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General Nilpotent and Solvable Approximations of Almost-Riemannian Structures

Yacine Chitour, Philippe Jouan, Ronald Manríquez

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Abstract

It is first shown that the nilpotent or the solvable approximation of an almost-Riemannian structure at a singular point is always a linear almost-Riemannian structure on a Lie group or a homogeneous space.

The generic properties of almost-Riemannian structures are then investigated in all dimensions and the generic nilpotent and solvable approximations are identified.

Keywords: Almost-Riemannian geometry; Nilpotent approximation; Solvable approximation; Genericity.

1 Introduction

The aim of this paper is twofold. On the first hand it is shown that the nilpotent or the solvable approximation of an almost-Riemannian structure at a singular point is always a linear almost-Riemannian structure on a Lie group or a homogeneous space. On the other hand the generic almost-Riemannian structures are described and used to exhibit the generic nilpotent and solvable approximations.

More precisely an almost-Riemannian structure, ARS in short, on an n -dimensional differential manifold is a rank-varying sub-Riemannian structure that can be locally defined by a set of n vector fields satisfying the Lie algebra rank condition and such that the singular locus \mathcal{Z} , that is the set of points where their rank is not full, is a proper but with empty interior subset (see [2], [3], [4], [6], [7], [12]).

In particular almost-Riemannian structures on Lie groups or homogeneous spaces that are defined by invariant and linear vector fields only are referred to as linear ARSs (a vector field on a Lie group is linear if its flow is a one-parameter group of automorphisms)(see [4]).

Following [5] and [10] the nilpotent approximation of such a structure at a point p of the singular locus has been used in [7] and [6] to investigate the behaviour around p in dimension respectively 2 and 3.

However it may happen that some vector fields of the nilpotent approximation vanish, changing the almost-Riemannian structure into a sub-Riemannian one. In that case the nilpotent approximation can be replaced by a solvable one, the definition of which was stated in [13] and is recalled in Section 2.2. In [13] the solvable approximation is used to study the distance around some singular points in the generic $3D$ -case.

The first aim is herein to prove that the nilpotent or solvable approximation of an ARS at a singular point is a linear almost-Riemannian structure on a Lie group or a homogeneous space, excepted in some very degenerated cases where neither the nilpotent approximation nor the solvable one defines an ARS. It is the purpose of Section 3, and the statements are illustrated by examples postponed to Section 5.

The generic properties of almost-Riemannian structures are then examined in Section 4. It is in particular shown that generically: (1) the singular set \mathcal{Z} is a union of submanifolds \mathcal{Z}_r of codimension r^2 where the rank is $n - r$; (2) the rank of $\Delta + [\Delta, \Delta]$ is everywhere full (Δ stands for the distribution). The structure of the points of \mathcal{Z}_r where $\dim(T_p\mathcal{Z}_r) + \dim(\Delta_p)$ is not maximal is described in Theorem 6. For example in \mathcal{Z}_1 these points are the so-called *tangency points* (see [6]), i.e. the points where $T_p\mathcal{Z}_1 = \Delta_p$. They are generically isolated in \mathcal{Z}_1 .

Thanks to these genericity results and with the help of local normal forms (see Section 4.2) it is finally shown that generically there are only two possibilities for the nilpotent/solvable approximation at a point $p \in \mathcal{Z}$:

1. At a tangency point p in \mathcal{Z}_1 one vector field of the nilpotent approximation vanishes, but the solvable approximation is not degenerated and defines a linear ARS.
2. At all other points, that is nontangency points of \mathcal{Z}_1 and all points in \mathcal{Z}_r with $r \geq 2$, the nilpotent approximation is not degenerated.

In conclusion the only *generic* points where the solvable approximation is useful are tangency points in \mathcal{Z}_1 .

As shown by the examples the picture is very different for nongeneric ARSs.

2 Basic definitions

2.1 Almost-Riemannian structures

For all that concern general sub-Riemannian geometry, including almost-Riemannian one, the reader is referred to [1].

Let M be a n -dimensional, connected, \mathcal{C}^∞ manifold. The \mathcal{C}^∞ -module of \mathcal{C}^∞ vector fields on M is denoted by $\Gamma(M)$. Let Δ be a submodule of $\Gamma(M)$. The flag of submodules

$$\Delta = \Delta^1 \subseteq \Delta^2 \subseteq \Delta^3 \subseteq \dots \subseteq \Delta^k \subseteq \dots$$

is defined by induction: $\Delta^2 = \Delta + [\Delta, \Delta]$ is the submodule of $\Gamma(M)$ generated by Δ and the Lie brackets of its elements, and $\Delta^{k+1} = \Delta^k + [\Delta, \Delta^k]$. The Lie algebra generated by Δ is $\mathcal{L}(\Delta) = \bigcup_{k \geq 1} \Delta^k$. The submodule Δ satisfies the rank condition if the evaluation of $\mathcal{L}(\Delta)$ at each point q is equal to $T_q M$.

Definition 1 *An almost-Riemannian structure (resp. distribution) on a smooth n -dimensional manifold M is a triple $(E, f, \langle \cdot, \cdot \rangle)$ (resp. a pair (E, f)) where E is a rank n vector bundle over M , $f : E \rightarrow TM$ is a morphism of vector bundles, and $(E, \langle \cdot, \cdot \rangle)$ is an Euclidean bundle, that is $\langle \cdot, \cdot \rangle_q$ is an inner product on the fiber E_q of E , smoothly varying w.r.t. q , assumed to satisfy the following properties:*

- (i) *The set of points $q \in M$ such that the restriction of f to E_q is onto is a proper open and dense subset of M ;*
- (ii) *The module Δ of vector fields of M , defined as the image by f of the module of smooth sections of E , satisfies the rank condition.*

The set of points of M where the rank of $f(E_q) = \Delta_q$ is less than n is called the singular locus of the ARS and denoted by \mathcal{Z} .

The inner product on E induces a bilinear symmetric and positive definite mapping, also denoted by $\langle \cdot, \cdot \rangle$, from $\Delta \times \Delta$ to $\mathcal{C}^\infty(M)$ ¹. Indeed an element $X \in \Delta$ (resp. $Y \in \Delta$) is the image by f of an unique section σ (resp. η) of E and we can set $\langle X, Y \rangle_q = \langle \sigma, \eta \rangle_q$.

Consequently an ARS can be alternately defined as follows.

¹i.e. $(X, Y) \mapsto \langle X, Y \rangle$ is $\mathcal{C}^\infty(M)$ -bilinear; $\langle X, X \rangle_q \geq 0$ for all $q \in M$ and $\langle X, X \rangle_q = 0 \implies X_q = 0$

Definition 2 *An almost-Riemannian structure on a smooth n -dimensional manifold M is a pair $(\Delta, \langle \cdot, \cdot \rangle)$ where Δ is a submodule of $\Gamma(M)$ that can be locally defined by n vector fields and satisfies the rank condition, and $\langle \cdot, \cdot \rangle$ is a bilinear symmetric and positive definite mapping from $\Delta \times \Delta$ to $C^\infty(M)$, such that the set \mathcal{Z} of points q where the dimension of Δ_q is less than n is nonempty but with empty interior.*

Around any point $p \in M$ the submodule Δ can be locally defined by an orthonormal frame (X_1, X_2, \dots, X_n) . It is enough to select a set of n sections (e_1, e_2, \dots, e_n) of E orthonormal in a neighborhood of p and define $X_i = f_*e_i$ where by definition $f_*e_i = f \circ e_i$.

Norm

The almost-Riemannian norm on Δ_q is defined by

$$\|v\| = \min\{\|u\|; u \in E_q \text{ and } f(u) = v\}.$$

2.2 Privileged coordinates and approximations

All the material of this section comes from [5] and [10], excepted the solvable approximations that are defined in [13].

Let p be a point of M and let $\Delta^k(p)$, $k \geq 1$ be the evaluation of the submodule Δ^k at p . Thanks to the rank condition these submodules verify $\Delta^1(p) \subset \Delta^2(p) \subset \dots \subset \Delta^r(p) = T_pM$ for some integer r referred to as the degree of nonholonomy at p . Let n_j stand for the dimension of $\Delta^j(p)$. The nonholonomic weights w_1, w_2, \dots, w_n at p are defined by $w_i = j \iff n_{j-1} < i \leq n_j$.

