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# Large deviations and the emergence of a logarithmic delay in a nonlocal Fisher-KPP equation

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## Abstract

We study a variant of the Fisher-KPP equation with nonlocal dispersal. Using the theory of large deviations, we show the emergence of a “Bramson-like” logarithmic delay for the linearised equation with step-like initial data. We conclude that the logarithmic delay emerges also for the solutions of the nonlinear equation. Previous papers found very precise results for the nonlinear equation with strong assumptions on the decay of the kernel. Our results are less precise, but they are valid for all continuous symmetric thin-tailed kernels.

## 1 Introduction

We are concerned with the spreading properties of the solution of the Fisher-KPP equation with nonlocal dispersal

$$\begin{cases} \partial_t v(t, x) = [J * v(t, x) - v(t, x)] + f(v(t, x)) & t > 0, x \in \mathbb{R}, \\ v(0, x) = \mathbb{1}_{(-\infty, 0]}(x) & x \in \mathbb{R}, \end{cases} \quad (1)$$

where

$$J * v(t, x) = \int_{\mathbb{R}} J(y)v(t, x - y) dy.$$

We shall look for the emergence of a “Bramson-like” logarithmic delay. Here, the reaction term  $f$  is monostable and satisfies the KPP condition, and the dispersal kernel  $J$  is thin-tailed. These assumptions are precised and explained below.

The Cauchy problem (1) is akin to the classical (local) Fisher-KPP equation [15, 23]

$$\begin{cases} \partial_t w(t, x) = \partial_{xx} w(t, x) + f(w(t, x)) & t > 0, x \in \mathbb{R}, \\ w(0, x) = \mathbb{1}_{(-\infty, 0]}(x) & x \in \mathbb{R}, \end{cases} \quad (2)$$

but the diffusion term  $\partial_{xx} w$  is replaced by the nonlocal dispersal term  $J * v - v$ .

In both Cauchy problems (1) and (2), under suitable assumptions on the reaction term  $f$ , the maximum principle holds and implies that  $0 \leq v \leq 1$  and  $0 \leq w \leq 1$  (see [23] for the local equation and [40] for the nonlocal equation). Therefore, in biological models, one can see  $v(t, x)$  and  $w(t, x)$  as the environment occupancy by an invading species at time  $t$  and at location  $x$ . For example, the Cauchy problem (1) arises in [16] as the infinite particle limit of a stochastic process modelling the range expansion of a plant. In this model, which stems from [5, 26], the reaction term  $f(v)$  corresponds to the demography (birth and death) of individuals subject to competition, while the dispersal term  $J * v - v$  (which is the generator of a jump process) describes the dispersal of the seeds. As  $\frac{1}{2}\Delta$  is the generator of Brownian motion, the local equation (2) usually models small and frequent movements, as in the model of Fisher [15]. See also [36, 38] for related models.

Although our primary interest lies in the Cauchy problem (1), we shall focus on the *linearised* equation

$$\begin{cases} \partial_t u(t, x) = [J * u(t, x) - u(t, x)] + ru(t, x) & t > 0, x \in \mathbb{R}, \\ u(0, x) = \mathbb{1}_{(-\infty, 0]}(x) & x \in \mathbb{R}, \end{cases} \quad (3)$$

where  $r := f'(0) > 0$ . The behaviour of the linear equation seems simpler to understand. From the study of the solutions of (3), we shall deduce a similar (weaker) result on the solutions of (1).

Now, let us focus on known results about the local equation, which has already been widely studied. We say that the reaction term  $f$  is *monostable* when it satisfies

$$f \in C^1([0, 1]), \quad f'(0) > 0, \quad f(0) = f(1) = 0, \quad \forall u \in (0, 1), f(u) > 0. \quad (4)$$

We assume throughout that the reaction term  $f$  is monostable. The solution  $w$  of the local Cauchy problem (2) is then known to propagate with a finite speed  $c^* > 0$  ([3], Theorems 4.1 and 4.3), in the sense that

$$\begin{aligned} \lim_{t \rightarrow +\infty} w(t, c't) &= 0 \quad \text{for all } c' > c^*, \\ \lim_{t \rightarrow +\infty} w(t, c''t) &= 1 \quad \text{for all } c'' \in (0, c^*). \end{aligned} \quad (5)$$

The speed  $c^*$  that satisfies (5) is called the *critical speed*. A *travelling wave* of speed  $c$  is a solution  $w$  of the local equation that can be written in the form  $w(t, x) := W(x - ct)$ , where  $W : \mathbb{R} \rightarrow [0, 1]$  satisfies  $W(-\infty) = 1$  and  $W(+\infty) = 0$ . The function  $W$  is called the profile of the travelling wave, and has the shape of an interface between the *invaded zone*, where  $W \simeq 1$ , and the *non-invaded zone*, where  $W \simeq 0$ . The solution of the Cauchy problem (2) converges “in shape” to the unique positive travelling wave of critical speed  $c^*$  [23, 39]. We thus say that an *invasion front* appears in the solution.

Now, we might wonder where exactly the invasion front is: is it really “near” the travelling wave? More precisely, given any level  $\rho \in (0, 1)$ , let  $\theta_{loc}^\rho(t)$  be the largest position such that  $w(t, \theta_{loc}^\rho(t)) = \rho$ . The goal is to compare  $\theta_{loc}^\rho(t)$ , which represents the position of the invasion front, to  $c^*t$ , which is the position of the travelling wave with minimal speed.

The *KPP condition* on the reaction term  $f$  is defined by

$$\forall u \in [0, 1], \quad f(u) \leq f'(0)u. \quad (6)$$

The KPP condition means that the per-capita growth rate  $f(u)/u$  is maximal at 0. In biological models, it implies that there is no Allee effect (e.g., [38]). When the KPP condition is satisfied, the critical speed is known to be  $c^* = 2\lambda$ , with  $\lambda = \sqrt{f'(0)}$  [23]. Bramson, in [9], showed that if  $f'(u)$  is maximal at  $u = 0$  (which implies that the KPP condition is satisfied), and if the initial condition is  $\mathbb{1}_{(-\infty, 0]}$ , then the invasion front is located well behind the position  $x = c^*t$ . More precisely, he showed

$$\theta_{loc}^\rho(t) = c^*t - s \ln(t) + O_{t \rightarrow +\infty}(1) \quad \text{with} \quad s = \frac{3}{2\lambda}, \quad (7)$$

where  $\lambda = \sqrt{f'(0)}$ . The term  $-s \ln(t)$  is called the (Bramson) logarithmic delay. The same year, Uchiyama [39] proved a slightly less precise result, but for much more general initial conditions and more general reaction term: the KPP condition is almost sufficient. A few years later, keeping the assumption that  $f'(u)$  is maximal at  $u = 0$ , Bramson showed that in fact the term  $O(1)$  can be turned into a term  $C + o(1)$  for some constant  $C \in \mathbb{R}$  (which depends on the initial condition). Moreover, he extended again the class of initial conditions for which (7) holds, and for which the solution converges to a travelling wave. His proofs use the so-called McKean representation with a branching Brownian motion [28, 37]. Lau [25] gave a new proof of Bramson's latter result, with the same assumptions, but using analytic techniques. Likewise, more recently, the emergence of the logarithmic delay in (7) has been proved thanks to interesting analytic techniques [22], for initial conditions with a support bounded from above and for reaction terms satisfying the KPP condition. For example, the techniques of [22] have been used in [29, 30] to prove refinements of (7) up to order  $\frac{1}{\sqrt{t}}$  (which before had been found formally in [13]).

