > 0$  by the contrapositive of Lemma A.7, because  $\mathcal{B}^c$  is not null. Lemma A.6(a) yields  $\mathcal{P}_2 \in \mathfrak{B}$  with  $\mathcal{P}_2 \subset \mathcal{B}^c$  and  $\delta(\mathcal{P}_2) = \delta(\mathcal{B}^c)/2$ .

Now,  $\mathcal{P}_2$  and  $\mathcal{B}$  are disjoint, so  $\mathcal{P}_2 \sqcup \mathcal{B} \in \mathfrak{B}$ . Thus, if  $\mathcal{P}_3 := (\mathcal{P}_2 \sqcup \mathcal{B})^c$ , then  $\mathcal{P}_3 \in \mathfrak{B}$ . Furthermore,  $\mathbb{N} = \mathcal{B} \sqcup \mathcal{P}_2 \sqcup \mathcal{P}_3$ , so  $1 = \delta(\mathbb{N}) = \delta(\mathcal{B}) + \delta(\mathcal{P}_2) + \delta(\mathcal{P}_3)$  so

$$\delta(\mathcal{P}_3) = 1 - \delta(\mathcal{B}) - \delta(\mathcal{P}_2) = \delta(\mathcal{B}^c) - \delta(\mathcal{B}^c)/2 = \delta(\mathcal{B}^c)/2 = \delta(\mathcal{P}_2).$$

Thus, Lemma A.6(b) yields some  $\gamma \in \Gamma$  such that  $\gamma(\mathcal{P}_2) = \mathcal{P}_3$ . Thus, Axiom A4 implies that  $\mathcal{P}_2$  is non-null if and only if  $\mathcal{P}_3$  is non-null. If  $\mathcal{P}_2$  and  $\mathcal{P}_3$  are both null, then  $\mathcal{P}_2 \sqcup \mathcal{P}_3 = \mathcal{B}^c$  is null, contradicting our hypothesis. Thus, at least one of them must be non-null, which means they are both non-null.  $\square$

The next lemma is a standard result about the uniqueness of additive utility representations. We include a proof for completeness.

**Lemma A.9** *Let  $\mathcal{X}$  be a connected topological space with a continuous order  $\geq_*$ . Let  $\geq$  be a continuous weak order on  $\mathcal{X} \times \mathcal{X}$ . Let  $u_1, u_2, v_1, v_2 : \mathcal{X} \rightarrow \mathbb{R}$  be continuous functions, all strictly increasing in  $\geq_*$ , such that the functions  $U(x_1, x_2) = u_1(x_1) + u_2(x_2)$  and  $V(x_1, x_2) = v_1(x_1) + v_2(x_2)$  are both representations for  $\geq$ . Then there is a constant  $a > 0$  and constants  $b_1, b_2 \in \mathbb{R}$  such that  $v_j = a u_j + b_j$  for  $j = 1, 2$ .*

*Proof:* Let  $\mathcal{U}_1 := u_1(\mathcal{X})$ ,  $\mathcal{U}_2 := u_2(\mathcal{X})$ ,  $\mathcal{V}_1 := v_1(\mathcal{X})$ , and  $\mathcal{V}_2 := v_2(\mathcal{X})$ . These are intervals in  $\mathbb{R}$  because  $u_1, u_2, v_1$ , and  $v_2$  are continuous and  $\mathcal{X}$  is connected. The functions  $u_1, u_2, v_1$ , and  $v_2$  are all strictly increasing in  $\geq_*$ ; thus, they are increasing transformations of one another. In particular, there are increasing bijections  $\phi_1 : \mathcal{U}_1 \rightarrow \mathcal{V}_1$  and  $\phi_2 : \mathcal{U}_2 \rightarrow \mathcal{V}_2$  such that  $v_1 = \phi_1 \circ u_1$  and  $v_2 = \phi_2 \circ u_2$ .

Let  $\mathcal{U} := U(\mathcal{X} \times \mathcal{X})$  and  $\mathcal{V} := V(\mathcal{X} \times \mathcal{X})$ . Again, these are intervals in  $\mathbb{R}$  because  $U$  and  $V$  are continuous and  $\mathcal{X} \times \mathcal{X}$  is connected. The functions  $U$  and  $V$  both represent  $\geq$ , so there is an increasing bijection  $\Phi : \mathcal{U} \rightarrow \mathcal{V}$  such that  $V = \Phi \circ U$ . In other words, for any  $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ , we have:

$$\begin{aligned} \Phi \left( u_1(x_1) + u_2(x_2) \right) &= \Phi \circ U(x_1, x_2) = V(x_1, x_2) \\ &= v_1(x_1) + v_2(x_2) = \phi_1 \circ u_1(x_1) + \phi_2 \circ u_2(x_2). \end{aligned} \quad (\text{A3})$$

Letting  $(x_1, x_2)$  range over all of  $\mathcal{X} \times \mathcal{X}$  in equation (A3), we get:

$$\Phi(r_1 + r_2) = \phi_1(r_1) + \phi_2(r_2), \quad \text{for all } (r_1, r_2) \in \mathcal{U}_1 \times \mathcal{U}_2. \quad (\text{A4})$$

This is a Pexider equation. Since  $\phi_1, \phi_2$  and  $\Phi$  are increasing bijections between intervals, they are continuous functions. Thus, there exist constants  $a > 0, b_1, b_2 \in \mathbb{R}$  and  $b = b_1 + b_2$  such that  $\phi_1(r_1) = ar_1 + b_1$  for all  $r_1 \in \mathcal{U}_1$ ,  $\phi_2(r_2) = ar_2 + b_2$  for all  $r_2 \in \mathcal{U}_2$ , and  $\Phi(r) = ar + b$ , for all  $r \in \mathcal{U}$  (Radó and Baker, 1987, Theorem 1 and Corollary 3).<sup>16</sup> In other words,  $v_1 = au_1 + b_1$  and  $v_2 = au_2 + b_2$ , as claimed.  $\square$

The proof of the main result adapts ideas from the proof of Theorem 16 of Wakker and Zank (1999) (hereafter referred to as WZ) to construct an “additive representation” for  $\geq$ . To do this, we will follow some steps in the proof of WZ Theorem 11 —the special case of WZ Theorem 16 when  $\mathcal{X} = \mathbb{R}$ . We will show that these steps still work when  $\mathcal{X}$  is a connected metric space and Axioms A1 - A4 are satisfied.

One complication is that WZ start with a weak order defined on the set  $\mathcal{F}$  of all acts that are measurable with respect to a Boolean algebra  $\mathfrak{A}$  of subsets of the state space. As we have already noted,  $\mathfrak{B}$  is *not* a Boolean algebra. And  $\text{Reg}(\mathcal{X})$  does not include all functions on  $\mathbb{N}$  that are “measurable” relative to  $\mathfrak{B}$ . However, the proof of WZ does not actually need  $\mathcal{F}$  to contain *all* measurable functions (it suffices that  $\mathcal{F}$  contain all simple functions). Also, the proof does not need all the properties of a Boolean algebra; what properties it does use, we can verify on an *ad hoc* basis for  $\mathfrak{B}$ .

For any  $\mathcal{B} \in \mathfrak{B}$ , let  $\text{SmpReg}(\mathcal{B}, \mathcal{X})$  denote the set of all simple regular sequences in  $\ell^\infty(\mathcal{B}, \mathcal{X})$ . Following the terminology of WZ, an *additive representation* for  $\geq$  on  $\text{SmpReg}(\mathcal{X})$  is a set of functions  $\{V_{\mathcal{B}} : \text{SmpReg}(\mathcal{B}, \mathcal{X}) \rightarrow \mathbb{R}\}_{\mathcal{B} \in \mathfrak{B}}$  such that

(AR1)  $\geq$  is represented by  $V := V_{\mathbb{N}}$ .

(AR2) For all  $\mathcal{B} \in \mathfrak{B}$ , the function  $V_{\mathcal{B}}$  is  $d_\infty$ -continuous on  $\text{SmpReg}(\mathcal{B}, \mathcal{X})$ . It is either constant, or it is increasing with respect to  $\geq_*$  when restricted to the constant elements of  $\text{SmpReg}(\mathcal{B}, \mathcal{X})$  (identified with  $\mathcal{X}$  in the obvious way).

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<sup>16</sup>See also Theorem 13.3.5, p.361 of Kuczma (2009) for the special case when  $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U} = \mathbb{R}$ .

(AR3) For any disjoint  $\mathcal{B}_1, \dots, \mathcal{B}_N \in \mathfrak{B}$  with  $\mathcal{B} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_N \in \mathfrak{B}$ , and any  $\mathbf{x} \in$

$$\text{SmpReg}(\mathcal{B}, \mathcal{X}), V_{\mathcal{B}}(\mathbf{x}) = \sum_{n=1}^N V_{\mathcal{B}_n}(\mathbf{x}_{\mathcal{B}_n}).$$

For any  $x \in \mathcal{X}$  and  $\mathcal{B} \subseteq \mathbb{N}$ , let  $[x]_{\mathcal{B}}$  be the all- $x$  element of  $\ell^\infty(\mathcal{B}, \mathcal{X})$ . In particular, let  $[x] := [x]_{\mathbb{N}}$ . The next result is analogous to WZ Proposition 3.