Let (x_1, x_2, \dots, x_n) be a system of coordinates centered at p . These coordinates are privileged if for each i there exist w_i vector fields in Δ such that the Lie derivative $X_{j_1}X_{j_2} \dots X_{j_{w_i}}x_i$ does not vanish at $p = 0$ but that any such Lie derivative of length smaller than w_i vanishes at 0 (see [5] or [10]).

Systems of privileged coordinates always exist (under the rank condition) and in such a system the weighted degree (homogeneous nonholonomic order) of the monomial $x_1^{\alpha_1}x_2^{\alpha_2} \dots x_n^{\alpha_n}$ is $\alpha_1w_1 + \alpha_2w_2 + \dots + \alpha_nw_n$, the weighted

degree of the vector field $\frac{\partial}{\partial x_j}$ is $-w_j$, and the weighted degree of the vector

field $x_1^{\alpha_1}x_2^{\alpha_2} \dots x_n^{\alpha_n} \frac{\partial}{\partial x_j}$ is $\alpha_1w_1 + \alpha_2w_2 + \dots + \alpha_nw_n - w_j$.

More generally the nonholonomic order at p of a function f (resp. a vector field X) is the minimum of the homogeneous nonholonomic orders of the monomials of its Taylor series.

It is important to notice that the nonholonomic degree $\text{ord}_p X$ of a vector field X at p cannot be less than $-r$, and that $\text{ord}_p[X, Y] = \text{ord}_p X + \text{ord}_p Y$ if X and Y are homogeneous and $[X, Y] \neq 0$ (see [5] or [10]).

The nonholonomic order of a vector field X belonging to Δ is at least equal to -1 . Consider a set X_1, X_2, \dots, X_n of vector fields that generates Δ around p . In privileged coordinates each X_j can be decomposed into

$$X_j = X_j^{(-1)} + X_j^{(0)} + X_j^{(1)} + \dots + X_j^{(s)} + \dots$$

where $X_j^{(s)}$ is the component of X_j of homogeneous order s .

The **nilpotent approximation** of $X_j \in \Delta$ is $\widehat{X}_j = X_j^{(-1)}$.

The Lie algebra generated by $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_n$ is nilpotent and finite dimensional. The rank condition is preserved: it is satisfied by $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_n$ as soon it is satisfied by X_1, X_2, \dots, X_n .

It may happen that some of the vector fields \widehat{X}_j globally vanish. In that case they can be replaced by $\widetilde{X}_j = X_j^{(0)}$. Let us assume that only m elements of $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_n$ are linearly independent (as vector fields). As explained in the next section we can assume without loss of generality that these vector fields are $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_m$ and that $\widehat{X}_{m+1}, \dots, \widehat{X}_n$ vanish. The set of vector fields

$$\widehat{X}_1, \dots, \widehat{X}_k, \dots, \widehat{X}_m, \widetilde{X}_{m+1}, \dots, \widetilde{X}_n,$$

is called the **solvable approximation** of X_1, X_2, \dots, X_n .

2.3 ARSs on Lie groups and homogeneous spaces

The purpose of this section is to recall the definition of linear ARSs on Lie groups and homogeneous spaces (see [4]).

Let G be a connected Lie group and \mathfrak{g} its Lie algebra (the set of left-invariant vector fields, identified with the tangent space at the identity). A vector field \mathcal{X} on G is **linear** if its flow is a one-parameter group of automorphisms or equivalently if $\mathcal{X}(e) = 0$ and for any left-invariant field Y the Lie bracket $[\mathcal{X}, Y]$ is also left-invariant.

A linear ARS on G is an almost-Riemannian structure defined by a set of:

- $n - s$ left-invariant vector fields Y_1, \dots, Y_{n-s} .
- $s > 0$ linear vector fields $\mathcal{X}_{n-s+1}, \dots, \mathcal{X}_n$.

assumed to satisfy the rank condition and to have full rank on a proper open and dense subset of G . The almost-Riemannian metric is defined by the orthonormality of this set of vector fields.

The singular set \mathcal{Z} where their rank is less than n cannot be empty because at least \mathcal{X}_n vanishes at the identity. On the other hand its interior is empty by analyticity as soon as the rank is full at one point.

A linear ARS is said to be simple if $s = 1$. For instance, the famous Grushin plane on the Abelian Lie group \mathbb{R}^2 is a simple ARS.

Consider a homogeneous space G/H of G by a closed and connected subgroup H (the elements of G/H are the right cosets of H). The projection of a left-invariant vector field Y onto G/H is well-defined and will be referred to as a left-invariant vector field (see details in [11]). On the other hand the projection of a linear field \mathcal{X} of G does exist on G/H if and only if H is invariant under its flow, or equivalently, because H is connected, if the Lie algebra of H is $\text{ad}(\mathcal{X})$ -invariant. This allows to define linear vector fields and linear ARSs on homogeneous spaces in exactly the same way than on Lie groups.

3 Nilpotent and solvable approximations are linear

Though this section deals with local questions, around a point p belonging to the singular locus, it will be more convenient to assume the ARS defined by a bundle E and a morphism f from E to TM as in Definition 1.

Firstly it is necessary to show that it is always possible to define the ARS locally, around the point $p = 0$ in local privileged coordinates, by a set of n orthonormal vector fields X_1, \dots, X_n such that the solvable approximation

$$\widehat{X}_1, \dots, \widehat{X}_k, \widehat{X}_{k+1}, \dots, \widehat{X}_m, \widetilde{X}_{m+1}, \dots, \widetilde{X}_n,$$

satisfies

- $\widehat{X}_i(0) \neq 0$ for $i = 1, \dots, k$;
- $\widehat{X}_i \neq 0$ but $\widehat{X}_i(0) = 0$ for $i = k + 1, \dots, m$;
- $\widehat{X}_i = 0$ for $i = m + 1, \dots, n$.

Let K_p be the kernel of the restriction of f to E_p , and let V_p be an orthogonal complement to K_p in E_p , that is $K_p \perp V_p$ and $K_p \oplus V_p = E_p$.

Let e_1, \dots, e_n be a set of n sections of E , orthonormal in a neighborhood of p , such that $e_j(p) \in V_p$ for $j = 1, \dots, k$ and $e_j(p) \in K_p$ for $j = k + 1, \dots, n$.

The vector fields $X_j = f_*(e_j)$, $j = 1, \dots, n$ define the ARS around p . Let (x_1, \dots, x_n) be a set of privileged coordinates and $\widehat{X}_1, \dots, \widehat{X}_n$ be the related nilpotent approximation. Let \mathcal{L} be the submodule of $\Gamma(E)$ generated by e_j for $j = k + 1, \dots, n$. Consider now the mapping $e \in \mathcal{L} \mapsto \widehat{f}_*e$.

Its rank is $m - k$ with $k \leq m \leq n$ and we can assume without loss of generality that e_{m+1}, \dots, e_n belong to the kernel of that linear map, and that e_{k+1}, \dots, e_m are orthogonal to that kernel.

The vector fields X_1, \dots, X_n satisfy the above conditions.

It may happen that the vector fields $\widetilde{X}_{m+1}, \dots, \widetilde{X}_n$ fail to be linearly independent. In that case neither the nilpotent approximation nor the solvable one define an almost-Riemannian structure.

For that reason we will always assume in what follows that the vector fields

$$\widehat{X}_1, \dots, \widehat{X}_k, \widehat{X}_{k+1}, \dots, \widehat{X}_m, \widetilde{X}_{m+1}, \dots, \widetilde{X}_n,$$

are linearly independent. Denote by \mathcal{Z}_a (a for approximation) the set of points where their rank is not full. It is not empty because at least one vector field vanishes at $p = 0$. On the other hand the approximating vector fields are polynomial and the interior of \mathcal{Z}_a is empty. Consequently \mathcal{Z}_a is a proper with empty interior subset of \mathbb{R}^n and the set of approximating vector fields defines an ARS.

Remarks.

