



**HAL**  
open science

# **A few remarks about the computer implementation and the verification of some hyperelastic constitutive laws and an illustration with the mechanical response of an artery**

Manon Blaise, Franz Chouly, Pierre-Yves Rohan

## ► **To cite this version:**

Manon Blaise, Franz Chouly, Pierre-Yves Rohan. A few remarks about the computer implementation and the verification of some hyperelastic constitutive laws and an illustration with the mechanical response of an artery. 2022. <hal-03637834>

**HAL Id: hal-03637834**

**<https://hal.science/hal-03637834v1>**

Preprint submitted on 11 Apr 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# A few remarks about the computer implementation and the verification of some hyperelastic constitutive laws and an illustration with the mechanical response of an artery

Manon BLAISE<sup>a</sup>, Franz CHOULY<sup>b,c,d</sup>, Pierre-Yves ROHAN<sup>e</sup>

<sup>a</sup>Laboratoire ICube, UMR CNRS 7357, 4 rue Bousingault, 67000 Strasbourg, France.

<sup>b</sup>Institut de Mathématiques de Bourgogne, UMR CNRS 5584, Université Bourgogne Franche-Comté, Faculté des Sciences Mirande, 9 avenue Alain Savary BP 47870, 21078 Dijon Cedex, France.

<sup>c</sup>Center for Mathematical Modeling and Department of Mathematical Engineering, University of Chile and IRL 2807 - CNRS, Santiago, Chile.

<sup>d</sup>Departamento de Ingeniería Matemática, CI2MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

<sup>e</sup>Institut de Biomécanique Humaine Georges Charpak, Arts et Métiers ParisTech, 151 bd de l'Hôpital, 75013 Paris, France.

---

## Abstract

In this short technical note, we are interested in the constitutive equations used to model macroscopically the mechanical function of soft tissues under mechanical loading. Soft tissues have the ability to undergo large elastic reversible deformations under quasi-static loading and are usually modelled using hyperelastic constitutive laws. Several constitutive equations have been defined in the literature by means of strain components or strain invariants isotropic hyperelastic models. This short technical note recalls how to derive the weak form for hyperelasticity equations, and how this weak form can be linearized. It also presents a semi-analytical solution for a simplified geometry representing the mechanical response of an artery modelled under pressure loading.

---

## 1. Introduction

As recalled in the book “Biomechanics of Living Organs: Hyperelastic Constitutive Laws for Finite Element Modeling” by Yohan Payan and Jacques Ohayon [10], the mechanics of human soft tissues has been an emerging research field since the publication, in 1981, of the book “Biomechanics: Mechanical Properties of Living Tissues” by Yuan-Cheng Fung [7]. Since that date, many groups in the world have proposed biomechanical models of soft organs to investigate both the underlying mechanisms that drive normal physiology of biological soft tissues and the mechanical factors that contribute to the onset and development of diseases such as tumour growth [15], atherosclerosis or aneurysms [12], or multilevel lumbar disc degenerative diseases [13], to name a few. Such organs indeed deform under physiological conditions (such as muscle activations or interactions with other tissues) or because of the mechanical interaction with the surgical gesture.

Assuming we know the governing ordinary and partial differential equations, finite element models can predict the behavior of the system from given initial and boundary conditions measured at a few selected points. This approach is incredibly powerful, but, to close the system of equations, we need constitutive equations that characterize the behavior of the system, which we need to calibrate with experimental data, for instance.

In the last decades, significant theoretical results, many confirmed by experiments, have projected considerable light on the mechanical behaviour of elastic materials subjected to large deformations. It has consistently been shown that hyperelasticity can be used to model macroscopically the mechanical function of soft tissues under mechanical loading.

In this document we focus on general hyperelasticity formulations, and detail the derivation of the weak form, and then of the tangent weak problem, for the deformation of an hyperelastic body. These are the two fundamental ingredients for an implementation within a finite element library. Nowadays modern finite element libraries such as FEniCS [8] or GetFEM++ [11] make use of high level assembling languages and even of symbolic

---

*Email addresses:* manon.blaise@etu.unistra.fr (Manon BLAISE), franz.chouly@u-bourgogne.fr (Franz CHOULY), pierre-yves.rohan@ensam.eu (Pierre-Yves ROHAN)

automatic differentiation, and are capable of solving an hyperelastic problem by specifying only the complete expression of the hyperelastic potential. Nevertheless, even in this case, it can be interesting to know how to carry out by hand this derivation, for instance to gain more control on the solving procedure. Furthermore, the linearized hyperelasticity problem is involved in adjoint-based solvers, that are useful in many applications, such as parameter calibration, shape optimization or dual-based error estimation, see, *e.g.*, [1, 2]. This first topic that concerns the weak form and its linearization is of course covered in many documents about hyperelasticity and the presentation here is mostly inspired from [3, 6].

The second part of this document details a semi-analytical solution for the hyperelastic deformation of a ring, that may represent an idealized artery. This can be helpful to verify numerical solutions with finite elements, in the same spirit as manufactured solutions [5]. This is a first step towards semi-analytical solutions with active constitutive laws such as presented in [10].

## 2. Hyperelasticity

We first recall that, when a material is purely elastic, all the energy that is supplied to it during loading can be regained after unloading. So, during the whole cycle of deformation, no energy is dissipated physically. It can be shown that the stress in such a material can be derived from a scalar functional representing the specific elastic energy density (elastic energy per unit undeformed volume).

### 2.1. Notations

Let  $\Omega \subset \mathbb{R}^d$  be an open bounded set that denotes the reference configuration of one deformable solid in a space of dimension  $d = 2$  or  $3$ . A deformed configuration  $\Omega^t$  can be defined through a transformation  $\varphi$  which maps any point  $X$  of the reference configuration to a point  $x$  of the deformed one:

$$\begin{aligned} \varphi : \bar{\Omega} &\longrightarrow \mathbb{R}^d \\ X &\longmapsto x = \varphi(X). \end{aligned}$$

We define the displacement  $u$  relatively to the reference configuration as:

$$u(X) := \varphi(X) - X.$$

The gradient of a quantity in the deformed (resp. reference) configuration will be noted  $\nabla$  (resp.  $\nabla_X$ ). To describe the deformation, we introduce as usual the identity matrix of size  $d \times d$ , denoted by  $\mathbf{I}$  and the deformation gradient  $\mathbf{F} = \mathbf{I} + \nabla_X u$ . We introduce as well

$$\boldsymbol{\epsilon}(u) := \text{sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T) = \frac{1}{2}((\nabla_X u)\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla_X u)^T) = \text{sym}((\nabla_X u)\mathbf{F}^{-1})$$

where  $\text{sym}(\cdot)$  denotes the symmetric part of a second order tensor. The jacobian of  $\varphi$  is denoted by  $J := \det \mathbf{F}$ . We introduce also the Cauchy-Green tensor  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ , and the Green-Lagrange tensor  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ . We will note  $\boldsymbol{\sigma}$  the Cauchy stress tensor,  $\boldsymbol{\Pi} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$  the first Piola-Kirchhoff stress tensor and  $\boldsymbol{\Sigma} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$  the second Piola-Kirchhoff stress tensor.

### 2.2. Hyperelastic constitutive law

We consider a general hyperelastic constitutive law, derived from a potential  $\mathcal{W}$  that depends on the deformation through  $\mathbf{E}$  (or  $\mathbf{C}$ ) (see, *e.g.*, [3, 6]), so that the second Piola-Kirchhoff stress is

$$\boldsymbol{\Sigma} = \frac{\partial \mathcal{W}}{\partial \mathbf{E}} = 2 \frac{\partial \mathcal{W}}{\partial \mathbf{C}},$$

with corresponding fourth-order elasticity tensor

$$\boldsymbol{\mathcal{C}} := \frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{E}} = \frac{\partial^2 \mathcal{W}}{\partial \mathbf{E} \partial \mathbf{E}}.$$

We will need as well the isotropic tensor

$$\mathcal{I} := \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i),$$

where  $\otimes$  denotes the tensor product of two vectors,  $(\mathbf{e}_i)_{i=1,\dots,d}$  is the canonical basis of  $\mathbb{R}^d$  and where Einstein's summation convention is used. The tensor  $\mathcal{I}$  has the property  $\mathcal{I} : \mathbf{A} = \mathbf{A}$  for any symmetric second-order tensor  $\mathbf{A}$  ( $:$  denotes the double-dot product between two tensors).

We will denote the global potential energy by  $\mathcal{J}(\cdot)$ . For example, if considering simple equilibrium under a gravity force, the potential energy is

$$\mathcal{J}(u) = \int_{\Omega} \mathcal{W}(\mathbf{E}(u)) \, dX - \int_{\Omega} \rho \, g \cdot u \, dX, \quad (1)$$

where  $\rho$  is the density in the reference configuration and  $g$  is the gravity acceleration vector. Of course, additional terms such as boundary loads, can be considered as well. Dirichlet conditions can also be prescribed, but, to simplify the formulation, the treatment of Dirichlet conditions will be omitted in the following.