**Lemma A.10** *Let  $\geq$  be a weak order on  $\text{SmpReg}(\mathcal{X})$  satisfying Axioms A1 - A4. Let  $o, l \in \mathcal{X}$  with  $o <_* l$ . Then  $\geq$  admits an additive representation on  $\text{SmpReg}(\mathcal{X})$ , such that  $V([l]) = 1$ , while  $V_{\mathcal{B}}([o]_{\mathcal{B}}) = 0$  for all  $\mathcal{B} \in \mathfrak{B}$ .*

*Proof:* Consider a  $\mathfrak{B}$ -measurable partition  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_K\}$  of  $\mathbb{N}$ . Let  $\text{SmpReg}(\mathfrak{P})$  be the set of all elements of  $\text{SmpReg}(\mathcal{X})$  subordinate to  $\mathfrak{P}$ . Then  $\text{SmpReg}(\mathfrak{P})$  is isomorphic to the Cartesian product  $\mathcal{X}^K$  in the obvious way, and the order  $\geq$  restricted to  $\text{SmpReg}(\mathfrak{P})$  thereby induces an order  $\geq_{\mathfrak{P}}$  on  $\mathcal{X}^K$ . Some cells might be null sets for  $\geq$ . By permuting the cells if necessary, suppose that  $\mathcal{P}_1, \dots, \mathcal{P}_J$  are non-null, and (if  $J < K$ ) that  $\mathcal{P}_{J+1}, \dots, \mathcal{P}_K$  are null. The order  $\geq_{\mathfrak{P}}$  is reducible to an order on  $\mathcal{X}^J$ , because it is independent of the values of coordinates  $\{J+1, \dots, K\}$ . We will refer to the order on  $\mathcal{X}^J$  as  $\geq'_{\mathfrak{P}}$ . (If  $J = K$ , then  $\geq'_{\mathfrak{P}}$  is the same as  $\geq_{\mathfrak{P}}$ .)

Assume  $J \geq 3$ . The order  $\geq'_{\mathfrak{P}}$  is  $\geq_*$ -increasing in each coordinate of  $\mathcal{X}^J$  by Axiom A1, continuous by Axiom A2 and separable by Axiom A3, so Debreu's (1960) theorem yields a collection of continuous functions  $u_{\mathcal{P}_1}^{\mathfrak{P}}, \dots, u_{\mathcal{P}_J}^{\mathfrak{P}} : \mathcal{X} \rightarrow \mathbb{R}$ , each increasing relative to  $\geq_*$ , such that the function  $V(x_1, \dots, x_J) := u_{\mathcal{P}_1}^{\mathfrak{P}}(x_1) + \dots + u_{\mathcal{P}_J}^{\mathfrak{P}}(x_J)$  is a representation of  $\geq_{\mathfrak{P}}$ .<sup>17</sup> This representation is unique up to positive affine transformations. By adding suitable (distinct) constants to  $u_{\mathcal{P}_1}^{\mathfrak{P}}, \dots, u_{\mathcal{P}_J}^{\mathfrak{P}}$  and then multiplying them by a suitable (common) scalar, one can ensure that

$$u_{\mathcal{P}_1}^{\mathfrak{P}}(o) = \dots = u_{\mathcal{P}_J}^{\mathfrak{P}}(o) = 0, \quad \text{while} \quad u_{\mathcal{P}_1}^{\mathfrak{P}}(l) + \dots + u_{\mathcal{P}_J}^{\mathfrak{P}}(l) = 1. \quad (\text{A5})$$

For all  $j \in \{J+1, \dots, K\}$ , define  $u_{\mathcal{P}_j}^{\mathfrak{P}} : \mathcal{X} \rightarrow \mathbb{R}$  to be the constant 0-valued function.

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<sup>17</sup>Debreu assumed  $\mathcal{X} \subseteq \mathbb{R}$  was an interval. But his theorem holds when  $\mathcal{X}$  is any connected topological space (Wakker, 1989, Theorem III.6.6, Remarks III.7.1 and A3.1). We have assumed  $\mathcal{X}$  is connected.

Thus, for any  $\mathbf{x}, \mathbf{y} \in \text{SmpReg}(\mathfrak{P})$ ,

$$\left(\mathbf{x} \geq \mathbf{y}\right) \iff \left(u_{\mathcal{P}_1}^{\mathfrak{P}}(x_1) + \cdots + u_{\mathcal{P}_J}^{\mathfrak{P}}(x_K) \geq u_{\mathcal{P}_1}^{\mathfrak{P}}(y_1) + \cdots + u_{\mathcal{P}_J}^{\mathfrak{P}}(y_K)\right), \quad (\text{A6})$$

where  $x_1, \dots, x_K$  and  $y_1, \dots, y_K$  are the values that  $\mathbf{x}$  and  $\mathbf{y}$  take on the cells of  $\mathfrak{P}$ .

For any non-null  $\mathcal{B} \in \mathfrak{B}$  such that  $\mathcal{B}^c$  is also non-null, Lemma A.8 yields a three-element  $\mathfrak{B}$ -partition  $\mathfrak{P}$  where one cell is  $\mathcal{B}$  and the other two cells are non-null. Thus, one can always find at least one  $\mathfrak{B}$ -partition  $\mathfrak{P}$  such that  $u_{\mathcal{B}}^{\mathfrak{P}}$  is well-defined. We now show that it is independent of the  $\mathfrak{B}$ -partition in which  $\mathcal{B}$  is embedded.<sup>18</sup>

**Claim 1:** *For all non-null  $\mathcal{B} \in \mathfrak{B}$  such that  $\mathcal{B}^c$  is also non-null, there is a function  $u_{\mathcal{B}} : \mathcal{X} \rightarrow \mathbb{R}$  such that  $u_{\mathcal{B}}^{\mathfrak{P}} = u_{\mathcal{B}}$  for any  $\mathfrak{B}$ -partition  $\mathfrak{P}$  in which  $\mathcal{B}$  is a cell.*

*Proof:* Let  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  and  $\mathfrak{Q} = \{\mathcal{Q}_1, \dots, \mathcal{Q}_K\}$  be two  $\mathfrak{B}$ -partitions, with  $\mathcal{P}_1 = \mathcal{B} = \mathcal{Q}_1$ . We claim that  $u_{\mathcal{B}}^{\mathfrak{P}} = u_{\mathcal{B}}^{\mathfrak{Q}}$ . To see this, define the functions  $v_{\mathfrak{P}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  and  $v_{\mathfrak{Q}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  as follows: for any  $x, y \in \mathcal{X}$ ,

$$v_{\mathfrak{P}}(x, y) := u_{\mathcal{B}}^{\mathfrak{P}}(x) + \sum_{j=2}^J u_{\mathcal{P}_j}^{\mathfrak{P}}(y) \quad \text{and} \quad v_{\mathfrak{Q}}(x, y) := u_{\mathcal{B}}^{\mathfrak{Q}}(x) + \sum_{k=2}^K u_{\mathcal{Q}_k}^{\mathfrak{Q}}(y)$$

Let  $\mathfrak{R}$  be the  $\mathfrak{B}$ -partition  $\mathfrak{R} := \{\mathcal{B}, \mathcal{B}^c\}$ . There is an obvious bijection from  $\text{SmpReg}(\mathfrak{R})$  to  $\mathcal{X} \times \mathcal{X}$ . Thus, the restriction of  $\geq$  to  $\text{SmpReg}(\mathfrak{R})$  induces a weak order on  $\mathcal{X} \times \mathcal{X}$ , and  $v_{\mathfrak{P}}$  and  $v_{\mathfrak{Q}}$  are both additive representations of this weak order. Thus, Lemma A.9 says they are equivalent up to positive affine transformation. But the normalization (A5) implies that  $u_{\mathcal{B}}^{\mathfrak{P}}(o) = u_{\mathcal{B}^c}^{\mathfrak{P}}(o) = u_{\mathcal{B}}^{\mathfrak{Q}}(o) = u_{\mathcal{B}^c}^{\mathfrak{Q}}(o) = 0$ , and  $u_{\mathcal{B}}^{\mathfrak{P}}(l) + u_{\mathcal{B}^c}^{\mathfrak{P}}(l) = 1 = u_{\mathcal{B}}^{\mathfrak{Q}}(l) + u_{\mathcal{B}^c}^{\mathfrak{Q}}(l)$ . Thus,  $u_{\mathcal{B}}^{\mathfrak{P}} = u_{\mathcal{B}}^{\mathfrak{Q}}$ . ◇ Claim 1

Meanwhile, for any null  $\mathcal{B} \in \mathfrak{B}$ , let  $u_{\mathcal{B}}$  be the constant 0-valued function on  $\mathcal{X}$ .

**Claim 2:** *For any disjoint  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$  such that  $(\mathcal{A} \sqcup \mathcal{B})^c$  is non-null, and any  $x \in \mathcal{X}$ ,  $u_{\mathcal{A}}(x) + u_{\mathcal{B}}(x) = u_{\mathcal{A} \sqcup \mathcal{B}}(x)$ .*

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<sup>18</sup>WZ prove this using the fact that any two partitions have a common refinement. Unfortunately, two  $\mathfrak{B}$ -partitions do *not* necessarily have a common  $\mathfrak{B}$ -partition refinement, so we cannot use this strategy.