Bramson and Uchiyama's results, with their extensions, induce that in the local equation, when the KPP condition is satisfied and when the initial condition decreases sufficiently fast, the shift between the position  $\theta_{loc}^\rho$  of the invasion front and the position  $c^*t$  is unbounded. Such a situation does not always occur. See the works of Rothe [34] and of Fife and McLeod [14] for the study of reaction terms that give rise to bounded shifts. See also the work of Giletti [20] for the study of general monostable reaction terms (that is, satisfying (4) but not necessarily (6)), which always give rise to unbounded shifts.

Much fewer articles have focused on nonlocal equations. The emergence of a logarithmic delay has been shown for other nonlocal variants of the classical Fisher-KPP equation. The case of a competition that is nonlocal in space has been studied in [31] (using probabilistic arguments) and in [6] (using analytic arguments). In the latter, the authors also showed that for a slowly-decaying competition kernel, the delay takes an algebraic form. The case of a phenotype-dependent equation with a competition that is nonlocal in phenotype and a diffusion rate that depends on the phenotype, the ‘‘cane-toad equation’’, has been studied in [7].

We now turn our attention to the results we have at hand for the nonlocal Cauchy problems of interest (1) and (3). Here, the nonlocality is on the movements, not on the competition. Throughout this work, we assume that  $J$  satisfies the assumptions:

$$J \in C^0(\mathbb{R}), \quad J \geq 0, \quad \int_{\mathbb{R}} J = 1, \quad \forall x \in \mathbb{R}, J(x) = J(-x), \quad (8)$$

together with the assumption:

$$\text{There exists } L > 0 \text{ such that } \int_{\mathbb{R}} e^{L|x|} J(x) dx < +\infty. \quad (9)$$

Assumption (9) means that the dispersal kernel is *thin-tailed*. It means that individuals usually do not move too far away from their origin, and it implies that travelling waves do exist. If the tail of  $J$  is too heavy, the invasion is accelerating [18, 24]. When the invasion is accelerating, auto-similar travelling waves cannot exist due to a flattening of the solutions [19]. The assumption that  $J$  is symmetric is made for convenience but is not essential (see [11]).

Under those assumptions (8) and (9) on the dispersal kernel  $J$  and the monostability assumptions (4) on the reaction term  $f$ , the Cauchy problem is well-posed (see e.g. the argument at the beginning of [40]) and, as in the local case, there exists a minimal speed for travelling waves, called the *critical speed* [10, 35]. For the nonlocal equation, we denote by  $c$  the critical speed. When the KPP condition (6) is satisfied, the expression of  $c$  is explicit:

$$c = \inf_{\lambda > 0} \frac{M(\lambda) + r - 1}{\lambda} \quad \text{with} \quad M(\lambda) = \int_{\mathbb{R}} e^{\lambda x} J(x) dx. \quad (10)$$

Since the function  $\lambda \mapsto (M(\lambda) + r - 1)/\lambda$  is strictly convex and goes to infinity as  $\lambda \rightarrow 0$  and as  $\lambda \rightarrow +\infty$ , the infimum is reached at a unique  $\lambda_r > 0$ , which thus satisfies

$$c = \frac{M(\lambda_r) + r - 1}{\lambda_r}. \quad (11)$$

Under mild additional assumptions on the kernel  $J$ , the critical speed  $c$  satisfies the propagation property (5) for all nonzero solutions  $v$  with an initial support bounded from above (replacing  $w$  by  $v$  in (5)). See [27], Theorem 3.2.

Recently, Graham [21] showed that the logarithmic delay (7) can also arise for the *nonlocal* equation (1). He assumes that  $f$  satisfies the KPP condition. He also makes a slightly technical assumption on the kernel  $J$ , which holds if  $J$  has a compact support or is Gaussian, but may fail for other thin-tailed kernels. Roquejoffre [33], assuming that  $J$  has a compact support, was able to turn the term  $O(1)$  into  $C + o(1)$  (for some  $C \in \mathbb{R}$ ). The techniques in both works are close to those of [22] (Graham combines them with a probabilistic argument, and Roquejoffre uses a refinement as in [29]). Our goal is to relax their conditions on  $J$  and to treat the general case of thin-tailed kernels. A first step towards this goal is to work on the *linear* Cauchy problem and, using the maximum principle, to find an upper bound for the position of the invasion front for the nonlinear Cauchy problem.

One specificity of this work is the use of the large deviation theory. In the works cited above, there are essentially two kinds of proofs: those that use analytic tools, as in [14, 15,

20, 22, 23, 25, 30, 33, 34] and those that use probabilistic tools, as in [8, 9, 28, 31, 39]. Graham and Uchiyama combine both kinds of proofs. The methods here are probabilistic. We shall use a Feynman-Kac representation of the solution. Then, we shall use the large deviation theory to estimate precisely the terms of the sum that will arise. In fact, it seems natural to use the large deviation theory in the problem we have raised, because we are mainly concerned with the extreme behaviour of individuals modelled by the solution of the Cauchy problem – much as the large deviation theory is concerned with the extreme values of random processes.

Note however that the use of the large deviation theory is not new in the study of local or nonlocal Fisher-KPP equations. Freidlin [17] uses a large deviation principle over the paths of a Brownian motion to determine the area covered by the solution in a heterogeneous environment. A recent preprint [2] deals with an equation close to ours, but with bistable or ignition reaction. The authors use the same Feynman-Kac representation as we do, the terms of which are also estimated thanks to the large deviation theory. With such estimates they are able to derive requirements on the initial condition so that the population persists. Their problem, therefore, is different from ours but their methods are very close. Finally, it is worth to mention [1], in which Addario-Berry and Reed deal with branching random walks in discrete time. In his work, Graham [21] explains quickly how, from Addario-Berry and Reed’s main result, one can deduce the emergence of the logarithmic delay (7) for the solution of the nonlinear nonlocal Cauchy problem (1) for a *restricted* class of monostable reaction terms and general dispersal kernels (he does not enter much into the details because he focuses on *general* monostable reaction term).

