Strategic stability of equilibria in multi-sender signaling games

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Strategic Stability of Equilibria in Multi-Sender Signaling Games*

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Abstract

The concept of unprejudiced beliefs equilibrium is simple: out-of-equilibrium beliefs should be consistent with the principle that multiple deviations are infinitely less likely than single deviations. Our questions are: does there always exist such an equilibrium and what can be done if there are multiple such equilibria?

To select a unique equilibrium, this notion is usually coupled with the intuitive criterion. The simultaneous usage of these concepts is ad hoc, unjustified, and again might eliminate all the equilibria.

We show that coupling these notions is legitimate, as both are implied by strategic stability (Kohlberg and Mertens (1986)), hence a desired equilibrium always exists. The intuitive criterion is trivially implied by stability. We show that in generic multi-sender signaling games stable outcomes can be supported with unprejudiced beliefs. It follows by forward induction that stable sets contain an equilibrium which is unprejudiced and intuitive at the same time.

In many applications where pooling is an issue, the senders have only two possible types. Our result offers a simple tool for analyzing games where the senders can have arbitrarily (but finitely) many types.

Keywords: multi-sender signaling, unprejudiced beliefs, strategic stability, forward induction

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1 Introduction

It is well known that in extensive form games restricting the out-of-equilibrium beliefs can eliminate equilibria which are not sensible. In this paper we investigate the usage of the extremely simple and powerful restriction of Bagwell and Ramey (1991), dubbed unprejudiced beliefs, in signaling games with multiple senders. In several applications, see e.g., Bagwell and Ramey (1991), Bester and Demuth (2015), Schultz (1996), (1999), and Hartman-Glaser and Hébert (2019), this restriction is used together with versions of the intuitive criterion (see Cho and Kreps (1987)), so as to be able to eliminate (or to justify) undesirable pooling, yet unprejudiced equilibria.[1] Some of these papers report the non-existence of pure equilibrium outcomes which can be supported both by unprejudiced and by intuitive beliefs. Unprejudiced beliefs and the intuitive criterion are seemingly unrelated concepts. Hence, their simultaneous usage, even though successful and frequent, seems to be ad hoc, unjustified and can yield to eliminate all the (pure) equilibria. Moreover, there can be undesirable equilibrium outcomes which can be supported by both types of beliefs but cannot be supported with a belief which is unprejudiced and intuitive at the same time (see our example in section 1.4).

Our question is: is it legitimate to couple these concepts? Does there always exist an equilibrium satisfying both of these concepts? Our answer and main contribution is: yes, in the sense that both are implied by strategic stability (see Kohlberg and Mertens (1986)). More precisely, a strategically stable set contains an equilibrium which is unprejudiced and intuitive (or D1 etc.).

A strategically stable set of equilibria (henceforth: stable) always exists, exhibits desirable properties and narrows down the set of equilibria. Although the consequences of stability are well understood when there is a single-sender (see Banks and Sobel (1987), Cho and Kreps (1987), and Cho and Sobel (1990)), the properties of stable outcomes of multi-sender games have not yet been analyzed.[2] It is obvious, by

[1] Other applications of unprejudiced beliefs where pooling is not an issue can be found for example in Emons and Fluet (2009) and in Zhang (2020).

[2] An exception is Honryo (2018), where the full power of stability is used in a multi-sender setting because unprejudiced beliefs have no bite on the multiple equilibria.

the application of forward induction, that the multi-sender versions of the intuitive criterion are also implied by stability. We show that stable equilibrium outcomes have the following additional desirable property. They can always be maintained with unprejudiced beliefs. It follows that the simultaneous usage of unprejudiced beliefs and the intuitive criterion is justified. The existence of an equilibrium (possibly in mixed strategies) satisfying both of these concepts is also ensured.

First, in sections 1.1, 1.2, 1.3 we explain by an example with economic content: the main difference between the structures of the out-of-equilibrium events in the single and the multiple sender case, what unprejudiced beliefs are, and the gist which is common in all the applications mentioned above. In this example we concentrate on pure equilibria, just as in the papers mentioned above. Pooling equilibria are eliminated solely by a version of the intuitive criterion, and then unprejudiced beliefs select a unique equilibrium among the separating equilibria. This method is successful, just because the senders are assumed to have only two types, exactly as in all the papers mentioned above.

Second, to provide further motivation and to show that something new can be done using our result, in section 1.4 we extend our previous example and allow for the possibility of a third type of the senders. We show that there is an undesirable equilibrium *outcome* which can be supported by unprejudiced beliefs and also by beliefs satisfying various versions of the intuitive criterion. However, none of these intuitive beliefs are unprejudiced. Hence, the outcome could only be eliminated by requiring that the supporting beliefs are unprejudiced and intuitive at the same time. Doing this elimination safely, i.e. without risking to lose the existence of a desirable equilibrium, is extremely subtle. In short, one must use our result and the full power of Proposition 6 in Kohlberg and Mertens (1986) (we discuss this in detail in section 1.4). It is not surprising that in the applications listed above the senders cannot have more than two types. Eliminating an equilibrium outcome solely on the ground of some versions of the intuitive criterion *x*or solely on the ground of unprejudiced beliefs is something that many authors have done already. Contrary to other scholars, we still believe that doing so is unjustified and risky without knowing our result.

But eliminating an equilibrium outcome because it cannot be supported with beliefs which are unprejudiced *and* intuitive *at the same time* has never been done. Yet, using our result, this can be safely done without risking the non-existence of a desired equilibrium.

Summing up, our result ensures that one can always find an equilibrium that can be supported by intuitive and unprejudiced beliefs and provides a safe and effective tool for analyzing and solving models having more than two types.

We conclude the introduction with section 1.5 in which we restate our main contribution and describe the structure of the paper.

1.1 The Single Sender Case

To understand the notion of unprejudiced beliefs and the main differences between the single- and the multi-sender case, consider a variant of the job market signaling model of Spence (1973) similar to that of in Cho and Kreps (1987). Let us start with a single worker whose marginal product t can be 0 (low) or 1 (high), with equal probabilities, and t is known to the worker but unknown to the firm. The worker chooses a level of education m, to signal his type, which changes his marginal product to $4tm$. The firm, after observing m, forms a belief about the worker's type t and pays a wage equal to his expected marginal product. On the equilibrium path beliefs must be formed using Bayes rule, out-of-equilibrium the beliefs can be arbitrary. Education is costly for the worker and its cost is given by $(3-t)m^2$. Hence, given m and that the firm's belief, conditional on m, about t is $\tau(m)(t)$, the type t worker's payoff is $4m\sum_t t\tau(m)(t) - (3-t)m^2$. It is well known that if the set of available signals is sufficiently rich, there is a unique equilibrium outcome that survives a version of the intuitive criterion (called D1) and that this is the unique stable outcome (see for example Cho and Sobel (1990)). In this equilibrium outcome the types fully separate: the low type chooses $m = 0$, gets 0 wage, and 0 utility. The high type's equilibrium signal is the lowest possible signal that the low type does not want to mimic. This is the signal 4/3, and the high type gets 16/3 wage and 16/9 utility. The firm's belief about t is 0 after observing the equilibrium signal 0, and it is 1 after the equilibrium signal 4/3. For any unsent, out-of-equilibrium signal $m \in (0, 4/3)$, the firm believes that the worker is of low type with high enough probability.[3] This is necessary for maintaining the equilibrium. To see this, for example, should the firm assign 0 probability to the low type after observing the out-of-equilibrium signal $m = 1$, the high type can deviate and achieve his first best utility level (given that the firm knows his type) $2 > 16/9$ by sending the signal 1.

1.2 The Multiple Sender Case, Unprejudiced Beliefs

In this section, we argue that in some situations the firm should indeed attach 0 probability to the low type at the unexpected event when observing the signal $m = 1$. Assume there is another worker with the same (perfectly, positively correlated) type, with the same equilibrium behavior as described in the single sender case? Suppose the firm observes the out-of-equilibrium signal 1 from one of the workers and the equilibrium signal 4/3 from the other at the same time? Should not the firm infer that the workers are high type from the fact that one of the workers is sending the equilibrium signal 4/3 of the high type? This brings us to the concept of unprejudiced beliefs.

First, we describe the game with two senders which we derive from the game with a single sender described above. Second, we describe an equilibrium of this game in which both senders and the receiver behave exactly as in the equilibrium we discussed in the single sender case. Third, we argue that this equilibrium is not sensible. We question the plausibility of the out-of-equilibrium beliefs with which this equilibrium can be maintained. We argue that the justification of these beliefs is too complicated (or prejudiced). We show that there are other simpler (or unprejudiced) beliefs which are much easier to justify given the putative equilibrium. However, with these simpler beliefs this equilibrium cannot be maintained, it falls apart.