*Proof:* If one of  $\mathcal{A}$  or  $\mathcal{B}$  is null, this is trivial. So assume that both are non-null. By hypothesis  $\mathcal{C} := (\mathcal{A} \sqcup \mathcal{B})^c$  is also non-null. Consider the three-element  $\mathfrak{B}$ -partition  $\mathfrak{P} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$  and the two-element partition  $\mathfrak{Q} = \{\mathcal{A} \sqcup \mathcal{B}, \mathcal{C}\}$ . Claim 1 says that  $u_{\mathcal{C}}^{\mathfrak{P}} = u_{\mathcal{C}} = u_{\mathcal{C}}^{\mathfrak{Q}}$  for some function  $u_{\mathcal{C}}$ . Likewise,  $u_{\mathcal{A}}^{\mathfrak{P}} = u_{\mathcal{A}}$ ,  $u_{\mathcal{B}}^{\mathfrak{P}} = u_{\mathcal{B}}$ , and  $u_{\mathcal{A} \sqcup \mathcal{B}}^{\mathfrak{Q}} = u_{\mathcal{A} \sqcup \mathcal{B}}$ .

Let  $\geq_{\mathfrak{P}}$  be the restriction of  $\geq$  to  $\text{SmpReg}(\mathfrak{P})$ ; then the functions  $\{u_{\mathcal{A}}, u_{\mathcal{B}}, u_{\mathcal{C}}\}$  yield an additive representation of  $\geq_{\mathfrak{P}}$  in the sense of formula (A6). Likewise, let  $\geq_{\mathfrak{Q}}$  be the restriction of  $\geq$  to  $\text{SmpReg}(\mathfrak{Q})$ ; then  $\{u_{\mathcal{A} \sqcup \mathcal{B}}, u_{\mathcal{C}}\}$  yields an additive representation of  $\geq_{\mathfrak{Q}}$ . But  $\geq_{\mathfrak{Q}}$  is also the restriction of  $\geq_{\mathfrak{P}}$  to  $\text{SmpReg}(\mathfrak{Q})$ . Thus, the two functions  $\{(u_{\mathcal{A}} + u_{\mathcal{B}}), u_{\mathcal{C}}\}$  yield *another* additive representation of  $\geq_{\mathfrak{Q}}$ . Lemma A.9 says these two representations are equivalent up to positive affine transformation. Invoking the normalization (A5) yields  $u_{\mathcal{A}}(x) + u_{\mathcal{B}}(x) = u_{\mathcal{A} \sqcup \mathcal{B}}(x)$  for all  $x \in \mathcal{X}$ .  $\diamond$  claim 2

Conspicuously, Claim 1 does not address the case when  $\mathcal{B}^c$  is null—in particular, when  $\mathcal{B} = \mathbb{N}$ . Given any  $\mathfrak{B}$ -partition  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  (with  $J \geq 3$ ), we could construct functions  $u_{\mathcal{P}_1}, \dots, u_{\mathcal{P}_J} : \mathcal{X} \rightarrow \mathbb{R}$  yielding an additive representation (A6) and satisfying the normalization conditions (A5), and then define  $u_{\mathbb{N}}^{\mathfrak{P}} : \mathcal{X} \rightarrow \mathbb{R}$  by setting

$$u_{\mathbb{N}}^{\mathfrak{P}}(x) := u_{\mathcal{P}_1}(x) + \dots + u_{\mathcal{P}_J}(x), \quad \text{for all } x \in \mathcal{X}. \quad (\text{A7})$$

However, if  $\mathfrak{Q} = \{\mathcal{Q}_1, \dots, \mathcal{Q}_K\}$  is another  $\mathfrak{B}$ -partition, then it is not obvious that  $u_{\mathbb{N}}^{\mathfrak{P}} = u_{\mathbb{N}}^{\mathfrak{Q}}$ . We cannot apply Claim 2, because this claim specifically does not address the case when  $(\mathcal{A} \sqcup \mathcal{B})^c$  is null—in particular, when  $\mathcal{A} \sqcup \mathcal{B} = \mathbb{N}$ . If  $\mathfrak{P}$  and  $\mathfrak{Q}$  had a common  $\mathfrak{B}$ -partition refinement  $\mathfrak{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_L\}$ , then we could apply Claim 2 to express the functions  $u_{\mathcal{P}_1}, \dots, u_{\mathcal{P}_J}$  and  $u_{\mathcal{Q}_1}, \dots, u_{\mathcal{Q}_K}$  in terms of sums of  $u_{\mathcal{R}_1}, \dots, u_{\mathcal{R}_L}$ ; from this, we could deduce that  $u_{\mathbb{N}}^{\mathfrak{P}} = u_{\mathbb{N}}^{\mathfrak{R}} = u_{\mathbb{N}}^{\mathfrak{Q}}$ . But as we have already noted,  $\mathfrak{B}$  is not a Boolean algebra, so  $\mathfrak{P}$  and  $\mathfrak{Q}$  do not necessarily have a common  $\mathfrak{B}$ -partition refinement. So this option is unavailable. Instead, we will use the following two claims.

**Claim 3:** *Let  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  and  $\mathfrak{Q} = \{\mathcal{Q}_1, \dots, \mathcal{Q}_J\}$  be two  $\mathfrak{B}$ -partitions, each with at least two non-null cells. Suppose there is some  $\gamma \in \Gamma$  such that  $\gamma(\mathcal{P}_j) = \mathcal{Q}_j$  for all  $j \in \{1, \dots, J\}$ . Then  $u_{\mathcal{P}_j} = u_{\mathcal{Q}_j}$  for all  $j \in \{1, \dots, J\}$ .*

*Proof:* Define  $\gamma^* : \text{SmpReg}(\Omega) \rightarrow \text{SmpReg}(\mathfrak{P})$  by setting  $\gamma^*(\mathbf{x}) := \mathbf{x}'$  where  $x'_n := x_{\gamma(n)}$  for all  $n \in \mathbb{N}$ . For all  $\mathbf{x}, \mathbf{y} \in \text{SmpReg}(\Omega)$ , Axiom A4 implies that  $\mathbf{x} \geq \mathbf{y}$  if and only if  $\gamma^*(\mathbf{x}) \geq \gamma^*(\mathbf{y})$ . Thus,  $\geq_{\mathfrak{P}}$  and  $\geq_{\Omega}$  are the same weak order on  $\mathcal{X}^J$ . But  $u_{\mathcal{P}_1}, \dots, u_{\mathcal{P}_J}$  and  $u_{\mathcal{Q}_1}, \dots, u_{\mathcal{Q}_J}$  both yield additive representations of this weak order. Thus, standard uniqueness results (e.g. Lemma A.9) say they are equal up to positive affine transformation. Thus, the normalization (A5) implies that  $u_{\mathcal{P}_j} = u_{\mathcal{Q}_j}$  for all  $j \in \{1, \dots, J\}$ . ◇ Claim 3

**Claim 4:** Let  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  and  $\Omega = \{\mathcal{Q}_1, \dots, \mathcal{Q}_K\}$  be two  $\mathfrak{B}$ -partitions of  $\mathbb{N}$ , each with at least two non-null cells. Then for all  $x \in \mathcal{X}$ ,  $u_{\mathcal{P}_1}(x) + \dots + u_{\mathcal{P}_J}(x) = u_{\mathcal{Q}_1}(x) + \dots + u_{\mathcal{Q}_K}(x)$ .

*Proof:* By merging some of the cells of  $\mathfrak{P}$  together and applying Claim 2, we can assume without loss of generality that  $\mathfrak{P}$  has only two cells (both non-null); say,  $\mathfrak{P} = \{\mathcal{P}_1, \mathcal{P}_2\}$ . Likewise, by applying Claim 2 we can assume that  $\Omega = \{\mathcal{Q}_1, \mathcal{Q}_2\}$ . Let  $p_j := \delta(\mathcal{P}_j)$  and  $q_j := \delta(\mathcal{Q}_j)$  for  $j = 1, 2$ ; then  $0 < p_1, p_2, q_1, q_2 < 1$  (by the contrapositive of Lemma A.7), and  $p_1 + p_2 = 1 = q_1 + q_2$ .

Lemma A.6(a) yields  $\mathcal{R}_{11} \subseteq \mathcal{P}_1$  with  $\mathcal{R}_{11} \in \mathfrak{B}$  such that  $\delta(\mathcal{R}_{11}) = p_1 q_1$ . Let  $\mathcal{R}_{12} := \mathcal{P}_1 \setminus \mathcal{R}_{11}$ ; then it is easily verified that  $\mathcal{R}_{12} \in \mathfrak{B}$  and  $\delta(\mathcal{R}_{12}) = p_1(1 - q_1) = p_1 q_2$  (because  $\delta(\mathcal{R}_{11}) + \delta(\mathcal{R}_{12}) = \delta(\mathcal{P}_1) = p_1$ ). Likewise, Lemma A.6(a) yields  $\mathcal{R}_{21} \subseteq \mathcal{P}_2$  with  $\mathcal{R}_{21} \in \mathfrak{B}$  and  $\delta(\mathcal{R}_{21}) = p_2 q_1$ ; if  $\mathcal{R}_{22} := \mathcal{P}_2 \setminus \mathcal{R}_{21}$ ; then  $\mathcal{R}_{22} \in \mathfrak{B}$  and  $\delta(\mathcal{R}_{22}) = p_2 q_2$ .