## 2 Main results

The first result is a proposition that gives a representation of the solution  $u(t, x)$  of the linear Cauchy problem (3). Let  $J$  be a kernel that satisfies the hypotheses (8) and (9). The kernel  $J$  is therefore the density of a probability law. We consider a sequence  $(X_k)_{k \geq 1}$  of real independent and identically distributed random variables, following the law of density  $J$ . Define the random walk

$$S_n = \sum_{k=1}^n X_k.$$

The following proposition is a Feynman-Kac representation of the solution of the linear Cauchy problem (3).

**Proposition 2.1.** *Let  $(S_n)_{n \geq 1}$  be the random walk defined above. Let  $u(t, x)$  be a solution of the linear Cauchy problem (3). Then, for all  $t \in [0, +\infty)$ , for all  $x \in \mathbb{R}$ ,*

$$u(t, x) = e^{(r-1)t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} \mathbb{P}(S_n \geq x). \quad (12)$$

As we look closely at each term of the sum (12), we observe that when  $x$  is near the position  $ct$ , there is a trade-off between the two factors  $\frac{t^n}{n!}$  and  $\mathbb{P}(S_n \geq x)$ :

- for  $t \simeq n$ , the factor  $\frac{t^n}{n!}$  is large, but the probability  $\mathbb{P}(S_n \geq x)$  is small;
- for  $t \gg n$ , the probability  $\mathbb{P}(S_n \geq x)$  is large but the factor  $\frac{t^n}{n!}$  is small;
- for  $t \ll n$ , both the probability  $\mathbb{P}(S_n \geq x)$  and the factor  $\frac{t^n}{n!}$  are small.

In fact, we shall understand in the following that for  $x = ct + o(\sqrt{t})$ , the dominant terms of the sum (12) are located around a position  $n \simeq \alpha t$  proportional to  $t$ . Therefore, when we study  $u(t, ct)$ , we will have to deal with probabilities of the form  $\mathbb{P}\left(\frac{S_n}{n} \geq \frac{c}{\alpha}\right)$ , where  $\frac{c}{\alpha}$  is a constant. The theory of large deviation provides an interesting framework to estimate precisely those very small probabilities as  $n \rightarrow +\infty$ , see e.g. the introduction of [12]. These observations will allow us to prove the main result.

**Theorem 2.2.** *Let  $J$  be a kernel that satisfies the hypotheses (8) and (9). Let  $u$  be a solution of the linear Cauchy problem (3). Take  $\rho \in (0, 1)$  and denote by*

$$\sigma^\rho(t) = \sup \{x \in \mathbb{R} \mid u(t, x) \geq \rho\}$$

*the position of the level  $\rho$  of  $u$  at time  $t$ . We have*

$$\sigma^\rho(t) = ct - s \ln(t) + O_{t \rightarrow +\infty}(1) \quad \text{with} \quad s = \frac{1}{2\lambda_r},$$

*where  $\lambda_r$  is defined by (11).*

*Remark 2.3.* Naturally, as we are currently considering the linear equation, we are not expecting to find exactly the same result as in Equation (7). However, we do get the same result with the local counterpart of the linear Cauchy problem (3),

$$\partial_t z = \partial_{xx} z + rz, \quad r = f'(0), \quad (13)$$

together with the initial condition  $z(t, 1) = \frac{1}{4\pi} e^{-x^2/4}$ . We then have a simple explicit expression for the solutions of Equation (13),  $z(t, x) = \frac{e^{rx}}{\sqrt{4\pi t}} e^{-x^2/4t}$ . We now look for the position  $\sigma_{loc}^\rho(t)$  such that  $z(t, \sigma_{loc}^\rho(t)) = \rho$ . A short computation yields, as  $t \rightarrow +\infty$ ,

$$\sigma_{loc}^\rho(t) = 2\sqrt{rt} - \frac{1}{2\sqrt{r}} \ln(t) + O(1) = c^*t - \frac{1}{2\lambda} \ln(t) + O(1),$$

which implies the same result as in Theorem 2.2 but for the local equation. Note also that the values  $\lambda = \sqrt{f'(0)}$  and  $\lambda_r$  play a symmetric role for, respectively, the local equation and the nonlocal equation.

Thanks to the maximum principle, we then deduce the following corollary about the nonlinear equation.

**Corollary 2.4.** *Let  $J$  be a kernel that satisfies the hypotheses (8) and (9). Let  $f$  be a reaction term that satisfies the monostability conditions (4) and the KPP condition (6). Let  $v$  be a solution of the nonlinear Cauchy problem (1). Take  $\rho \in (0, 1)$  and denote by*

$$\theta^\rho(t) = \sup \{x \in \mathbb{R} \mid v(t, x) \geq \rho\}$$

*the position of the level  $\rho$  of  $v$  at time  $t$ . We have*

$$\theta^\rho(t) \leq ct - s \ln(t) + O_{t \rightarrow +\infty}(1) \quad \text{with} \quad s = \frac{1}{2\lambda_r},$$

*where  $\lambda_r$  is defined by (11).*

*Remark 2.5.* This corollary is consistent with the main result of Graham [21] which deals with the situation when  $J$  has compact support (see above).

Section 3 gives preliminary results about the theory of large deviations; we shall use these results in Subsection 4.2 to prove Theorem 2.2. Subsection 4.1 is devoted to the proof of Proposition 2.1. Finally, Subsection 4.3 is devoted to the proof of Corollary 2.4.

### 3 Preliminary results about large deviations

We first focus on the useful notions from large deviations theory. Let  $J$  be a dispersal kernel satisfying the hypotheses (8) and (9) and let  $X$  be a real random variable following the law  $J$ . Let  $D = \{\zeta \in \mathbb{R} \mid \mathbb{E}[e^{\zeta X}] < +\infty\}$ . The *cumulant generating function* of the law of density  $J$  is defined at all  $\zeta \in D$  by

$$\Lambda(\zeta) = \ln \left( \mathbb{E} \left[ e^{\zeta X} \right] \right) = \ln \left( \int_{\mathbb{R}} J(x) e^{\zeta x} dx \right).$$

As  $J$  decreases at least exponentially fast at  $\pm\infty$ , the set  $D$  contains a neighbourhood of 0. The Legendre-Fenchel transform of  $\Lambda$  is denoted by  $\Lambda^*$  and plays a fundamental role in the statement of the theorem of Bahadur-Rao. It is defined, for all  $z \in \mathbb{R}$ , by

$$\Lambda^*(z) = \sup_{\zeta \in D} [\zeta z - \Lambda(\zeta)] \in [0, +\infty].$$

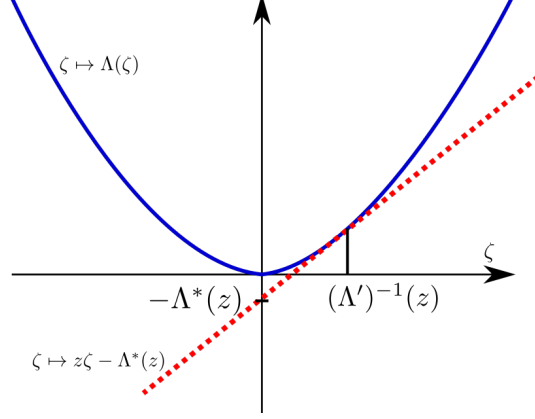
Figure 1 gives a graphical interpretation of the Legendre-Fenchel transform of a (symmetric) convex function.