Hence, assume that there are two workers, husband and wife, who are assortatively matched, i.e., their types are perfectly, positively correlated which we denote also by $t \in \{0, 1\}$. That is, either both have low types or both have high types. Assume that

[3] For signals higher than 4/3 the beliefs can be chosen arbitrarily.

the husband's preference is the same as described above and it does not depend on the signal sent by the wife neither it depends on the wage the wife receives. Assume the same for the wife's preferences. Further assume that signals are sent simultaneously and that the firm forms a belief about t and pays each worker his and her own expected marginal product. Clearly, the firm cannot ever believe that the workers have different types. The signaling strategy profile, in which both workers signal according to the equilibrium described above in the single sender case, can be maintained as a sequential equilibrium outcome. Namely, there is an equilibrium in which the firm either observes the signal pair (0,0) and concludes that both of them have low type or observes the signal pair (4/3, 4/3) and concludes that both of them have high type. Individual deviations from this equilibrium behavior are deterred just as in the single sender case. Namely, after any out-of-equilibrium signal pair, say for example after (1, 4/3), the firm believes that the workers are of low type with high enough probability and pays a sufficiently low wage which makes these deviations unattractive.

We do not believe in the plausibility of this equilibrium because we find the out-of-equilibrium beliefs of the firm rather strange (prejudiced). To demonstrate this, consider an unilateral deviation and suppose that the high type husband sends the signal 1 instead of his equilibrium signal 4/3, while, of course, the high type wife sends her equilibrium signal 4/3. As discussed above in the single sender case, to deter such a deviation of the husband, the firm must believe that the couple has low type with positive probability after observing the out-of-equilibrium signal pair (1, 4/3) , where the first coordinate is the signal of the husband.[4] We as many others (see the references above) question the plausibility of such a belief, which is called prejudiced, as the wife is clearly signaling that their type is high. To recapitulate, the firm cannot believe that the wife has high type and the husband has low. Types are perfectly and positively correlated. Clearly, this situation cannot emerge in the single sender case.

Unprejudiced beliefs, as defined in Bagwell and Ramey (1991), require that the firm should not attach positive probability to double deviations when an out-of-

[4]Consistency of such a belief is demonstrated in the example right after Proposition 1.

equilibrium event can be explained by a single deviation.[5] Given the equilibrium above and given the out-of-equilibrium signal pair (1, 4/3), such a simpler, unprejudiced explanation is that the couple is of high type and only the husband was deviating. Believing the couple is of low type is prejudiced, more complicated, and less likely. For such a prejudiced belief, the firm must assume that both workers were deviating at the same time, i.e. they are low type, the husband has sent the signal 1 instead of his equilibrium signal 0 and the wife has sent the signal 4/3 instead of her equilibrium signal 0. This equilibrium can only be maintained with prejudiced beliefs and cannot be maintained by simple, unprejudiced beliefs. Hence, the restriction to unprejudiced beliefs eliminates this equilibrium.

1.3 Selecting a Unique Equilibrium: the Two-type Case

In our example, if the signal space is sufficiently rich, the unique pure strategy stable equilibrium outcome is the same as the unique pure equilibrium outcome which survives a version of the intuitive criterion and where the beliefs are unprejudiced. This outcome is the efficient one: low types are sending the signal 0 and high types are sending the signal 1 and achieve their first best utility level (conditional on that their type is known to the firm).

Notice that just as in the single sender case, solely a version of the intuitive criterion (a multi-sender version of D1) eliminates all the pooling equilibria. We give a formal definition of an even stronger criterion in the next subsection. For now, it is enough if one thinks of a pooling equilibrium in which both players pool in pure strategies and considers a deviation of a single player. Then the intuitive criterion or D1, as defined and applied in the single sender case, directly applies in the very same way here as well. This is because the signal of the other player does not change with his or her type, hence the incentives of the different types of a sender are clearly comparable.

[5] In their recent paper, Milgrom and Mollner (2018) introduce a new refinement of the Nash equilibrium, called test-set equilibrium, which is strongly related to the notion of unprejudiced beliefs. As Milgrom and Mollner (2018) state: the concept "... formalizes the idea that players contemplate only deviations from equilibrium play in which a single competitor plays a non-equilibrium best response."

Now we show how unprejudiced beliefs select a unique equilibrium among the separating equilibria. Assume w.l.o.g. that the husband separates. Then the low type wife must send the signal 0, otherwise deviating to 0, by unprejudiced beliefs, she must get her first best outcome (since the firm must believe that their type is low given the low type husband's separating signal). Also, for the same reason, the high type wife must send the signal 1. Given that the wife also separates the husband's separating equilibrium signals must be also 0 and 1. To recapitulate, clearly, if one sender separates the other must choose his or her first best given that the firm knows his or her type (given that the spouse separates, unprejudiced belief dictates that the firm concentrates its belief on the true type). Hence, in the unique equilibrium low types must send the signal 0 and high type must send the signal 1.[6] Any deviation to a non-equilibrium signal is deterred by the unique unprejudiced belief formed by using the equilibrium signal of the other player who fully separates and sends his or her equilibrium signal. Deviations to the equilibrium signal of another type are deterred by for example the lowest possible unprejudiced belief which is the belief that the senders' type is 0 with probability 1.

1.4 An Example with Three Types

Consider the same game as in section 1.2 with the difference that now the senders may have 3 different types, $t = 0, 1$ and also 1.5, with equal probabilities. Let us also restrict the signal space of both senders to $[0, 2]$. We claim that the following partially pooling equilibrium outcome can be supported with a version of intuitive beliefs and also with unprejudiced beliefs, however, it cannot be supported with beliefs which are intuitive and unprejudiced at the same time.

The equilibrium outcome is as follows: Type 0 senders send the signal 0, type 1 and type 1.5 senders pool by sending the signal 1, hence separate from type 0 senders. The firm uses Bayes rule to calculate its belief τ on the equilibrium path and payments

[6]The situation is a bit different when education does not increase the marginal product. In that case the first best education levels are 0 for both types. In the unique equilibrium selected, the low types now choose 0 education level and the high types choose the lowest level different from 0. Then deviating to 0 for the high types can be deterred by the, in this case, unprejudiced belief that the senders are of low type.

are made accordingly. Type 0 senders obtain 0 utility, type 1 senders obtain 3 utility, and type 1.5 senders obtain a utility of 3.5. For example, for any out-of-equilibrium signal pair, choosing the lowest unprejudiced belief supports the given outcome as an equilibrium.

1.4.1 Unprejudiced Beliefs are not Intuitive

Notice that in any unprejudiced beliefs supporting this outcome, the firm must believe with positive probability that the type of the senders is 1 after the out-of-equilibrium signal pair (2,1), i.e. $\tau(2,1)(1) > 0$. Clearly, by unprejudiced beliefs it must be that $\tau(2,1)(0) = 0$. Furthermore, if we had that $\tau(2,1)(1) = 0$ then $\tau(2,1)(1.5) = 1$ in which case type 1.5 husband finds it profitable to deviate to signal 2 and obtains $6 = 4 \cdot 2 \cdot 1.5 - (3 - 1.5) \cdot 2^2 > 4 \cdot 1 \cdot 1.25 - (3 - 1.5) \cdot 1^2 = 3.5$. However, such a belief does not survive the multi-sender version of the D1 criterion. We can simply compare the incentives of type 1 and type 1.5 husbands, as those types of the wife pool. Clearly, whenever type 1 husband is weakly better off by sending signal 2 relative to his equilibrium payoff, type 1.5 husband always strictly prefers to send the signal 2 rather than his equilibrium signal. Hence, there are no unprejudiced beliefs which are intuitive and support the outcome at the same time.

1.4.2 An Intuitive but Prejudiced Belief

Yet, while believing that the type of senders is 0 after the signal pair (2,1) is clearly prejudiced, it does satisfy the multi-sender version of the D1 criterion and deters type 1.5 husband from deviating and sending the signal 2. To see this, we invoke and tailor to our games the definition of Never a Weak Best Response (NWBR) criterion used in the single sender case in Cho and Kreps (1986).[7]

An equilibrium outcome together with supporting beliefs τ (i.e. an equilibrium) is called intuitive if for all out-of equilibrium signal pair (m_H, m_W), where at least one of the signals is sent in equilibrium, $\tau(m_H, m_W)(t) = 0$ if either m_H or m_W is never

[7] We could have come up with definitions for the intuitive criterion or for D1 and D2 in an expense of extra notation. We know that some scholars do not think that the NWBR criterion is intuitive, we used this here only for simplicity.

a weak best response for the husband or for the wife of type t.[8]

It is easy to see, using Proposition 6 in Kohlberg and Mertens (1986), that any stable outcome is an intuitive equilibrium outcome. Of course, not all intuitive equilibrium outcomes are stable as we will see it in the next subsection.