Now let  $\mathcal{Q}'_1 := \mathcal{R}_{11} \sqcup \mathcal{R}_{21}$  and  $\mathcal{Q}'_2 := \mathcal{R}_{12} \sqcup \mathcal{R}_{22}$ . Then  $\mathcal{Q}'_1, \mathcal{Q}'_2 \in \mathfrak{B}$ , so  $\Omega' := \{\mathcal{Q}'_1, \mathcal{Q}'_2\}$  is another  $\mathfrak{B}$ -partition. Furthermore,  $\delta(\mathcal{Q}'_1) = \delta(\mathcal{R}_{11}) + \delta(\mathcal{R}_{21}) = p_1 q_1 + p_2 q_1 = q_1 = \delta(\mathcal{Q}_1)$ . Thus, Lemma A.6(b) yields some  $\gamma \in \Gamma$  such that  $\gamma(\mathcal{Q}_1) = \mathcal{Q}'_1$ . Since  $\mathcal{Q}_2 = \mathcal{Q}'_1^c$  and  $\mathcal{Q}'_2 = (\mathcal{Q}'_1)^c$ , it follows that  $\gamma(\mathcal{Q}_2) = \mathcal{Q}'_2$ . Thus, Claim 3 implies that

$$u_{\mathcal{Q}'_1} = u_{\mathcal{Q}_1} \quad \text{and} \quad u_{\mathcal{Q}'_2} = u_{\mathcal{Q}_2}. \tag{A8}$$

Thus, for any  $x \in \mathcal{X}$ , we have:

$$\begin{aligned} u_{\mathcal{P}_1}(x) + u_{\mathcal{P}_2}(x) &\stackrel{(*)}{=} u_{\mathcal{R}_{11}}(x) + u_{\mathcal{R}_{12}}(x) + u_{\mathcal{R}_{21}}(x) + u_{\mathcal{R}_{22}}(x) \\ &\stackrel{(\dagger)}{=} u_{\mathcal{Q}'_1}(x) + u_{\mathcal{Q}'_2}(x) \stackrel{(\circ)}{=} u_{\mathcal{Q}_1}(x) + u_{\mathcal{Q}_2}(x), \end{aligned}$$

as desired. Here, (\*) is by Claim 2 because  $\mathcal{P}_1 = \mathcal{R}_{11} \sqcup \mathcal{R}_{12}$  and  $\mathcal{P}_2 = \mathcal{R}_{21} \sqcup \mathcal{R}_{22}$ , while (†) is by Claim 2 because  $\mathcal{Q}'_1 = \mathcal{R}_{11} \sqcup \mathcal{R}_{21}$  and  $\mathcal{Q}'_2 = \mathcal{R}_{12} \sqcup \mathcal{R}_{22}$ . Finally, (◇) is by equation (A8). ◇ Claim 4

We can now define  $u_{\mathbb{N}}$  as in formula (A7); Claim 4 guarantees that this definition is independent of the choice of partition  $\mathfrak{B}$ . But there is still the question of how to define  $u_{\mathcal{B}}$  when  $\mathcal{B} \in \mathfrak{B}$  is a proper subset of  $\mathbb{N}$  such that  $\mathcal{B}^c$  is null.

**Claim 5:** *Let  $\mathcal{B} \in \mathfrak{B}$  such that  $\mathcal{B}^c$  is null. Let  $\mathfrak{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_J\}$  be a  $\mathfrak{B}$ -partition of  $\mathbb{N}$  with at least two non-null cells. For all  $j \in \{1, \dots, J\}$ , let  $\mathcal{Q}_j := \mathcal{P}_j \cap \mathcal{B}$ . Then  $\mathcal{Q}_1, \dots, \mathcal{Q}_J \in \mathfrak{B}$ , and for all  $x \in \mathcal{X}$ ,  $u_{\mathcal{Q}_1}(x) + \dots + u_{\mathcal{Q}_J}(x) = u_{\mathbb{N}}(x)$ .*

*Proof:* Let  $\mathcal{Q}_0 := \mathcal{B}^c$ ; then  $\delta(\mathcal{Q}_0) = 0$  by Lemma A.7. Thus, any subset of  $\mathcal{Q}_0$  also has zero density, and hence is an element of  $\mathfrak{B}$ . In particular, for all  $j \in \{1, \dots, J\}$ , we have  $\mathcal{Q}_0 \cap \mathcal{P}_j \in \mathfrak{B}$ , and thus  $\mathcal{Q}_j \in \mathfrak{B}$ , because  $\mathcal{Q}_j = \mathcal{P}_j \setminus (\mathcal{Q}_0 \cap \mathcal{P}_j)$ . Thus,  $\mathfrak{Q} := \{\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_J\}$  is a  $\mathfrak{B}$ -partition of  $\mathbb{N}$ .

Let  $\geq_{\mathfrak{P}}$  and  $\geq_{\mathfrak{Q}}$  be the weak orders induced on  $\mathcal{X}^J$  and  $\mathcal{X}^{J+1}$  respectively by the restrictions of  $\geq$  to  $\text{SmpReg}(\mathfrak{P})$  and  $\text{SmpReg}(\mathfrak{Q})$ . Then  $\{u_{\mathcal{P}_1}, \dots, u_{\mathcal{P}_J}\}$  yields an additive representation of  $\geq_{\mathfrak{P}}$ , while  $\{u_{\mathcal{Q}_0}, u_{\mathcal{Q}_1}, \dots, u_{\mathcal{Q}_J}\}$  yields an additive representation of  $\geq_{\mathfrak{Q}}$ . (Of course,  $u_{\mathcal{Q}_0} = 0$  because  $\mathcal{Q}_0$  is null.)

For any  $\mathbf{x} = (x_0, x_1, \dots, x_J)$  and  $\mathbf{y} = (y_0, y_1, \dots, y_J)$  in  $\mathbb{R}^{J+1}$ , if we define  $\mathbf{x}' := (x_1, \dots, x_J)$  and  $\mathbf{y}' := (y_1, \dots, y_J)$  in  $\mathbb{R}^J$ , then it is easily verified that  $\mathbf{x} \geq_{\mathfrak{Q}} \mathbf{y}$  if and only if  $\mathbf{x}' \geq_{\mathfrak{P}} \mathbf{y}'$  (because  $\mathcal{Q}_0$  is null). Thus,  $\{u_{\mathcal{Q}_1}, \dots, u_{\mathcal{Q}_J}\}$  is another additive representation of  $\geq_{\mathfrak{P}}$ . Thus, the two representations are equal up to positive affine transformation. Thus, the normalization (A5) implies that  $u_{\mathcal{Q}_j} = u_{\mathcal{P}_j}$  for all  $j \in \{1, \dots, J\}$ . Thus, for all  $x \in \mathcal{X}$ , we have  $u_{\mathcal{Q}_1}(x) + \dots + u_{\mathcal{Q}_J}(x) = u_{\mathcal{P}_1}(x) + \dots + u_{\mathcal{P}_J}(x) = u_{\mathbb{N}}(x)$ , where the second equality is by (A7). ◇ Claim 5

Now, for any  $\mathcal{B} \in \mathfrak{B}$  such that  $\mathcal{B}^c$  is null, define  $u_{\mathcal{B}}$  as follows. Let  $\mathfrak{Q} := \{\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_J\}$  be any  $\mathfrak{B}$ -partition of  $\mathbb{N}$  with  $\mathcal{Q}_0 = \mathcal{B}^c$ , and define  $u_{\mathcal{B}}(x) := u_{\mathcal{Q}_1}(x) + \dots + u_{\mathcal{Q}_J}(x)$ . Claim 5

says that this is independent of the choice of  $\mathfrak{Q}$ , and furthermore,  $u_{\mathcal{B}} = u_{\mathbb{N}}$ .

**Claim 6:** For any disjoint  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$  and any  $x \in \mathcal{X}$ ,  $u_{\mathcal{A}}(x) + u_{\mathcal{B}}(x) = u_{\mathcal{A} \sqcup \mathcal{B}}(x)$ .

*Proof:* Let  $\mathcal{C} := (\mathcal{A} \sqcup \mathcal{B})^c$ . If  $\mathcal{C}$  is non-null, this follows from Claim 2. If  $\mathcal{C}$  is null, then  $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$  is a  $\mathfrak{B}$ -partition, and  $u_{\mathcal{A} \sqcup \mathcal{B}} = u_{\mathcal{A}} + u_{\mathcal{B}}$  by the definition just above.  $\diamond$  claim 6

At this point, we have defined the functions  $u_{\mathcal{B}}$  for all  $\mathcal{B} \in \mathfrak{B}$ . Now, for any  $\mathcal{B} \in \mathfrak{B}$ , define  $V_{\mathcal{B}} : \text{SmpReg}(\mathcal{B}, \mathcal{X}) \rightarrow \mathbb{R}$  as follows. Let  $\mathbf{x} \in \text{SmpReg}(\mathcal{B}, \mathcal{X})$ , and suppose  $\mathbf{x}$  is subordinate to the  $\mathfrak{B}$ -partition  $\mathfrak{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_J\}$  of  $\mathcal{B}$ . Thus, there exist  $X_1, \dots, X_J \in \mathcal{X}$  such that  $x_c = X_j$  for all  $c \in \mathcal{C}_j$  and all  $j \in \{1, \dots, J\}$ . Define

$$V_{\mathcal{B}}(\mathbf{x}) := u_{\mathcal{C}_1}(X_1) + \dots + u_{\mathcal{C}_J}(X_J). \quad (\text{A9})$$

Of course,  $\mathbf{x}$  is subordinate to many different  $\mathfrak{B}$ -partitions of  $\mathcal{B}$ . In spite of this,  $V_{\mathcal{B}}(\mathbf{x})$  is well-defined by (A9), by the next claim.