The following lemma contains general properties of the functions  $\Lambda$  and  $\Lambda^*$  (see [12], Lemma 2.2.5).

**Lemma 3.1.** *Assume that  $J$  satisfies the hypotheses (8) and (9). We then have:*

1. *The function  $\Lambda$  is smooth (i.e.  $C^\infty$ ) over  $D$ . Let  $A = \sup \{\text{supp}(J)\}$ . The function  $\Lambda^*$  is finite and smooth over  $D^* = (-A, A)$ ;*





**Figure 1:** Graphical interpretation of the Legendre-Fenchel transform of  $\Lambda$  (inspired from [12]). The curve in blue looks like  $\Lambda$ , the dotted line in red is *the* line tangent to the blue curve and with slope  $z$ . The dotted line in red crosses the vertical axis at  $-\Lambda^*(z)$ , the opposite of the Legendre-Fenchel transform of  $\Lambda$  at  $z$ .

2. The functions  $\Lambda$  and  $\Lambda^*$  are nonnegative and strictly convex on, respectively,  $D$  and  $D^*$ . Moreover,  $\Lambda(0) = \Lambda^*(0) = 0$ ;
3. The function  $\Lambda'$  is a bijection from  $D$  to  $D^*$ . We define, for  $z \in D^*$ :  $\zeta_m(z) := (\Lambda')^{-1}(z)$ . We have, for all  $z \in D^*$ ,

$$\Lambda^*(z) = \zeta_m(z)z - \Lambda(\zeta_m(z)) \quad (14)$$

and

$$(\Lambda^*)'(z) = \zeta_m(z).$$

*Proof.* We sketch the proof of these classical properties (see also [12], Lemma 2.2.5).

- The nonnegativity of  $\Lambda$  comes from Jensen's inequality. Since  $J$  is continuous and thin-tailed,  $\Lambda$  is smooth in  $D$ ;
- The strict convexity of  $\Lambda$  follows: for  $\zeta \in D$ ,

$$\begin{aligned} \Lambda''(\zeta) &= \frac{\mathbb{E}[X^2 e^{\zeta X}] \mathbb{E}[e^{\zeta X}] - \mathbb{E}[X e^{\zeta X}]^2}{\mathbb{E}[e^{\zeta X}]^2} \\ &= \frac{\mathbb{E}[(X e^{\zeta X/2})^2] \mathbb{E}[(e^{\zeta X/2})^2] - \mathbb{E}[(X e^{\zeta X/2})(e^{\zeta X/2})]^2}{\mathbb{E}[e^{\zeta X}]^2} > 0, \end{aligned}$$

by the Cauchy-Schwarz inequality (the inequality is strict because  $X$  is nonconstant);

- We have

$$\Lambda'(\zeta) = \frac{\int_{\mathbb{R}} x e^{\zeta x} J(x) dx}{\int_{\mathbb{R}} e^{\zeta x} J(x) dx} \xrightarrow{\zeta \rightarrow +\infty} \sup \{\text{supp}(J)\},$$

thus:  $\sup_{\zeta \in D} \Lambda'(\zeta) = \sup \{\text{supp}(J)\}$ . Moreover,  $\Lambda'$  is strictly increasing and smooth. Therefore,  $\Lambda'$  is a smooth bijection from  $D$  to  $D^*$  with smooth converse  $(\Lambda')^{-1}$ ;

- For  $z \in D^*$ , the supremum in the definition of  $\Lambda^*(z)$  is reached when  $\Lambda'(\zeta) = z$ . We get (14). We then deduce that  $\Lambda^*$  is finite and smooth over  $D^*$ . Using (14), we get:  $(\Lambda^*)'(z) = \zeta_m(z)$ . Finally,  $\zeta_m(z)$  is increasing, so  $\Lambda^*$  is convex.

□

Let  $(X_k)_{k \geq 1}$  be a sequence of independent and identically distributed random variables, following the law  $J$ , and define the corresponding random walk  $(S_n)_{n \geq 1}$  as in the introduction. We are now ready to state, in our particular case, the theorem of Bahadur-Rao [4]. This theorem will be useful in the proof of Theorem 2.2. Recall that, for  $z \in D^*$ , we have defined  $\zeta_m(z) = (\Lambda')^{-1}(z)$ .

**Theorem 3.2** (Bahadur-Rao [4]). *Take  $M_1, M_2 \in D^*$  with  $M_2 > M_1 > 0$ . The random walk defined above satisfies*

$$\mathbb{P}(S_n \geq nz) = \frac{e^{-n\Lambda^*(z)}}{\zeta_m(z) \sqrt{2n\pi\Lambda''(\zeta_m(z))}} (1 + o(1)) \quad \text{as } n \rightarrow +\infty, \quad (15)$$

uniformly in  $z \in [M_1, M_2]$ .

The uniformity is not present in the original paper of Bahadur and Rao, but it has been proved by Petrov in [32].

## 4 Proofs of the main results

### 4.1 Proof of Proposition 2.1

*Proof of Proposition 2.1.* If  $u$  is a solution of the linear Cauchy problem (3), and if we set  $\tilde{u} := e^{-rt}u$ , then  $\tilde{u}$  is a solution of the linear equation  $\partial_t \tilde{u} = J * \tilde{u} - \tilde{u}$  with the same initial condition. Therefore, we may assume  $r = 0$ .

Let  $(Z_t)_{t \geq 0}$  be a Poisson process with rate 1 and jump law  $J$ , and such that  $Z_0 = 0$  almost surely. The infinitesimal generator of the process  $(Z_t)_{t \geq 0}$  is

$$\mathcal{G}f = J * f - f$$

and, therefore, the function

$$u(t, x) := \mathbb{E}[Z_t \geq x]$$

solves  $\partial_t u = J * u - u$  with initial condition  $u(0, \cdot) = \mathbb{1}_{(-\infty, 0]}$ . Upon partitioning events according to the number of jumps made by the process in  $[0, t]$ , we conclude

$$u(t, x) = \sum_{n=0}^{+\infty} \frac{e^{-t} t^n}{n!} \mathbb{P}(X_1 + \dots + X_n \geq x),$$

where  $\frac{e^{-t} t^n}{n!}$  is the probability that exactly  $n$  jumps occurred in  $[0, t]$ . This yields the result for  $r = 0$ . The conclusion follows by multiplying by  $e^{rt}$ . □