Moreover, we consider the following notations to simplify the mathematical presentation. The directional derivative of a quantity  $A$  with respect to the displacement  $u$  in direction  $\delta u$  will be denoted by  $A'(u; \delta u)$ . This directional derivative is defined as

$$A'(u; \delta u) := \lim_{\varepsilon \rightarrow 0} \frac{A(u + \varepsilon \delta u) - A(u)}{\varepsilon} \left( = \frac{d}{d\varepsilon} A(u + \varepsilon \delta u)|_{\varepsilon=0} \right)$$

when this limit exists.

### 2.3. Having some fun with differential calculus

We need first the following results on the directional derivatives of deformation tensors, that are obtained after simple computations (see, e.g. [6, Chapter 14]):

**Proposition 2.1.** *The directional derivatives of  $\mathbf{F}$  and  $\mathbf{E}$  are:*

$$\begin{aligned} \mathbf{F}'(u; \delta u) &= \nabla_X(\delta u), \\ \mathbf{E}'(u; \delta u) &= \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u)) = \mathbf{F}^T(u) \boldsymbol{\epsilon}(\delta u) \mathbf{F}(u). \end{aligned}$$

*Proof:* The expression of  $\mathbf{F}'(u; \delta u)$  is a direct consequence of the definition of  $\mathbf{F}$ . Then let us compute

$$\begin{aligned} \mathbf{E}'(u; \delta u) &= \frac{1}{2} \mathbf{C}'(u; \delta u) \\ &= \frac{1}{2} ((\mathbf{F}^T)')(u; \delta u) \mathbf{F}(u) + \mathbf{F}^T(u) \mathbf{F}'(u; \delta u) \\ &= \frac{1}{2} ((\nabla_X^T)')(\delta u) \mathbf{F}(u) + \mathbf{F}^T(u) \nabla_X(\delta u) \end{aligned}$$

where we used first the definition of  $\mathbf{E}$  and  $\mathbf{C}$ , then the product rule and finally the expression of  $\mathbf{F}'$ . As a result we recover the first expression. To get the second expression we can do as follows

$$\begin{aligned} \mathbf{E}'(u; \delta u) &= \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u)) \\ &= \text{sym}(\mathbf{F}^T(u) \nabla(\delta u) \mathbf{F}(u)) \\ &= \mathbf{F}^T(u) \text{sym}(\nabla(\delta u)) \mathbf{F}(u) \end{aligned}$$

where we used  $\nabla_X(\delta u) = \nabla(\delta u) \mathbf{F}(u)$  and the properties of the symmetric part.  $\square$

The computation of the directional derivatives of stress tensors  $\mathbf{\Pi}$  and  $\mathbf{\Sigma}$  is more involved and we recall their expression below (see as well [6, Chapter 14]):

**Proposition 2.2.** *The directional derivatives of  $\Sigma$ , resp.  $\Pi$ , are:*

$$\Sigma'(u; \delta u) = \mathcal{C}(u) : \mathbf{F}^T(u) \nabla_X(\delta u) = \mathcal{C}(u) : \mathbf{F}^T(u) \epsilon(\delta u) \mathbf{F}(u). \quad (2)$$

$$\Pi'(u; \delta u) = \nabla_X(\delta u) \Sigma(u) + \mathbf{F}(u) (\mathcal{C}(u) : \mathbf{F}^T(u) \nabla_X(\delta u)). \quad (3)$$

*Proof:* First, for an hyperelastic law, there holds in fact  $\Sigma(u) = \Sigma(\mathbf{E}(u))$  and we apply the chain rule:

$$\Sigma'(u; \delta u) = \Sigma'(\mathbf{E}(u); \mathbf{E}'(u; \delta u)) = \frac{\partial \Sigma}{\partial \mathbf{E}}(\mathbf{E}(u)) : \mathbf{E}'(u; \delta u).$$

Since  $\mathcal{C} = \partial \Sigma / \partial \mathbf{E}$  and using Proposition 2.1 we get:

$$\Sigma'(u; \delta u) = \mathcal{C}(u) : \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u)).$$

Then (2) is obtained with the above formula and the symmetry properties of  $\mathcal{C}$ . Using the relationship  $\Pi = \mathbf{F} \Sigma$  and applying the product rule yield:

$$\Pi'(u; \delta u) = \mathbf{F}'(u; \delta u) \Sigma(u) + \mathbf{F}(u) \Sigma'(u; \delta u).$$

We use (2) and once again Proposition 2.1 to obtain (3). □

#### 2.4. Weak formulation

Now we are ready to find the expression of the weak form for an hyperelastic material. This weak form is the first order optimality condition associated to the minimization of (1) and reads, for any virtual displacement  $\delta u$ :

$$0 = \mathcal{J}'(u; \delta u) = \int_{\Omega} (\mathcal{W} \circ \mathbf{E})'(u; \delta u) \, dX - \int_{\Omega} \rho \, g \cdot \delta u \, dX.$$

We compute, using the chain rule and Proposition 2.1:

$$(\mathcal{W} \circ \mathbf{E})'(u; \delta u) = \mathcal{W}'(\mathbf{E}(u); \mathbf{E}'(u; \delta u)) = \frac{\partial \mathcal{W}}{\partial \mathbf{E}}(\mathbf{E}(u)) : \mathbf{E}'(u; \delta u) = \Sigma(u) : \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u)).$$

Using the symmetry property of  $\Sigma$  and the relationship  $\Pi = \mathbf{F} \Sigma$ , we obtain:

$$(\mathcal{W} \circ \mathbf{E})'(u; \delta u) = \Sigma(u) : \mathbf{F}^T(u) \nabla_X(\delta u) = \mathbf{F}(u) \Sigma(u) : \nabla_X(\delta u) = \Pi(u) : \nabla_X(\delta u).$$

Finally we define (substituting  $v$  to  $\delta u$ )

$$a(u; v) = \int_{\Omega} \Pi(u) : \nabla_X v \, dX, \quad L(v) = \int_{\Omega} \rho \, g \cdot v \, dX,$$

and the weak form for an hyperelastic material reads, for all  $v$ :

$$a(u; v) = L(v). \quad (4)$$

This is the kind of expression that can be implemented in GetFEM++ or FEniCS, as an alternative to the expression of  $\mathcal{W}$ .

### 3. Linearization for hyperelasticity

We want to linearize the weak form (4). This linearization can be helpful for different purposes. Particularly it is involved in iterative procedures for numerical solving, such as the Newton method, that needs the exact jacobian of the weak form. It can also be useful for the theoretical study of linearized systems around some states or prestress calculations. It has also an interest for adjoint-based solvers when doing dual based a posteriori error estimation, sensitivity analysis, parameter calibration or optimization. As already mentioned, some computer software has now the capability of carrying out this linearization automatically, but it can be necessary sometimes, or just more amusing, to do this by hand. We can proceed as follows.

### 3.1. Details of the linearization procedure

Since  $L(\cdot)$  is linear, there only remains to compute the directional derivative of  $a(\cdot; \cdot)$ . We fix  $u$  and  $v$  and consider a direction of differentiation  $\delta u$ , and first note that:

$$a'(u; \delta u, v) = \int_{\Omega} \mathbf{\Pi}'(u; \delta u) : \nabla_X v \, dX.$$

There just remains to apply (3):

$$a'(u; \delta u, v) = \int_{\Omega} (\nabla_X(\delta u) \boldsymbol{\Sigma}(u) + \mathbf{F}(u)(\mathbf{C}(u) : \mathbf{F}^T(u) \nabla_X(\delta u))) : \nabla_X v \, dX.$$

Splitting the sum, there holds:

$$a'(u; \delta u, v) = \int_{\Omega} \nabla_X(\delta u) \boldsymbol{\Sigma}(u) : \nabla_X v \, dX + \int_{\Omega} \mathbf{F}(u)(\mathbf{C}(u) : \mathbf{F}^T(u) \nabla_X(\delta u)) : \nabla_X v \, dX.$$

This can be rewritten as (using symmetries of  $\mathbf{C}$ ):

$$a'(u; \delta u, v) = \int_{\Omega} \boldsymbol{\Sigma}(u) : \nabla_X(\delta u)^T \nabla_X v \, dX + \int_{\Omega} \mathbf{F}^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \mathbf{F}(u)^T \nabla_X v \, dX. \quad (5)$$