**Claim 7:** The value of  $V_{\mathcal{B}}(\mathbf{x})$  in formula (A9) is independent of the choice of subordinating partition.

*Proof:* Let  $\mathfrak{C}$  be the coarsest  $\mathfrak{B}$ -partition to which  $\mathbf{x}$  is subordinate. (It is easily verified that this exists.) Every subordinating partition is a refinement of  $\mathfrak{C}$ . Thus, Claim 6 ensures that the value obtained by applying formula (A9) with any other subordinating partition is equal to the value obtained by applying (A9) to  $\mathfrak{C}$ .  $\diamond$  claim 7

We have now constructed a collection of functions  $\{V_{\mathcal{B}} : \text{SmpReg}(\mathcal{B}, \mathcal{X}) \rightarrow \mathbb{R}\}_{\mathcal{B} \in \mathfrak{B}}$ . It is straightforward to verify that this collection satisfies properties (AR1)-(AR3). Thus, it yields an additive representation on  $\text{SmpReg}(\mathcal{X})$ . Furthermore the normalization (A5) implies that  $V([\mathit{l}]) = 1$ , while  $V_{\mathcal{B}}([\mathit{o}]_{\mathcal{B}}) = 0$  for all  $\mathcal{B} \in \mathfrak{B}$ .  $\square$

*Proof of the Theorem.* It is easily verified that the Cesàro average utility representation (3) implies that  $\geq$  satisfies Axioms A1 - A4. So we will focus on the converse implication.

Suppose  $\geq$  satisfies condition (B) and Axioms A1 - A4. Fix  $o, l \in \mathcal{X}$  with  $o <_* l$ . Let  $\{V_{\mathcal{B}}\}_{\mathcal{B} \in \mathfrak{B}}$  be the additive representation of  $\geq$  on  $\text{SmpReg}(\mathcal{X})$  from Lemma A.10. So  $V([l]) = 1$ , while  $V_{\mathcal{B}}([o]_{\mathcal{B}}) = 0$  for all  $\mathcal{B} \in \mathfrak{B}$ .

**Claim 1:** For any  $\mathbf{x}, \mathbf{y} \in \text{SmpReg}(\mathcal{X})$ , if  $x_n \geq_* y_n$  for all  $n \in \mathbb{N}$ , then  $\mathbf{x} \geq \mathbf{y}$ .

*Proof:* This follows from Lemma A.10, exactly as in WZ Corollary 18. ◇ claim 1

**Claim 2:** For all  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ , there exist  $x, z \in \mathcal{X}$  such that  $[x] \leq \mathbf{y} \leq [z]$ .

*Proof:* We will prove the existence of  $z$ . The proof for  $x$  is similar. The set  $\{y_n\}_{n=1}^{\infty}$  is totally bounded, because  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X}) \subseteq \ell^{\infty}(\mathbb{N}, \mathcal{X})$ . Thus, Condition (B) yields  $z \in \mathcal{X}$  such that  $y_n \leq_* z$  for all  $n \in \mathbb{N}$ .

For any  $k \in \mathbb{N}$ , Lemma A.3 yields some  $\mathbf{w} \in \text{SmpReg}(\mathcal{X})$  such that  $d_{\infty}(\mathbf{w}, \mathbf{y}) < \frac{1}{2k}$ . Suppose  $\mathbf{w}$  is subordinate to the  $\mathfrak{B}$ -partition  $\{\mathcal{B}_1, \dots, \mathcal{B}_J\}$ , and takes the values  $W_1, \dots, W_J$  on these cells. For all  $j \in \mathcal{J}$ , let  $n_j \in \mathcal{B}_j$  be arbitrary; then  $d(y_{n_j}, W_j) < \frac{1}{2k}$ . Also,  $y_{n_j} \leq_* z$ . So, define  $\mathbf{y}^k \in \text{SmpReg}(\mathcal{X})$  as follows: for all  $j \in \{1, \dots, J\}$  and all  $b \in \mathcal{B}_j$ , set  $y_b^k := y_{n_j}$ . It follows that  $y_b^k \leq_* z$  and  $d(y_b^k, w_b) = d(y_{n_j}, W_j) < \frac{1}{2k}$ . Thus,

(a) For all  $n \in \mathbb{N}$ ,  $y_n^k \leq_* z$ .

(b) For all  $n \in \mathbb{N}$ ,  $d(y_n^k, y_n) \leq d(y_n^k, w_n) + d(w_n, y_n) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$ .

Observation (b) means that  $d_{\infty}(\mathbf{y}^k, \mathbf{y}) \leq \frac{1}{k}$ . Meanwhile, observation (a) and Claim 1 imply that  $\mathbf{y}^k \leq [z]$ . Repeat this construction for all  $k \in \mathbb{N}$ , to obtain a sequence  $\{\mathbf{y}^k\}_{k=1}^{\infty}$  with  $\mathbf{y}^k \leq [z]$  for all  $k \in \mathbb{N}$ , such that  $\lim_{k \rightarrow \infty} d_{\infty}(\mathbf{y}^k, \mathbf{y}) = 0$ . Thus, Axiom A2 implies that  $\mathbf{y} \leq [z]$ . ◇ Claim 2

Let  $\mathbf{x} \in \overline{\text{Reg}}(\mathcal{X})$ . We shall say that  $y$  is a *constant equivalent* for  $\mathbf{x}$  if  $[y] \approx \mathbf{x}$ .<sup>19</sup>

**Claim 3:** Every element of  $\overline{\text{Reg}}(\mathcal{X})$  has a constant equivalent.

*Proof:* (Following WZ Lemma 8) Let  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ . Let  $\mathcal{X}^- := \{x \in \mathcal{X}; [x] \leq \mathbf{y}\}$  and let  $\mathcal{X}^+ := \{z \in \mathcal{X}; [z] \geq \mathbf{y}\}$ . These sets are nonempty by Claim 2, and closed by

<sup>19</sup>For comparison, Wakker and Zank (1999) call this a *certainty equivalent*.

Axiom A2. Finally,  $\mathcal{X} = \mathcal{X}^- \cup \mathcal{X}^+$ , since  $\geq$  is a complete relation. Thus,  $\mathcal{X}^-$  and  $\mathcal{X}^+$  cannot be disjoint, since  $\mathcal{X}$  is connected. Let  $y \in \mathcal{X}^- \cap \mathcal{X}^+$ . Then  $[y] \approx \mathbf{x}$ .  $\diamond$  claim 3

For all  $x \in \mathcal{X}$ , let  $u(x) := V([x])$ . This defines a function  $u : \mathcal{X} \rightarrow \mathbb{R}$ . This function is continuous and increasing with respect to  $\geq_*$ , by property (AR2) from Lemma A.10. Also,  $u(l) = 1$  and  $u(o) = 0$ . For any  $\mathbf{x} \in \overline{\text{Reg}}(\mathcal{X})$ , let

$$V(\mathbf{x}) := u(\bar{x}), \quad \text{where } \bar{x} \in \mathcal{X} \text{ is a constant equivalent of } \mathbf{x}, \quad (\text{A10})$$

where this constant equivalent  $\bar{x}$  exists by Claim 3. If  $\bar{x}$  and  $\hat{x}$  are both constant equivalents for  $\mathbf{x}$ , then  $[\bar{x}] \approx \mathbf{x} \approx [\hat{x}]$ , so that  $u(\bar{x}) = V([\bar{x}]) = V([\hat{x}]) = u(\hat{x})$  by property (AR1). Thus,  $V(\mathbf{x})$  is well-defined in formula (A10).

**Claim 4:**  $V$  represents  $\geq$ .

*Proof:* Let  $\mathbf{x}, \mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$  with  $\mathbf{x} \geq \mathbf{y}$ . If  $\bar{x}$  and  $\bar{y}$  are the constant equivalents of  $\mathbf{x}$  and  $\mathbf{y}$ , then  $[\bar{x}] \approx \mathbf{x} \geq \mathbf{y} \approx [\bar{y}]$ , so  $[\bar{x}] \geq [\bar{y}]$ , so  $V([\bar{x}]) \geq V([\bar{y}])$ , by (AR1) from Lemma A.10. Thus,  $V(\mathbf{x}) \geq V(\mathbf{y})$  by formula (A10).  $\diamond$  claim 4

**Claim 5:**  $V$  is  $d_\infty$ -continuous on  $\overline{\text{Reg}}(\mathcal{X})$ .

*Proof:* (Following WZ Lemma 19) Let  $\mathcal{R} := \{V(\mathbf{y}); \mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})\}$ . Then  $\mathcal{R} = \{u(x); x \in \mathcal{X}\}$  by (A10). Thus,  $\mathcal{R}$  is connected, because  $u$  is continuous and  $\mathcal{X}$  is connected.

