Problems of Buffon type with multiple intersections for lattices of isoscele trapeziums

Giuseppe Caristi

Pub. Inst. Stat. Univ. Paris
LI, fasc. 3, 2007, 17 à 26

Problems of Buffon type with multiple intersections for lattices of isoscele trapeziums

Giuseppe Caristi

Abstract

In this paper we study problems of Bufoon type with multiple intersections for a lattice of isoscele trapeziums, having angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, and body test a line segment.

AMS2000 subject Classification: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry.

AMS Classification: 60D05, 52A22.

1 Preliminaries

In the two works [1] and [2] are studied problems of Buffon type with multiple intersections for lattices of parallelograms and of equilater triangles taking as test body a line segment of constant length s. For the same test body we consider a lattice $\Re$ with elementary tile a trapezium $\Im$ of parallels sides A and $A-B$ and of sides B and B, that form an angle equal to $\frac{\pi}{3}$ with the side A.

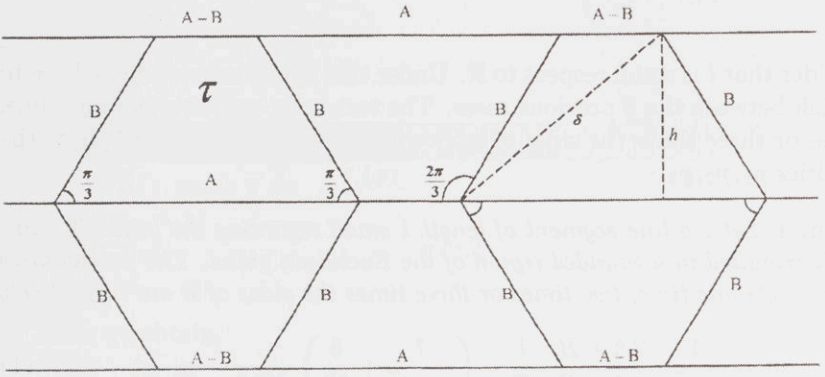

*Fig.*1

We want to determine the probability of multiple intersections of the test body s with the sides of lattice $\Re$, that is the probability $p_1, p_2, \ldots$ that the test body intersects respective one time, two times, ... the sides of lattice.

Because of the different relations between the geometric objects of the trapezium $\Im$, that is between sides A, B, the heights $h = \frac{B\sqrt{3}}{2}$ and $H = \frac{A\sqrt{3}}{2}$, and the diagonal $\delta = \sqrt{A^2 - AB + B^2}$ we must consider the following cases:

Case 1 : $B \leq A \leq \frac{2B}{\sqrt{3}}$. In this case we have:

$$A - B \leq h \leq H \leq B < \delta < A;$$

Case 2 : $\frac{2B}{\sqrt{3}} \leq A \leq \left(1 + \frac{\sqrt{3}}{2}\right) B$. In this case we have:

$$A - B \leq h \leq B \leq H < \delta < A;$$

Case 3 : $\left(1 + \frac{\sqrt{3}}{2}\right) B \leq A \leq 2B$. In this case we have:

$$h \leq A - B \leq B \leq H \leq \delta < A;$$

Case 4 : $2B \leq A \leq \left(4 + 2\sqrt{3}\right) B$. In this case we have:

$$h \leq B \leq A - B \leq H \leq \delta < A;$$

Case 5 : $\left(4 + 2\sqrt{3}\right) B \leq A$. In this case we have:

$$h \leq B \leq H \leq A - B < \delta < A.$$

The length needle s is small (in the given sense in [3]) respectively if and only if $l \leq A - B$ in the cases 1 and 2 and $l \leq B\frac{\sqrt{3}}{2}$ in the cases 3, 4 and 5.

2 The length needle

We consider that l is small respect to $\Re$. Under this hypothesis we don't have to distinguish between the 5 previous cases. The test body can intersect one time, two times or three times the sides of lattice; therefore we want to calculate the probabilities p_1, p_2, p_3.