Our equilibrium is intuitive. For example, the firm can believe after the out-of-equilibrium signal pair (2,1) that the type of the senders is 0, i.e. we can choose $\tau(2,1)(0) = 1$. Indeed, we can set $\tau_H(2,0)(1.5) = 1, \tau_W(0,1)(1) = 3/4, \tau_W(0,1)(0) = 1/4$, and one can set the rest of the beliefs arbitrarily for both τ_H and τ_W while supporting the outcome. Indeed, type 0 husband and type 0 wife both expect 0 by sending the signal 2 and the signal 1 under τ_H and τ_W, respectively, and there are no profitable deviations. One can similarly justify the belief concentrated on type 0 after any out-of-equilibrium signal pair.

1.4.3 Eliminating the Putative Equilibrium

Can we safely eliminate this equilibrium by the fact that the equilibrium cannot be supported with beliefs which are both unprejudiced and intuitive at the same time? It is still not obvious that the above, partially pooling equilibrium outcome is not a stable outcome. The subtle point is the following. It is not trivial to rule out the possibility that stable sets contain some unprejudiced equilibria and some others which are intuitive but do not contain an equilibrium which is unprejudiced and intuitive at the same time. We argue now that this indeed cannot be the case. The trick is to apply Proposition 6 in Kohlberg and Mertens (1986). It states that a stable set contains a *stable set* of the game obtained by the deletion of all the strategies which are never a weak best responses. In the resulting set all the equilibria are intuitive. Applying our theorem to this smaller stable set, we can find an unprejudiced equilibrium which is also intuitive. Notice, that we used the full force of forward induction of Proposition 6 in that after deleting NWBR strategies we still find a *stable* set. It follows that

[8] I.e. either there is no τ_H, supporting the outcome, such that type t of husband expects exactly his equilibrium payoff when sending m_H or there is no τ_W, supporting the outcome, such that type t of wife expects exactly her equilibrium payoff when sending m_W. In the case this criterion can be satisfied only with $\tau(m_H, m_W)(t) = 0$ for all t for some (m_H, m_W), we also allow the equilibrium outcome to pass our test if no choice of $\tau(m_H, m_W)$ induces profitable deviation from the outcome.

eliminating an equilibrium outcome with the combination of these concepts is safe, and that in any stable set one can find such a desired equilibrium. The partially pooling equilibrium outcome can be then eliminated by simply observing that no unprejudiced equilibrium is intuitive. On the one hand, there is no need to test and compare the incentives of types which are outside the pool because the belief set at (2,1) is irrelevant for the incentives of these types. On the other hand, testing and comparing types from the pool is just as simple as in the single sender case.

After elimination of the partially pooling equilibria, one can select again the efficient, non distorted equilibrium which is the unique pure strategy equilibrium with unprejudiced and intuitive beliefs. The strategies of types 0 and 1 are just like in the two type case, and types 1.5 send the signal 2 which is their first best signal given that the receiver knows their type as required by unprejudiced beliefs. Any deviation to a non-equilibrium signal is deterred by the unique unprejudiced belief formed by using the equilibrium signal of the other player who fully separates and sends his or her equilibrium signal. Deviations to the equilibrium signal of another type are deterred by the lowest possible unprejudiced belief ($\tau(2,1)(1) = \tau(1,2)(1) = 1, \tau(0,1)(0) = \tau(1,0)(0) = 1, \tau(2,0)(0) = \tau(0,2)(0) = 1$).

1.5 Our Main Contribution

Neither consistency nor the intuitive criterion nor forward induction or properness necessarily imply unprejudiced beliefs, but we show that stability does.

The converse is not true. First, in pooling equilibria any belief is unprejudiced. Second, these pooling outcomes are not stable in our example or in the applications mentioned above, once the signal space is sufficiently rich. This is exactly the reason why in the applications cited above unprejudiced beliefs have to be coupled with a version of the intuitive criterion, i.e., to be able to eliminate pooling equilibria.

By showing that stable outcomes can be supported with unprejudiced beliefs and observing that stable outcomes satisfy the various versions of the intuitive criterion, we can justify the coupling of these concepts. It follows that instead of working with the complicated machinery of strategic stability one can reach sharp predictions in games by the *safe* and simultaneous application of these simple concepts, even if there are more than two types of the senders. We repeat, in our example with three types above one can select again the efficient, non distorted equilibrium which is the unique pure strategy equilibrium with unprejudiced and intuitive beliefs. The strategies of types 0 and 1 are just like in the two type case, and types 1.5 send the signal 2 which is their first best signal given that the receiver knows their type as required by unprejudiced beliefs.

In our example, types are perfectly correlated. If the prior probability that the type of the husband is low or the type of the wife is high and vice versa is positive, then consistency already implies unprejudiced beliefs. We prove (see our Proposition 1) that any sequential equilibrium has unprejudiced beliefs if the prior has full support. For general supports, when some type profiles have 0 probability under the prior this is not the case. Sequential or even proper equilibria well can have prejudiced beliefs.

In our Theorem 1, we show that in generic multi-sender signaling games every stable outcome can be supported by unprejudiced beliefs.[9]

The paper is structured as follows. In Section 2, we define finite multi-sender signaling games, sequential equilibria, and its refinements: unprejudiced sequential equilibrium and stable sets of equilibria, and we also state Proposition 1 and Theorem 1. In Section 3, we provide the proof of Theorem 1. In section 4 we conclude and give a hint how and to what extent our result can be directly generalized for arbitrary extensive form games with perfect recall and generic payoffs.

[9]Although our example is not generic, we find it important to highlight (as suggested by Hari Govindan) that there are generic games possessing (even proper) equilibria which can be maintained *only* with prejudiced beliefs. We provide such an example in Section 5.2 of the Online Appendix. Our example in the introduction is not generic in the sense that the husband's payoff does not depend on the education level of the wife and vice versa. The class of games investigated in Bagwell and Ramey (1991) is not generic either because the entrant's payoff does not depend on the first period prices. But these features do not necessarily imply that our theorem does not hold for these games. In fact, we were unable to find a signalling game for which our Theorem does not hold. Nevertheless, our proof hinges on the qualification that the game under consideration satisfies a given property (see Definition 4 in the proof of Theorem 1) which holds for and open and dense set of games (see footnote 15). The reader is referred to Section 5.1 of the Online Appendix for further discussion of this issue.

2 The Games and the Solution Concepts

There are three players: two senders $S = \{1, 2\}$, and a receiver.[10] A typical sender is denoted by $i \in S$ and the other sender by $-i$. First, senders learn their private type $t_i \in T_i$ which is not known by the receiver nor by the other sender. Senders' types $t = (t_1, t_2)$ are drawn from some probability distribution $\pi \in \Delta T = \Delta(T_1 \times T_2)$.[11] $\pi(t)$ denotes the probability that the realized type profile is t, and $\pi(t_{-i}|t_i)$ is the conditional probability of t_{-i} given t_i. Second, each sender i simultaneously sends a signal m_i to the receiver. The set of possible signals for sender i is M_i, and we denote by m an element of $M_1 \times M_2 = M$. Finally, the receiver responds to the senders' signals by taking an action a from a set A. All the sets: M_1, M_2, T_1, T_2 and A are finite. The players have von Neumann-Morgenstern utility functions defined over type profiles, signal profiles and actions of the receiver. Sender i's payoff function is $u_i(t, m, a)$, and the receiver's payoff function is $v(t, m, a)$. A behavioral strategy for sender i is $\sigma_i = (\sigma_i(\cdot|t_i))_{t_i \in T_i}$, where for each t_i, $\sigma_i(\cdot|t_i) \in \Delta M_i$. A behavioral strategy for the receiver is $e = (e(\cdot|m))_{m \in M}$, where for each m, $e(\cdot|m) \in \Delta A$. The expected utility of sender t_i of choosing m_i is:

$$u_i(t_i, m_i, \sigma_{-i}, e) \doteq \sum_{t_{-i} \in T_{-i}} \pi(t_{-i}|t_i) \sum_{m_{-i} \in M_{-i}} \sum_{a \in A} u_i(t_i, t_{-i}, m_i, m_{-i}, a)\sigma_{-i}(m_{-i}|t_{-i})e(a|m_i, m_{-i}).$$

The receiver's beliefs about the types of the senders after signal profiles is a collection $\mu = (\mu(\cdot|m))_{m \in M}$ such that $\mu(\cdot|m) \in \Delta T$ for all $m \in M$. The receiver's pure best response correspondence is $\hat{e}(\mu, m) \doteq \arg\max_{a \in A} \sum_{t=1}^{T} v(t, m, a)\mu(t|m)$.