Now let  $r \in \mathcal{R}$ ; we will show that the preimage set  $V^{-1}[r, \infty)$  is  $d_\infty$ -closed. Let  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$  such that  $V(\mathbf{y}) = r$ . Then  $V^{-1}[r, \infty) := \{\mathbf{x} \in \overline{\text{Reg}}(\mathcal{X}); \mathbf{x} \geq \mathbf{y}\}$  by Claim 4, and this set is  $d_\infty$ -closed, by Axiom A2. By a similar argument,  $V^{-1}(-\infty, r]$  is  $d_\infty$ -closed. This holds for all  $r \in \mathcal{R}$ ; thus,  $V$  is  $d_\infty$ -continuous.  $\diamond$  claim 5

Recall from Lemma A.10 that

$$V_{\mathcal{B}}([o]_{\mathcal{B}}) = 0, \quad \text{for all } \mathcal{B} \in \mathfrak{B}. \quad (\text{A11})$$

For any  $x \in \mathcal{X}$ , define the function  $\mu^x : \mathfrak{B} \rightarrow \mathbb{R}$  as follows:

$$\mu^x(\mathcal{B}) := V_{\mathcal{B}}([x]_{\mathcal{B}}), \quad \text{for all } \mathcal{B} \in \mathfrak{B}. \quad (\text{A12})$$

This function is finitely additive: for any disjoint  $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$ , property (AR3) and definition (A12) yield  $\mu^x[\mathcal{A} \sqcup \mathcal{B}] = \mu^x[\mathcal{A}] + \mu^x[\mathcal{B}]$ .<sup>20</sup> The next claim says that  $\mu^x$  is  $\Gamma$ -invariant.

**Claim 6:** For all  $x \in \mathcal{X}$ , all  $\mathcal{B} \in \mathfrak{B}$ , and all  $\gamma \in \Gamma$ ,  $\mu^x[\gamma^{-1}(\mathcal{B})] = \mu^x[\mathcal{B}]$ .

*Proof:* Let  $\mathcal{B}' := \gamma^{-1}(\mathcal{B})$ . Define  $\mathbf{x}, \mathbf{x}' \in \text{Reg}(\mathcal{X})$  as follows:  $x_n = x$  for all  $n \in \mathcal{B}$  and  $x_n = o$  for all  $n \notin \mathcal{B}$ . Likewise,  $x'_n = x$  for all  $n \in \mathcal{B}'$  and  $x'_n = o$  for all  $n \notin \mathcal{B}'$ . Thus, for all  $n \in \mathbb{N}$ ,

$$x'_n = \begin{cases} x & \text{if } n \in \mathcal{B}'; \\ o & \text{if } n \notin \mathcal{B}' \end{cases} = \begin{cases} x & \text{if } \gamma(n) \in \mathcal{B}; \\ o & \text{if } \gamma(n) \notin \mathcal{B} \end{cases} = x_{\gamma(n)}. \quad (\text{A13})$$

In other words,  $\mathbf{x}' = \gamma(\mathbf{x})$ . Thus,  $\mathbf{x} \approx \mathbf{x}'$ , by Axiom A4. Thus,

$$V(\mathbf{x}) = V(\mathbf{x}'), \quad (\text{A14})$$

by property (AR1) from Lemma A.10. Now let  $\mathcal{C} := \mathbb{N} \setminus \mathcal{B}$  and  $\mathcal{C}' := \mathbb{N} \setminus \mathcal{B}'$ . Then

$$V(\mathbf{x}) \stackrel{(*)}{=} V_{\mathcal{B}}(\mathbf{x}_{\mathcal{B}}) + V_{\mathcal{C}}(\mathbf{x}_{\mathcal{C}}) \stackrel{(\dagger)}{=} V_{\mathcal{B}}([x]_{\mathcal{B}}) + V_{\mathcal{C}}([o]_{\mathcal{C}}) \stackrel{(\diamond)}{=} \mu^x[\mathcal{B}], \quad (\text{A15})$$

$$\begin{aligned} \text{and } V(\mathbf{x}') &\stackrel{(*)}{=} V_{\mathcal{B}'}(\mathbf{x}'_{\mathcal{B}'}) + V_{\mathcal{C}'}(\mathbf{x}'_{\mathcal{C}'}) \stackrel{(\dagger)}{=} V_{\mathcal{B}'}([x]_{\mathcal{B}'}) + V_{\mathcal{C}'}([o]_{\mathcal{C}'}) \\ &\stackrel{(\diamond)}{=} \mu^x[\mathcal{B}'] = \mu^x[\gamma^{-1}(\mathcal{B})]. \end{aligned} \quad (\text{A16})$$

Here, both (\*) are by (AR3) from Lemma A.10, both (†) are by the definitions of  $\mathbf{x}$  and  $\mathbf{x}'$ , and both (◇) are by equations (A11) and (A12). Combining equations (A14)-(A16), we get  $\mu^x[\gamma^{-1}(\mathcal{B})] = \mu^x[\mathcal{B}]$ , as claimed. This works for any  $x, \mathcal{B}$ , and  $\gamma$ .

◇ **Claim 6**

For any  $x \in \mathcal{X}$ , Claim 6 and Lemma A.6(c) yield  $r \in \mathbb{R}_+$  such that  $\mu^x[\mathcal{B}] = r \delta(\mathcal{B})$  for all  $\mathcal{B} \in \mathfrak{B}$ . In fact,  $r = u(x)$ , because  $\mu^x[\mathbb{N}] = V([x]) = u(x)$ , while  $\delta(\mathbb{N}) = 1$ . Thus,

$$\mu^x[\mathcal{B}] = u(x) \delta(\mathcal{B}), \quad \text{for all } \mathcal{B} \in \mathfrak{B}. \quad (\text{A17})$$

In particular,  $\mu^l[\mathcal{B}] = \delta(\mathcal{B})$ , for all  $\mathcal{B} \in \mathfrak{B}$ , because  $u(l) = 1$ .

**Claim 7:** If  $\mathbf{y} \in \text{SmpReg}(\mathcal{X})$ , then  $V(\mathbf{y}) = \text{CesAve}_{n \in \mathbb{N}} u(y_n)$ .

<sup>20</sup>We do not call  $\mu^x$  a “measure” because  $\mathfrak{B}$  is not a Boolean algebra.

*Proof:* If  $\mathbf{y} \in \text{SmpReg}(\mathcal{X})$ , then there exist a  $\mathfrak{B}$ -partition  $\{\mathcal{B}_1, \dots, \mathcal{B}_M\}$  and  $Y^1, \dots, Y^M \in \mathcal{X}$  such that  $y_b = Y^m$  for all  $b \in \mathcal{B}_m$  and all  $m \in \{1, \dots, M\}$ . In other words,  $\mathbf{y}_{\mathcal{B}_m} = [Y^m]_{\mathcal{B}_m}$ . Thus,

$$\begin{aligned}
V(\mathbf{y}) &\stackrel{(\S)}{=} \sum_{m=1}^M V_{\mathcal{B}_m}(\mathbf{y}_{\mathcal{B}_m}) = \sum_{m=1}^M V_{\mathcal{B}_m}([Y^m]_{\mathcal{B}_m}) \\
&\stackrel{(*)}{=} \sum_{m=1}^M \mu^{Y^m}(\mathcal{B}_m) \stackrel{(\dagger)}{=} \sum_{m=1}^M u(Y^m) \cdot \delta(\mathcal{B}_m) = \sum_{m=1}^M u(Y^m) \text{CesAve}_{n \in \mathbb{N}} \mathbf{1}_{\mathcal{B}_m}(n) \\
&\stackrel{(\ddagger)}{=} \text{CesAve}_{n \in \mathbb{N}} \sum_{m=1}^M u(Y^m) \cdot \mathbf{1}_{\mathcal{B}_m}(n) \stackrel{(\diamond)}{=} \text{CesAve}_{n \in \mathbb{N}} u(y_n),
\end{aligned}$$

as claimed. Here,  $(\S)$  is by property (AR3) from Lemma A.10,  $(*)$  is by equation (A12),  $(\dagger)$  is by (A17),  $(\ddagger)$  is by the linearity of the Cesàro average operator, and  $(\diamond)$  is because  $u(y_n) = \sum_{m=1}^M u(Y^m) \cdot \mathbf{1}_{\mathcal{B}_m}(n)$  for all  $n \in \mathbb{N}$ .  $\diamond$  Claim 7

**Claim 8:** Let  $\mathbf{y} \in \text{Reg}(\mathcal{X})$ , let  $\mathbf{x} \in \text{SmpReg}(\mathcal{X})$ , and let  $\epsilon > 0$ .