Remark that for a Newton-Raphson iteration, at each step, and for a given  $u$ , we will find an increment  $\delta u$  solution to the tangent system:

$$a'(u; \delta u, v) = L(v) - a(u; v).$$

### 3.2. Re-interpretation “à la Schotté” of the tangent stiffness weak form

We can provide a beautiful decomposition of the tangent stiffness system, following [14]. First we use the relationship  $\mathbf{F} = \mathbf{I} + \nabla_X$ , and reformulate the term:

$$\begin{aligned} & \int_{\Omega} \mathbf{F}^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \mathbf{F}(u)^T \nabla_X v \, dX \\ &= \int_{\Omega} (\mathbf{I} + \nabla_X^T(u)) \nabla_X(\delta u) : \mathbf{C}(u) : (\mathbf{I} + \nabla_X^T(u)) \nabla_X v \, dX \\ &= \int_{\Omega} \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X v \, dX \\ & \quad + \int_{\Omega} \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X^T(u) \nabla_X v \, dX \\ & \quad + \int_{\Omega} \nabla_X^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X v \, dX \\ & \quad + \int_{\Omega} \nabla_X^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X^T(u) \nabla_X v \, dX. \end{aligned}$$

With the above calculation, we can rewrite the tangent weak form as:

$$a'(u; \delta u, v) = a'_G(u; \delta u, v) + a'_L(u; \delta u, v) + a'_D(u; \delta u, v),$$

where

$$a'_G(u; \delta u, v) := \int_{\Omega} \boldsymbol{\Sigma}(u) : \nabla_X(\delta u)^T \nabla_X v \, dX$$

is the *geometric stiffness* contribution,

$$a'_L(u; \delta u, v) := \int_{\Omega} \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X v \, dX$$

is the *linear stiffness* contribution and

$$\begin{aligned} a'_D(u; \delta u, v) &:= \int_{\Omega} \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X^T(u) \nabla_X v \, dX \\ &+ \int_{\Omega} \nabla_X^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X v \, dX \\ &+ \int_{\Omega} \nabla_X^T(u) \nabla_X(\delta u) : \mathbf{C}(u) : \nabla_X^T(u) \nabla_X v \, dX. \end{aligned}$$

is the *predeformation stiffness* contribution. For the terminology and more explanations, see [14].

### 3.3. Saint-Venant-Kirchhoff material

As an example, and as in [9], suppose that the constitutive law is those of a Saint-Venant-Kirchhoff material, i.e., that:

$$\mathcal{W}(\mathbf{E}) = \frac{\lambda}{2} (\text{tr}(\mathbf{E}))^2 + \mu \text{tr}(\mathbf{E}^2),$$

where  $\lambda$  and  $\mu$  are material parameters, see, e.g., [3, Chapter 5]. The associated second Piola-Kirchhoff stress tensor and elasticity tensor are:

$$\mathbf{\Sigma} = \lambda \text{tr}(\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}, \quad \mathbf{C} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathcal{I}.$$

Let us detail the expression

$$\begin{aligned} \mathbf{\Pi}'(u; \delta u) &= \nabla_X(\delta u) \mathbf{\Sigma}(u) + \mathbf{F}(u) (\mathbf{C}(u) : \mathbf{F}^T(u) \nabla_X(\delta u)) \\ &= \nabla_X(\delta u) (\lambda \text{tr}(\mathbf{E}(u)) \mathbf{I} + 2\mu \mathbf{E}(u)) + \mathbf{F}(u) ((\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathcal{I}) : \mathbf{F}^T(u) \nabla_X(\delta u)). \end{aligned}$$

We compute separately

$$\mathbf{I} \otimes \mathbf{I} : \mathbf{F}^T(u) \nabla_X(\delta u) = (\mathbf{I} : \mathbf{F}^T(u) \nabla_X(\delta u)) \mathbf{I} = \text{tr}(\mathbf{F}^T(u) \nabla_X(\delta u)) \mathbf{I},$$

and

$$\mathcal{I} : \mathbf{F}^T(u) \nabla_X(\delta u) = \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u)).$$

This yields

$$\begin{aligned} \mathbf{\Pi}'(u; \delta u) &= \nabla_X(\delta u) (\lambda \text{tr}(\mathbf{E}(u)) \mathbf{I} + 2\mu \mathbf{E}(u)) \\ &+ \mathbf{F}(u) (\lambda \text{tr}(\mathbf{F}^T(u) \nabla_X(\delta u)) \mathbf{I} + 2\mu \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u))) \\ &= \lambda (\text{tr}(\mathbf{E}(u)) \nabla_X(\delta u) + \text{tr}(\mathbf{F}^T(u) \nabla_X(\delta u)) \mathbf{F}(u)) \\ &+ 2\mu (\nabla_X(\delta u) \mathbf{E}(u) + \mathbf{F}(u) \text{sym}(\mathbf{F}^T(u) \nabla_X(\delta u))). \end{aligned}$$

## 4. Inflation of an artery modelled as a hyperelastic cylinder : semi-analytical solution

The prediction of the stress distribution within arteries has been a major focus of cardiovascular biomechanics. The complexity of the physical system presents a formidable modelling challenge. The walls of blood vessels are mixtures of solid and fluid constituents, with a very sizable fraction of fluid content. The solid constituent of soft tissues is comprised of extra-cellular matrix containing various types of collagen and elastin, and cells which also contain fluids. Blood vessels are anisotropic and the anisotropy changes from point to point, and complicating matters further, blood vessels are inhomogeneous. To make some initial progress, idealisations of various degrees of realism have necessarily been made in the literature. The larger arteries are commonly idealised as being cylindrical in shape in cardiovascular biomechanics and assumed to be both incompressible and non-linearly elastic. The complex loadings experienced by arteries in situ are typically modelled using the approximation of uniform internal inflation. It is commonly assumed that the artery is cylindrically isotropic.

All of these idealisations are adopted in this section. The artery is modelled using a strain-energy function per unit undeformed volume. We derive here a semi-analytical solution for a simplified geometry (a ring).

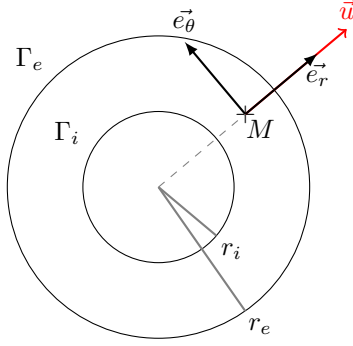


Figure 1: Reference configuration  $\Omega_0$

#### 4.1. Geometry and boundary conditions

We consider a perfect ring that represents the transverse section of an idealized arterial wall, see Figure 1. This configuration is established as the reference one  $\Omega$ . The notations  $\Gamma_i$  and  $\Gamma_e$  define internal and external boundaries, respectively. A predefined pressure  $p_{\text{ext}}$  is imposed on the external boundary  $\Gamma_e$  (where  $\vec{x} = r_e \vec{e}_r$ ):

$$\boldsymbol{\sigma} \vec{n} = p_{\text{ext}} \vec{n}. \quad (6)$$

On  $\Gamma_i$  (where  $\vec{x} = r_i \vec{e}_r$ ) we can impose either a Neumann (pressure) or Dirichlet (displacement) boundary condition, as:

$$\boldsymbol{\sigma} \vec{n} = p_{\text{int}} \vec{n}, \quad (7)$$

$$\vec{u} = u_{\text{int}} \vec{n}, \quad (8)$$

where  $p_{\text{int}}$  and  $u_{\text{int}}$  are fixed internal pressure and displacement, and  $\vec{n}$  normal vector pointing outward to the boundary, which is either  $\vec{e}_r$  for  $\Gamma_e$  or  $-\vec{e}_r$  for  $\Gamma_i$ .

#### 4.2. Incompressible neo-hookean material

In biomechanical studies, arterial tissue is commonly assumed to be incompressible. A challenge of this study is also to determine relevance of this incompressibility condition. Indeed, although ex-vivo experiences confirm it, in-vivo tests - more invasive - could be in favour of a quasi-incompressible model (see, e.g. [4]). Considering an incompressible neo-hookean material, the associated constitutive law yields:

$$\mathcal{W}(\mathbf{C}) = \frac{\mu}{2} (\text{tr}(\mathbf{C}) - d).$$

Second Piola-Kirchhoff tensor is computed as follows:

$$\boldsymbol{\Sigma} = 2 \frac{\partial \mathcal{W}}{\partial \mathbf{C}} - p \mathbf{C}^{-1},$$

where  $p$  is the hydrostatic pressure defined in each point of  $\Omega$ . Plus, we introduce the following condition for incompressibility (conservation of volume):

$$J = \det(\mathbf{F}) = 1. \quad (9)$$

#### 4.3. Equilibrium equation

Supposing the absence of body forces on the whole domain  $\Omega$ , the fundamental principle of statics is:

$$-\text{div}(\boldsymbol{\sigma}) = 0. \quad (10)$$

#### 4.4. First calculations in cylindrical coordinates

The gradient of the displacement  $u$  is given in cylindrical coordinates by:

$$\nabla u = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r} \left( \frac{\partial u_z}{\partial \theta} \right) & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

In the present case, the displacement  $u$  is unidirectional and depends only on  $r$ . Hence as sketched in Fig. 1, we assume that:

$$\forall M(r, \theta, z), \quad \vec{u}(M) = u_r(r) \vec{e}_r.$$

The gradient of the displacement can be directly simplified as follows:

$$\nabla u = \begin{pmatrix} \frac{du_r}{dr} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The tensors of deformation  $\mathbf{F} = \mathbf{I} + \nabla u$ , Cauchy  $\mathbf{C}$  and Green-Lagrange  $\mathbf{E}$  are derived in Eqs. (11) to (13) respectively.

$$\mathbf{F} = \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

$$\mathbf{C} = \begin{bmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (12)$$

$$\mathbf{E} = \frac{1}{2} \begin{bmatrix} 2 \frac{du_r}{dr} + \left( \frac{du_r}{dr} \right)^2 & 0 & 0 \\ 0 & 2 \frac{u_r}{r} + \left( \frac{u_r}{r} \right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{du_r}{dr} + \frac{1}{2} \left( \frac{du_r}{dr} \right)^2 & 0 & 0 \\ 0 & \frac{u_r}{r} + \frac{1}{2} \left( \frac{u_r}{r} \right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (13)$$

where  $\lambda_r = 1 + \frac{du_r}{dr}$  and  $\lambda_\theta = 1 + \frac{u_r}{r}$ .