(σ, e) is a sequential equilibrium (SE) if there is a μ such that:

(1) sequential rationality: (a) for all $m \in M$: $e(\cdot|m) \in \Delta\hat{e}(\mu, m)$, and (b) for all $i, t_i : \sigma_i(m_i'|t_i) > 0$ implies $m_i' \in \arg\max_{m_i \in M_i} u_i(t_i, m_i, \sigma_{-i}, e)$;

(2) consistency: There is a justifying sequence $(\sigma^n)_{n \in \mathbb{N}}$ such that $\sigma_i^n(m_i|t_i) > 0$ for all $n, i, t_i \in T_i, m_i \in M_i$ converging to σ, where μ^n, which is calculated according to Bayes' rule using σ^n, converges to μ.

[10] All of our results extend to games with more than two senders.
[11] ΔX denotes the set of probability distributions over any finite set X.

Definition 1. *Given σ, a message pair m for which there is an i, t_i such that $\sigma_i(m_i|t_i) > 0$ is called important. μ is* unprejudiced *given σ if for every important m we have that $\mu(t|m) > 0$ only if there is an i such that $\sigma_i(m_i|t_i) > 0$. An SE with unprejudiced μ is called an USE.*[12]

Proposition 1. *If π has full support then every SE is a USE.*

Proof. Consider a σ, which is part of an SE, and the justifying sequence σ^n. Since for equilibrium message pairs the proof is trivial, consider an important signal pair (x, m_2), where $\sigma_1(x|t_1) = 0$ for all $t_1 \in T_1$. The roles of the players can be exchanged. Define $T_2(m_2) = \{t_2' \in T_2 | \sigma_2(m_2|t_2') > 0\}$ and $T_1(x) = \{t_1 \in T_1 | \nexists t_1' \in T_1 : \lim_{n\to\infty} \frac{\sigma_1^n(x|t_1)}{\sigma_1^n(x|t_1')} = 0\}$ sets which are not empty, as the game is finite. Since $\pi(t) > 0$ for all $t \in T_1 \times T_2$, we have that $\lim_{n\to\infty} \mu^n(t|x, m_2) > 0$ if and only if $t \in T_1(x) \times T_2(m_2)$. □

However, USE is a powerful refinement of SE when π does not have full support. To see this, we recall the structure driving our example in the introduction and show that a prejudiced belief well may be consistent. Suppose that $M_i = \{l, h, x\}$ and $T_i = \{l, h\}$ for $i = 1, 2$, and that types are perfectly correlated, i.e., $\pi(l, h) = \pi(h, l) = 0$. Let us write l and h for the only possible type profiles (l, l) and (h, h), respectively. Suppose that payoffs are such that the following strategy profile σ is part of an SE: $\sigma_i(t_i|t_i) = 1$ for $i = 1, 2$. Hence, only signal pairs (l, l) and (h, h) are on the equilibrium path (see the example in the introduction). USE requires that $\mu(l|x, l) = 1$. We show that $\mu(h|x, l) > 0$ is possible in an SE. Consider the justifying sequence σ^n converging to σ for which $\sigma_1^n(x|h) = \sigma_2^n(l|h) = \varepsilon$ and $\sigma_1^n(x|l) = \varepsilon^3$ and $\varepsilon = 1/n$. Simple calculation shows that $\lim_{n\to\infty} \mu^n(h|x, l) = 1$. Proposition 1 fails now because $T_1(x) \times T_2(l) = \{(h, l)\}$ but $\pi(h, l) = 0$. [13]

[12] Our definition is weaker then that of Bagwell and Ramey (1991) in which $\mu(t|m) > 0$ *if and only if* there is an i such that $\sigma_i(m_i|t_i) > 0$ for all important m. This extra requirement, which they call open-mindedness, does not make any difference in generic games since $\mu(t|m) > 0$ is allowed to be arbitrarily small. In fact, it directly follows from the proof of Theorem 1 that generically open-mindedness is also implied by stability. Proposition 1 also remains to be true with this stronger concept of USE for generic games. We decided, however, for the sake of clarity, simplicity and brevity to work here with this weaker but simpler concept.

[13] Even though consistency allows for prejudiced beliefs it still restricts the set of possible beliefs, as it rules out for example that $\mu(h|x, l) > 0$ and $\mu(l|x, h) > 0$ hold at the same time.

To state our main result, we define now the notion of stable sets of equilibria à la Kohlberg and Mertens (1986) for multi-sender signaling games. Consider the (reduced) normal form Γ of a multi-sender signaling game. Let $\sigma = (\sigma_1, \sigma_2)$, where σ_i is a completely mixed-strategy of sender $i \in S$.[14] For $\delta > 0$, consider the set of all normal form games Γ' that have the same strategy space as Γ and for which for all $i \in S$ there exists $\delta_i \in (0, \delta)$, such that if some strategy profile $(\sigma_1^*, \sigma_2^*, e^*)$ is played in Γ', then the payoffs are the same as when each sender $i \in S$ plays $(1-\delta_i)\sigma_i^* + \delta_i \sigma_i$ and the receiver plays e^* in Γ. A game in this set is called a (σ, δ) perturbation of Γ.

Definition 2. *A set of Nash equilibria of* Γ *is* stable *if it is minimal with respect to the following property:* $\mathcal{N}$ *is a closed set of Nash equilibria of* Γ *satisfying: for each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that for any completely mixed* σ *the* (σ, δ) *perturbations of* Γ *have a Nash equilibrium* ε*-close to* $\mathcal{N}$.

A strategy profile induces a distribution over $T \times M \times A$ which is called an outcome. It is well known that generic extensive form games have a stable outcome, namely a stable set in which all strategy profile induces the same outcome.

Theorem 1. *Generically, any stable outcome is a USE outcome.*[15][16]

3 Proof of Theorem 1

Before describing the structure of the proof, let us set our most important definition.

Definition 3. *Fix a completely mixed* σ. *A Nash equilibrium* (σ^*, e^*) *is* $\sigma-$*perfect if for each* $\varepsilon > 0$, *there is a* $\delta > 0$ *for which any* (σ, δ) *perturbation of* Γ *has a Nash equilibrium* ε*-close to* (σ^*, e^*).[17]

[14]For simplicity, we perturb only the strategies of the senders (just as in the literature of the single-sender case), as we are interested in the beliefs generated by the stabilization of these trembles. Abusing notation slightly, we can identify mixed and behavioral strategies.

[15]A property of multi-sender signaling games with extensive game form G holds generically if there is an open and dense set $D \subseteq \mathbb{R}^{\dim G}$, such that the property holds for all games in D, where $\dim G = (|S|+1)|T \times M \times A|$.

[16]The stronger statement that all the equilibria in a stable set are USE does not hold. It is easy to construct an example where a stable set contains equilibria which are prejudiced. We thank Hari Govindan for pointing this out for us.

[17]In any game for any completely mixed σ there is a $\sigma-$perfect equilibrium of that game. The

The Structure of the Proof of Theorem 1

Consider a stable outcome and the corresponding stable set which we denote by (σ^*, E^*), where E^* is a set of strategies of the receiver which may differ only off the equilibrium path. Fix any completely mixed σ. First we show that the stable set contains a $\sigma-$perfect equilibrium. Consider the sequence of $\varepsilon^k, \delta^k, (\sigma^k, e^k), (\sigma^*, e^{*k})$, which exists by definition of stability, where (σ^k, e^k) is the Nash equilibrium of the (σ, δ^k) perturbed game which is ε^k-close to the stable set, namely to some (σ^*, e^{*k}), where $e^{*k} \in E^*$. It is apparent that (σ^*, e^*), where $e^* = \lim_{k\to\infty} e^{*k}$, is a $\sigma-$perfect equilibrium. Clearly, for any ε consider the δ^k for which (σ^*, e^{*k}) is sufficiently close to (σ^*, e^*) and to which (σ^k, e^k) is also sufficiently close. Hence, (σ^k, e^k) will be sufficiently close to (σ^*, e^*).