1. It will be shown in the second part of the paper that generically $m = n$ or $m = n - 1$ and that in the second case $\widetilde{X}_n \neq 0$.
2. In the case where some of the \widetilde{X}_j vanish or are linearly dependent it seems difficult, if not impossible, to go one step further by considering homogeneous approximations of nonholonomic order $s > 0$ because, as explained in the sequel, two important properties could be lost. First a homogeneous vector field of degree $s > 0$ need not be complete. Second the Lie algebra generated by the approximating vector fields would not be finite dimensional in general. These two drawbacks are related, see [14] or [11].

3.1 The generated Lie algebra

In view of the next sections it is very important to notice that all involved vector fields are complete, because of their triangular form.

This fact is well-known for the \widehat{X}_i (see [5] or [10]).

It is as well true for the order zero homogeneous fields \tilde{X}_i . Indeed such a field writes:

$$\tilde{X}_i = \sum_{j=1}^n P_{ij}(x) \frac{\partial}{\partial x_j}$$

where P_{ij} is polynomial of nonholonomic degree w_j . In particular P_{ij} depends linearly on the coordinates $(x_{j_1}, \dots, x_{j_s})$ of weight w_j , is polynomial in the coordinates of weight smaller than w_j , and does not depend on the coordinates of weight greater than w_j . The associated differential equation is consequently triangular: the coordinates of weight 1 satisfy a linear homogeneous equation, the coordinates of weight 2 satisfy a linear equation with a second member that depends on the the coordinates of weight 1 only, and so on. All solutions are therefore defined on \mathbb{R} and \tilde{X}_i is complete. We can state:

Proposition 1 *The vector fields \hat{X}_j and \tilde{X}_j defined above are complete.*

This important property does not hold for homogeneous vector fields of positive degree, for example the first coordinate such a vector field could be $x_1^2 \frac{\partial}{\partial x_1}$, but $\dot{x}_1 = x_1^2$ is not complete.

The second feature we will use in the next subsections is the finiteness of the generated Lie algebra;

Proposition 2 *The Lie algebra \mathcal{L} generated by*

$$\hat{X}_1, \dots, \hat{X}_k, \hat{X}_{k+1}, \dots, \hat{X}_m, \tilde{X}_{m+1}, \dots, \tilde{X}_n,$$

is finite dimensional.

Proof. The nonholonomic order of these vector fields is 0 or -1 , and the nonholonomic order of their brackets is in the range $-r, \dots, 0$ where r is the nonholonomic degree of the set of vector fields. Consequently their components are polynomials of degree less than or equal to r . The Lie algebra \mathcal{L} is thus a subspace of a finite dimensional vector space of polynomials. ■

3.2 The nilpotent case

It is the case where the vector fields $\hat{X}_1, \dots, \hat{X}_n$ are linearly independent and the vectors $\hat{X}_1(0), \dots, \hat{X}_k(0)$ are independent in \mathbb{R}^n . In particular no vector field \hat{X}_i vanishes, and $m = n$.

For $j = k + 1, \dots, n$ let D_j stand for $\text{ad}(\widehat{X}_j)$ and for any multi-index $J = (j_1, \dots, j_s)$ let $D_J = D_{j_s} \circ \dots \circ D_{j_1}$ (here $k + 1 \leq j_u \leq n$ and $s \geq 0$). Let

$$\mathcal{D} = \text{Span}\{D_J(\widehat{X}_i) / i = 1, \dots, k; J \text{ as above}\}.$$

Lemma 1 *The Lie algebra \mathfrak{g} generated by \mathcal{D} is D_j -invariant for $j = k + 1, \dots, n$.*

Proof. Let $D_{J_1}(\widehat{X}_{i_1})$ and $D_{J_2}(\widehat{X}_{i_2})$ in \mathcal{D} . Then

$$D_j[D_{J_1}(\widehat{X}_{i_1}), D_{J_2}(\widehat{X}_{i_2})] = [D_j \circ D_{J_1}(\widehat{X}_{i_1}), D_{J_2}(\widehat{X}_{i_2})] + [D_{J_1}(\widehat{X}_{i_1}), D_j \circ D_{J_2}(\widehat{X}_{i_2})]$$

belongs to \mathfrak{g} . ■

Let \mathcal{L} stand for the Lie algebra generated by $\widehat{X}_1, \dots, \widehat{X}_n$. It is a well-known fact that this Lie algebra is nilpotent and finite dimensional (see [5] or [10]).

Theorem 1 *1. The ideal generated in \mathcal{L} by $\widehat{X}_1, \dots, \widehat{X}_k$ is \mathfrak{g} . It is a nilpotent Lie algebra.*

2. The vector fields $\widehat{X}_{k+1}, \dots, \widehat{X}_n$ do not belong to \mathfrak{g} and act on \mathfrak{g} as derivations.

3. The rank at $p = 0$ of the elements of \mathfrak{g} is full.

Proof.

1. Since \mathfrak{g} is D_j -invariant for $j > k$ it is clear that it is an ideal of \mathcal{L} that contains $\widehat{X}_1, \dots, \widehat{X}_k$. The ideal generated by these vector fields contains all the $D_J(\widehat{X}_i)$ hence is equal to \mathfrak{g} .

2. Let us assume that \widehat{X}_j belongs to \mathfrak{g} for some $j > k$. Because of the rule about the nonholonomic order of brackets of homogeneous vector fields, all the elements of \mathfrak{g} of order -1 are linear combinations of $\widehat{X}_1, \dots, \widehat{X}_k$. The vector field \widehat{X}_j is homogeneous of order -1 and can consequently be written as:

$$\widehat{X}_j = \sum_{i=1}^k \lambda_i \widehat{X}_i.$$

But \widehat{X}_j vanishes at 0 and the vectors $\widehat{X}_i(0)$ are independent by assumption, so that the λ_i 's are all equal to 0, a contradiction.

3. Let $Y \in \mathcal{L} \setminus \mathfrak{g}$. It can be obtained only as brackets of $\widehat{X}_{k+1}, \dots, \widehat{X}_n$, hence $Y(0) = 0$ because all these fields vanish at 0.

If \mathfrak{g} did not satisfy the rank condition then the Lie algebra \mathcal{L} neither would satisfy it. This proves item 3. ■

After this analysis at the algebra level we can turn our attention to the Lie group level.

Let G be the simply connected Lie group whose Lie algebra is \mathfrak{g} . Since \mathfrak{g} is nilpotent the underlying manifold of G is \mathbb{R}^N , $N = \dim(\mathfrak{g})$. The first task is to show that \mathbb{R}^n is a homogeneous space of G . This is mainly due to the fact that \mathfrak{g} is generated by homogeneous vector fields.

Lemma 2 *The set \mathbb{R}^n is a homogeneous space of G . More accurately if H stands for the connected subgroup of G whose Lie algebra is the set of elements of \mathfrak{g} that vanish at 0, then \mathbb{R}^n is diffeomorphic to the quotient G/H .*

Proof.

Since the elements of \mathfrak{g} are complete vector fields of \mathbb{R}^n the group G acts naturally on \mathbb{R}^n as a group of diffeomorphisms, and it is enough to show that this action is transitive. More accurately G is the set of $\exp(t_s Y_s) \dots \exp(t_1 Y_1)$ where $Y_i \in \mathfrak{g}$ is a vector field on \mathbb{R}^n and $t_i \in \mathbb{R}$, i.e. a group of diffeomorphisms of \mathbb{R}^n .