#### 4.5. Expression for the displacement

First, the displacement  $u_r$  is obtained by solving the differential equation of incompressibility:

$$0 = \det(\mathbf{F}) - 1 = \lambda_r \lambda_\theta - 1 = \frac{du_r}{dr} \frac{u_r}{r} + \frac{du_r}{dr} + \frac{u_r}{r}.$$

We assume that a solution is

$$u_r(r) = \sqrt{c + r^2} - r,$$

where  $c$  is a constant which will be noted  $c = r_o^2$ . Let us proceed with intermediate calculations:

$$\begin{aligned}
\frac{du_r}{dr} &= \frac{2r}{2\sqrt{c+r^2}} - 1 \\
\det(\mathbf{F}) - 1 &= \left( \frac{r}{\sqrt{c+r^2}} - 1 \right) \left( \frac{\sqrt{c+r^2}}{r} - 1 \right) + \frac{r}{\sqrt{c+r^2}} - 1 + \frac{\sqrt{c+r^2}}{r} - 1 \\
&= -\frac{r}{\sqrt{c+r^2}} + 1 - \frac{\sqrt{c+r^2}}{r} + 1 + \frac{r}{\sqrt{c+r^2}} + \frac{\sqrt{c+r^2}}{r} - 2 \\
&= 0
\end{aligned}$$

So we checked that the expression for  $u_r$  is correct. Then the constant  $c$  is determined using the boundary conditions.

- In the case where  $u_r(r_i)$  is known, we consider Eq. (8), so that:

$$u_r(r_i) = -u_{\text{int}} = \sqrt{r_o^2 + r_i^2} - r_i.$$

Then we find

$$r_o^2 = u_{\text{int}}(u_{\text{int}} - 2r_i).$$

Finally, response of the displacement is defined for  $r_i \leq r \leq r_e$  and given by:

$$u_r(r) = \sqrt{u_{\text{int}}(u_{\text{int}} - 2r_i) + r^2} - r. \quad (14)$$

- In the case where  $u_r(r_i)$  is unknown,  $r_o^2$  is determined by pressure conditions Eqs. (6) and (7). Hence,  $u_r$  is noted:

$$u_r(r) = \sqrt{r^2 + r_o^2} - r.$$

#### 4.6. Stress and equilibrium

We compute the Cauchy stress tensor  $\boldsymbol{\sigma}$  under the incompressibility hypothesis Eq. (9):

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \boldsymbol{\Sigma} \mathbf{F}^T = \mathbf{F} \boldsymbol{\mu} \mathbf{I} \mathbf{F}^T - p \mathbf{F} \mathbf{C}^{-1} \mathbf{F}^T = \boldsymbol{\mu} \mathbf{C} - p \mathbf{F} \mathbf{F}^{-1} \mathbf{F}^{-T} \mathbf{F}^T = \boldsymbol{\mu} \mathbf{C} - p \mathbf{I}, \quad (15)$$

*i.e.:*

$$\boldsymbol{\sigma} = \begin{bmatrix} \mu \lambda_r^2 - p & 0 & 0 \\ 0 & \mu \lambda_\theta^2 - p & 0 \\ 0 & 0 & \mu - p \end{bmatrix}. \quad (16)$$

Then remember that the divergence of a second order tensor  $A$  in cylindrical coordinates is given by:

$$\overrightarrow{\text{div}}(A) = \begin{pmatrix} \frac{\partial A_{rr}}{\partial r} + \frac{1}{r} \frac{\partial A_{r\theta}}{\partial \theta} + \frac{\partial A_{rz}}{\partial z} + \frac{A_{rr} - A_{\theta\theta}}{r} \\ \frac{\partial A_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{\theta z}}{\partial z} + \frac{A_{r\theta} + A_{\theta r}}{r} \\ \frac{\partial A_{zr}}{\partial r} + \frac{1}{r} \frac{\partial A_{z\theta}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z} + \frac{A_{zr}}{r} \end{pmatrix}.$$

For the Cauchy stress tensor, invariant in  $\theta$  and  $z$ :

$$\overrightarrow{\text{div}}(\boldsymbol{\sigma}) = \begin{pmatrix} \mu \frac{d\lambda_r^2}{dr} - \frac{dp}{dr} + \frac{1}{r} \mu (\lambda_r^2 - \lambda_\theta^2) \\ 0 \\ 0 \end{pmatrix}.$$

Applying the fundamental principle of statics Eq. (10) and incompressible equation Eq. (9), assuming that  $\lambda_r = \frac{1}{\lambda_\theta}$ :

$$\begin{aligned}\frac{dp}{dr} &= \mu \frac{d\lambda_r^2}{dr} + \frac{1}{r} \mu (\lambda_r^2 - \lambda_\theta^2) \\ &= \mu \left[ \frac{d}{dr} \left( \frac{r^2}{r^2 + r_o^2} \right) + \frac{1}{r} \left( \frac{r^2}{r^2 + r_o^2} - 1 + \frac{r_o^2}{r^2} \right) \right] \\ &= \mu \left[ \frac{d}{dr} \left( \frac{r^2}{r^2 + r_o^2} \right) + \frac{2r}{2(r^2 + r_o^2)} - \frac{1}{r} - \frac{r_o^2}{r^3} \right].\end{aligned}$$

By integration, we obtain then:

$$p_r(r) = \mu \left[ \frac{r^2}{r^2 + r_o^2} + \frac{1}{2} \ln(r^2 + r_o^2) - \ln(r) + \frac{r_o^2}{2r^2} \right] + p_o = \mu \left[ \frac{2r^4 + r_o^2(r^2 + r_o^2)}{2r^2(r^2 + r_o^2)} + \ln \left( \frac{\sqrt{r^2 + r_o^2}}{r} \right) \right] + p_o, \quad (17)$$

with  $p_o$  the constant of integration.

From Eqs. (6) and (7) on boundary conditions, with an external pressure  $p_{\text{ext}} = 0$ :

$$\begin{aligned}-(\mu\lambda_{r_i}^2 - p(r_i)) &= p_{\text{int}}, \\ \mu\lambda_{r_e}^2 - p(r_e) &= 0.\end{aligned}$$

Thus

$$p_{\text{int}} = -\mu \left[ \frac{r_i^2}{r_i^2 + r_o^2} - \frac{2r_i^4 + r_o^2(r_i^2 + r_o^2)}{2r_i^2(r_i^2 + r_o^2)} - \ln \left( \frac{\sqrt{r_i^2 + r_o^2}}{r_i} \right) \right] + p_o \quad (18)$$

with

$$p_o = \mu \left[ \frac{r_e^2}{r_e^2 + r_o^2} - \frac{2r_e^4 + r_o^2(r_e^2 + r_o^2)}{2r_e^2(r_e^2 + r_o^2)} - \ln \left( \frac{\sqrt{r_e^2 + r_o^2}}{r_e} \right) \right]. \quad (19)$$

We sum Eq. (18) + Eq. (19) to get the expression of  $p_{\text{int}}$  in function of  $r_i$ ,  $r_e$  and  $r_o$ :

$$p_{\text{int}} = -\mu \left( \frac{r_i^2}{r_i^2 + r_o^2} - \frac{r_e^2}{r_e^2 + r_o^2} \right) + \frac{\mu}{2} \left( \frac{2r_i^4 + r_o^2(r_i^2 + r_o^2)}{r_i^2(r_i^2 + r_o^2)} - \frac{2r_e^4 + r_o^2(r_e^2 + r_o^2)}{r_e^2(r_e^2 + r_o^2)} \right) + \mu \left( \ln \left( \frac{r_e}{r_i} \sqrt{\frac{r_i^2 + r_o^2}{r_e^2 + r_o^2}} \right) \right).$$

Finally, from:

$$p_r(r) = \mu \left[ \frac{2r^4 + r_o^2(r^2 + r_o^2)}{2r^2(r^2 + r_o^2)} + \ln \left( \frac{\sqrt{r^2 + r_o^2}}{r} \right) \right] + p_o,$$

the final expression of  $p_r$  is given by:

$$p_r(r) = \mu \left[ \frac{2r^4 + r_o^2(r^2 + r_o^2)}{2r^2(r^2 + r_o^2)} + \frac{r_e^2}{r_e^2 + r_o^2} - \frac{2r_e^4 + r_o^2(r_e^2 + r_o^2)}{2r_e^2(r_e^2 + r_o^2)} + \ln \left( \frac{\sqrt{r^2 + r_o^2}}{r} \right) - \ln \left( \frac{\sqrt{r_e^2 + r_o^2}}{r_e} \right) \right].$$

Hence, we obtain the expressions of the principal radial stress tensor (cf Eq. (15)), and therefore derived expressions such as the Von Mises stress, etc. Wonderful, isn't it ?