Theorem 1 *Let s a line segment of length l small regarding the lattice $\Re$, uniformly distribuited in a bounded region of the Euclidean plane. The probabilities that s interscts one time, two times or three times the sides of $\Re$ are respectively*

$$p_1 = \frac{4}{\pi\sqrt{3}} \cdot \frac{2A + B}{2A - B} \cdot \frac{l}{B} - \left(\frac{7}{2\pi\sqrt{3}} + \frac{5}{18}\right) \cdot \frac{l^2}{(2A - B)\, B}; \tag{1}$$

$$p_2 = \left(\frac{4}{9} + \frac{2}{\pi\sqrt{3}}\right) \cdot \frac{l^2}{(2A - B)\, B} \tag{2}$$

$$p_3 = \left(\frac{1}{\pi\sqrt{3}} - \frac{1}{9}\right) \cdot \frac{l^2}{(2A - B)\, B}. \tag{3}$$

Proof. Let φ the angle formed from s with the the straight parallels of the lattice, we observe that for reasons of simmetry is sufficient to consider the direction of φ in the interval $\left[0,\ \frac{\pi}{2}\right]$. Let M the set of the length segments l that have the barycentre m in the trapezium $\Im$ and the arbitrary direction $\varphi \in \left[0,\ \frac{\pi}{2}\right]$. For $k = 0, 1, 2, 3$ let $\aleph_k$ set of segments s length l that have the barycentre m in the trapezium $\Im$, the direction φ such that che s intersects $\Re$ k times. Therefore $M = \aleph_1 \cup \aleph_2 \cup \aleph_3$ is a three-dimensional set and if φ is the direction s, then $s \in \aleph_k$ if and only if the centre m of s is founded in a set of $\Im$, that will be denote by $\Im_k(\varphi)$. If μ is the Lebesgue measure in the Euclidean space, then Stoka's formula gives to us

$$p_k = \frac{\mu(\aleph_k)}{\mu(M)} = \frac{\int_0^{\frac{\pi}{2}} Area\ \Im_k(\varphi)\, d\varphi}{\frac{\pi}{2}\ Area\ \Im} = \frac{4\int_0^{\frac{\pi}{2}} Area\ \Im_k(\varphi)\, d\varphi}{\pi\left(AB\sqrt{3} - B^2\frac{\sqrt{3}}{2}\right)}, \tag{4}$$

hence $Area\ \Im = \frac{1}{2}(2A - B)\cdot h = \frac{AB\sqrt{3}}{2} - \frac{B^2\sqrt{3}}{4}$.

For $\varphi \in \left[0, \frac{\pi}{3}\right[$ the set $\Im_3(\varphi)$ is empty; the set $\Im_2(\varphi)$ is the union of two parallelograms and of two trapezioms with height $\frac{l}{2}\sin\varphi$ and sides $\frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right) =: l_1(\varphi)$, $\frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} + \varphi\right) =: l_2(\varphi)$, like in Fig. 2.

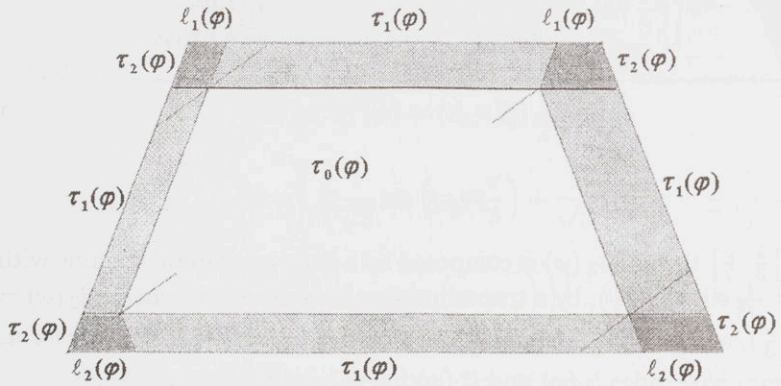

Fig.2

Then we obtain:

$$Area\ \Im_2(\varphi) = \left(l_1(\varphi) + l_2(\varphi) + 2\cdot\frac{l_1(\varphi) + l_2(\varphi)}{2}\right)\frac{l}{2}\sin\varphi =$$

$$2\frac{l}{\sqrt{3}}\left(\sin\left(\frac{\pi}{3} - \varphi\right) + \sin\left(\frac{\pi}{3} + \varphi\right)\right)\frac{l}{2}\sin\varphi = l^2\sin\varphi\cos\varphi. \tag{5}$$

The set $\Im_1(\varphi)$ is composed from four parallelograms; the two "horizontal" have the height $\frac{l}{2}\sin\varphi$ and horizontal sides $A - 2l_2(\varphi)$ respectively $A - B -$