Let us begin by a simple remark. Let $Y = \sum_{j=1}^n y_j(x) \frac{\partial}{\partial x_j}$ be an element of \mathfrak{g} , and assume that Y is homogeneous and that $Y(0) \neq 0$. There exists an index i such that $y_i(0) \neq 0$, and by homogeneity y_i is constant: $\forall x \in \mathbb{R}^n$, $y_i(x) = a_i \neq 0$. By homogeneity again the polynomials y_j are homogeneous of order $w_j - w_i$. Consequently

$$Y = \sum_{w_j=w_i} a_j \frac{\partial}{\partial x_j} + \sum_{w_j>w_i} y_j(x) \frac{\partial}{\partial x_j} \quad \text{where } a_i \neq 0.$$

Since \mathfrak{g} is generated by homogeneous elements and its rank is full at 0, we can choose in \mathfrak{g} a set n homogeneous elements Y_1, Y_2, \dots, Y_n linearly independent at 0. We can also assume that the nonholonomic order at p of Y_1, \dots, Y_{n_1} is -1 , the one of $Y_{n_1+1}, \dots, Y_{n_2}$ is -2 and so on. Up to linear combinations we can assume that Y_1, \dots, Y_{n_1} have the following form:

$$Y_i = \frac{\partial}{\partial x_i} + \sum_{w_j>1} y_j(x) \frac{\partial}{\partial x_j}.$$

More generally we can assume that if the order of Y_i is w_i then

$$Y_i = \frac{\partial}{\partial x_i} + \sum_{w_j > w_i} y_j(x) \frac{\partial}{\partial x_j}.$$

This way it is clear that the rank of the set of vector fields Y_1, Y_2, \dots, Y_n is full everywhere in \mathbb{R}^n . This implies that the action of G on \mathbb{R}^n is transitive. \blacksquare

To complete the construction we associate to the derivation $D_j = \text{ad}(\widehat{X}_j)$ of \mathfrak{g} a linear vector field \mathcal{X}_j on G for $j > k$ (\mathcal{X}_j does exist because G is simply connected). It is clear that the projection of \mathcal{X}_j on \mathbb{R}^n is \widehat{X}_j (see [11] for details). Finally the vector fields $\widehat{X}_1, \dots, \widehat{X}_k$ are invariant, and $\widehat{X}_{k+1}, \dots, \widehat{X}_n$ are linear vector fields on the homogeneous space $\mathbb{R}^n = G/H$.

We can state:

Theorem 2 *The space \mathbb{R}^n is a homogeneous space of the nilpotent Lie group G whose Lie algebra is \mathfrak{g} .*

The vector fields $\widehat{X}_1, \dots, \widehat{X}_k$ are projections of invariant vector fields of G and $\widehat{X}_{k+1}, \dots, \widehat{X}_n$ are projections of linear vector fields of G .

Consequently the set $\widehat{X}_1, \dots, \widehat{X}_n$ defines a linear ARS on the homogeneous space \mathbb{R}^n .

3.3 The non-nilpotent case

We set $D_j = \text{ad}(\widehat{X}_j)$ for $j = k+1, \dots, m$ and $D_j = \text{ad}(\widetilde{X}_j)$ for $j = m+1, \dots, n$. As well as in the nilpotent case we set $D_J = D_{j_s} \circ \dots \circ D_{j_1}$ for any multi-index $J = (j_1, \dots, j_s)$ where $s \geq 0$ and $k+1 \leq j_u \leq n$, and

$$\mathcal{D} = \text{Span}\{D_J(X_i) / i = 1, \dots, k; J \text{ as above}\}.$$

The Lie algebra \mathfrak{g} generated by \mathcal{D} is again D_j -invariant for $j = k+1, \dots, n$, which shows that \mathfrak{g} is the ideal generated in \mathcal{L} by $\widehat{X}_1, \dots, \widehat{X}_k$.

Theorem 3 *1. The ideal generated in \mathcal{L} by $\widehat{X}_1, \dots, \widehat{X}_k$ is \mathfrak{g} . It is a nilpotent Lie algebra.*

2. The vector fields $\widetilde{X}_{m+1}, \dots, \widetilde{X}_n$ do not belong to \mathfrak{g} and act on \mathfrak{g} as derivations.

3. The vector fields \widehat{X}_j , with $k+1 \leq j \leq m$ that do not belong to \mathfrak{g} act on \mathfrak{g} as derivations.

4. The rank at $p = 0$ of the elements of \mathfrak{g} is full.

Proof. The proofs that \mathfrak{g} is the ideal generated in \mathcal{L} by $\widehat{X}_1, \dots, \widehat{X}_k$ and that its rank at $p = 0$ is full are identical to the nilpotent case.

The Lie algebra \mathfrak{g} is generated by homogeneous vector fields of order at most -1 . Since the order of a Lie bracket is the sum of the orders of the factors and a vector field of order less than $-r$ vanishes, all brackets of length larger than r vanish, which shows that \mathfrak{g} is nilpotent.

The points 2. and 3. are clear. ■

Opposite to the nilpotent case we cannot assert that the vector fields $\widehat{X}_{k+1}, \dots, \widehat{X}_m$ do not belong to \mathfrak{g} . Because of this phenomenon, illustrated by Example 3 in Section 5, we are lead to introduce one more index. Up to a re-ordering we can assume that $\widehat{X}_{k+1}, \dots, \widehat{X}_l$ belong to \mathfrak{g} and that $\widehat{X}_{l+1}, \dots, \widehat{X}_m$ do not belong to \mathfrak{g} , where $k + 1 \leq l \leq m$.

Theorem 4 *The space \mathbb{R}^n is a homogeneous space of the nilpotent Lie group G whose Lie algebra is \mathfrak{g} .*

The vector fields $\widehat{X}_1, \dots, \widehat{X}_l$ are projections of invariant vector fields of G . The vector fields $\widehat{X}_{l+1}, \dots, \widehat{X}_m$ are projections of linear or affine vector fields of G and $\widetilde{X}_{m+1}, \dots, \widetilde{X}_n$ are projections of linear ones.

Consequently the set of vector fields $\widehat{X}_1, \dots, \widehat{X}_m, \widetilde{X}_{m+1}, \dots, \widetilde{X}_n$ defines a linear ARS on the homogeneous space \mathbb{R}^n .

Proof. Similar to the one of Theorem 2. ■

Remark.

As shown by Example 4 in Section 5 the Lie algebra \mathcal{L} generated by $\widehat{X}_1, \dots, \widehat{X}_m, \widetilde{X}_{m+1}, \dots, \widetilde{X}_n$ need not be solvable when $m \leq n - 2$.

However it is solvable if $m = n - 1$ and it will be proven in the next section that generically $m = n - 1$ or $m = n$.

It is why we call *solvable* the approximations of the previous kind.

4 Genericity

The examples of Section 5 show that many different, complicated structures may arise and the aim of this section is to determine the generic ones.

In what follows we will say that a property of almost-Riemannian distributions (resp. structures) on a manifold M is **generic** if for any rank n

vector bundle E (resp. Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$) over M the set of smooth morphisms of vector bundles from E to TM for which this property is satisfied is open and dense in the \mathcal{C}^2 Whitney topology.

Let U be an open subset of M on which E and TM are trivializable, and let Π be the projection from E onto M . Then the restriction to $\Pi^{-1}(U)$ of a vector bundle morphism f is equivalent to a smooth mapping X from U to the set $\mathcal{M}_n(\mathbb{R})$ of $n \times n$ square matrices.

Alternately X can be viewed as a mapping (X_1, X_2, \dots, X_n) from U to the set $\Gamma(U)^n$ of n vector fields on U .

It is not useful to assume that f satisfies the properties of almost-Riemannian distributions because these conditions will turn out to be generic.

The first two theorems deal with distributions only, and do not require neither metric, nor normal forms.

In what follows we denote by $\mathcal{M}_{n \times m}(\mathbb{R})$ the set of real $n \times m$ matrices (simply $\mathcal{M}_n(\mathbb{R})$) if $m = n$) and by L^r the set of elements of $\mathcal{M}_{n \times m}(\mathbb{R})$ of corank r . It is a submanifold of $\mathcal{M}_{n \times m}(\mathbb{R})$ of codimension $(n - q + r)(m - q + r)$ where $q = \min\{n, m\}$ (see [8]).

Recall from Section 2 that f being given, Δ stands for the submodule of $\Gamma(M)$ it defines.

4.1 Generic distributions

Theorem 5 *The following properties are generic:*

1. *Let R be the largest integer such that $R^2 \leq n$. For $1 \leq r \leq R$ let \mathcal{Z}_r be the set of points where the rank of f_p , or locally the rank of $\{X_1, X_2, \dots, X_n\}$, is $n - r$. Each \mathcal{Z}_r is a codimension r^2 submanifold and the singular locus \mathcal{Z} is the union of these disjoint submanifolds.*
2. *The submanifold \mathcal{Z}_{r+1} is included in the closure $\overline{\mathcal{Z}_r}$ of \mathcal{Z}_r for $r = 1, \dots, R - 1$.*
3. *For any local representation X of the distribution the mapping $x \mapsto \det(X(x))$ is a submersion at all points $x \in \mathcal{Z}_1$.*
4. *For $n \geq 3$ the rank of $\Delta + [\Delta, \Delta]$ is full at all points.*

Proof.