*In order to find a numerical value of  $r_o$ :*

We introduce the following function  $\Phi$ , such as  $\Phi(r_o) = 0$ :

$$\Phi(x) = -p_{\text{int}} - \mu \left( \frac{r_i^2}{r_i^2 + x^2} - \frac{r_e^2}{r_e^2 + x^2} \right) + \frac{\mu}{2} \left( \frac{2r_i^4 + x^2(r_i^2 + x^2)}{r_i^2(r_i^2 + x^2)} - \frac{2r_e^4 + x^2(r_e^2 + x^2)}{r_e^2(r_e^2 + x^2)} \right) + \mu \left( \ln \left( \frac{r_e}{r_i} \sqrt{\frac{r_i^2 + x^2}{r_e^2 + x^2}} \right) \right).$$

By dichotomy, a numerical value of  $r_o$  can be obtained for  $r_e = 2.25 \text{ mm}$  and  $r_i = 1.93 \text{ mm}$ :

$$r_o = 0.01707\dots$$

#### 4.7. Numerical application

Let us take a very simple example of an expansion movement. The value for the parameters are:  $r_i = 4.0 \text{ mm}$ ,  $r_e = 7.0 \text{ mm}$ , and  $u_{\text{int}} = -2.0 \text{ mm}$ . See configuration Fig. 2:

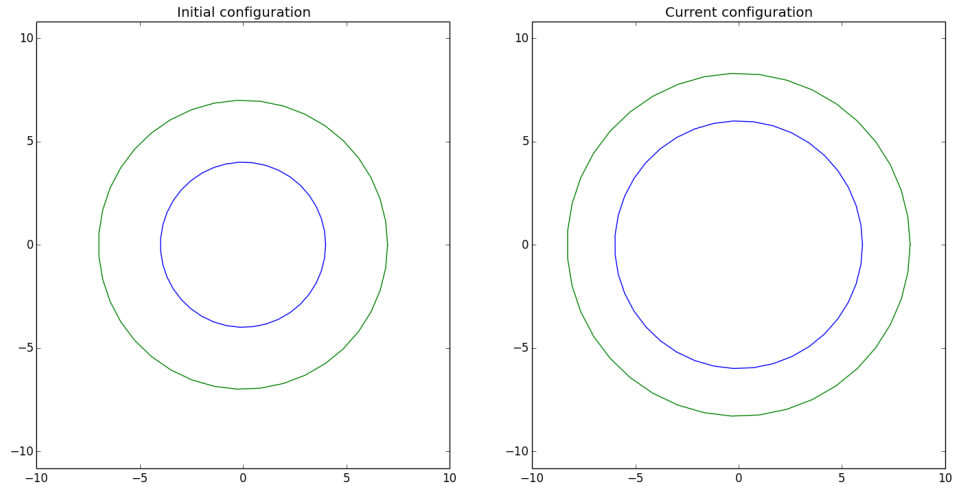


Figure 2: Visualisation of displacement from the reference to the "current" configuration under pressure.

For the next figure 3, we use following values:  $r_i = 1.93 \text{ mm}$ ,  $r_e = 2.25 \text{ mm}$ , and  $\mu = 27.9 \text{ kPa}$ , see [10].

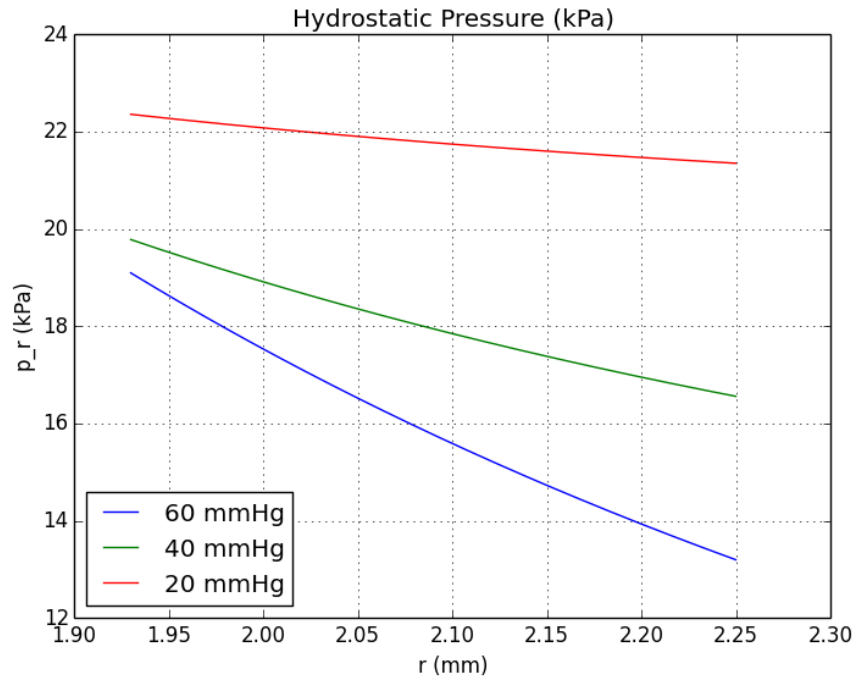


Figure 3: Hydrostatic pressure at 20, 40 and 60 mmHg

#### 4.8. Saint-Venant-Kirchhoff material

As an other example, suppose that the constitutive law is those of a Saint-Venant-Kirchhoff material. According to the constitutive law adapted to this case

$$\boldsymbol{\Sigma} = \lambda \operatorname{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E},$$

Second Piola-Kirchhoff stress is written:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \frac{du_r}{dr} + \mu \left(\frac{du_r}{dr}\right)^2 & 0 & 0 \\ 0 & \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \frac{u_r}{r} + \mu \left(\frac{u_r}{r}\right)^2 & 0 \\ 0 & 0 & \lambda \operatorname{tr}(\mathbf{E}) \end{pmatrix}$$

and allows to find the first Piola-Kirchhoff stress tensor  $\mathbf{\Pi} = \mathbf{F}\boldsymbol{\Sigma}$ :

$$\mathbf{F}\boldsymbol{\Sigma} = \begin{pmatrix} \left(1 + \frac{du_r}{dr}\right) \left[\lambda \operatorname{tr}(\mathbf{E}) + 2\mu \frac{du_r}{dr} + \mu \left(\frac{du_r}{dr}\right)^2\right] & 0 & 0 \\ 0 & \left(1 + \frac{u_r}{r}\right) \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \frac{u_r}{r} + \mu \left(\frac{u_r}{r}\right)^2 & 0 \\ 0 & 0 & \lambda \operatorname{tr}(\mathbf{E}) \end{pmatrix}.$$

Thus

$$\vec{\operatorname{div}}(\mathbf{F}\boldsymbol{\Sigma}) = \begin{pmatrix} \frac{\partial \mathbf{F}\boldsymbol{\Sigma}_{11}}{\partial r} + \frac{1}{r}(\mathbf{F}\boldsymbol{\Sigma}_{11} - \mathbf{F}\boldsymbol{\Sigma}_{22}) \\ 0 \\ 0 \end{pmatrix}$$

and

$$\operatorname{tr}(\mathbf{E}) = \frac{du_r}{dr} + \frac{1}{2} \left(\frac{du_r}{dr}\right)^2 + \frac{u_r}{r} + \frac{1}{2} \left(\frac{u_r}{r}\right)^2.$$