$2l_1(\varphi)$; the two "oblique" have the oblique sides $B - \frac{2}{\sqrt{3}} l \sin\varphi$ and the height $\frac{\sqrt{3}}{2} l_1(\varphi)$ respectively $\frac{\sqrt{3}}{2} l_2(\varphi)$. Then:

$$Area\ \Im_1(\varphi) = 2A - B - 2(l_1(\varphi) + l_2(\varphi))\frac{l}{2}\sin\varphi +$$

$$\frac{\sqrt{3}}{2}\left(B - \frac{2l}{\sqrt{3}}\sin\varphi\right)(l_1(\varphi) + l_2(\varphi)) =$$

$$Al\sin\varphi + Bl\sin\left(\frac{\pi}{3} - \varphi\right) - 2l^2\sin\varphi\cos\varphi. \tag{6}$$

For $\varphi \in \left[0, \frac{\pi}{3}\right[$ and $\varphi \in \left]\frac{\pi}{3}, \frac{\pi}{2}\right]$ we have two different geometric situation as we can see in Fig. 2 and 3.

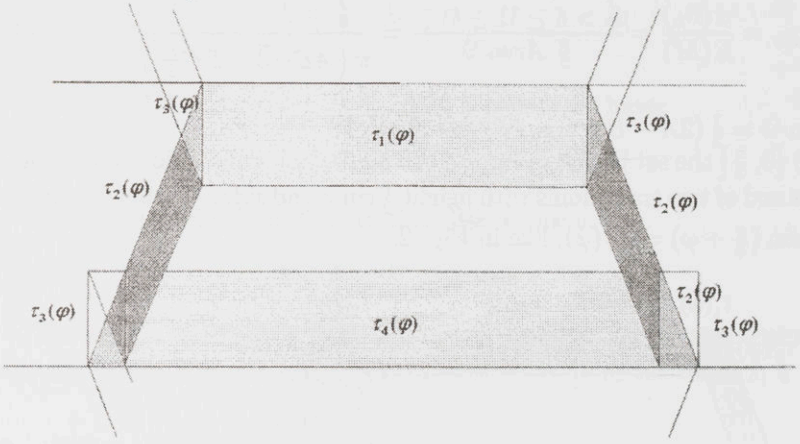

Fig.3

For $\varphi \in \left]\frac{\pi}{3}, \frac{\pi}{2}\right]$ the set $\Im_3(\varphi)$ is composed by a little equilateral triangle with side $l_1^*(\varphi) = \frac{l}{\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)$, by a trapezium that have the sides $l\cos\varphi$ e $l_2(\varphi) = \frac{l}{\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right)$, and height $\frac{l}{2}\sin\left(\varphi - \frac{\pi}{3}\right)$, and by two triangle that entirety forms a parallelogram with sides $l_2(\varphi)$ and $l_1^*(\varphi)$ havig height $\frac{l}{2}\sin\left(\varphi - \frac{\pi}{3}\right)$.

$$Area\Im_3(\varphi) = \frac{l^2}{4\sqrt{3}}\sin^2\left(\varphi - \frac{\pi}{3}\right) + \frac{l^2}{4}\sin\left(\varphi - \frac{\pi}{3}\right)\left[\cos\varphi + \frac{1}{\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right)\right] +$$

$$\frac{l^2}{2\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi + \frac{\pi}{3}\right) = \frac{l^2}{4\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi - \frac{\pi}{3}\right) +$$

$$\frac{l^2}{4\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi + \frac{\pi}{3}\right) + \frac{l^2}{2\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi + \frac{\pi}{3}\right) =$$

$$\frac{l^2}{\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi + \frac{\pi}{3}\right) = -\frac{l^2}{2\sqrt{3}}\left[\frac{1}{2} + \cos 2\varphi\right]. \tag{7}$$

The set $\Im_2(\varphi)$ is the union of an equilater of side $l\cos\varphi$, a trapezium with parallel sides $l_1^*(\varphi)$ and $2l_1^*(\varphi)$ and height $\frac{l}{2}\sin\left(\varphi-\frac{\pi}{3}\right)$, of an equilateral triangle with side $l_2(\varphi)$, and of a pentagon that with a little equilateral triangle with side $l_1^*(\varphi)$ they form a parallelogram with sides $l_2(\varphi)$, $l_3(\varphi):=\frac{l}{\sqrt{3}}\sin\varphi$ and angles $\frac{\pi}{3}$, $\frac{2\pi}{3}$.