To complete the proof of Theorem 1 it is now sufficient to show that for any completely mixed σ, a $\sigma-$perfect equilibrium is USE generically. We establish this result in Lemma 1. This statement may not be true for any game. We show in the proof of Lemma 1 that if a $\sigma-$perfect equilibrium is not an USE then we can locally increase the support of the outcome of the given equilibrium. But the support of the Nash equilibrium outcomes are generically locally constant. This motivates the following definition of genericity.[18] Hence, we start by fixing the generic set of games for which our statements (Lemma 1, hence Theorem 1) hold, i.e. those in which the support of the Nash equilibrium outcomes are locally constant, and then state and prove Lemma 1.

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proof of this statement goes exactly along the lines of the proof of existence of perfect equilibria and taking the converging test sequence (which can be arbitrary see e.g. in Selten (1975)) so that each of its element is induced by the same perturbation σ. Notice that in general the set of equilibria which are $\sigma-$perfect for some σ is a strict subset of the perfect equilibria as defined by Selten (1975). The reason is that Selten allows the σ to depend on ε. When σ is the product of uniform distributions over the strategy spaces of the players then $\sigma-$perfection is equivalent to uniform perfection as defined in Hársanyi (1982).

[18]Further clarification about the role of the genericity assumption can be found in Section 5.1 of the Online Appendix.

3.1 The Generic Set of Games

Definition 4. *A Nash equilibrium (σ, e) of a game γ is* not nice *if there is a sequence of γ^n such that $\gamma^n \to \gamma$ and a sequence of Nash equilibria of the corresponding games converging to (σ, e) such that supports of the outcomes induced by the Nash equilibria along the sequence is strictly larger than that of induced by (σ, e). We say that a game is* nice *if it does not have a not nice equilibrium. We denote the set of nice games by D.*[19][20]

Claim 1: D is open and dense in $\mathbb{R}^{\dim G}$.

Proof of Claim 1: We show that the complement of D is closed and has an empty interior. Given a converging sequence of games which are not nice, one can choose a converging sequence of games $\tilde{\gamma}^n$ and not nice equilibria $(\tilde{\sigma}^n, \tilde{e}^n)$ which make the games in the sequence not nice. By upper-hemicontinuity the limit strategy profile, denote it by (σ, e), is an equilibrium of the limit game, which we denote by γ. We show that (σ, e) is not nice in γ and makes it not nice relative to V. First, the support of the outcome of (σ, e) cannot be larger than that of $(\tilde{\sigma}^n, \tilde{e}^n)$ for n large enough. Then one can choose the converging set of games γ^n and the corresponding equilibria (σ^n, e^n) as follows. For each $\tilde{\gamma}^n, (\tilde{\sigma}^n, \tilde{e}^n)$ choose the nearby game γ^n and an equilibrium of this nearby game (σ^n, e^n) which induces an outcome with larger support than $(\tilde{\sigma}^n, \tilde{e}^n)$. Choose γ^n and (σ^n, e^n) so that they converge to γ and (σ, e). This sequence of games and the corresponding equilibria justify that the limit game is not nice, because (σ, e) does not induce larger support than $(\tilde{\sigma}^n, \tilde{e}^n)$, while (σ^n, e^n) induces strictly larger supports than $(\tilde{\sigma}^n, \tilde{e}^n)$. Finally, it is easy to see that there cannot be an open set of games which are not nice since there are only finitely many possible supports of the outcome distributions.□

[19]The support of the outcome is a subset of $T \times M \times A$, containing those elements for which the induced distribution (the outcome) is positive.

[20]We thank Hari Govindan for suggesting this line of the proof, which significantly shortened and simplified our original proof and made our statement stronger.

3.2 Lemma 1: $\sigma-$perfection implies USE

Lemma 1. *For any game in D and for any completely mixed σ, any $\sigma-$perfect equilibrium (σ^*, e^*) is USE.*

Proof of Lemma 1:

The Main Idea of the Proof of Lemma 1

The idea of the proof is as follows. Given a sequence of $\varepsilon^k \to 0$ and the corresponding sequence of δ^k, σ^k, let us denote by $\bar{\sigma}^k = \max_{i,t_i,m_i|\sigma_i^*(m_i|t_i)=0} \sigma_i^k(m_i|t_i)$ the largest voluntary mixing in the equilibrium of the perturbed game on a message which is not sent by a given type in the original equilibrium. Then, generically it must be that $\frac{\delta^k}{\bar{\sigma}^k} \nrightarrow 0$ (see Claim 2 below). The intuition is that generically the perturbation σ must be used with strictly positive weight when forming beliefs at the limit. To put it differently, there can be no important out-of-equilibrium signal pair m where only the limit of the voluntary mixing of the nearby equilibria determines the out-of-equilibrium belief.

The Nash equilibria of the perturbed games together with the perturbations constitute a sequence of completely mixed strategies converging to the original equilibrium which can be considered as a justifying sequence. If $\frac{\delta^k}{\bar{\sigma}^k} \nrightarrow 0$, then this justifying sequence results in an unprejudiced μ. This is because then the probability that any type of any sender sends any out of equilibrium message converges in the same order of magnitude (δ^k) to 0 (cf. the example above with perfectly correlated types, where this is not true for the given justifying sequence).[21] Given this discussion, we only have to prove that:

[21] Suppose that there is a sequence of $\varepsilon^k, \delta^k, \sigma^k$ such that σ^k is part of an equilibrium of the (σ, δ^k) perturbed game; it is ε^k-close to σ^*, $\lim_{k\to\infty} \varepsilon^k = \lim_{k\to\infty} \delta^k = 0$ and $\lim_{k\to\infty} \frac{\delta^k}{\bar{\sigma}^k} > 0$. Consider an important signal pair m and a t which must get 0 probability under unprejudiced beliefs, i.e., $\sigma_i^*(m_i|t_i) = 0$ for $i = 1, 2$. Consider $((1-\delta^k)\sigma_i^k + \delta^k \sigma_i)_{i\in S}$ as a justifying sequence. Type t can get extra probability weight (beyond the perturbation) along the sequence only via $\bar{\sigma}^k > \sigma_i^k(m_i|t_i) > 0$ for $i = 1, 2$. On the other hand, by the definition of an important signal pair, there must be a t' for which there is an i such that $\sigma_i^*(m_i|t_i') > 0$. Along the sequence, such a t' sends the message m at least with probability arbitrarily close to $\delta^k c$, where $c = \sigma_i^*(m_i|t_i')\sigma_{-i}(m_{-i}|t_{-i}') > 0$ is a constant given that the perturbation σ is fixed(!). On the other hand, t sends the message m only at most with probability $(\bar{\sigma}^k + \delta^k)^2$. Using that $\lim_{k\to\infty} \frac{\delta^k}{\bar{\sigma}^k} > 0$, one obtains that conditionally on observing m, t' is infinitely more likely than t in the limit, hence $\mu(t|m) = 0$, as required by USE.

Claim 2: Given a $\sigma-$perfect equilibrium (σ^*, e^*) of a nice game γ the sequence of ε^k and the corresponding δ^k can be chosen to be so that $\frac{\delta^k}{\bar{\sigma}^k} \nrightarrow 0$.

Proof of Claim 2:

The Structure of the Proof of Claim 2

We proceed by contradiction and assume that there is a nice game γ in D such that for some $\sigma-$perfect equilibrium (σ^*, e^*), we must have that $\frac{\delta^k}{\sigma_i^k(m_i|t_i)} \to 0$ due to some m_i which is part of some out-of-equilibrium signal profile $m = (m_1, m_2)$, i.e., some type t_i must send the message m_i with relatively high (though vanishing) probability in the perturbed games. We are going to reach a contradiction by finding another nice game γ' in D which admits a variant of (σ^*, e^*) as a Nash equilibrium and show that the support of its outcome is locally increasing. Namely, we construct a sequence of games converging to γ' and a corresponding sequence of equilibria converging to the variant of (σ^*, e^*) such that in these equilibria of these nearby games m will also be sent with positive probability, whereas in the variant of (σ^*, e^*) it is sent with probability 0, implying that the equilibrium is not nice, hence we reach a contradiction.