1. Let E be a rank n vector bundle over M and let $\mathcal{H}om$ be the vector bundle over M whose fiber at $p \in M$ is the vector space $\mathcal{H}om_p = \mathcal{H}om(E_p, T_pM)$ of homomorphisms from E_p to T_pM .

A smooth vector bundle morphism f from E to TM can be viewed as a smooth section of $\mathcal{H}om$.

In each fiber $\mathcal{H}om_p$ the set of morphisms of corank $r > 0$ is a submanifold of $\mathcal{H}om_p$ of codimension r^2 . Their union over M defines a codimension r^2 submanifold S^r of $\mathcal{H}om$. Since the union of the S^r for $r = 1, \dots, n$ is closed, the set of smooth sections of $\mathcal{H}om$ that are transversal to all the S^r is open and dense in the set of smooth sections of $\mathcal{H}om$ endowed with the \mathcal{C}^2 Whitney topology.

The codimension of S^r is r^2 and transversality means nonintersection if $r^2 > n$. If $f^{-1}(S^r)$ is not empty transversality implies that it is a submanifold of M of codimension r^2 . This proves 1.

2. *In what follows we will always assume that the distributions under consideration are transversal to the manifolds S^r for $r = 1, \dots, n$.*

Let such a distribution be locally defined by X around a point p assumed to belong to \mathcal{Z}_r with $r \geq 2$. The rank of $\Lambda = X(p)$ is $n-r$ and there exists an invertible matrix P such that $P\Lambda P^{-1} = \begin{pmatrix} A_\Lambda & B_\Lambda \\ C_\Lambda & D_\Lambda \end{pmatrix}$ where $A_\Lambda \in \mathcal{M}_{n-r}(\mathbb{R})$ is invertible. As in the proof of the product of coranks Theorem we consider $T = \begin{pmatrix} I & 0 \\ -C_\Lambda A_\Lambda^{-1} & I \end{pmatrix}$ so that $TP\Lambda P^{-1} = \begin{pmatrix} A_\Lambda & B_\Lambda \\ 0 & D_\Lambda - C_\Lambda A_\Lambda^{-1} B_\Lambda \end{pmatrix}$.

Let the matrix P be fixed. The set Ω of matrices Q such that, with obvious notations, A_Q is invertible, hence $\Phi(Q) = D_Q - C_Q A_Q^{-1} B_Q$ is well defined, is open and the mapping Φ is a submersion from this open subset of $\mathcal{M}_n(\mathbb{R})$ onto $\mathcal{M}_r(\mathbb{R})$. Clearly $\text{rank}(Q) = n - r + \text{rank}(\Phi(Q))$ and for $0 \leq s \leq r$ we have $L_{\mathcal{M}_n(\mathbb{R})}^s = \Phi^{-1}(L_{\mathcal{M}_r(\mathbb{R})}^s)$ (with obvious notations again).

Since the distribution is generic we have $X \bar{\cap}_p L^r$ which implies that $\Phi \circ X$ is a submersion at p (see [8], Lemma 4.3). Let V be an open neighborhood of p and $W = V \cap X^{-1}(\Omega)$. Then $\Phi \circ X(W)$ is a neighborhood of 0 in $\mathcal{M}_r(\mathbb{R})$ that encounters $L_{\mathcal{M}_r(\mathbb{R})}^s$ for $0 \leq s \leq r$. Consequently

$$\forall s, \quad 0 < s < r, \quad V \cap \mathcal{Z}_s \neq \emptyset.$$

3. Consider now the determinant mapping, denoted by \det , from $\mathcal{M}_n(\mathbb{R})$ to \mathbb{R} . The differential of \det at Λ applied to H is $d\det(\Lambda).H = \text{Trace}(\tilde{\Lambda}H)$ where $\tilde{\Lambda}$ stands for the transpose of the matrix of cofactors of Λ . Consequently $d\det(\Lambda) \neq 0$ if and only if $\text{rank}(\Lambda) \geq n-1$. This shows that \det is

a submersion on $\mathcal{M}_n(\mathbb{R}) \setminus \left(\bigcup_{r \geq 2} L^r \right)$. In particular \det is a submersion in a neighborhood W of L^1 small enough for $L^1 = \{\det = 0\} \cap W$.

Let us consider a distribution locally defined by X around a point p assumed to belong to \mathcal{Z}_1 . Since $X \bar{\pi}_p L^1$ and according to [8] (Lemma 4.3) $\det \circ X$ is a submersion at p . In other words $q \mapsto \det(X(q))$ is a submersion at p .

4. It remains to show that the rank of $\Delta + [\Delta, \Delta]$ is generically full everywhere. Let $X = (X_1, \dots, X_n)$ be a set of n vector fields on an open subset U of M that define locally the distribution. Up to a system of coordinates X is a mapping from U to $(\mathbb{R}^n)^n \equiv \mathcal{M}_n(\mathbb{R})$. Let J^1 be a typical fiber of the space of 1-jets of sections of $T^n U$, identified with $(\mathbb{R}^n)^n \times (\mathcal{M}_n(\mathbb{R}))^n$. The set F_i of elements $\bar{X} = (X, dX)$ of J^1 such that $X_i = 0$ and $\text{rank}(dX_i) < n$ is closed with empty interior: it is a finite union of submanifolds of codimension $n + \rho^2$, where ρ is the corank of dX_i . Consequently the set $\mathcal{O} = J^1 \setminus (\bigcup_{i=1}^n F_i)$ is the open and dense subset of J^1 of elements that verify $X_i = 0 \implies dX_i$ invertible.

Let Ψ be the mapping from J^1 to $\mathcal{M}_{n \times \frac{n(n+1)}{2}}(\mathbb{R})$ defined by

$$\Psi(\bar{X}) = \left(X, ([X_i, X_j] = dX_j \cdot X_i - dX_i \cdot X_j)_{1 \leq i < j \leq n} \right).$$

The mapping Ψ is a submersion on \mathcal{O} .

On the other hand the set L^r of elements of $\mathcal{M}_{n \times \frac{n(n+1)}{2}}(\mathbb{R})$ of corank r is a submanifold of codimension $r \left(\frac{n(n-1)}{2} + r \right)$ and $\Psi^{-1}(L^r) \cap \mathcal{O}$ is a submanifold of \mathcal{O} of the same codimension.

But $n < r \left(\frac{n(n-1)}{2} + r \right)$ excepted in the particular case $n = 2$ and $r = 1$ where there is equality (this case has been studied in [3]).

To finish the set of X that are transversal to the F_i , the union of which is closed, is open and dense. Such distributions take their values in \mathcal{O} , and the set of X that are moreover transversal to all the $\Psi^{-1}(L^r) \cap \mathcal{O}$ is open and dense.

Excepted in the case $n = 2$ and $r = 1$ transversality means nonintersection and implies that the rank of $\Delta + [\Delta, \Delta]$ is full at all points. ■

Two subspaces of $T_p M$ are attached to a point p belonging to the strata \mathcal{Z}_r of the singular locus, namely Δ_p , the distribution at p , and $T_p \mathcal{Z}_r$, the tangent subspace to \mathcal{Z}_r at p . Their dimensions being respectively $n - r$ and $n - r^2$, the dimension of $T_p \mathcal{Z}_r + \Delta_p$ is at most equal to $\min(n, 2n - r^2 - r)$. We are interested in the cases where the actual dimension of $T_p \mathcal{Z}_r + \Delta_p$ is less than $\min(n, 2n - r^2 - r)$. For example in \mathcal{Z}_1 this means that $\Delta_p = T_p \mathcal{Z}_r$ (tangency points).

In what follows we note $s = \min(n, 2n - r^2 - r) - \dim(T_p\mathcal{Z}_r + \Delta_p)$, and $\lfloor \alpha \rfloor$ stands for the integer part of the real number α .