Term by term, calculations show that:

$$\begin{aligned} \mathbf{F}\boldsymbol{\Sigma}_{11} - \mathbf{F}\boldsymbol{\Sigma}_{22} &= \left(1 + \frac{du_r}{dr}\right) \left[\lambda \operatorname{tr}(\mathbf{E}) + \mu \left(2 \frac{du_r}{dr} + \left(\frac{du_r}{dr}\right)^2\right)\right] - \left(1 + \frac{u_r}{r}\right) \left[\lambda \operatorname{tr}(\mathbf{E}) + \mu \left(2 \frac{u_r}{r} + \left(\frac{u_r}{r}\right)^2\right)\right] \\ &= \lambda \operatorname{tr}(\mathbf{E}) \left(\frac{du_r}{dr} - \frac{u_r}{r}\right) + \mu \left[2 \frac{du_r}{dr} + \left(\frac{du_r}{dr}\right)^2 - 2 \frac{u_r}{r} - \left(\frac{u_r}{r}\right)^2\right] + \mu \left[2 \left(\frac{du_r}{dr}\right)^2 + \left(\frac{du_r}{dr}\right)^3 - 2 \left(\frac{u_r}{r}\right)^2 - \left(\frac{u_r}{r}\right)^3\right] \\ &= \lambda \left[\left(\frac{du_r}{dr}\right)^2 + \frac{1}{2} \left(\frac{du_r}{dr}\right)^3 + \frac{u_r}{r} \frac{du_r}{dr} + \frac{1}{2} \left(\frac{u_r}{r}\right)^2 \frac{du_r}{dr} - \frac{u_r}{r} \frac{du_r}{dr} - \frac{1}{2} \left(\frac{du_r}{dr}\right)^2 \frac{u_r}{r} - \left(\frac{u_r}{r}\right)^2 - \frac{1}{2} \left(\frac{u_r}{r}\right)^3\right] \\ &+ \mu \left[2 \frac{du_r}{dr} + 3 \left(\frac{du_r}{dr}\right)^2 + \left(\frac{du_r}{dr}\right)^3 - 2 \frac{u_r}{r} - 3 \left(\frac{u_r}{r}\right)^2 - \left(\frac{u_r}{r}\right)^3\right] \\ &= \left(2\mu - \frac{\lambda}{2} \frac{u_r}{r} \frac{du_r}{dr}\right) \left(\frac{du_r}{dr} - \frac{u_r}{r}\right) \\ &+ (\lambda + 3\mu) \left(\left(\frac{du_r}{dr}\right)^2 - \left(\frac{u_r}{r}\right)^2\right) \\ &+ \left(\frac{\lambda}{2} + \mu\right) \left(\left(\frac{du_r}{dr}\right)^3 - \left(\frac{u_r}{r}\right)^3\right). \end{aligned}$$

Furthermore

$$\begin{aligned} \mathbf{F}\boldsymbol{\Sigma}_{11} &= \left(1 + \frac{du_r}{dr}\right) \left[\lambda \left(\frac{du_r}{dr} + \frac{1}{2} \left(\frac{du_r}{dr}\right)^2 + \frac{u_r}{r} + \frac{1}{2} \left(\frac{u_r}{r}\right)^2\right) + \mu \left(2 \frac{du_r}{dr} + \left(\frac{du_r}{dr}\right)^2\right)\right] \\ &= \lambda \left[\frac{u_r}{r} + \frac{1}{2} \left(\frac{u_r}{r}\right)^2 + \frac{du_r}{dr} + \frac{3}{2} \left(\frac{du_r}{dr}\right)^2 + \frac{1}{2} \left(\frac{du_r}{dr}\right)^3 + \frac{u_r}{r} \frac{du_r}{dr} + \frac{1}{2} \left(\frac{u_r}{r}\right)^2 \frac{du_r}{dr}\right] \\ &+ \mu \left[2 \frac{du_r}{dr} + 3 \left(\frac{du_r}{dr}\right)^2 + \left(\frac{du_r}{dr}\right)^3\right]. \end{aligned}$$

Some intermediates calculations could be helpful:

$$\begin{aligned}
\frac{d}{dr}\left(\frac{u_r}{r}\right) &= \frac{1}{r}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right), \\
\frac{d}{dr}\left(\frac{u_r}{r}\right)^2 &= 2\frac{u_r}{r^2}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right), \\
\frac{d}{dr}\left(\frac{du_r}{dr} \frac{u_r}{r}\right) &= \frac{u_r}{r} \frac{d^2u_r}{dr^2} + \frac{1}{r}\left(\left(\frac{du_r}{dr}\right)^2 - \left(\frac{du_r}{dr} \frac{u_r}{r}\right)\right), \\
\frac{d}{dr}\left(\left(\frac{u_r}{r}\right)^2 \frac{du_r}{dr}\right) &= \left(\frac{u_r}{r}\right)^2 \frac{d^2u_r}{dr^2} + 2\frac{u_r}{r^2}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\mathbf{F}\Sigma_{11}}{dr} &= \frac{\lambda}{r}\left(1 + \frac{u_r}{r}\right)\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) + (2\mu + \lambda)\frac{d^2u_r}{dr^2} \\
&+ (6\mu + 3\lambda)\left(\frac{du_r}{dr} \frac{d^2u_r}{dr^2}\right) + 3\left(\frac{\lambda}{2} + \mu\right)\left(\frac{du_r}{dr}\right)^2 \frac{d^2u_r}{dr^2} \\
&+ \lambda\left(\frac{u_r}{r} \frac{d^2u_r}{dr^2} + \frac{1}{r}\left(\left(\frac{du_r}{dr}\right)^2 - \frac{u_r}{r} \frac{du_r}{dr}\right) + \frac{1}{2}\left(\frac{u_r}{r}\right)^2 \frac{d^2u_r}{dr^2} + \frac{1}{r} \frac{u_r}{r} \frac{du_r}{dr} \left(\frac{du_r}{dr} - \frac{u_r}{r}\right)\right).
\end{aligned}$$

We finally obtain:

$$\begin{aligned}
\vec{\text{div}}(\mathbf{F}\Sigma) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= -\left(\frac{2\mu + \lambda}{r}\right)\frac{u_r}{r} + \left(\frac{2\mu + \lambda}{r}\right)\frac{du_r}{dr} + (2\mu + \lambda)\left(\frac{du_r}{dr}\right)^2 \\
&- \left(\frac{2\lambda + 3\mu}{r}\right)\left(\frac{u_r}{r}\right)^2 - \frac{1}{r}\left(\frac{\lambda}{2} + \mu\right)\left(\frac{u_r}{r}\right)^3 \\
&+ \left(\frac{2\lambda + 3\mu}{r}\right)\left(\frac{du_r}{dr}\right)^2 + \frac{1}{r}\left(\frac{\lambda}{2} + \mu\right)\left(\frac{du_r}{dr}\right)^3 \\
&- \frac{1}{r}\left(1 - \frac{\lambda}{2}\right)\left(\left(\frac{u_r}{r}\right)^2 \frac{du_r}{dr}\right) + \frac{1}{r}\left(1 - \frac{\lambda}{2}\right)\left(\frac{u_r}{r}\left(\frac{du_r}{dr}\right)^2\right) \\
&+ \lambda\frac{u_r}{r} \frac{d^2u_r}{dr^2} + \frac{\lambda}{2}\left(\frac{u_r}{r}\right)^2 \frac{d^2u_r}{dr^2} \\
&+ (6\mu + 3\lambda)\frac{du_r}{dr} \frac{d^2u_r}{dr^2} + \frac{1}{2}(6\mu + 3\lambda)\left(\frac{du_r}{dr}\right)^2 \frac{d^2u_r}{dr^2}
\end{aligned} \tag{20}$$

Then, using the above expression and the equilibrium equation Eq. (10), an approximated numerical solution could be obtained.

## 5. Conclusion

In this short technical note, after a brief review of the general hyperelasticity framework and the detail of the derivation of the weak form, the theory was applied to a the case of the mechanical response of an artery modelled under pressure loading. This can be helpful to verify numerical solutions with finite elements, in the same spirit as manufactured solutions [5]. This is a first step towards semi-analytical solutions with active constitutive laws such as presented in [10].

## Appendix A. Computations using the first Piola-Kirchhoff stress tensor

### Appendix A.1. Boundary conditions

A pressure  $p_{\text{ext}}$  is imposed on  $\Gamma_e$ , where  $\vec{x} = r_e \vec{e}_r$ :

$$\mathbf{\Pi} \vec{n}(\vec{x}) = p_{\text{ext}}, \tag{A.1}$$

On  $\Gamma_i$ , where  $\vec{x} = r_i \vec{e}_r$ , we apply either:

$$\mathbf{\Pi} \vec{n}(\vec{x}) = -p_{\text{int}} \vec{n} \quad (\text{A.2})$$

or

$$\vec{u}_r(r_i) = u_{\text{int}} \vec{n}, \quad (\text{A.3})$$

where  $p_{\text{int}}$  and  $u_{\text{int}}$  are fixed internal pressure and displacement, and  $\vec{n}$  normal vector pointing outward to the boundary, which is either  $\vec{e}_r$  for  $\Gamma_e$  or  $-\vec{e}_r$  for  $\Gamma_i$ .