*Remark 4.1.* An important feature of Proposition 2.1 is that it allows us to work on the discrete-time random walk  $(X_1 + \dots + X_n)_{n \geq 0} = (S_n)_{n \geq 0}$  rather than on the continuous-time random walk  $(Z_t)_{t \geq 0}$ . In this situation, the theory of large deviations applies more easily. Another method to do the transformation from continuous time to discrete time is to consider the continuous-time random walk  $(Z_t)_{t \geq 0}$  restricted to integer times, which gives the discrete-time random walk  $(Z_n)_{n \geq 0}$ . The jump law of the process  $(Z_n)_{n \geq 0}$  is found by conditioning on the number of jumps made by the process  $(Z_t)_{t \geq 0}$  between the times 0 and 1. Hence the sum arising in Proposition 2.1 is directly incorporated into the jump law of  $(Z_n)_{n \geq 0}$ . This second method, which is inspired from the introduction of [1], should lead to easier computations: one simply has to estimate the probabilities  $\mathbb{P}\left(Z_n \geq cn - \frac{1}{2\lambda_r} \ln(n)\right)$ , and no more sum is involved. We shall rather concentrate on the first method, which seems more interesting from the point of view of modelling: indeed, it explicitly counts the jumps, i.e. the generations. Such a record can be helpful if one wants to take into account the fact, for example, that mutations can arise at each generation.

## 4.2 Proof of Theorem 2.2

Throughout Subsection 4.2, we assume for convenience that  $\sup\{\text{supp}J\} = +\infty$ , so that  $\Lambda^*$  is defined on  $D^* = \mathbb{R}$  (the proof is almost the same if  $\sup\{\text{supp}J\} < +\infty$ ). We consider a nonnegative function  $m(t) = o(\sqrt{t})$ , and we note

$$x_t = ct - m(t).$$

The function  $m$  is intended to represent the delay, while  $x_t$  is intended to represent the position of the front at time  $t$ . Our goal is to estimate  $u(t, x_t)$  as  $t$  grows to infinity. The idea of the proof is to cut the sum expressing  $u(t, x_t)$  given by Proposition 2.1,

$$u(t, x_t) = e^{(r-1)t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} \mathbb{P}(S_n \geq x_t) \quad (16)$$

into partial sums  $S_{a,b}(t, x_t)$  of the form

$$S_{a,b}(t, x_t) := e^{(r-1)t} \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} \frac{t^n}{n!} \mathbb{P}(S_n \geq x_t),$$

and to estimate each partial sum independently. We will show that there exists a value  $\alpha > 0$  such that the dominant terms of the total sum are located around  $\lfloor \alpha t \rfloor$ . A large part of the proof is devoted to the estimation of those terms, that is, to the estimation of  $S_{\alpha-\varepsilon, \alpha+\varepsilon}(t, x_t)$ .

We begin with a lemma that estimates the partial sums  $S_{a,b}(t, x_t)$ . We use a version of the theorem of Bahadur-Rao (Theorem 3.2) to turn the probability  $\mathbb{P}(S_n \geq x_t)$  into a more tractable expression. In the proofs, we will sometimes use the notation

$$p(t) = \Theta(q(t))$$

to say that there exist  $0 < K_- < K_+$  and  $t_0 > 0$  such that for all  $t \geq t_0$ ,  $K_-q(t) \leq p(t) \leq K_+q(t)$ .

**Lemma 4.2.** *Let  $b > a > 0$ . Define*

$$g(y) := y \left( \ln \left( \frac{e}{y} \right) - \Lambda^* \left( \frac{c}{y} \right) \right) + r - 1$$

and

$$h(t, y) := \frac{e^{(\Lambda^*)'(c/y)m(t)}}{t} e^{tg(y)}.$$

There exist  $K_-(a, b), K_+(a, b) > 0$  such that for  $t$  large enough,

$$K_-(a, b) \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} h \left( t, \frac{n}{t} \right) \leq S_{a,b}(t, x_t) \leq K_+(a, b) \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} h \left( t, \frac{n}{t} \right).$$

*Proof.* We shall use the following property, resulting from Theorem 3.2: for every subset  $V \subset (0, +\infty)$ , there exists  $n_0(V)$  such that for all  $n \geq n_0(V)$ , for all  $z \in V$ ,

$$\frac{e^{-n\Lambda^*(z)}}{2\zeta_m(z)\sqrt{2n\pi\Lambda''(\zeta_m(z))}} \leq \mathbb{P}(S_n \geq nz) \leq 2 \frac{e^{-n\Lambda^*(z)}}{\zeta_m(z)\sqrt{2n\pi\Lambda''(\zeta_m(z))}}. \quad (17)$$

Recall that  $\zeta_m(z)$  is defined in Lemma 3.1.

**Step 1: Application of (17).** Take  $V = \left[ \frac{c}{2b}, \frac{2c}{a} \right]$ . Recall that  $x_t = ct - m(t)$  where  $m(t) = o(\sqrt{t})$  is nonnegative. Therefore, for  $t$  large enough and for all integer  $n$  between  $\lfloor at \rfloor + 1$  and  $\lfloor bt \rfloor$ , we have  $x_t/n \in V$ . Let  $n_0(V)$  be the index given by Property (17). Consider  $t_0 \geq 0$  such that for all  $t \geq t_0$ , both  $\lfloor at \rfloor + 1 \geq n_0(V)$  and  $x_t/n \in V$  hold. Take  $t \geq t_0$  and apply Property (17) to each integer  $n$  between  $\lfloor at \rfloor + 1$  and  $\lfloor bt \rfloor$ , each time with  $z_n = \frac{x_t}{n} \in V$ . We find that there exist two constants  $0 < K_0 < K_1$  independent of  $t$  such that for each integer  $n$  between  $\lfloor at \rfloor + 1$  and  $\lfloor bt \rfloor$ ,

$$K_0 \frac{e^{-n\Lambda^*\left(\frac{x_t}{n}\right)}}{\zeta_m(x_t/n)\sqrt{2n\pi\Lambda''(\zeta_m(x_t/n))}} \leq \mathbb{P}(S_n \geq x_t) \leq K_1 \frac{e^{-n\Lambda^*\left(\frac{x_t}{n}\right)}}{\zeta_m(x_t/n)\sqrt{2n\pi\Lambda''(\zeta_m(x_t/n))}}.$$

By Lemma 3.1, the function  $\zeta_m$  is continuous and positive over  $(0, +\infty)$ . Since  $V$  is a compact subset of  $(0, +\infty)$ , we have

$$0 < \inf_{z \in V} \frac{1}{\zeta_m(z)\sqrt{2\pi\Lambda''(\zeta_m(z))}} \leq \sup_{z \in V} \frac{1}{\zeta_m(z)\sqrt{2\pi\Lambda''(\zeta_m(z))}} < +\infty.$$

Thus, summing over  $n$ , as  $t \rightarrow +\infty$ ,

$$S_{a,b}(t, x_t) = \Theta(T_{a,b}(t, x_t)) \quad (18)$$

where

$$T_{a,b}(t, x_t) := e^{(r-1)t} \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} \frac{t^n}{n! \sqrt{n}} e^{-n\Lambda^*\left(\frac{x_t}{n}\right)}.$$

Our goal is now to estimate the sum  $T_{a,b}(t, x_t)$ .