In the figure 3 we see again that the set $\Im_1(\varphi)$ is the union of a pentagon and of three parallelograms with angles $\frac{\pi}{3}$, $\frac{2\pi}{3}$. The sides of the parallelogram are respectively $l_1^*(\varphi)$ and $B-2l_3(\varphi)$, $A-B$ and $\frac{l}{2}$, and $l_2(\varphi)$ and $B-2l_3(\varphi)$. The pentagon is the difference between a trapezium with parallel sides $A-[l_1^*(\varphi)+l_2(\varphi)]$ and $A-2[l_1^*(\varphi)+l_2(\varphi)]$ and height $\frac{l}{2}\sin\varphi$ and a little equilateral triangle with sides $l_1^*(\varphi)$. Consequently:

$$Area\ \Im_2(\varphi)=\left[\frac{\sqrt{3}}{4}\cos^2\varphi+\frac{1}{\sqrt{3}}\sin^2\left(\varphi-\frac{\pi}{3}\right)+\frac{1}{4\sqrt{3}}\sin^2\left(\varphi+\frac{\pi}{3}\right)\right.$$

$$\left.+\frac{1}{2\sqrt{3}}\sin\varphi\sin\left(\varphi+\frac{\pi}{3}\right)-\frac{1}{4\sqrt{3}}\sin^2\left(\varphi-\frac{\pi}{3}\right)\right]l^2=$$

$$\left[\frac{1}{\sqrt{3}}+\frac{\sqrt{3}}{8}\cos 2\varphi-\frac{\sqrt{3}}{8}\cos\left(2\varphi-\frac{2\pi}{3}\right)-\right.$$

$$\left.\frac{1}{8\sqrt{3}}\cos\left(2\varphi+\frac{2\pi}{3}\right)-\frac{1}{4\sqrt{3}}\cos\left(2\varphi+\frac{\pi}{3}\right)\right]l^2 \tag{8}$$

and

$$Area\ \Im_1(\varphi)=(A-B)\frac{l}{2}\sin\varphi+$$

$$\left(B\frac{\sqrt{3}}{2}-l\sin\varphi\right)\left(\frac{l}{\sqrt{3}}\sin\left(\varphi-\frac{\pi}{3}\right)+\frac{l}{\sqrt{3}}\sin\left(\varphi+\frac{\pi}{3}\right)\right)+$$

$$\left[A-\frac{3}{2}\left(\frac{l}{\sqrt{3}}\sin\left(\varphi-\frac{\pi}{3}\right)+\frac{l}{\sqrt{3}}\sin\left(\varphi+\frac{\pi}{3}\right)\right)\right]\frac{l}{2}\sin\varphi-(l\cos\varphi)^2\frac{\sqrt{3}}{4}=$$

$$Al\sin\varphi-\frac{7}{4\sqrt{3}}l^2\sin^2\varphi-\frac{\sqrt{3}}{4}l^2\cos^2\varphi=Al\sin\varphi+\frac{\cos 2\varphi}{2\sqrt{3}}l^2-\frac{5l^2}{4\sqrt{3}}. \tag{9}$$

We calculate the integrals of these areas:

$$\int_0^{\frac{\pi}{2}}Area\ \Im_1(\varphi)\,d\varphi=\int_0^{\frac{\pi}{3}}Area\ \Im_1(\varphi)\,d\varphi+\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}Area\ \Im_1(\varphi)\,d\varphi=$$

$$\left(-Al\cos\varphi+Bl\cos\left(\frac{\pi}{3}-\varphi\right)-l^2\sin^2\varphi\right)\Big|_0^{\frac{\pi}{3}}+\left(-Al\cos\varphi+\frac{\sin 2\varphi}{4\sqrt{3}}l^2-\frac{5l^2}{4\sqrt{3}}\varphi\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}=$$

$$Al+\frac{1}{2}Bl-\left(\frac{7}{8}+\frac{5\pi}{24\sqrt{3}}\right)l^2. \tag{10}$$

$$\int_0^{\frac{\pi}{2}} Area\ \Im_2(\varphi)\, d\varphi = \int_0^{\frac{\pi}{3}} Area\ \Im_2(\varphi)\, d\varphi + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} Area\ \Im_2(\varphi)\, d\varphi =$$

$$\frac{l^2}{2}\sin^2\varphi\Big|_0^{\frac{\pi}{3}} + \frac{\pi}{6\sqrt{3}}l^2 + l^2\left[\left(\frac{\sqrt{3}}{16}\sin 2\varphi\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \right.$$

$$\frac{\sqrt{3}}{16}\sin\left(2\varphi - \frac{2\pi}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \frac{1}{16\sqrt{3}}\sin\left(2\varphi + \frac{2\pi}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} -$$

$$\left.\left|\frac{1}{8\sqrt{3}}\sin\left(2\varphi + \frac{\pi}{3}\right)\right|_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right] = \left(\frac{\pi}{6\sqrt{3}} + \frac{1}{4}\right)l^2, \tag{11}$$

$$\int_0^{\frac{\pi}{2}} Area\Im_3(\varphi)\, d\varphi = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} Area\Im_3(\varphi)\, d\varphi =$$

$$-\frac{l^2}{2\sqrt{3}}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{4}\right] = \left(\frac{1}{8} - \frac{\pi}{24\sqrt{3}}\right)l^2. \tag{12}$$