We start with a short introduction of some terms and notation, where we identify the relevant signal-type pairs which will increase the support of the original outcome locally. Then, although, we could continue and do the proof at once (as outlined above), we split it into two parts. In the first part, we provide the proof of Claim 2 for a simple case. In this simple case we assume that there is a relevant out-of-equilibrium signal profile m such that the receiver "knows for sure" that sender i was deviating. This is the case when m_i is never sent under σ_i^*. We refer to this case as "The Deviator is Known". In this case γ' can be chosen to be γ, hence the variant of (σ^*, e^*) is just our original equilibrium (σ^*, e^*). In the second part, we go to the more complicated case where "The Deviator is not Known", i.e., when all the relevant out-of-equilibrium signal profiles are such that both signals are sent by some type of the given sender but this type profile has 0 prior probability. This completes the proof of Claim 2, hence the proof of Lemma 1 is completed.

Some Terms and Notations, Relevant Signal-Type Pairs

To clarify the wording used above and to introduce some notation, fix some $\hat{\sigma}$. We say that m_i is never sent under $\hat{\sigma}$ if $\hat{\sigma}_i(m_i) \doteq \sum_{t\in T}\pi(t)\hat{\sigma}_i(m_i|t_i) = 0$ and we define $\hat{\sigma}(m) \doteq \sum_{t\in T}\pi(t)\hat{\sigma}_1(m_1|t_1)\hat{\sigma}_1(m_2|t_2)$. We say that the deviator is known at m if $\hat{\sigma}(m) = 0$ and there is an i such that $\hat{\sigma}_i(m_i) = 0$ and $\hat{\sigma}_{-i}(m_{-i}) > 0$. We say that the deviator is not known at m if $\hat{\sigma}(m) = 0$ and for all $i \in S$ we have that $\hat{\sigma}_i(m_i) > 0$.

Consider a $\sigma-$perfect equilibrium (σ^*, e^*) of a nice game $\gamma \in D$. Consider a sequence $\varepsilon^k \to 0$ and assume that there must be an i, t_i, m_i such that $\lim_{k\to\infty}\sigma_i^k(m_i|t_i) = 0$ and $\lim_{k\to\infty}\frac{\delta^k}{\sigma_i^k(m_i|t_i)} = 0$, where σ^k is a part of the ε^k-nearby equilibrium of a (σ, δ^k) perturbed game. Let $O = \cup_{i\in S}\{(m_i, t_i)|\sigma_i^*(m_i|t_i) = 0\}$ and consider the set:

$$N = \cup_{i\in S}\{(m_i, t_i) \in O | \forall j \in S, \forall (m_j, t_j) \in O : \lim_{k\to\infty}\frac{\sigma_i^k(m_i|t_i)}{\sigma_j^k(m_j|t_j)} > 0\}.$$

We say that (m_i, t_i) is relevant if it is in N. These are the strongest or largest voluntary mixings in the nearby equilibria in the sense that their speed of convergence to 0 is the slowest, and even slower than that of δ^k by assumption. We choose a $\xi \in (0, 1)$, sufficiently small (to be set later), and an $\eta : N \to (\xi^2, \xi)$ such that for all $(m_i, t_i), (m'_j, t'_j) \in N$ we have that:

$$\lim_{k\to\infty}\frac{\sigma_i^k(m_i|t_i)}{\sigma_j^k(m'_j|t'_j)} = \frac{\eta(m_i, t_i)}{\eta(m'_j, t'_j)}.$$

Clearly, by the definition of N, there is a $\bar{\xi}$, which can be taken to be smaller than $\min_{(m_i,t_i)\notin O}\sigma_i^*(m_i|t_i)$, such that for all $\xi \in (0, \bar{\xi})$ there exists such an $\eta(\cdot\cdot)$. For all $\xi \in (0, \bar{\xi})$ let us construct a new strategy profile for the senders denoted by σ^ξ for which $\lim_{\xi\to 0}\sigma^\xi = \sigma^*$ and the support induced by σ^ξ is strictly larger than that of σ^*. We would like to have σ^ξ as part of a Nash equilibrium of some nearby game and to reach contradiction, i.e., to conclude that our original equilibrium is not nice.

The Case when the Deviator is Known

Let us first consider the simple case in which N contains an (m_i, t_i) such that m_i is never sent under σ_i^*. So let us fix this m_i and for all t_i for which $(m_i, t_i) \in N$ set

$\sigma_i^{\xi}(m_i|t_i) = \eta(m_i, t_i)$ and for all $m'_i \neq m_i$ set $\sigma_i^{\xi}(m'_i|t_i) = \sigma_i^*(m'_i|t_i)(1 - \eta(m_i, t_i))$. For $(m'_j, t'_j) \notin N$ set $\sigma_j^{\xi}(m'_j|t'_j) = \sigma_j^*(m'_j|t'_j)$ for all $j \in S$. It is easy to see that $e^*(\cdot|m)$ is exactly sequentially rational under σ^{ξ} for all m such that $\sigma^{\xi}(m) > 0$ and $\sigma^*(m) = 0$. For the signal pairs such that $\sigma^*(m) > 0$ one can slightly modify the receiver's payoff to restore sequential rationality of $e^*(\cdot|m)$ and slightly modify the payoffs of sender $-i$ to restore indifferences and make sure that $-i$ does not want to send any message which he has never sent under σ^*_{-i}. Hence, (σ^{ξ}, e^*) becomes a Nash equilibrium of a game γ^{ξ} arbitrarily close to γ (by choosing ξ to be sufficiently small), having a strictly larger induced support. Moreover, these equilibria converge to (σ^*, e^*) as the neighborhood around γ becomes smaller. This shows that our equilibrium is not nice, which is a contradiction.□

The Case when the Deviator is not Known

Increasing the support locally is more complicated if there is no relevant m_i which was never sent by i under σ_i^*. In that case one has to consider *all* the pairs in N when defining σ^{ξ} because all these voluntary mixings of the two senders may interplay in general. Hence, consider the following σ^{ξ}. For all i, t_i let $N(t_i) = \{m_i | (m_i, t_i) \in N\}$, and for every $m_i \in N(t_i)$ set $\sigma_i^{\xi}(m_i|t_i) = \eta(m_i, t_i)$ and for $m'_i \notin N(t_i)$ set $\sigma_i^{\xi}(m'_i|t_i) = \sigma_i^*(m'_i|t_i)(1 - \sum_{m_i \in N(t_i)} \eta(m_i, t_i))$. It follows that there might be an m which is important under σ^* but for which $0 < \sigma^{\xi}(m) < \xi^2$ (and $\sigma^*(m) = 0$). For these signal pairs e^* may be far from sequential rationality under σ^{ξ}. This is because the beliefs of the receiver at these message pairs induced by the $\sigma-$perfect equilibrium might be very different from those induced by σ^{ξ}.[22] So setting $e^{\xi}(\cdot|m)$ to be sequentially rational under σ^{ξ} at these pairs is not sufficient in itself as it could induce unilateral deviations from σ^{ξ}. Nevertheless, let us define e^{ξ} by changing e^*, *only* for those $m-$s for which $0 < \sigma^{\xi}(m) < \xi^2$, in such a way that e^{ξ} becomes sequentially rational under

[22] For example, suppose the speed of convergence to 0 of $\sigma_i^k(m_i|t_i)$ and $\sigma_j^k(m_j|t_j)$, for $i \neq j$ are in the order of $\sqrt{\delta^k}$ for some, hence it is so for all $(m_i, t_i), (m_j, t_j) \in N$. Then it is possible that $\sigma^*(m) = 0$ for $m = (m_i, m_j)$ and that the belief under e^* stems from the convex combination of the perturbation and that of the voluntary mixings since $\sigma_i^k(m_i|t_i)\sigma_j^k(m_j|t_j)$ converges to 0 in the order of the speed of δ^k. Under σ^{ξ} however, the perturbation gets 0 weight and $\sigma_i^{\xi}(m_i|t_i)\sigma_j^{\xi}(m_j|t_j)$ is strictly positive.

σ^ξ at these m-s. Since we need that (σ^ξ, e^ξ) converges to (σ^*, e^*), let us replace e^* with $e_* = \lim_{\xi\to 0} e^\xi$. Clearly, (σ^*, e_*) may not be a Nash equilibrium of γ any more because an m with $0 < \sigma^\xi(m) < \xi^2$ may be important under σ^*.

Hence, we have to modify our original nice game γ. We do this in such a way that the resulting game γ' is still in D and (σ^*, e_*) becomes a Nash equilibrium of this new game. We will show that for every neighborhood of γ' there is a ξ such that (σ^ξ, e^ξ) is a Nash equilibrium of some game in the neighborhood. This completes the proof by reaching the contradiction that γ' is not nice.