### Appendix A.2. Equilibrium

We start from

$$\vec{\text{div}}(\mathbf{\Pi}) = \vec{\text{div}}(\mathbf{F}\mathbf{\Sigma}) = \begin{pmatrix} \frac{\partial \mathbf{F}\mathbf{\Sigma}_{11}}{\partial r} + \frac{1}{r}(\mathbf{F}\mathbf{\Sigma}_{11} - \mathbf{F}\mathbf{\Sigma}_{22}) \\ 0 \\ 0 \end{pmatrix} \quad (\text{A.4})$$

and compute second Piola-Kirchhoff tensor as follows:

$$\mathbf{\Sigma} = 2 \frac{\partial W}{\partial \mathbf{C}} - p \mathbf{C}^{-1} = 2 \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\mu}{2} (\text{tr}(\mathbf{C}) - d) \right] - p \mathbf{C}^{-1} = \mu \frac{\partial (\mathbf{C}_{kk} - d)}{\partial \mathbf{C}} - p \mathbf{C}^{-1} = \mu \mathbf{I} - p \mathbf{C}^{-1}$$

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\left(1 + \frac{du_r}{dr}\right)^2} & 0 & 0 \\ 0 & \frac{1}{\left(1 + \frac{u_r}{r}\right)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} \mu - \frac{p}{\left(1 + \frac{du_r}{dr}\right)^2} & 0 & 0 \\ 0 & \mu - \frac{p}{\left(1 + \frac{u_r}{r}\right)^2} & 0 \\ 0 & 0 & \mu - p \end{bmatrix}$$

in order to obtain:

$$\mathbf{F}\mathbf{\Sigma} = \begin{bmatrix} \mu \left(1 + \frac{du_r}{dr}\right) - \frac{p}{\left(1 + \frac{du_r}{dr}\right)} & 0 & 0 \\ 0 & \mu \left(1 + \frac{u_r}{r}\right) - \frac{p}{\left(1 + \frac{u_r}{r}\right)} & 0 \\ 0 & 0 & \mu - p \end{bmatrix}$$

$$\frac{d\mathbf{F}\mathbf{\Sigma}_{11}}{dr} + \frac{1}{r}(\mathbf{F}\mathbf{\Sigma}_{11} - \mathbf{F}\mathbf{\Sigma}_{22}) = \mu \frac{d^2 u_r}{dr^2} + p \frac{d^2 u_r}{dr^2} \frac{1}{\left(1 + \frac{du_r}{dr}\right)^2} - \frac{dp}{dr} \frac{1}{1 + \frac{du_r}{dr}} + \frac{1}{r} \left[ \mu \left( \frac{du_r}{dr} - \frac{u_r}{r} \right) - p \left( \frac{1}{\left(1 + \frac{du_r}{dr}\right)} - \frac{1}{\left(1 + \frac{u_r}{r}\right)} \right) \right] \quad (\text{A.5})$$

It results from Equilibrium equation Eq. (10) combined with Eqs. (A.4) and (A.5):

$$\left( \mu + \frac{p}{\left(1 + \frac{du_r}{dr}\right)^2} \right) \frac{d^2 u_r}{dr^2} - \frac{dp}{dr} \frac{1}{\left(1 + \frac{du_r}{dr}\right)} + \frac{\mu}{r} \left( \frac{du_r}{dr} - \frac{u_r}{r} \right) - \frac{p}{r} \left( \frac{1}{\left(1 + \frac{du_r}{dr}\right)} - \frac{1}{\left(1 + \frac{u_r}{r}\right)} \right) = 0 \quad (\text{A.6})$$

This equation will give the differential equation of  $u_r$ , as a function of hydrostatic pressure  $p$ . Let us finally take the result of the fundamental principle of statics given by Eq. (10):

$$p \left[ \frac{\frac{d^2 u_r}{dr^2}}{\left(1 + \frac{du_r}{dr}\right)^2} + \frac{1}{r} \left( \frac{1}{1 + \frac{u_r}{r}} - \frac{1}{1 + \frac{du_r}{dr}} \right) \right] - \frac{dp}{dr} \frac{1}{1 + \frac{du_r}{dr}} = \frac{\mu}{r} \left( \frac{u_r}{r} - \frac{du_r}{dr} \right) - \mu \frac{d^2 u_r}{dr^2}.$$

## Appendix B. A few lines of Python

We provide here a Python script for the numerical application.

Listing 1: ro po

```
import matplotlib.pyplot as plt
import math
import numpy as np
from pylab import *

# Il s'agit de determiner des valeurs numériques approchées des constantes ro et po

# Ceci permettant ensuite de calculer en tout point du domaine :
# le déplacement, la pression hydrostatique,
# et d'autres paramètres tels que les contraintes principales,
# contrainte de Von Mises, déformation...

# --- Paramètres (unité SI)
ri = 1.93e-3 # en m. Rayon interne du cylindre parfait
re = 2.25e-3 # rayon externe
Ri = ri*ri
Re = re*re

r1 = 1.99e-3 # points de calcul du modele numerique
r2 = 2.06e-3
r3 = 2.12e-3
r4 = 2.19e-3

mu = 27.9e3 # en Pa.
# En realite, cette valeur de 27.9kPa
# correspond a une donnee pour l'intima, et non la media

# Conditions de bords en pression.
# on etudie une difference de pression : 100-80 = 20 mmHg
p_int = 2.666e3 #13.33e3 = 100 mmHg
p_ext = 0 #10.664e3 = 80 mmHg

# --- Fonction pour determiner ro --- (Phi(ro) = 0)
def Phi(x): #tq Phi(ro)=0

    terme_mu = - mu*( (Ri/(Ri+x**2)) - (Re/(Re+x**2)) );
    terme_mu12 = (mu/2)*( (2*Ri*Ri + x**2*(Ri+x**2)) / (Ri*(Ri+x**2))
    - (2*Re*Re + x**2*(Re+x**2)) / (Re*(Re+x**2)) );
    terme_ln = mu* log( sqrt( (Re*(Ri+x**2)) / (Ri*(Re+x**2)) ) );
    return( - p_int - p_ext + terme_mu + terme_mu12 + terme_ln);

# Fonction dichotomie pour trouver le zero d'une fonction f
def dichotomie(f,a,b,prec):

    while b-a>prec:
        c = (a+b)/2
        if f(a)*f(c) <= 0:
            b = c
        else:
            a = c
    return a

# Determination des constantes --ro-- et --po--
ro = dichotomie(Phi,0,0.1,1e-15) #attention, ro est une constante en m
Ro = ro**2

po = mu*(Re/(Re+Ro)) - (mu/2)*(2*Re**2 + Ro*(Re+Ro)) / (Re*(Re+Ro))
- mu*log( sqrt(Re+Ro)/re ) + p_ext

alpha= -mu*(Ri/(Ri+Ro)) + (mu/2)*(2*Ri**2 + Ro*(Ri+Ro)) / (Ri*(Ri+Ro))
+ mu*log( sqrt(Ri+Ro)/ri ) + po # verification : alpha = pint
```

```

# graphe permettant de situer ro, avec Phi(ro)=0
plt.figure(1)
x_def = np.linspace(-0.4,0.4,500)
y_def = [ Phi(i) for i in x_def]
plt.plot(x_def, y_def, marker="t")
#plt.xlim(-0.1,0.1)
#plt.ylim(-10000,10000)
plt.title("Phi(x) - Phi(ro) = 0")
plt.legend()
xlabel("x")
ylabel("Phi")
plt.grid(True)
plt.show()

# -----

# Determination du champ de déplacement, visualisation

def u_r(r): #attention, r en m, donc u_r en m
    return( sqrt(r**2 + ro**2) - r)

plt.figure(2)
plt.grid(True)
x_def = np.linspace(ri*1e3, re*1e3, 10000)
y_def = [ u_r(i*1e-3)*1e3 for i in x_def]
plt.plot(x_def, y_def, label = "20 mmHg")
x = np.array([1.93, 1.99, 2.06, 2.12, 2.19, 2.25]) # numerical model
#[ri*1e3, r1*1e3, r2*1e3, r3*1e3, r4*1e3, re*1e3]
y = np.array([0.484669, 0.476807, 0.469517, 0.462759, 0.456496, 0.450694]) #
plt.plot(x, y, 'ro')
plt.xlim(1.9, 2.3)
plt.title("Displacement_u_r(r)")
plt.legend()
xlabel("r(mm)")
ylabel("u(mm)")
plt.grid(True)
plt.show()

# -----

# Determination de la pression hydrostatique, visualisation, kPa

def p_r(r):
    p = mu * (2*r**4 + ro**2*(r**2+ro**2))/(2*r**2*(r**2+ro**2))
    + mu * log( sqrt(r**2+ro**2)/r) + po
    return(p)

plt.figure(3)
plt.grid(True)
x_def = np.linspace(ri*1e3, re*1e3, 10000)
y_def = [ (p_r(i*1e-3))*1e-3 for i in x_def]
plt.plot(x_def, y_def, label = "20 mmHg")
x2 = np.array([1.93, 1.99, 2.06, 2.12, 2.19, 2.25])
y2 = np.array([6.2, 5.9, 5.8, 5.8, 5.7, 5.6])
plt.plot(x2, y2, 'ro')
plt.xlim(1.9, 2.3)
plt.title("Hydrostatic_Pressure_(kPa)")
plt.legend(loc = 'best')
xlabel("r(mm)")
ylabel("p_r_(kPa)")
plt.show()

# -----

# Determination des contraintes, principales et Von Mises, visualisation

def sigma_rr(r):
    lambd_r2 = r**2/(r**2+ro**2) #lambda_r au carré
    return( mu*lambd_r2 - p_r(r) )

def sigma_tt(r):
    lambd_t2 = 1 + ro**2/r**2 #lambda_theta au carré
    return( mu*lambd_t2 - p_r(r) )

def sigma_zz(r):