**Step 2: Estimation of  $\Lambda^*$ .** Take  $t \geq t_0$  and an integer  $n$  such that  $\lfloor at \rfloor + 1 \leq n \leq \lfloor bt \rfloor$ . Recall that  $x_t = ct - m(t)$  where  $m(t) = o(\sqrt{t})$ . We have

$$\Lambda^* \left( \frac{x_t}{n} \right) = \Lambda^* \left( \frac{ct}{n} - \frac{m(t)}{n} \right) = \Lambda^* \left( \frac{ct}{n} \right) - (\Lambda^*)' \left( \frac{ct}{n} \right) \frac{m(t)}{n} + O_{m/n \rightarrow 0} \left( \frac{m(t)^2}{n^2} \right).$$

Thus there exists a time  $t_1 \geq t_0$  and a constant  $K_2 > 0$  such that for all  $t \geq t_1$  and for every integer  $n$  such that  $\lfloor at \rfloor + 1 \leq n \leq \lfloor bt \rfloor$ , we have

$$\left| \Lambda^* \left( \frac{x_t}{n} \right) - \left( \Lambda^* \left( \frac{ct}{n} \right) - (\Lambda^*)' \left( \frac{ct}{n} \right) \frac{m(t)}{n} \right) \right| \leq K_2 \frac{m(t)^2}{n^2}.$$

We rewrite this as

$$\begin{aligned} & \Lambda^* \left( \frac{ct}{n} \right) - (\Lambda^*)' \left( \frac{ct}{n} \right) \frac{m(t)}{n} - K_2 \frac{m(t)^2}{n^2} \\ & \leq \Lambda^* \left( \frac{x_t}{n} \right) \\ & \leq \Lambda^* \left( \frac{ct}{n} \right) - (\Lambda^*)' \left( \frac{ct}{n} \right) \frac{m(t)}{n} + K_2 \frac{m(t)^2}{n^2}. \end{aligned} \quad (19)$$

**Step 3: Estimation of each single term of the sum  $T_{a,b}(t, x_t)$ .** By Stirling's formula, as  $n \rightarrow +\infty$ ,

$$n! = \Theta \left( \sqrt{n} \exp(n \ln(n/e)) \right).$$

Using the second inequality in (19), we obtain a constant  $K_3 > 0$  and a time  $t_2 \geq t_1$  such that, for all  $t \geq t_2$ , for all integer  $n$  such that  $\lfloor at \rfloor + 1 \leq n \leq \lfloor bt \rfloor$ ,

$$\begin{aligned} & \frac{e^{(r-1)t} t^n}{n! \sqrt{n}} e^{-n\Lambda^*(x_t/n)} \\ & \geq \frac{K_3}{n} \exp \left[ n \left( \ln \left( e \frac{t}{n} \right) - \Lambda^*(ct/n) \right) + (r-1)t \right] \exp \left[ (\Lambda^*)'(ct/n)m(t) - K_2 \frac{m(t)^2}{n} \right] \\ & = \frac{K_3}{n} \exp \left[ tg \left( \frac{n}{t} \right) \right] \exp \left[ (\Lambda^*)'(ct/n)m(t) - K_2 \frac{m(t)^2}{n} \right]. \end{aligned}$$

Since  $m(t) = o(\sqrt{t})$ , we have  $m(t)^2 = o(t)$ . Therefore there exist  $K_4 > 0$  and  $t_3 \geq t_2$  such that for all  $t \geq t_3$ , for all integer  $n$  such that  $\lfloor at \rfloor + 1 \leq n \leq \lfloor bt \rfloor$ ,

$$\frac{e^{(r-1)t} t^n}{n! \sqrt{n}} e^{-n\Lambda^*(x_t/n)} \geq \frac{K_4}{n} \exp \left[ tg \left( \frac{n}{t} \right) \right] \exp [(\Lambda^*)'(ct/n)m(t)] \geq \frac{K_4}{b} h \left( t, \frac{n}{t} \right).$$

Thus, summing on all  $n$  between  $\lfloor at \rfloor + 1$  and  $\lfloor bt \rfloor$ , we obtain an estimation of  $T_{a,b}(t, x_t)$  that yields, together with Equation (18), the existence of a constant  $K_-(a, b)$  such that for all  $t \geq t_3$ ,

$$S_{a,b}(t, x_t) \geq K_-(a, b) \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} h \left( t, \frac{n}{t} \right).$$

Using the first inequality in (19), we proceed to the same reasoning with reversed inequalities, and we obtain the existence of a constant  $K_+(a, b)$  and a time  $t'_3 \geq t_2$  such that for all  $t \geq t'_3$ ,

$$S_{a,b}(t, x_t) \leq K_+(a, b) \sum_{n=\lfloor at \rfloor + 1}^{\lfloor bt \rfloor} h\left(t, \frac{n}{t}\right).$$

The lemma is proved.  $\square$

Now, we introduce the value  $\alpha$ , which is constructed so that the most important terms of the sum (16) expressing  $u(t, x_t)$  are located around the position  $n \simeq \alpha t$ .

**Lemma 4.3.** *Set*

$$\alpha = c\lambda_r - (r - 1).$$

Then  $\alpha > 0$  and

$$\lambda_r = (\Lambda^*)' \left( \frac{c}{\alpha} \right). \quad (20)$$

Moreover, the function  $g$  defined in Lemma 4.2 is strictly concave on  $(0, +\infty)$  and is maximal at  $\alpha$ . Finally, we have  $g(\alpha) = 0$ ,  $g'(\alpha) = 0$ , and, for  $y > 0$ ,  $y \neq \alpha$ , we have  $g(y) < 0$ .

*Remark 4.4.* We can write  $h(t, y)$  in the form  $h(t, y) = C(t, y)e^{tg(y)}$ , where  $C(t, y)$  is not too large as  $t \rightarrow +\infty$ . Hence, Lemmas 4.2 and 4.3 entail that when  $a < b < \alpha$  or  $b > a > \alpha$ , the partial sum  $S_{a,b}(t, x_t)$  contains only terms that are exponentially small in  $t$ , thus the partial sum  $S_{a,b}(t, x_t)$  is also exponentially small in  $t$ . Therefore, the dominant terms of the whole sum (16) expressing  $u(t, x_t)$  are located around  $n \simeq \alpha t$ .

