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PRODUCT MOMENTS OF DOWNTON'S BIVARIATE EXPONENTIAL DISTRIBUTION

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ABSTRACT: Downton's bivariate exponential distribution has been around since the 1940s. However, it seems that no explicit expressions for its product moments are available. In this short note, we provide a direct derivation of a general expression for the product moments and also derive several particular forms.

1 INTRODUCTION

Suppose X and Y are correlated exponential random variables with the joint pdf given by

$$f(x, y) = \frac{1}{1-\rho} \exp\left(-\frac{x+y}{1-\rho}\right) I_0\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) \quad (1)$$

for $x > 0$, $y > 0$ and $0 \leq \rho < 1$, where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k.$$

This distribution is due to Downton (1970) and therefore known as Downton's bivariate exponential distribution. The mathematical properties of this distribution (including joint cumulative distribution function, moment generating function, marginal properties and conditional properties) have been studied by several authors. It has also received applications in several areas. Some of them are:

- Queueing systems: Consider a single-server queueing system such that the inter-arrival time X and the service time Y have exponential distributions, as is a common assumption in this context. If it is desired to introduce positive correlation (arising from cooperative service) into the model, Downton's distribution is a suitable choice (Conolly and Choo, 1979). Langaris (1986) applied it to queueing system with infinitely many servers.

- As a model for Wold's Markov dependent process (Lai, 1978): Let X_i denote the time interval between the i th and $(i + 1)$ th events. Assuming that $\{X_i\}$ is a Markov chain, then $N(t) =$ number of events that occur in $(0, t]$ is a Wold's point process. Lai (1978) used Downton's bivariate exponential model to describe the joint distribution of the lengths of successive time intervals.
- Hydrology: Nagao and Kadoya (1971) claimed that Downton's distribution can be used for such pairs of hydrological quantities as streamflow at two points on a river, or rainfall at two locations.
- As a model for the intensity and duration of a storm of rainfall: Cordova and Rodriguez-Iturbe (1985) claim that exponential distributions for these variables have been shown to be sufficiently realistic. They find that independence should not be assumed - firstly, it is empirically not true; secondly, some important quantities are highly dependent on the correlation: quantities such as the mean and variance of storm depth (product of intensity and duration) and the probability of non-zero storm surface runoff (if the soil is sufficiently dry and the storm is sufficiently small, there is no surface runoff).
- Height of water waves: Kimura and Seyama (1985) assumed that the Rayleigh-marginals version of Downton's distribution described the joint distribution of successive wave heights. Their concern was overtopping of a sea wall that may occur when a group of high waves attacks it.

However, it seem that no direct expressions for the product moments of (1) are available in the literature. In this short note, we derive a general expression for $E(X^m Y^n)$ for $m \geq 1$ and $n \geq 1$. The mathematical derivation is direct and does not use the moment generating function approach. The calculations involve the Jacobi polynomial defined by

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left\{ (1-x)^{\alpha+n} (1+x)^{\beta+n} \right\},$$

and, the modified Laguerre polynomial defined by

$$L_n^\nu(x) = \frac{x^{-\nu} \exp(x)}{n!} \frac{d^n}{dx^n} \left\{ x^{n+\nu} \exp(-x) \right\}.$$

We also need the following important lemmas.

LEMMA 1 (Equation (2.15.5.4), Prudnikov et al., 1986, volume 2) For $p > 0$ and $\nu > -n - 1$,

$$\int_0^\infty x^{\nu+2n+1} \exp(-px^2) I_\nu(cx) dx = \frac{n! c^\nu}{2^{\nu+1} p^{n+\nu+1}} \exp\left(\frac{c^2}{4p}\right) L_n^\nu\left(-\frac{c^2}{4p}\right).$$

LEMMA 2 (Equation (2.19.3.2), Prudnikov et al., 1986, volume 2) For $p > 0$ and $\alpha > 0$,

$$\int_0^\infty x^{\alpha-1} \exp(-px) L_n^\lambda(cx) dx = \frac{\Gamma(\alpha)}{p^\alpha} P_n^{\lambda, \alpha-\lambda-n-1}\left(1 - \frac{2c}{p}\right).$$

The properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 PRODUCT MOMENTS

The main result of this note is provided in the following theorem. It derives an explicit expression for the product moment in terms of the Jacobi polynomials.