To this end, assume w.l.o.g. that all the payoffs of the senders are strictly positive in γ and set the terminal payoffs of all the types of all the senders below 0 after exactly those m-s for which $0 < \sigma^\xi(m) < \xi^2$ in such a way that this new game γ' is still in D. This is possible as D is open and dense in $\mathbb{R}^{\dim G}$. Clearly, (σ^*, e_*) is a Nash equilibrium of γ'. Notice that e^ξ is almost sequentially rational under σ^ξ at those m-s for which $\xi^2 < \sigma^\xi(m)$. This is because the belief of the receiver at these messages can get arbitrarily close to the beliefs generated in the limit by the original $\sigma-$perfect equilibrium if ξ is sufficiently small, and because $e^\xi(\cdot|m) = e_*(\cdot|m) = e^*(\cdot|m)$ at these messages by definition. At these signal pairs, sequential rationality of e^ξ under σ^ξ can then be restored by slightly changing the receiver's payoff in the tree (the smaller ξ is the smaller is the necessary modification). What is left is to restore the indifferences of the senders under (σ^ξ, e^ξ) and make sure that the senders do not want to send messages which they have never sent under σ^ξ.[23] This can be achieved by slightly modifying the tree payoffs (the smaller ξ is the smaller is the necessary modification). Hence, for any neighborhood of γ' we can found a game in this neighborhood such that (σ^ξ, e^ξ) is a Nash equilibrium of this game, having a strictly larger induced support than that of (σ^*, e_*). Moreover, these equilibria converge to (σ^*, e_*) as the neighborhood around γ' becomes smaller. This contradicts to the niceness of γ'.□

[23] Note, that the senders' payoff can be adjusted even in the case when their payoffs do not depend on the other sender's signal (see footnote 9).

4 Conclusion

The concept of Nash equilibrium expresses the robustness of a strategy profile against unilateral deviations. This suggests that one should be able to explain sequentially rational out of equilibrium behavior with unilateral deviations, i.e. with unprejudiced beliefs, whenever it is possible. We have shown that for generic multi-sender signaling games indeed this is the case.

We conjecture that Theorem 1 and its proof can be generalized to arbitrary extensive-form games with perfect recall.[24]

However, in the general case one may wish to further restrict beliefs also on information sets which are "not important" in our short game, namely on those that are at least two deviations away from the equilibrium play (cf. Definition 1 for important message pairs). We left this work for future research. The main difficulty is that it is unclear how the current proof generalizes when the beliefs are restricted in *all* (and not only on important) information sets in such a way that only those nodes can get positive probability which can be reached by the smallest sets (in terms of cardinality) of deviating agents.

An open question is whether USE outcomes always exist, namely, even for non-generic games. Recall that $\sigma-$perfect equilibria exists for any completely mixed σ and for any game (see footnote 17). But a $\sigma-$perfect equilibrium might not be USE in those non-generic games where the support of the Nash equilibrium outcomes are locally not constant, i.e. when the game is not nice (see Definition 4). We were unable to find an example which does not have an USE. However, we show now by a highly non-generic example that it well can be the case that no USE are perfect. The example is an extension of the 3 player game given by Milgrom and Mollner (2018) on their Figure 1 which is as follows: player 1 chooses row, 2 column and 3 the matrix.

[24] The key step is to apply Mertens's notion of stability (see Mertens (1989, 1991)) or that of Hillas (1990) for generic games. This allows one to look at $\sigma-$perfect equilibria of the agent normal form and then carefully modify the original extensive form game similarly as in the proof of Theorem 1. To find a $\sigma-$perfect equilibrium of the agent normal form, first, one finds "arbitrary" quasi-perfect equilibria (strong backward induction in the sense of Govindan and Wilson (2006)) in any Mertens-stable set. Second, one observes that some of these are $\sigma-$perfect equilibria of the agent normal form, since generically quasi-perfect and agent normal form perfect equilibria coincide (see Pimienta and Shen (2014)).

	L	C	R
U	$0,\ 0,\ 0$	$0,\ 0,\ 0$	$0,\ 0,\ 0$
D	$0,\ 0,\ 0$	$-1,\ 1,\ 0$	$1,-1,\ 0$

W

	L	C	R
U	$0,\ 0,\ 0$	$0,-1,\ 1$	$0,\ 1,\ 0$
D	$-1,\ 0,\ 0$	$-1,-1,\ 0$	$1,-1,-1$

E

Consider a fourth player who has two actions X and Y but his actions do not affect the payoffs of the other players. By choosing X player 4 gets 0 independently of the others' actions. By choosing Y player 4 gets 0 unless players 1 and 2 chooses U and C or D and R in which cases his payoff is 1 and -2, respectively. The four player game has a continuum of Nash equilibria in all of which players 1,2 and 3 chooses U, L, W and player 4 chooses an arbitrary mixture of X and Y. However, there is a unique perfect equilibrium which is U, L, W, X. Clearly, W must be a best reply against the completely mixed strategy profiles converging to the perfect equilibrium. But this means that the probability of (U, C) must be smaller than or equal to the probability of (D, R) in which case player 4 must choose X. Consider the extensive form game in which first players 1,2 and 3 choose actions simultaneously. Player 4, without observing the choices of the other players, chooses an action only if (U, C) or (D, R) was chosen by 1 and 2. Player 4's belief is prejudiced in every perfect equilibrium in the sense that it attaches positive probability to (D, R), which is a double deviation relative to (U, L), whereas in any USE only (U, C) could get positive probability. Clearly, the support of the Nash equilibrium outcome of this game is locally not constant. Note however, that there is still an USE in which player 4 chooses Y.

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5 Online Appendix (Not for publication)

5.1 Genericity

The following discussion is about our Definition 4 of genericity and whether our Theorem holds or not for a larger set of possibly "non-generic" games, where the payoff of some players cannot depend on the entire history. We show that in a discrete version of Bagwell and Ramey (1991) our theorem holds on a (relatively) open and dense set of games within that "non-generic" class. To this end, we refine the notion of nice game and tailor it to a class of games, so that the payoff perturbations must remain within this class.

Definition 5. *We say that a game $\gamma \in V$ is* nice *relative to $V \subseteq \mathbb{R}^{\dim G}$ if it does not have a Nash equilibrium (σ, e) such that there is a sequence of $\gamma^n \in V$ such that $\gamma^n \to \gamma$ and a sequence of Nash equilibria of the corresponding games converging to (σ, e) such that supports of the outcomes induced by the Nash equilibria along the sequence is strictly larger than that of induced by (σ, e).*

We show now by an example, that niceness relative to the entire payoff space is not really necessary for our Theorem to hold. When γ already lives in a proper subspace V, it is enough that γ is nice relative to this V.

An Example à la Bagwell and Ramey (1991)

Our class of games is as follows. Suppose there are only two types for both senders $T_i = \{l, h\}$ and $\pi(l, h) = \pi(h, l) = 0$ and $\pi(l, l) = \pi(h, h)$, so types are perfectly, positively correlated. We write l and h for (l, l) and (h, h), respectively. Players utilities are given by $u_i : T \times M \times A \to \mathbb{R}$ and most importantly $v : T \times A \to \mathbb{R}$. That is, our class of games is non generic in the sense that these games live in a proper subspace V of $\mathbb{R}^{dimG}$. Let us, however, assume that $v(t, a) \neq v(t', a')$ whenever $(t, a) \neq (t', a')$.

Consider a $\sigma-$perfect equilibrium of a game which is nice relative to V. Clearly, the induced beliefs are unprejudiced whenever the deviator is not known since there are only two perfectly correlated types. Hence, assume by contradiction that the belief is prejudiced when the deviator is known. Suppose w.l.o.g., that the only relevant signal-type pair for sender one is (m_1, l). Fix this m_1 and note that the only problem can be that the belief induced by the $\sigma-$perfect equilibrium is prejudiced after some (m_1, m_2^h) for some (w.l.o.g. unique) m_2^h for which $\sigma_2^*(m_2^h|h) > 0$ but $\sigma_2^*(m_2^h|l) = 0$, i.e., that $\mu(l|m_1, m_2^h) > 0$. It follows that (m_2^h, l) must be also relevant as otherwise the belief induced by the $\sigma-$perfect equilibrium is unprejudiced after (m_1, m_2^h). We want sender one of type l to send m_1 or sender two of type l to send m_2^h with probability $\xi > 0$ in equilibrium of some nearby game in V and reach a contradiction. The challenge is to win back ξ from other messages of the given type of the given sender in such a way that the receiver's original strategy remains sequentially rational. The reason is that now we cannot implement small changes in the receiver's payoff depending on the message profiles as we a restricted to stay within V.