*Proof of Lemma 4.3.* Let  $M(\lambda) = \int_{\mathbb{R}} e^{\lambda x} J(x) dx$ . Let  $A(\lambda) = \frac{M(\lambda) + r - 1}{\lambda}$ . With this notation,  $c = \inf_{\lambda > 0} A(\lambda) = A(\lambda_r)$  and

$$\alpha = c\lambda_r - (r - 1) = M(\lambda_r).$$

Hence  $\alpha > 0$ . Since  $\lambda_r$  is positive and  $A$  is smooth and minimal at  $\lambda_r$ , we have  $A'(\lambda_r) = 0$ , that is

$$\frac{1}{\lambda_r^2} (\lambda_r M'(\lambda_r) - M(\lambda_r) - (r - 1)) = 0.$$

Recall  $\Lambda = \ln M$ . Thus, differentiating,

$$\lambda_r \Lambda'(\lambda_r) = \lambda_r \frac{M'(\lambda_r)}{M(\lambda_r)} = 1 + \frac{r - 1}{M(\lambda_r)},$$

which implies

$$\Lambda'(\lambda_r) = \frac{1}{\lambda_r} + \frac{r - 1}{\lambda_r M(\lambda_r)} = \frac{M(\lambda_r) + r - 1}{\lambda_r M(\lambda_r)} = \frac{c}{M(\lambda_r)} = \frac{c}{\alpha}.$$

Lemma 3.1 tells us that  $(\Lambda^*)' = \zeta_m = (\Lambda')^{-1}$ . Equality (20) follows.

We also have

$$\Lambda^* \left( \frac{c}{\alpha} \right) = \zeta_m \left( \frac{c}{\alpha} \right) \frac{c}{\alpha} - \Lambda \left( \zeta_m \left( \frac{c}{\alpha} \right) \right) = \lambda_r \frac{c}{\alpha} - \Lambda(\lambda_r)$$

so, using the fact that  $\alpha = M(\lambda_r)$ ,

$$\Lambda^* \left( \frac{c}{\alpha} \right) + \ln(\alpha) - \frac{c\lambda_r}{\alpha} = 0. \quad (21)$$

With equalities (20) and (21) at hand, we are ready to conclude. Recall that

$$g(y) = y \left( \ln \left( \frac{e}{y} \right) - \Lambda^* \left( \frac{c}{y} \right) \right) + r - 1.$$

We have, by (21),

$$g(\alpha) = \alpha (1 - \ln(\alpha) - \Lambda^*(c/\alpha)) + r - 1 = \alpha \left( 1 - \frac{c\lambda_r}{\alpha} \right) + r - 1.$$

Hence, since  $\alpha = c\lambda_r - (r - 1)$ , we conclude that  $g(\alpha) = 0$ . Furthermore,

$$g'(y) = -\ln(y) - \Lambda^* \left( \frac{c}{y} \right) + \frac{c}{y} (\Lambda^*)' \left( \frac{c}{y} \right).$$

Therefore, by (20) and (21), we get  $g'(\alpha) = 0$ . Finally, we have, for all  $y > 0$ ,

$$g''(y) = -\frac{1}{y} - \frac{c^2}{y^3} (\Lambda^*)'' \left( \frac{c}{y} \right) < 0.$$

These elements allow us to conclude. □

**Lemma 4.5.** *The following asymptotic properties hold as  $t \rightarrow +\infty$ .*

1. For  $0 < a < b < \alpha$ , there exists  $\chi > 0$  such that

$$S_{a,b}(t, x_t) = o(e^{-\chi t});$$

2. For  $B > A > \alpha$ , there exists  $\chi > 0$  such that

$$S_{A,B}(t, x_t) = o(e^{-\chi t});$$

3. There exists  $\varepsilon > 0$ , two constants  $K_-, K_+ > 0$  and a time  $t_0 > 0$  such that for all  $t \geq t_0$ ,

$$K_- \frac{e^{\lambda_r m(t)}}{\sqrt{t}} \leq S_{\alpha-\varepsilon, \alpha+\varepsilon}(t, x_t) \leq K_+ \frac{e^{\lambda_r m(t)}}{\sqrt{t}}.$$

*Proof.* Let  $0 < a < b < \alpha$  and let  $\chi = -\frac{1}{2} \sup_{y \in [a,b]} g(y)$ . Then, by Lemma 4.3,  $\chi$  is positive. From Lemma 4.2 and the fact that  $m(t) = o(\sqrt{t})$ , we deduce that the first point holds for this value of  $\chi$ . The same reasoning is valid for the second point as well. Now we prove the third point, which is more involved due to the equality  $g(\alpha) = 0$ .

**Step 1: Estimation of the partial sum  $S_{\alpha-\varepsilon, \alpha+\varepsilon}(t, x_t)$ ; simple integrals arise.** Take  $\varepsilon \in (0, \alpha/2)$ . By Lemma 4.2, there exist  $K_0, K_1 > 0$  such that for  $t$  large enough,

$$K_0 t \left( \frac{1}{t} \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} h\left(t, \frac{n}{t}\right) \right) \leq S_{\alpha-\varepsilon, \alpha+\varepsilon}(t, x_t) \leq K_1 t \left( \frac{1}{t} \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} h\left(t, \frac{n}{t}\right) \right).$$

By Lemma 4.3, there exists a neighbourhood  $W$  of  $\alpha$  such that for all  $t > 0$ , the function  $y \mapsto h(t, y)$  is decreasing on  $W$ . Up to reducing  $\varepsilon > 0$ , we may assume that  $[\alpha - 2\varepsilon, \alpha + 2\varepsilon]$  is included in  $W$ . Therefore, for all  $t > 0$ ,

$$\frac{1}{t} \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} h\left(t, \frac{n}{t}\right) \geq \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} \int_{\frac{n-1}{t}}^{\frac{n}{t}} h(t, y) dy$$

and

$$\frac{1}{t} \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} h\left(t, \frac{n}{t}\right) \leq \sum_{n=\lfloor(\alpha-\varepsilon)t\rfloor+1}^{\lfloor(\alpha+\varepsilon)t\rfloor} \int_{\frac{n}{t}}^{\frac{n+1}{t}} h(t, y) dy.$$

Thus, there exist  $K_2, K_3 > 0$  such that for  $t$  large enough,

$$K_2 t \int_{-\varepsilon}^{\varepsilon} h(t, \alpha + y) dy \leq S_{\alpha-\varepsilon, \alpha+\varepsilon}(t, x_t) \leq K_3 t \int_{-\varepsilon}^{\varepsilon} h(t, \alpha + y) dy. \quad (22)$$

**Step 2: Estimation of the integral arising in (22) and conclusion.** At the light of Equation (22), we wish to estimate  $h(t, \alpha + y)$  for  $y$  close to 0. By Lemma 4.3, we have  $g''(\alpha) < 0$  and

$$g(\alpha + y) = O(y^2).$$

Therefore we can take  $A', A'' > 0$ , and reduce  $\varepsilon > 0$  if necessary, so that for all  $y \in (-\varepsilon, \varepsilon)$ ,

$$-A'y^2 \leq g(\alpha + y) \leq -A''y^2.$$

Then, we have for some constants  $K_5, K_6 > 0$  and for  $t$  large enough,

$$K_5 \int_{-\varepsilon}^{\varepsilon} \frac{e^{(\Lambda^*)' \left(\frac{c}{\alpha+y}\right) m(t)}}{t} e^{-tA'y^2} dy \leq \int_{-\varepsilon}^{\varepsilon} h(t, y) dy \leq K_6 \int_{-\varepsilon}^{\varepsilon} \frac{e^{(\Lambda^*)' \left(\frac{c}{\alpha+y}\right) m(t)}}{t} e^{-tA''y^2} dy. \quad (23)$$