Theorem 6 *The following properties are generic:*

- $r = 1$. The points $p \in \mathcal{Z}_1$ where $T_p\mathcal{Z}_1 = \Delta_p$ are isolated in \mathcal{Z}_1 .
- $r \geq 2$. Let $m(n, r)$ be the largest dimension that $T_p\mathcal{Z}_r + \Delta_p$ may reach, that is $m(n, r) = \min(n, 2n - r^2 - r)$, and let $s = m(n, r) - \dim(T_p\mathcal{Z}_r + \Delta_p)$. Then
 1. The set of points $p \in \mathcal{Z}_r$ where $s = 1$ is a submanifold of \mathcal{Z}_r for $n \geq r^2 + r - \lfloor \frac{r-1}{2} \rfloor$. It is empty if $n < r^2 + r - \lfloor \frac{r-1}{2} \rfloor$.
 2. The set of points $p \in \mathcal{Z}_r$ where $s \geq 2$ and $s^2 \leq r$ is a submanifold of \mathcal{Z}_r for $r^2 + r - \lfloor \frac{r-s^2}{s-1} \rfloor \leq n \leq r^2 + r + \lfloor \frac{r-s^2}{s-1} \rfloor$. It is empty if n is not in this interval.
 3. The set of points $p \in \mathcal{Z}_r$ where $s \geq 2$ and $s^2 > r$ is empty.

Proof. In this proof all distributions are assumed to satisfy the transversality conditions of Theorem 5.

1. Let us consider such a distribution and a point p in \mathcal{Z}_r for some $r > 0$, locally defined by X on an open set U containing p .

Moreover the set L^r is a codimension r^2 submanifold of $\mathcal{M}_n(\mathbb{R})$ and it can be locally defined, in a neighborhood V of the point $X(p) \in L^r$, by a submersion Φ from V to \mathbb{R}^{r^2} , so that $L^r \cap V = \Phi^{-1}(0)$.

2. Let us denote by $d(p)$ the dimension of the quotient $(T_p\mathcal{Z}_r + \Delta_p) / T_p\mathcal{Z}_r$. Since the dimension of \mathcal{Z}_r is $n - r^2$ and the one of Δ_p is $n - r$ we have $n - r \leq \dim(T_p\mathcal{Z}_r + \Delta_p) \leq \min(n, 2n - r^2 - r)$, and:

$$r^2 - r \leq d(p) \leq \min(n - r, r^2).$$

The distribution being assumed to be transversal to L^r at p the restriction of dX at any supplementary subspace to $T_p\mathcal{Z}_r$ in T_pU is an isomorphism onto its image. On the other hand $dX(p)(T_p\mathcal{Z}_r) \subset \ker(d\Phi(X_p))$ because $\mathcal{Z}_r = X^{-1}(L^r)$, and since Φ is a submersion at p we obtain $d(p) = \text{rank } d(\Phi \circ X)(p).X(p)$.

Moreover there are exactly $n - r$ indices such that the vector fields $X_{i_1}, X_{i_2}, \dots, X_{i_{n-r}}$ are independant at p so that

$$d(p) = \text{rank } (d(\Phi \circ X)(p).X_{i_1}(p), \dots, d(\Phi \circ X)(p).X_{i_{n-r}}(p)).$$

3. In a neighborhood of p where the vector fields $X_{i_1}, X_{i_2}, \dots, X_{i_{n-r}}$ are independant at all points we define:

$$x \longmapsto \Theta(x) = (X(x); d(\Phi \circ X)(x).X_{i_1}(x), \dots, d(\Phi \circ X)(x).X_{i_{n-r}}(x))$$

The mapping Θ takes its values in $\mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{r^2 \times (n-r)}(\mathbb{R})$ and $\Theta(x)$ belongs to $L^r \times L^s$ if and only if $x \in \mathcal{Z}_r$ and $d(x) = \min(r^2, n-r) - s$.

This shows that s measures the gap between $d(p)$ and the maximal value it can take. Consequently we are interested in the values of s that verify:

$$0 \leq s \leq \min(r, n - r^2).$$

4. The sets $L^r \times L^s$ are submanifolds in the typical fibers of the space of 1-jets of smooth sections of $\mathcal{H}om$. Since their union is closed the set of smooth sections of $\mathcal{H}om$ that are transversal to all these submanifolds is open and dense in the set of smooth sections of $\mathcal{H}om$ endowed with the Whitney \mathcal{C}^2 topology.
5. Let us first assume that $n - r \geq r^2$. The codimension of $L^r \times L^s$ in $\mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{r^2 \times (n-r)}(\mathbb{R})$ is $r^2 + s(n - r^2 - r + s)$.

For $s = 1$ it is $n - r + 1$. Since $n \geq n - r + 1$ the set of points of \mathcal{Z}_r where $s = 1$ is generically a submanifold of codimension $n - r + 1$. In particular for $r = 1$ the codimension of this submanifold is n and it consists in isolated points.

For $s \geq 2$ transversality means nonintersection if $n < r^2 + s(n - r^2 - r + s)$. But the two conditions $n \geq r^2 + s(n - r^2 - r + s)$ and $n - r \geq r^2$ are equivalent to $0 \leq n - r^2 - r \leq \frac{r - s^2}{s - 1}$, hence to:

$$r^2 + r \leq n \leq r^2 + r + \lfloor \frac{r - s^2}{s - 1} \rfloor. \quad (1)$$

Notice that $s^2 \leq r$ is a necessary condition for this inequality to hold.

6. Let us now assume that $n - r \leq r^2$. The codimension of $L^r \times L^s$ is $r^2 + s(r^2 + r + s - n)$. Again transversality means nonintersection if $n < r^2 + s(r^2 + r + s - n)$. The two conditions $n - r \leq r^2$ and $n \geq r^2 + s(r^2 + r + s - n)$ are equivalent to:

$$r^2 + r - \lfloor \frac{r - s^2}{s - 1} \rfloor \leq n \leq r^2 + r \quad (2)$$

Again $s^2 \leq r$ is necessary for this inequality to hold.

7. The results of items 4 and 5 provide all the statements of the theorem.

4.2 Normal forms of almost-Riemannian structures

We consider an almost-Riemannian structure defined by a Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$ and a vector bundle morphism f , and we are interested in local normal forms of orthonormal vector fields defining the structure in a neighborhood of a point p that we can assume to be $p = 0$ in local coordinates.

These normal forms turn out to be the key of the next section.

First we follow the lines of [2] (also used in [7] and [6]).

Let W be a codimension 1 submanifold transversal to the distribution. We can define a coordinate system $y = (x_2, \dots, x_n)$ in W , and choose an orientation transversal to W . Let γ_y be the family of normal geodesics parametrized by arclength, transversal to W at y , and positively oriented. The mapping $(x_1, y) \mapsto \gamma_y(x_1)$ is a local diffeomorphism and the geodesics $x_1 \mapsto \gamma_y(x_1)$ realize the minimal distance between $W = \{x_1 = 0\}$ and the surfaces $\{x_1 = c\}$ for c small enough. The transversality conditions of the PMP are consequently satisfied along all these surfaces: if $\lambda(x_1)$ is a covector associated to one of these geodesics then the tangent space to $\{x_1 = c\}$ at $\gamma_y(x_1)$ is $\ker(\lambda(x_1))$.

Now let $X_1 = \partial_{x_1}$ be the vector field defined by $X_1(q) = \frac{d}{dx_1} \gamma_y(x_1)$ at the point $q = \gamma_y(x_1)$. It is a unitary vector field belonging to Δ . Let X_2, \dots, X_n be $n-1$ vector fields such that $\{X_1, X_2, \dots, X_n\}$ be an orthonormal frame of Δ . For geodesics the control functions (u_1, u_2, \dots, u_n) from the PMP satisfy $u_j = \langle \lambda, X_j \rangle$. But here $(u_1, u_2, \dots, u_n) = (1, 0, \dots, 0)$ so that $\langle \lambda, X_j \rangle = 0$ for $j = 2, \dots, n$ and the vector fields X_2, \dots, X_n are tangent to the surfaces $\{x_1 = c\}$. Consequently the vector fields have the following form:

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad X_j = \begin{pmatrix} 0 \\ a_{2,j} \\ \cdot \\ \cdot \\ a_{n,j} \end{pmatrix} \quad \text{for } 1 < j < n$$

for any choice of the coordinates in W and any choice of X_2, \dots, X_n , under the condition that they provide an orthonormal frame related to the sub-Riemannian metric.

Notice that thanks to the transversality conditions the vectors fields X_2, \dots, X_n are not only orthogonal to X_1 for the sub-Riemannian metric but also for the canonical inner product of \mathbb{R}^n for the chosen coordinates.