We distinguish three cases: (1) there is an m_1^l such that $\sigma_1^*(m_1^l|l) > 0$ but $\sigma_1^*(m_1^l|h) = 0$, (2) there is an m_1^h such that $\sigma_1^*(m_1^h|h) > 0$ but $\sigma_1^*(m_1^h|l) = 0$ or (3) none of the types of sender one separates with positive probability.

In case (1), instead of decreasing all the equilibrium probabilities of sender one of type l with the factor $(1 - \xi)$ we only set $\sigma_1^\xi(m_1^l|l) = \sigma_1^*(m_1^l|l) - \xi$ and $\sigma_1^\xi(m_1|l) = \xi$ and leave everything else unchanged. It is easy to see that the beliefs of the receiver are unaffected and $e^*(\cdot|m)$ is sequentially rational for all m under σ^ξ as well. One only has to change slightly the payoffs of sender two to restore the equilibrium.

In case (2), we can suppose that case (1) does not hold which means that $1 > \sigma_1^*(m_1^h|h)$. In this case we can decrease the probabilities of type l with the factor $(1-\xi)$ (so we can set $\sigma^\xi(m_1|l) = \xi$) as in the general proof but we have now the possibility to decrease the probabilities for type h with the same factor and increase $\sigma_1^*(m_1^h|h)$ accordingly by ξ. It is easy to see that the beliefs of the receiver are unaffected and $e^*(\cdot|m)$ is sequentially rational for all m under σ^ξ as well. Again, one only has to change slightly the payoffs of sender two to restore the equilibrium.

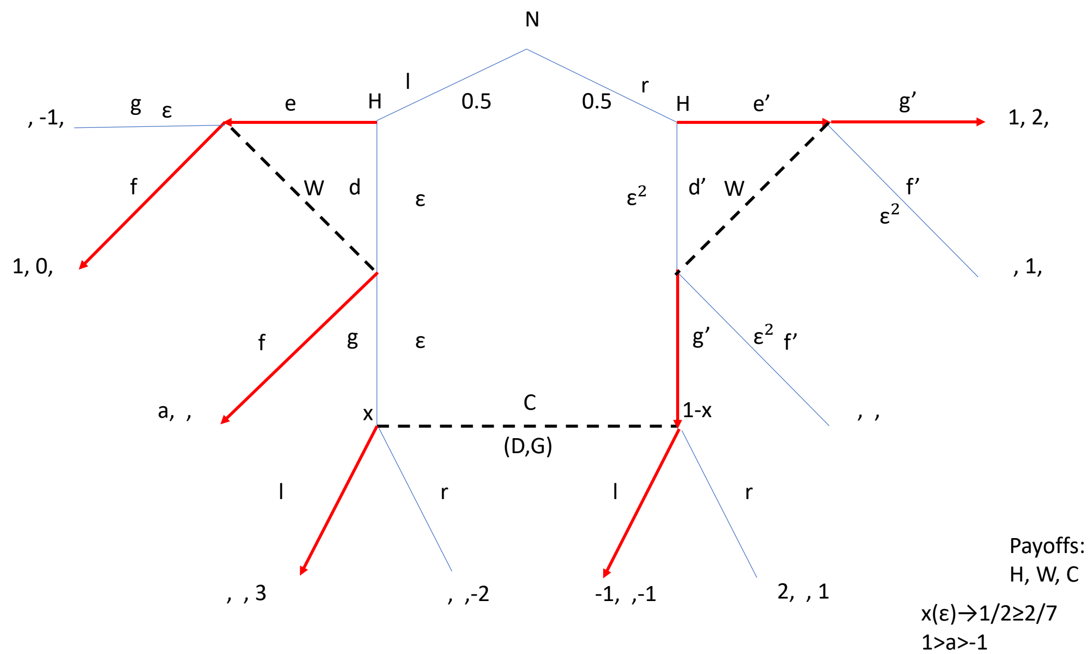
In case (3), we set $\sigma_1^\xi = \sigma_1^*$ and we change the strategy of sender two by setting $\sigma_2^\xi(m_2^h|l) = \xi$. The belief induced by the $\sigma-$perfect equilibrium can be prejudiced only after (m_1, m_2^h) (remember that sender one's types are pooling). We have to consider two different subcases and win back ξ from other messages of sender two of type l.

3(a) If there is an m_2^l for which $\sigma_2^*(m_2^l|l) > 0$ but $\sigma_2^*(m_2^l|h) = 0$ then simply set $\sigma_2^\xi(m_2^l|l) = \sigma_2^*(m_2^l|l) - \xi$. The beliefs of the receiver may change a bit only after (m_1^p, m_2^h) by putting an arbitrarily small (depending on ξ) probability on type l, where m_1^p denotes an arbitrary message on which sender one pools. But by the assumption that $v(t,a) \neq v(t',a')$ whenever $(t,a) \neq (t',a')$ this will not change the unique optimal pure action a of the receiver $e^*(a|(m_1^p, m_2^h)) = 1$ if ξ is sufficiently small. Hence, again, there is no need to change the receiver's payoff. One only has to change slightly the payoffs of sender one now to restore the equilibrium.

Finally 3(b), we can assume that sender two pools on every equilibrium signal m_2^p except that type h sometimes separates himself by sending m_2^h. Then it must be the case that $1 > \sigma_2^\xi(m_2^h|h)$ and we can set $\sigma_2^\xi(m_2^p|l) = \sigma_2^*(m_2^p|l)(1-\xi)$ and $\sigma_2^\xi(m_2^p|h) = \sigma_2^*(m_2^p|h)(1-\xi)$ for every m_2^p on which sender two pools and set $\sigma_2^\xi(m_2^h|h) = \sigma_2^*(m_2^h|h) + \xi$. It is easy to see that the beliefs of the receiver change only after message pairs (m_1^p, m_2^h) as in case 3(a) above, hence there is no need to change the receiver's payoff. One only has to change slightly the payoffs of sender one now again to restore the equilibrium. Notice that one may think that in case (3) the support induced by σ^ξ is not larger than that of σ^* because the receiver observes the same set of signal pairs on the path. Nevertheless, under σ^ξ the signal pair (m_1^p, m_2^h) is sent by types l as well, so the induced distributions (outcomes) do have different supports on $T \times M \times A$. $\square$

5.2 A Generic Game with a Proper but Prejudiced Outcome

In this section we give an example of a generic game which possesses a proper equilibrium outcome which cannot be maintained with unprejudiced beliefs. We call this a prejudiced outcome.



The description of the signaling game is as follows: nature (N) chooses the state l or r with equal probabilities and both the Husband (H) and the wife (W) are informed about the state. Then H and W sends a signal simultaneously to the firm (C) from the signal sets $\{E, D\}, \{F, G\}$ for H and W respectively. For example the strategy of H sending E in state l and sending D in state r is indicated on the picture by the actions e and d'. C must take an action only if the signal profile that he observes is (D, G), i.e. the state is either l and H was choosing the action d and W was choosing the action g or the state is r and H was choosing the action d' and W was choosing the action g'. A three-tuple at the terminal nodes indicate the payoffs in the order of H,W, and C. A blanc space for a payoff means that it can be chosen arbitrarily. The red arrows indicate an equilibrium in which H always sends the signal E and W sends the signal F in state l and the signal G in state r. Hence, C must choose an action l or

r only off the equilibrium path after observing (D, G). There is a single unprejudiced belief for which C assigns 0 probability to the left node of his information set (D, G), i.e. setting $x = 0$. This is because to reach the left node it must be that the state is l and both H and W were deviating, while being in the right node requires only the deviation of H in state r. However, for this belief C chooses r sequentially rationally in which case when the state is r, H would deviate and send the signal D. There is no way to maintain this outcome with unprejudiced beliefs. However, it is a proper equilibrium (and also extensive form proper) as long as $1 > a > -1$ in which C's induced belief at the limit is prejudiced at puts probability 0.5 on the left node of (D, G) and chooses action l sequentially rationally which deters H from deviating. The $\varepsilon-$proper equilibrium is indicated by the $\varepsilon, \varepsilon^2$ behavioral mixing (take the red arrows to 1 and then normalize). The normal form sequence can be obtained by simple multiplications. The trembling of C is irrelevant. Notice that if the outcome is a $\sigma-$perfect equilibrium outcome then it must be that $a = 1$ which is a non generic tree payoff and indeed the support of the outcome can then be locally increased (for an open set of payoffs of W) by letting H to send the signal D in state l with sufficiently small probability, i.e. choosing the action d.