```

```

    return( mu - p_r(r) )

def sigma_VM(r):
    return ( 1/(sqrt(2))* sqrt( (sigma_rr(r)-sigma_tt(r))**2
    + (sigma_tt(r)-sigma_zz(r))**2 + (sigma_zz(r)-sigma_rr(r))**2 ) )

# Contrainte radiale Sigma_rr (Principal radial stress)
plt.figure(4)
#plt.subplot(121)
plt.grid(True)
x_def = np.linspace(ri*1e3,re*1e3,1000)
y_def = [ -sigma_rr(i*1e-3)*1e-3 for i in x_def]
#y3_def = [ sigma_tt(i*1e-3) for i in x_def]
#y4_def = [ sigma_zz(i*1e-3) for i in x_def]
plt.plot(x_def, y_def, label = "Analytical_S11")
#plt.plot(x_def, y3_def, label = "sigma_tt")
#plt.plot(x_def, y4_def, label = "sigma_zz")
plt.xlim(ri*1e3-0.05,re*1e3+0.05)
x = np.array([1.93,1.99,2.06,2.12,2.19,2.25])
# numerical model
y = np.array([0.00252897e3,0.00205152e3,0.00148911e3,0.000959144e3,0.000459518e3,0.0000857227e3])
#
plt.plot(x, y, 'ro', label = "Numerical_S11")
plt.title("Principal_Radial_Stress_(kPa)")
plt.legend(loc = 'best') #met la legende au meilleur endroit
xlabel("r_(mm)")
ylabel("sigma_11_(kPa)")
plt.show()

# Contrainte principale selon theta
#
# plt.figure(4)
# plt.subplot(122)
# plt.grid(True)
# x_def = np.linspace(ri*1e3,re*1e3,1000)
# y3_def = [ sigma_tt(i*1e-3) for i in x_def]
# plt.plot(x_def, y3_def, label = "sigma_tt")
# #plt.xlim(ri*1e3-0.05,re*1e3+0.05)
# #plt.ylim(sigma_rr(ri)-1000,sigma_zz(re)+1000)
# plt.title("stress (r)")
# plt.legend(loc = 'best')
# xlabel("r (mm)")
# ylabel("sigma_tt (Pa)")
# plt.show()

# Affichage contrainte Von Mises en MPa
plt.figure(5)
plt.grid(True)
x_def = np.linspace(ri*1e3,re*1e3,1000)
y5_def = [ (sigma_VM(i*1e-3))*1e-3 for i in x_def]
plt.plot(x_def, y5_def, label = "20_mmHg")
plt.xlim(1.9,2.27)
x = np.array([1.93,1.99,2.06,2.12,2.19,2.25])
# numerical model
y = np.array([0.0282709e3,0.0267656e3,0.0253366e3,0.0240268e3,0.0228226e3,0.0216665e3])
#
plt.plot(x, y, 'ro')
plt.title("Von_Mises_Stress_(kPa)")
plt.legend(loc = 'best')
xlabel("r_(mm)")
ylabel("sigma_VM_(kPa)")
plt.show()

```

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Hyperelasticity</b>	<b>2</b>
2.1	Notations . . . . .	2
2.2	Hyperelastic constitutive law . . . . .	2
2.3	Having some fun with differential calculus . . . . .	3
2.4	Weak formulation . . . . .	4

<b>3</b>	<b>Linearization for hyperelasticity</b>	<b>4</b>
3.1	Details of the linearization procedure . . . . .	5
3.2	Re-interpretation “à la Schotté” of the tangent stiffness weak form . . . . .	5
3.3	Saint-Venant-Kirchhoff material . . . . .	6
<b>4</b>	<b>Inflation of an artery modelled as a hyperelastic cylinder : semi-analytical solution</b>	<b>6</b>
4.1	Geometry and boundary conditions . . . . .	7
4.2	Incompressible neo-hookean material . . . . .	7
4.3	Equilibrium equation . . . . .	7
4.4	First calculations in cylindrical coordinates . . . . .	8
4.5	Expression for the displacement . . . . .	8
4.6	Stress and equilibrium . . . . .	9
4.7	Numerical application . . . . .	11
4.8	Saint-Venant-Kirchhoff material . . . . .	12
<b>5</b>	<b>Conclusion</b>	<b>13</b>
<b>Appendix A</b>	<b>Computations using the first Piola-Kirchhoff stress tensor</b>	<b>13</b>
Appendix A.1	Boundary conditions . . . . .	13
Appendix A.2	Equilibrium . . . . .	14
<b>Appendix B</b>	<b>A few lines of Python</b>	<b>15</b>

## References

- [1] G. Allaire. A review of adjoint methods for sensitivity analysis, uncertainty quantification and optimization in numerical codes. *Ingénieurs de l’Automobile*, 836:33–36, July 2015.
- [2] R. Becker and R. Rannacher. An optimal control approach to a posteriori error estimation in finite element methods. *Acta Numerica*, 10:1–102, 2001. ISSN 0962-4929.
- [3] J. Bonet and R. D. Wood. *Nonlinear continuum mechanics for finite element analysis*. Cambridge University Press, Cambridge (UK), New York, 2008.
- [4] L. Bouten. *Identification des propriétés mécaniques des tissus constitutifs du mollet pour l’étude mécanique de la contention*. PhD thesis, Ecole Nationale Supérieure des Mines de Saint-Etienne, France, 2009.
- [5] É. Chamberland, A. Fortin, and M. Fortin. Comparison of the performance of some finite element discretizations for large deformation elasticity problems. *Computers & Structures*, 88(11-12):664–673, 2010.
- [6] A. Fortin and A. Garon. *Les éléments finis: de la théorie à la pratique*. Université Laval, 2011.
- [7] Y. C. Fung. *Biomechanics: mechanical properties of living tissues*. Springer Science & Business Media, 2013.
- [8] A. Logg, K.-A. Mardal, and G. Wells. *Automated solution of differential equations by the finite element method: The FEniCS book*, volume 84. Springer Science & Business Media, 2012.
- [9] R. Mlika, Y. Renard, and F. Chouly. An unbiased Nitsche’s formulation of large deformation frictional contact and self-contact. *Computer Methods in Applied Mechanics and Engineering*, 325:265–288, 2017.
- [10] Y. Payan and J. Ohayon. *Biomechanics of Living Organs - Hyperelastic Constitutive Laws for Finite Element Modeling*. World Bank Publications, 2017.
- [11] Y. Renard and K. Poullos. GetFEM: automated FE modeling of multiphysics problems based on a generic weak form language. *ACM Trans. Math. Software*, 47(1):Art. 4, 31, 2021. ISSN 0098-3500.
- [12] A. Romo, P. Badel, A. Duprey, J.-P. Favre, and S. Avril. In vitro analysis of localized aneurysm rupture. *Journal of Biomechanics*, 47(3):607–616, 2014. ISSN 0021-9290.

- [13] H. Schmidt, F. Galbusera, A. Rohlmann, T. Zander, and H.-J. Wilke. Effect of multilevel lumbar disc arthroplasty on spine kinematics and facet joint loads in flexion and extension: a finite element analysis. *European Spine Journal*, 21(5):663–674, 2012. ISSN 0940-6719, 1432-0932.
- [14] J.-S. Schotté and R. Ohayon. Various modelling levels to represent internal liquid behaviour in the vibration analysis of complex structures. *Computer Methods in Applied Mechanics and Engineering*, 198(21-26):1913–1925, 2009.
- [15] T. E. Yankeelov, N. Atuegwu, D. Hormuth, J. A. Weis, S. L. Barnes, M. I. Miga, E. C. Rericha, and V. Quaranta. Clinically relevant modeling of tumor growth and treatment response. *Science Translational Medicine*, 5(187):187ps9–187ps9, 2013. ISSN 1946-6234.