- (a) Suppose  $y_n \geq_* x_n$  for all  $n \in \mathbb{N}$ . Then there exists  $\mathbf{y}' \in \text{SmpReg}(\mathcal{X})$  such that  $d_\infty(\mathbf{y}', \mathbf{y}) < \epsilon$  and  $V(\mathbf{y}') \geq V(\mathbf{x}) - \epsilon$ .
- (b) Suppose  $y_n \leq_* x_n$  for all  $n \in \mathbb{N}$ . Then there exists  $\mathbf{y}' \in \text{SmpReg}(\mathcal{X})$  such that  $d_\infty(\mathbf{y}', \mathbf{y}) < \epsilon$  and  $V(\mathbf{y}') \leq V(\mathbf{x}) + \epsilon$ .

*Proof:* We will prove (a); the proof of (b) is analogous. Let  $\mathcal{Y} = \{y_n\}_{n=1}^\infty$ . Then  $\mathcal{Y}$  is totally bounded because  $\mathbf{y} \in \text{Reg}(\mathcal{X}) \subseteq \ell^\infty(\mathbb{N}, \mathcal{X})$ . So there exist  $t_1, \dots, t_J \in \mathcal{X}$  such that  $\mathcal{Y} \subseteq \bigcup_{j=1}^J \mathcal{B}(t_j, \frac{\epsilon}{2})$ . For all  $j \in \{1, \dots, J\}$ , let  $\mathcal{Y}_j := \mathcal{Y} \cap \mathcal{B}(t_j, \frac{\epsilon}{2})$ .

Condition (B) yields  $x, z \in \mathcal{X}$  such that  $x \leq_* y \leq_* z$  for all  $y \in \mathcal{Y}$ . So  $u(x) \leq u(y) \leq u(z)$  for all  $y \in \mathcal{Y}$ . Let  $M := \lceil (u(z) - u(x))/\epsilon \rceil$ . Then for each  $j \in \{1, \dots, J\}$ , there is a finite subset  $\mathcal{W}_j \subseteq \mathcal{Y}_j$  (with  $|\mathcal{W}_j| \leq M$ ) such that for all  $y \in \mathcal{Y}_j$  there is some  $w \in \mathcal{W}_j$  with  $|u(y) - u(w)| < \epsilon$ . Note that  $d(y, w) \leq d(y, t_j) + d(t_j, w) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . Thus,  $y \in \mathcal{B}(w, \epsilon)$ . So for all  $w \in \mathcal{W}_j$ , let  $\mathcal{O}_w := \{x \in \mathcal{B}(w, \epsilon); |u(x) - u(w)| < \epsilon\}$ . This is an open subset of  $\mathcal{X}$  (because  $u$  is continuous and  $\mathcal{B}(w, \epsilon)$  is open). We have just seen



that  $\mathcal{Y}_j \subseteq \bigcup_{w \in \mathcal{W}_j} \mathcal{O}_w$ . Repeating this construction for all  $j \in \{1, \dots, J\}$ , we deduce that

$$\mathcal{Y} \subseteq \bigcup_{j=1}^J \bigcup_{w \in \mathcal{W}_j} \mathcal{O}_w = \bigcup_{w \in \mathcal{W}} \mathcal{O}_w, \quad \text{where } \mathcal{W} := \bigcup_{j=1}^J \mathcal{W}_j.$$

$\mathcal{W}$  is a finite union of finite sets, hence finite. By repeating the construction of Lemma A.3, we obtain  $\mathbf{y}' \in \text{SmpReg}(\mathbb{N}, \mathcal{W})$  such that for all  $w \in \mathcal{W}$  and all  $n \in \mathbb{N}$ , if  $y'_n = w$ , then  $y_n \in \mathcal{O}_w$ . Thus,  $d(y_n, y'_n) < \epsilon$  and  $|u(y_n) - u(y'_n)| < \epsilon$ , hence  $u(y_n) < u(y'_n) + \epsilon$ . By hypothesis,  $x_n \leq_* y_n$ , so  $u(x_n) \leq u(y_n)$ ; thus, we get  $u(x_n) < u(y'_n) + \epsilon$ .

Thus,  $u(y'_n) > u(x_n) - \epsilon$  for all  $n \in \mathbb{N}$ , so  $V(\mathbf{y}') \geq V(\mathbf{x}) - \epsilon$  by Claim 7 and monotonicity of the Cesàro average. Also,  $d(y_n, y'_n) < \epsilon$  for all  $n \in \mathbb{N}$ , so  $d_\infty(\mathbf{y}, \mathbf{y}') < \epsilon$ .

◇ Claim 8

The next claim shows that  $V$  satisfies a limited form of pointwise monotonicity.<sup>21</sup>

**Claim 9:** *Let  $\mathbf{y} \in \text{Reg}(\mathcal{X})$  and  $\mathbf{x} \in \text{SmpReg}(\mathcal{X})$ .*

(a) *If  $x_n \leq_n y_n$  for all  $n \in \mathbb{N}$ , then  $V(\mathbf{x}) \leq V(\mathbf{y})$ .*

(b) *If  $x_n \geq_n y_n$  for all  $n \in \mathbb{N}$ , then  $V(\mathbf{x}) \geq V(\mathbf{y})$ .*

*Proof:* We will prove (a). The proof of (b) is analogous. Claim 8(a) yields a sequence

$\{\mathbf{y}^k\}_{k=1}^\infty$  in  $\text{SmpReg}(\mathcal{X})$  with  $\lim_{k \rightarrow \infty} d(\mathbf{y}^k, \mathbf{y}) = 0$ , such that for any  $\epsilon > 0$ , there is some  $K \in \mathbb{N}$  such that  $V(\mathbf{y}^k) \geq V(\mathbf{x}) - \epsilon$  all  $k \geq K$ . Claim 5 yields  $\lim_{k \rightarrow \infty} V(\mathbf{y}^k) = V(\mathbf{y})$ .

Thus,  $V(\mathbf{y}) \geq V(\mathbf{x}) - \epsilon$ , for any  $\epsilon > 0$ . Thus,  $V(\mathbf{y}) \geq V(\mathbf{x})$ . ◇ Claim 9

**Claim 10:** *For any  $\mathbf{y} \in \text{Reg}(\mathcal{X})$ ,  $V(\mathbf{y}) = \text{CesAve}_{n \in \mathbb{N}} u(y_n)$ .*

*Proof:* The set  $\{y_n\}_{n=1}^\infty$  is totally bounded, because  $\text{Reg}(\mathcal{X}) \subseteq \ell^\infty(\mathbb{N}, \mathcal{X})$ . So (B) yields

$x, z \in \mathcal{X}$  with  $x \leq_* y_n \leq_* z$  for all  $n \in \mathbb{N}$ . Thus,  $u(x) \leq u(y_n) \leq u(z)$  for all  $n \in \mathbb{N}$ .

Let  $\epsilon > 0$ . Let  $M := \lceil |u(z) - u(x)|/\epsilon \rceil$ . Since  $\mathcal{X}$  is connected and  $u$  is continuous, there exist  $w_0, \dots, w_M \in \mathcal{X}$  with  $x = w_0 < w_1 < \dots < w_M = z$  and  $|u(w_m) -$

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<sup>21</sup>The corresponding result in WZ is Lemma 9. But their proof uses the fact that  $\mathbb{R}$  is linearly ordered and its order topology is the same as its metric topology. So it is not suitable in the present context.

$u(w_{m-1})| < \epsilon$  for all  $m \in \{1, \dots, M\}$ . Define  $\mathcal{B}_M := \{n \in \mathbb{N}; w_{M-1} \leq y_n \leq w_M\}$ , and for all  $m \in [1 \dots M)$ , define  $\mathcal{B}_m := \{n \in \mathbb{N}; w_{m-1} \leq y_n < w_m\}$ . Then  $\mathbb{N} = \mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_M$ .

**Claim 10A:**  $\mathcal{B}_m \in \mathfrak{B}$  for all  $m \in \{1, \dots, M\}$ .

*Proof:* First suppose  $m \in [1 \dots M)$ . Let  $\mathcal{O}_m := \{x \in \mathcal{X}; w_{m-1} \leq_* x <_* y_m\}$ . Axiom A2 implies that the order  $\geq_*$  is continuous on  $\mathcal{X}$ . Thus,  $\mathcal{O}_m$  is the intersection of the open set  $\{x \in \mathcal{X}; x <_* y_m\}$  and the closed set  $\{x \in \mathcal{X}; w_{m-1} \leq_* x\}$ , hence it is an element of  $\mathfrak{D}(\mathcal{X})$ . But  $\mathcal{B}_m := \{n \in \mathbb{N}; y_n \in \mathcal{O}_m\}$ ; thus  $\mathcal{B}_m \in \mathfrak{B}$  because  $\mathbf{y}$  is regular. This argument works for  $m \in [1 \dots M)$ . A similar argument shows that  $\mathcal{B}_M \in \mathfrak{B}$  (but in this case, the set  $\mathcal{O}_M$  is closed in  $\mathcal{X}$ ). ∇ Claim 10A

Now define  $\underline{\mathbf{y}}, \bar{\mathbf{y}} \in \text{SmpReg}(\mathcal{X})$  as follows:

For all  $m \in \{1, \dots, M\}$  and all  $b \in \mathcal{B}_m$ ,  $\underline{y}_b := w_{m-1}$  and  $\bar{y}_b := w_m$ .