Now, note that

$$(\Lambda^*)' \left( \frac{c}{\alpha + y} \right) m(t) = \lambda_r m(t) + O_{y \rightarrow 0}(ym(t)).$$

Thus, there are random variables  $Z_t \sim \mathcal{N}\left(O\left(\frac{m(t)}{t}\right), \frac{1}{2tA'}\right)$  such that as  $t$  goes to infinity,

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} \frac{e^{(\Lambda^*)' \left(\frac{c}{\alpha+y}\right) m(t)}}{t} e^{-tA'y^2} dy &= \Theta \left( \frac{e^{\lambda_r m(t)}}{t} \int_{-\varepsilon}^{\varepsilon} \exp \left[ -tA' \left( y + O\left(\frac{m(t)}{t}\right) \right) \right]^2 dy \right) \\ &= \Theta \left( \frac{e^{\lambda_r m(t)}}{t\sqrt{t}} \times \mathbb{P}(-\varepsilon < Z_t < \varepsilon) \right) \\ &= \Theta \left( \frac{e^{\lambda_r m(t)}}{t\sqrt{t}} \right). \end{aligned}$$



The last line holds because  $m(t) = o(t)$ , so that  $Z_t$  converges to 0 in probability, as  $t \rightarrow +\infty$ . Injecting twice this estimation into (23) (once as such and once replacing  $A'$  by  $A''$ ), we deduce that there exist constants  $K_7, K_8 > 0$  such that for  $t$  large enough,

$$K_7 \frac{e^{\lambda_r m(t)}}{t\sqrt{t}} \leq \int_{-\varepsilon}^{\varepsilon} h(t, y) dy \leq K_8 \frac{e^{\lambda_r m(t)}}{t\sqrt{t}}. \quad (24)$$

Equation (24), together with Equation (22), proves that the third point of the statement holds for the values of  $\varepsilon$  that we have selected in the beginning of Step 2.  $\square$

We are now ready to conclude the proof of the main theorem.

*Proof of Theorem 2.2.* Let  $\varepsilon > 0$  be defined as in Lemma 4.5. Take  $a, B$  such that  $0 < a < \alpha < B$ . By Proposition 2.1, we have

$$u(t, x_t) = e^{(r-1)t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} \mathbb{P}(S_n \geq x_t),$$

so, cutting the sum into five parts, we have

$$u(t, x_t) = \left( \sum_{n=0}^{\lfloor at \rfloor} + \sum_{n=\lfloor at \rfloor+1}^{\lfloor (\alpha-\varepsilon)t \rfloor} + \sum_{n=(\alpha-\varepsilon)+1}^{\lfloor (\alpha+\varepsilon)t \rfloor} + \sum_{n=\lfloor (\alpha+\varepsilon)t \rfloor+1}^{\lfloor Bt \rfloor} + \sum_{n=\lfloor Bt \rfloor+1}^{+\infty} \right) \frac{e^{(r-1)t} t^n}{n!} \mathbb{P}(S_n \geq x_t). \quad (25)$$

Now, we estimate the first and last part of the sum (the other three will be estimated thanks to Lemma 4.5). First, for  $a > 0$  small enough and  $\chi_1 > 0$  small enough, we have, as  $t \rightarrow +\infty$ ,

$$\sum_{n=0}^{\lfloor at \rfloor} \frac{e^{(r-1)t} t^n}{n!} \mathbb{P}(S_n \geq x_t) \leq e^{rt} \mathbb{P}(S_{\lfloor at \rfloor} \geq x_t) = o(e^{-\chi_1 t}).$$

The last estimation is obtained thanks to the Theorem 3.2 with  $z = c/a$  (which becomes large when  $a > 0$  becomes small). Second, we have, as  $t \rightarrow +\infty$ ,

$$\begin{aligned} \sum_{n=\lfloor Bt \rfloor+1}^{+\infty} \frac{e^{(r-1)t} t^n}{n!} \mathbb{P}(S_n \geq x_t) &\leq \sum_{n=\lfloor Bt \rfloor+1}^{+\infty} \frac{e^{(r-1)t} t^n}{n!} \\ &\leq \frac{e^{(r-1)t} t^{\lfloor Bt \rfloor}}{\lfloor Bt \rfloor!} \sum_{n=0}^{+\infty} \frac{t^n}{n!} = e^{rt} \frac{t^{\lfloor Bt \rfloor}}{\lfloor Bt \rfloor!} = o(e^{-\chi_2 t}) \end{aligned}$$

for  $B > 0$  large enough and  $\chi_2 > 0$  small enough.

Therefore, thanks to Lemma 4.5 and the decomposition (25), we conclude that there exist constants  $K_-, K_+ > 0$  and  $\chi_3 > 0$  such that for  $t$  large enough,

$$K_- \frac{e^{\lambda_r m(t)}}{\sqrt{t}} \leq u(t, x_t) \leq K_+ \frac{e^{\lambda_r m(t)}}{\sqrt{t}} + o(e^{-\chi_3 t}).$$

Hence, there exists  $t_0 > 0$  such that for all  $t \geq t_0$ ,

$$\frac{1}{2}K_- \frac{e^{\lambda_r m(t)}}{\sqrt{t}} \leq u(t, x_t) \leq 2K_+ \frac{e^{\lambda_r m(t)}}{\sqrt{t}}.$$

Hence, upon choosing  $m(t) = \frac{1}{2\lambda_r} \ln(t) \pm C$  for a large  $C$ , we have:

$$\begin{aligned} u\left(t, ct - \frac{1}{2\lambda_r} \ln(t) + C\right) &\ll \rho, \\ u\left(t, ct - \frac{1}{2\lambda_r} \ln(t) - C\right) &\gg \rho, \end{aligned}$$

as  $t \rightarrow +\infty$ . (Recall that  $\rho$  is the level in which we are interested). Thus for  $t$  large enough,  $\left|\sigma^\rho(t) - \frac{1}{2\lambda_r} \ln(t)\right| < C$ . The conclusion of the theorem follows.  $\square$

### 4.3 Proof of Corollary 2.4

*Proof of Corollary 2.4.* We denote by  $v$  the solution of the nonlinear Cauchy problem (1) and by  $u$  the solution of the linear Cauchy problem (3) with  $r = f'(0)$ . As the reaction term satisfies the KPP condition (6), the function  $v$  is a subsolution of the linear Cauchy problem (3). As  $u$  and  $v$  have the same initial condition, the maximum principle tells us that  $v \leq u$ . Finally, when we apply Theorem 2.2 to the function  $u$ , we get the conclusion of Corollary 2.4.  $\square$

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