Let us assume now that $p = 0$ belongs to \mathcal{Z}_r with $r \geq 1$ and $r^2 \leq n$. We want to show that the coordinates and X_2, \dots, X_n can be chosen in such a way that they write:

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 + b_2(x) \\ a_{3,2}(x) \\ \cdot \\ \cdot \\ a_{n,2}(x) \end{pmatrix} \quad \dots \quad X_{n-r} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 1 + b_{n-r}(x) \\ a_{n-r+1,n-r} \\ \cdot \\ a_{n,n-r}(x) \end{pmatrix} \quad \text{and}$$

$$(X_{n-r+1} \quad \dots \quad X_n) = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \text{---} \\ D(x) \end{pmatrix} \quad \text{where } D(x) \in \mathcal{M}_r(\mathbb{R}),$$

$b_j(0) = a_{i,j}(0) = 0$, and $D(0) = 0$.

Firstly we can assume that $X_j(0) = \partial_{x_j}$ for $j = 2, \dots, n-r$ and $X_j(0) = 0$ for $j = n-r+1, \dots, n$. Indeed X_2, \dots, X_n are tangent to W where we can choose freely the coordinates (x_2, \dots, x_n) .

If $r = n-1$, which is generically possible only if $n = 2$ and $r = 1$, it is finished. Otherwise we can first replace X_2 by the normalization of $\sum_{j=2}^n a_{2j} X_j$. Then we replace X_j by $X_j - \frac{a_{2j}}{a_{22}} X_2$ for $j > 2$. These vector fields belong to Δ , their first two coordinates vanish and they are orthonormal to (the new) X_2 . It remains to orthonormalize these $n-2$ vector fields. This cannot be done directly because some of them vanish in the singular locus. However they are images by the vector bundle morphism f of locally nonvanishing smooth sections of E that can be orthonormalized.

The desired form of the vector fields is obtained by induction.

4.3 Nilpotent and solvable approximations of generic distributions

Theorem 7 *For a generic distribution holds:*

- (i) *Let p be a tangency point in \mathcal{Z}_1 , that is a point where $T_p \mathcal{Z}_1 = \Delta_p$. Then $\widehat{X}_n = 0$ but $\widetilde{X}_n \neq 0$, in normal form.*

(ii) At all other points, including all points in \mathcal{Z}_r with $r \geq 2$, the nilpotent approximation $\widehat{X}_1, \widehat{X}_2, \dots, \widehat{X}_n$ is a set of n linearly independent vector fields.

Proof.

Consider a generic distribution, a point p in \mathcal{Z}_1 and privileged coordinates centered at p such that

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 \\ 1 + b_2(x) \\ a_{3,2}(x) \\ \cdot \\ \cdot \\ a_{n,2}(x) \end{pmatrix} \dots X_{n-1} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 + b_{n-1}(x) \\ a_{n,n-1}(x) \end{pmatrix} X_n = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ a_n(x) \end{pmatrix}$$

where $b_j(0) = a_{i,j}(0) = 0$. The determinant of X is $a_n(x) \prod_{j=2}^{n-1} (1 + b_j(x))$ and the singular locus is locally $\mathcal{Z} = \mathcal{Z}_1 = \{a_n = 0\}$.

Let $a_n(x) = \sum_{i=1}^n \alpha_i x_i + o(\|x\|)$. Since the determinant is a submersion at p there exists i such that $\alpha_i \neq 0$.

1. If there exists $i_0 \leq n-1$ such that $\alpha_{i_0} \neq 0$ then $\widehat{X}_n \neq 0$. Moreover $da_n(0).X_{i_0} = \alpha_{i_0} \neq 0$. Hence $X_{i_0}(0) \notin T_0\mathcal{Z}_1$ and $T_0\mathcal{Z}_1 + \Delta_0 = \mathbb{R}^n$.
2. If $\alpha_i = 0$ for $i = 1, \dots, n-1$, then $\widehat{X}_n = 0$, but $\alpha_n \neq 0$ and $\widetilde{X}_n = (\alpha_n x_n + q(x_1, \dots, x_{n-1})) \frac{\partial}{\partial x_n}$, where $q(x_1, \dots, x_{n-1})$ is quadratic.

Moreover $T_0\mathcal{Z}_1 = \ker da_n(0) = \{x_n = 0\}$, hence $X_j(0) \in T_0\mathcal{Z}_1$ for $j = 1, \dots, n-1$ and $\Delta_0 = T_0\mathcal{Z}_1$.

Let now $p \in \mathcal{Z}_r$ with $r \geq 2$. We can choose privileged coordinates centered at p such that:

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 \\ 1 + b_2(x) \\ a_{3,2}(x) \\ \cdot \\ \cdot \\ a_{n,2}(x) \end{pmatrix} \dots X_{n-r} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 1 + b_{n-r}(x) \\ a_{n-r+1,n-r} \\ \cdot \\ a_{n,n-r}(x) \end{pmatrix} \quad \text{and}$$

$$(X_{n-r+1} \quad \dots \quad X_n) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline D(x) \end{pmatrix}$$

where $b_j(0) = a_{i,j}(0) = 0$ and $D(x) \in \mathcal{M}_r(\mathbb{R})$.

Let $x = (y, z)$ where $y = (x_1, \dots, x_{n-r})$ and $z = (x_{n-r+1}, \dots, x_n)$. The assumption $p = 0 \in \mathcal{Z}_r$ implies $D(0) = 0$ and we can write:

$$D(x) = l(y) + k(z) + \mathcal{Q}(y, z)$$

where l and k are linear and \mathcal{Q} contains all the terms of degree greater than one. Here $l(y) = (l_{ij}(y))_{n-r+1 \leq i, j \leq n}$ belongs to $\mathcal{M}_r(\mathbb{R})$ and each entry l_{ij} is linear w.r.t. $y = (x_1, \dots, x_{n-r})$.

As in the proof of Theorem 6 we denote by $d(p)$ the dimension of the quotient $(T_p \mathcal{Z}_r + \Delta_p) / T_p \mathcal{Z}_r$. It is clear that here $d(0) = \dim(dD(0) \cdot \Delta_0)$, at $p = 0$. But $\Delta_0 = \text{Span}\{X_1(0), \dots, X_{n-r}(0)\}$ and for $j = 1, \dots, n-r$:

$$dD(0) \cdot X_j(0) = \frac{\partial D}{\partial x_j}(0) = \frac{\partial l}{\partial x_j}(0).$$

Consequently $d(0) = \text{rank}(l)$ (as a mapping from \mathbb{R}^{n-r} to $\mathcal{M}_r(\mathbb{R})$).

For $j = n-r+1, \dots, n$ the nilpotent approximation \widehat{X}_j of X_j is

$$\widehat{X}_j = \sum_{i=n-r+1}^n l_{ij}(y) \frac{\partial}{\partial x_i}.$$

This is due to the fact that the weights of the coordinates are 1 for $i = 1, \dots, n-r$ and 2 for $i = n-r+1, \dots, n$.

Let us assume that the vector fields $\widehat{X}_{n-r+1}, \dots, \widehat{X}_n$ are not linearly independant. Then the mapping l takes its values in a subspace of $\mathcal{M}_r(\mathbb{R})$ of dimension $r^2 - r$ and $d(p)$ is smaller than or equal to $r^2 - r$. But $r^2 - r \leq d(p) \leq \min(n-r, r^2)$ according to the proof of Theorem 6 and the equality can hold for $r = 1$ only. Indeed let $s = \min(n-r, r^2) - d(p)$ as in the proof of Theorem 6. Then:

1. If $n-r \geq r^2$ then $s = r^2 - (r^2 - r) = r$. But $r \geq 2$ implies $s \geq 2$ and $s^2 > r$ which is impossible according to Theorem 6.
2. If $n-r < r^2$ then $s = n-r - (r^2 - r) = n - r^2$. But n, r and s must satisfy $n \geq r^2 + s(r^2 + r + s - n)$ according to the proof of Theorem 6. For $s = n - r^2$ this is $s \geq sr$ which is impossible for $r \geq 2$.

■

5 Examples

The following four examples use the notations of Section 3.