Thus, for all  $n \in \mathbb{N}$ , we have  $\underline{y}_n \leq_* y_n \leq_* \bar{y}_n$ . Thus Claim 9 yields

$$V(\underline{\mathbf{y}}) \leq V(\mathbf{y}) \leq V(\bar{\mathbf{y}}). \quad (\text{A18})$$

Meanwhile, Claim 7 says that

$$V(\underline{\mathbf{y}}) = \text{CesAve}_{n \in \mathbb{N}} u(\underline{y}_n) \quad \text{and} \quad V(\bar{\mathbf{y}}) = \text{CesAve}_{n \in \mathbb{N}} u(\bar{y}_n). \quad (\text{A19})$$

Combining statements (A18) and (A19), we get

$$\text{CesAve}_{n \in \mathbb{N}} u(\underline{y}_n) \leq V(\mathbf{y}) \leq \text{CesAve}_{n \in \mathbb{N}} u(\bar{y}_n), \quad (\text{A20})$$

Also, for all  $n \in \mathbb{N}$ , we have  $|u(\bar{y}_n) - u(\underline{y}_n)| \leq \epsilon$ . Thus,

$$\text{for all } n \in \mathbb{N}, \quad \left| u(y_n) - u(\underline{y}_n) \right| \leq \epsilon \quad \text{and} \quad \left| u(\bar{y}_n) - u(y_n) \right| \leq \epsilon.$$

Thus,

$$\left| \text{CesAve}_{n \in \mathbb{N}} u(\bar{y}_n) - \text{CesAve}_{n \in \mathbb{N}} u(y_n) \right| \leq \epsilon \quad \text{and} \quad \left| \text{CesAve}_{n \in \mathbb{N}} u(y_n) - \text{CesAve}_{n \in \mathbb{N}} u(\underline{y}_n) \right| \leq \epsilon. \quad (\text{A21})$$

Combining (A20) and (A21) yields

$$\text{CesAve}_{n \in \mathbb{N}} u(y_n) - \epsilon \leq V(\mathbf{y}) \leq \text{CesAve}_{n \in \mathbb{N}} u(y_n) + \epsilon,$$

This holds for any  $\epsilon > 0$ . The claim follows. ◇ Claim 10

We now extend Claim 10 to  $\overline{\text{Reg}}(\mathcal{X})$ , using  $d_\infty$ -density.

**Claim 11:** For any  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ ,  $V(\mathbf{y}) = \text{CesAve}_{n \in \mathbb{N}} u(y_n)$ .

*Proof:* Let  $\mathbf{y} \in \overline{\text{Reg}}(\mathcal{X})$ . If  $\mathbf{y} \in \text{Reg}(\mathcal{X})$  then Claim 10 immediately yields the above equation. Otherwise, there is a compact subset  $\mathcal{K} \subseteq \mathcal{X}$  and a sequence  $(\mathbf{y}^j)_{j=1}^\infty$  of elements of  $\text{Reg}(\mathcal{K})$  such that  $\lim_{j \rightarrow \infty} d_\infty(\mathbf{y}^j, \mathbf{y}) = 0$ . Thus, Claim 5 yields

$$\lim_{j \rightarrow \infty} V(\mathbf{y}^j) = V(\mathbf{y}). \quad (\text{A22})$$

For all  $n \in \mathbb{N}$ , we have  $y_n = \lim_{j \rightarrow \infty} y_n^j$ , so  $y_n \in \mathcal{K}$ , because  $\mathcal{K}$  is closed. Now,  $u$  is continuous and  $\mathcal{K}$  is compact, so the restricted function  $u|_{\mathcal{K}}$  is uniformly continuous. Thus, for any  $\epsilon > 0$  there is some  $\eta > 0$  such that  $|u(x) - u(y)| < \epsilon$  for all  $x, y \in \mathcal{K}$  with  $d(x, y) < \eta$ . But  $\lim_{j \rightarrow \infty} d_\infty(\mathbf{y}^j, \mathbf{y}) = 0$ , so there is some  $J_\epsilon \in \mathbb{N}$  such that for all  $j \geq J_\epsilon$ , we have  $d_\infty(\mathbf{y}^j, \mathbf{y}) < \eta$ ; hence for all  $n \in \mathbb{N}$ ,  $d(y_n^j, y_n) < \eta$  and thus  $|u(y_n^j) - u(y_n)| < \epsilon$ . From this it follows that

$$\left| \text{CesAve}_{n \in \mathbb{N}} u(y_n^k) - \text{CesAve}_{n \in \mathbb{N}} u(y_n) \right| < \epsilon, \quad \text{for all } j \geq J_\epsilon.$$

We can find such a  $J_\epsilon$  for any  $\epsilon > 0$ . Thus,

$$\lim_{j \rightarrow \infty} \text{CesAve}_{n \in \mathbb{N}} u(y_n^j) = \text{CesAve}_{n \in \mathbb{N}} u(y_n). \quad (\text{A23})$$

Finally, for all  $j \in \mathbb{N}$ , Claim 10 says  $V(\mathbf{y}^j) = \text{CesAve}_{n \in \mathbb{N}} u(y_n^j)$ . Combine this fact with equations (A22) and (A23) to conclude that  $V(\mathbf{y}) = \text{CesAve}_{n \in \mathbb{N}} u(y_n)$ .  $\diamond$  **Claim 11**

The theorem now follows by combining Claims 4 and 11.  $\square$

**Remarks.** (a) We have required Axiom A3 (*Separability*) to apply on all of  $\text{Reg}(\mathcal{X})$ . But an inspection of the proof shows that we really only need A3 to hold on  $\text{SmpReg}(\mathcal{X})$ .

(b) In the proof of the theorem, the first step was to show that  $\geq$  has a “subjective expected utility” (SEU) representation involving a finitely additive “measure”  $\mu$  on  $\mathfrak{B}$ . We then used Axiom A4 to show that  $\mu$  is  $\Gamma$ -invariant, and then invoked Theorem 2 of **Obata**

(1988a) to deduce that  $\mu$  is the natural density measure  $\delta$ . From this, we deduced that the aforementioned SEU representation is the Cesàro average utility representation (3). To achieve the initial SEU representation, we used Axioms A1 - A4 and the proof strategy of Wakker and Zank (1999). But one could use another axiomatic characterization of SEU.

One complication is that  $\mathfrak{B}$  is not a Boolean algebra. So the original theorem of Savage (1954) is not applicable. However,  $\mathfrak{B}$  is a mosaic; thus, the axiomatization of Kopylov (2007) is applicable.<sup>22</sup> Thus, one could state and prove a version of the main result using Kopylov's formulation of Savage's axioms in place of Axioms A1 - A4.

(c) A version of Cesàro average can be defined on any countable amenable group (or monoid) using a Følner sequence.<sup>23</sup> The standard Cesàro average on  $\mathbb{N}$  is just the special case corresponding to the Følner sequence  $\{\{1, \dots, N\}\}_{N=1}^{\infty}$ . Except for Lemma A.6 (based on results of Obata (1988a)), the proof of the Theorem does not use any special properties of  $\mathbb{N}$ . The rest of the proof generalizes immediately to any other countable monoid (indeed, any countable set). Thus, given an extension of Obata's results to some other countable amenable monoid  $\mathbb{M}$ , one could immediately obtain a corresponding extension of the Theorem to  $\mathbb{M}$ . Unfortunately, to our knowledge, there are no such extensions. This is an interesting avenue for future research.

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<sup>22</sup>A *mosaic* is any collection  $\mathfrak{B}$  of subsets satisfying statement (A2).

<sup>23</sup>A countable group (or monoid)  $(\mathcal{G}, \cdot)$  is *amenable* if there is a finitely additive probability measure  $\mu$  defined on the power set  $\wp(\mathcal{G})$  that is invariant under left-multiplication—that is, for any  $\mathcal{A} \subseteq \mathcal{G}$  and  $g \in \mathcal{G}$ ,  $\mu(g \cdot \mathcal{A}) = \mu(\mathcal{A})$  (see e.g. Fremlin 2006a, §449). For example,  $\mathbb{N}$  and  $\mathbb{Z}$  are amenable, as is any Cartesian product  $\mathbb{N}^J \times \mathbb{Z}^K$  for any  $J, K \in \mathbb{N}$ . Any finitely generated abelian group is amenable. So is any solvable group, or any group with a Cayley digraph having subexponential growth.

A *Følner sequence* for  $\mathcal{G}$  is an increasing sequence of finite subsets  $\mathfrak{F} = \{\mathcal{F}_n\}_{n=1}^{\infty}$  of  $\mathcal{G}$  such that

(1)  $\mathfrak{F}$  *exhausts*  $\mathcal{G}$ : For all  $g \in \mathcal{G}$  there is some  $N \in \mathbb{N}$  such that  $g \in \mathcal{F}_n$  for all  $n \geq N$ ; and

(2)  $\mathfrak{F}$  is *approximately invariant*: For all  $g \in \mathcal{G}$ ,  $\lim_{n \rightarrow \infty} \frac{|(g \cdot \mathcal{F}_n) \cap \mathcal{F}_n|}{|\mathcal{F}_n|} = 1$ .

A countable group is amenable if and only if it has a Følner sequence (Følner 1955; Fremlin 2006a, 449X(n)). For monoids, the situation is similar but more complicated; see e.g. Gray and Kambites (2017).

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