THEOREM *If X and Y are jointly distributed according to (1) then*

$$E(X^m Y^n) = m!n!(1-\rho)^n P_n^{(0, m-n)}\left(\frac{1+\rho}{1-\rho}\right) \quad (2)$$

for $m \geq 1$ and $n \geq 1$.

PROOF: One can express

$$\begin{aligned} E(X^m Y^n) &= \frac{1}{1-\rho} \int_0^\infty x^m \exp\left(-\frac{x}{1-\rho}\right) \int_0^\infty y^n \exp\left(-\frac{y}{1-\rho}\right) I_0\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) dy dx \\ &= \frac{2}{1-\rho} \int_0^\infty x^m \exp\left(-\frac{x}{1-\rho}\right) \int_0^\infty w^{2n+1} \exp\left(-\frac{w^2}{1-\rho}\right) I_0\left(\frac{2\sqrt{x\rho w}}{1-\rho}\right) dw dx \\ &= n!(1-\rho)^n \int_0^\infty x^m \exp(-x) L_n^0\left(-\frac{x\rho}{1-\rho}\right) dx, \end{aligned} \quad (3)$$

which follows after setting $w = \sqrt{y}$ and applying Lemma 4. The integral in (3) can be calculated by direct application of Lemma 5 to yield

$$\int_0^\infty x^m \exp(-x) L_n^0\left(-\frac{x\rho}{1-\rho}\right) dx = \Gamma(m+1) P_n^{(0, m-n)}\left(\frac{1+\rho}{1-\rho}\right). \quad (4)$$

The result of the lemma follows by combining (3) and (4). \blacktriangle

Using special properties of the Jacobi polynomials, one can derive elementary forms for (2) for particular values of m and n . This is shown in the corollaries below.

COROLLARY 1 *If X and Y are jointly distributed according to (1) then*

$$E(X^n Y^n) = (n!)^2 (1-\rho)^n P_n^{(0,0)}\left(\frac{1+\rho}{1-\rho}\right) \quad (5)$$

for $n \geq 1$.

COROLLARY 2 *If X and Y are jointly distributed according to (1) then*

$$\begin{aligned} E(XY) &= 1 + \rho, \\ E(X^2 Y) &= 2 + 4\rho, \\ E(X^3 Y) &= 6 + 18\rho, \\ E(X^4 Y) &= 24 + 96\rho, \\ E(X^5 Y) &= 120 + 600\rho, \\ E(X^2 Y^2) &= 4 + 16\rho + 4\rho^2, \end{aligned}$$

$$E(X^3Y^2) = 12 + 72\rho + 36\rho^2,$$

$$E(X^4Y^2) = 48 + 384\rho + 288\rho^2,$$

$$E(X^5Y^2) = 240 + 2400\rho + 2400\rho^2,$$

$$E(X^3Y^3) = 36 + 324\rho + 324\rho^2 + 36\rho^3,$$

$$E(X^4Y^3) = 144 + 1728\rho + 2592\rho^2 + 576\rho^3,$$

$$E(X^5Y^3) = 720 + 10800\rho + 21600\rho^2 + 7200\rho^3,$$

$$E(X^4Y^4) = 576 + 9216\rho + 20736\rho^2 + 9216\rho^3 + 576\rho^4,$$

$$E(X^5Y^4) = 2880 + 57600\rho + 172800\rho^2 + 115200\rho^3 + 14400\rho^4,$$

$$E(X^5Y^5) = 14400 + 360000\rho + 1440000\rho^2 + 1440000\rho^3 + 360000\rho^4 + 14400\rho^5.$$

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