The first one is a standard example of a solvable approximation on the group Heisenberg. Example 2 shows that the elements of the Lie algebra \mathcal{L} of nonholonomic order smaller than -1 are not necessarily in \mathfrak{g} . We exhibit a vector field of the distribution that vanishes at p but belongs to the ideal \mathfrak{g} in Example 3 (which implies that the rank of the nilpotent approximation is not full).

To finish the Lie algebra \mathcal{L} of Example 4 is not solvable, it contains a semi-simple subalgebra. Recall that this is not generic.

In these four examples the vector fields are equal to their nilpotent or solvable approximations at 0. It is of course possible to add terms of higher nonholonomic order without modifying the conclusions.

Example 1

Consider in \mathbb{R}^3 the almost-Riemannian structure defined by the vector fields:

$$X_1 = \partial_x, \quad X_2 = \partial_y + x\partial_z, \quad X_3 = x\partial_y + \frac{1}{2}x^2\partial_z.$$

At $p = (0, 0, 0)$ the coordinates (x, y, z) are privileged with weights $(1, 1, 2)$, the vector fields X_1 and X_2 are homogeneous of order -1 and X_3 is homogeneous of order 0, so that $\hat{X}_1 = X_1$, $\hat{X}_2 = X_2$, $\hat{X}_3 = 0$ and $\tilde{X}_3 = X_3$. The nilpotent approximation at p is not an almost-Riemannian structure, it is the constant rank 2 sub-Riemannian structure defined by X_1 and X_2 .

The Lie algebra generated by $\hat{X}_1, \hat{X}_2, \tilde{X}_3$ is $\mathcal{L} = \text{Span}\{\hat{X}_1, \hat{X}_2, \tilde{X}_3, \partial_z\}$, the ideal $\mathfrak{g} = \text{Span}\{\hat{X}_1, \hat{X}_2, \partial_z\}$ is here the Heisenberg Lie algebra, and \tilde{X}_3 is a linear vector field on \mathfrak{g} . Finally $\hat{X}_1, \hat{X}_2, \tilde{X}_3$ is a linear ARS on the Heisenberg group.

Example 2

The almost-Riemannian structure is here defined in \mathbb{R}^3 by:

$$X_1 = \partial_x, \quad X_2 = x\partial_y, \quad X_3 = y^2\partial_z.$$

The Lie algebra \mathcal{L} contains X_1, X_2, X_3 and

$$\begin{aligned} X_4 &= [X_1, X_2] = \partial_y, & X_5 &= \frac{1}{2}[X_4, X_3] = y\partial_z, \\ X_6 &= [X_2, X_5] = x\partial_z, & X_7 &= [X_1, X_6] = [X_4, X_5] = \partial_z \\ X_8 &= \frac{1}{2}[X_2, X_3] = xy\partial_z, & X_9 &= [X_2, X_8] = x^2\partial_z. \end{aligned}$$

Therefore:

$$\begin{aligned} \Delta^1 &= \Delta = \{X_1, X_2, X_3\}, & \Delta^2 &= \Delta^1 + \{X_4, X_8\} \\ \Delta^3 &= \Delta^2 + \{X_5, X_9\}, & \Delta^4 &= \Delta^3 + \{X_6\}, & \Delta^5 &= \Delta^4 + \{X_7\}. \end{aligned}$$

The canonical coordinates are privileged with weights $(1, 2, 5)$ and the vector fields X_1 , X_2 , and X_3 are homogeneous of order -1 hence equal to their nilpotent approximations.

The algebra \mathfrak{g} is here the ideal of \mathcal{L} generated by X_1 that is

$$\mathfrak{g} = \text{Span}\{X_1, X_4, X_5, X_6, X_7\}.$$

The vector fields X_2 and X_3 are linear, as well as $X_8 = \frac{1}{2}[X_2, X_3]$ and $X_9 = [X_2, X_8]$.

The orders of X_8 and X_9 are respectively -2 and -3 which shows that the vector fields of order smaller than -1 are not necessarily in \mathfrak{g} .

Notice that the singular locus is here $\mathcal{Z} = \{xy = 0\}$. ■

Example 3

Consider in \mathbb{R}^4 the almost-Riemannian structure defined by the vector fields:

$$X_1 = \partial_x, \quad X_2 = \partial_y + x\partial_z, \quad X_3 = y\partial_w, \quad X_4 = \frac{1}{2}x^2\partial_z + \frac{1}{2}y^2\partial_w.$$

Since $[X_1, X_2] = \partial_z$ and $[X_2, X_3] = \partial_w$, the coordinates (x, y, z, w) are privileged with weights $(1, 1, 2, 2)$ at $p = (0, 0, 0, 0)$. The vector fields X_1 , X_2 are homogeneous of order -1 and independent at 0 , and the vector field X_3 is homogeneous of order -1 but vanishes at 0 . The last field X_4 is homogeneous of order 0 . Consequently the first three are equal to their nilpotent approximation and $X_4 = \tilde{X}_4$. According to the notations of Section 3 we have $k = 2$ and $m = 3$.

The Lie algebra \mathcal{L} is spanned by X_1, X_2, X_3, X_4 and

$$X_5 = [X_1, X_2] = \partial_z, \quad X_6 = [X_2, X_3] = \partial_w, \quad X_7 = [X_1, X_4] = x\partial_z.$$

Despite the fact that $\widehat{X}_3(0) = X_3(0) = 0$ we cannot assert as in the nilpotent case that X_3 does not belong to \mathfrak{g} (see Section 3.3 after Theorem 3). Indeed the ideal generated in \mathcal{L} by X_1 and X_2 is here

$$\mathfrak{g} = \text{Span}\{X_1, X_2, X_3, X_5, X_6, X_7\}$$

because $X_3 = [X_2, X_4]$.

As explained in Section 3.3 this may happen when $k < m < n$.

Notice that the determinant of X_1, X_2, X_3, X_4 is $-\frac{1}{2}x^2y$. Therefore the singular locus is $\mathcal{Z} = \{xy = 0\}$ which shows that the structure is not generic.

■

In the general case the Lie algebra \mathcal{L} need not be solvable. Indeed it is a subalgebra of the semi-direct product of \mathfrak{g} by its algebra of derivations. But the algebra of derivations of a nilpotent Lie algebra is not solvable in general. For instance the derivations of the Heisenberg algebra is the set of endomorphisms the matrix of which writes in the canonical basis:

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a + d \end{pmatrix}$$

The subalgebra of such derivations that moreover satisfy $e = f = a + d = 0$ is equal to \mathfrak{sl}_2 hence semisimple.

Example 4 illustrates that possibility.

Example 4

Consider in \mathbb{R}^5 , with coordinates (x, y, z, w, t) :

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_y + x\partial_w + z\partial_t, & X_3 &= \partial_z + x\partial_t, \\ X_4 &= x\partial_y + \frac{1}{2}x^2\partial_w, & X_5 &= y\partial_x + \frac{1}{2}y^2\partial_w \end{aligned}$$

The Lie algebra \mathcal{L} contains also

$$\begin{aligned} X_6 &= [X_1, X_2] = \partial_w, & X_7 &= [X_1, X_3] = [X_3, X_2] = \partial_t, \\ X_8 &= [X_3, X_5] = -y\partial_t, & X_9 &= [X_8, X_4] = x\partial_t, \\ X_{10} &= [X_4, X_5] = x\partial_x - y\partial_y. \end{aligned}$$

The coordinates (x, y, z, w, t) are privileged with weights $(1, 1, 1, 2, 2)$ at the origin. At this point the vector fields X_1, X_2, X_3 (resp. X_4, X_5) are homogeneous of order -1 (resp. 0), hence equal to their nilpotent approximations (resp. $\tilde{X}_4 = X_4$ and $\tilde{X}_5 = X_5$).

Since $X_{10} = [X_4, X_5]$, $[X_{10}, X_4] = 2X_4$ and $[X_{10}, X_5] = -2X_5$ the vector fields X_4, X_5 and X_{10} , that do not belong to \mathfrak{g} , generate a semi-simple Lie algebra isomorphic to \mathfrak{sl}_2 . Consequently the algebra \mathcal{L} is not solvable.

The singular locus is here $\mathcal{Z} = \{xyz = 0\}$, and again the structure is not generic.

■

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