Considering the formulas (4), (10), (11) and (12) we obtain the probabilities (1), (2) e (3). ■

3 The not small needle

Now we consider that the needle of length l is not little respect to $\Re$, but minor or equal to A. For each of the five previous cases we must disitinguish others five undercases, for example in the case 1 the undercases are: $A - B \leq l \leq h$; $h \leq l \leq H$; $H \leq l \leq B$; $B \leq l \leq \delta$; $\delta \leq l \leq A$. The cases 1 and 2 have got two identical undercases ($A - B \leq l \leq he\delta \leq l \leq A$) and the cases 2 and 3 have got three identical undercases ($B \leq l \leq H, H \leq l \leq \delta$ e $\delta \leq l \leq A$), and so on. We observe that for $l \leq A$ exactly 16 undercases are different. In order to simplify we will make the calculations on the undercase $A - B \leq l \leq h$. In this case we must distinguish between the following situations:

$$i)\ h \leq 2(A - B); \qquad ii)\ 2(A - B) \leq h.$$

In this last situation two possibilities exist:

$$i)_1\ A - B\ \leq l \leq 2(A - B); \qquad ii)_2\ 2(A - B) \leq l \leq h.$$

Now we consider the situations $i)$ and $ii)_1$. The interval $\left[0, \frac{\pi}{2}\right]$ is the union of $[0, \varphi_0]$, $\left[\varphi_0, \frac{\pi}{3}\right]$ and $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, where $\varphi_0 \in \left[0, \frac{\pi}{3}\right]$ is defined by the relation

$$\sin\left(\frac{\pi}{3} - \varphi_0\right) = \frac{\sqrt{3}}{2} \cdot \frac{A-B}{l}.$$

So for $\varphi \in \left[\varphi_0, \frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ we can be useful for the calculations of the areas maked for the little case, it remain to determine *Area* $\Im_1(\varphi)$, *Area* $\Im_2(\varphi)$ and *Area* $\Im_3(\varphi)$ when $\varphi \in [0, \varphi_0]$, and for this we observe the Fig. 4:

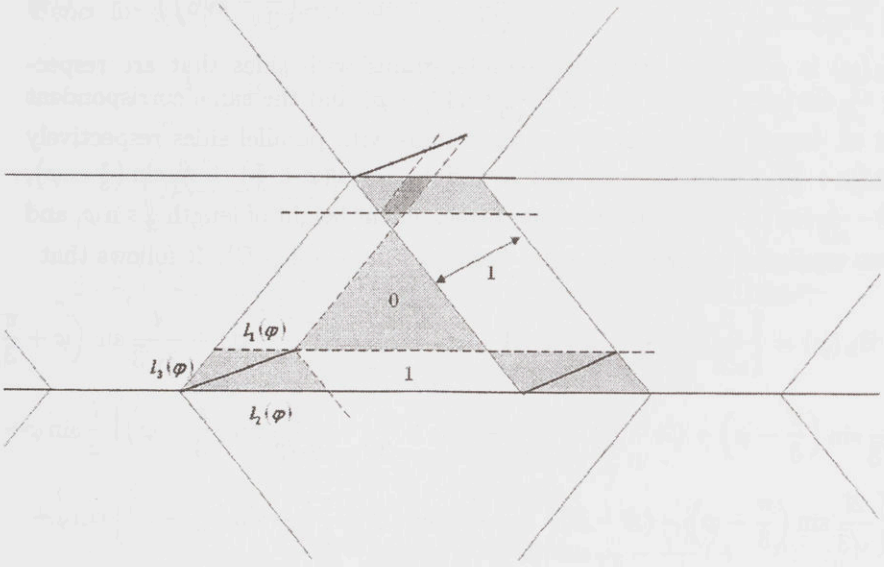

Fig.4

Area $\Im_3(\varphi)$ is the area of parallelogram with sides $\frac{2l}{\sqrt{3}} \sin\left(\frac{\pi}{3} - \varphi\right) - (A-B)$, $\frac{l}{\sqrt{3}} \sin\varphi$ and angles $\frac{\pi}{3}$, $\frac{2\pi}{3}$; therefore we have:

$$Area\ \Im_3(\varphi) = \frac{l}{2} \sin\varphi \left[\frac{2l}{\sqrt{3}} \sin\left(\frac{\pi}{3} - \varphi\right) - (A-B)\right] =$$

$$\frac{l^2}{2\sqrt{3}} \cos\left(\frac{\pi}{3} - 2\varphi\right) - \frac{l^2}{4\sqrt{3}} - \frac{l(A-B)}{2} \sin\varphi,$$

$$\int_0^{\varphi_0} Area \Im_3(\varphi)\, d\varphi = \frac{l(A-B)}{2} (\cos\varphi_0 - 1) + .$$

$$\frac{l^2}{8} - \frac{l^2}{4\sqrt{3}} \left(\varphi_0 + \sin\left(\frac{\pi}{3} - 2\varphi_0\right)\right). \qquad (13)$$

From (7) and (13) we obtain

$$\int_0^{\frac{\pi}{2}} Area\ \Im_3(\varphi)\,d\varphi = \left(\int_0^{\varphi_0} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right) Area\ \Im_3(\varphi)\,d\varphi = \frac{l(A-B)}{2}(\cos\varphi_0 - 1) +$$

$$\frac{l^2}{8} - \frac{l^2}{4\sqrt{3}}\left(\varphi_0 + \sin\left(\frac{\pi}{3} - 2\varphi_0\right)\right) + l^2\left(\frac{1}{8} - \frac{\pi}{24\sqrt{3}}\right) =$$

$$\frac{l(A-B)}{2}(\cos\varphi_0 - 1) + \frac{l^2}{4} - \frac{l^2}{4\sqrt{3}}\left(\frac{\pi}{6} + \varphi_0 + \sin\left(\frac{\pi}{3} - 2\varphi_0\right)\right). \qquad (14)$$

$\Im_2(\varphi)$ is composed from two parallelograms with sides that are respectively $\frac{l}{\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right)$ and $A - B - \frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right)$ and the same corrispondent height of length $\frac{l}{2}\sin\varphi$, from two trapeziums with parallel sides respectively $\frac{l}{\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right)$, $\frac{l}{\sqrt{3}}\sin\left(\varphi - \frac{\pi}{3}\right)$ and $A - B + \frac{l}{\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right) - \frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right)$, $A - B - \frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right)$ and the same corrispondent height of length $\frac{l}{2}\sin\varphi$, and from an equilater triangle with side $\frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right) - (A - B)$. It follows that

$$Area\ \Im_2(\varphi) = \left[\frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} + \varphi\right) + (A-B) - \frac{l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right) + \frac{l}{2\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right)\right.$$

$$\left. + \frac{l}{2\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right) + (A-B) + \frac{l}{2\sqrt{3}}\sin\left(\varphi + \frac{\pi}{3}\right) - \frac{3l}{2\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right)\right]\frac{l}{2}\sin\varphi +$$

$$\frac{\sqrt{3}}{4}\left[\frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \varphi\right) - (A-B)\right]^2 = \frac{l^2}{\sqrt{3}}\left[\sin\left(\varphi + \frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} - \varphi\right)\right]\sin\varphi +$$

$$(A-B)\,l\left[\sin\varphi - \sin\left(\frac{\pi}{3} - \varphi\right)\right] + \frac{\sqrt{3}}{4}(A-B)^2 + \frac{l^2}{\sqrt{3}}\sin^2\left(\frac{\pi}{3} - \varphi\right) =$$

$$\frac{\sqrt{3}}{4}(A-B)^2 + \sqrt{3}(A-B)\,l\sin\left(\varphi - \frac{\pi}{6}\right) + \frac{l^2}{\sqrt{3}}\left[1 - \frac{1}{2}\cos\left(\frac{\pi}{3} - 2\varphi\right)\right],$$

$$\int_0^{\frac{\pi}{2}} \Im_2(\varphi)\,d\varphi = \left|\frac{\sqrt{3}}{4}(A-B)^2\varphi_0 - \sqrt{3}(A-B)\,l\cos\left(\varphi - \frac{\pi}{6}\right)\right|_0^{\varphi_0} +$$

$$\frac{l^2}{\sqrt{3}}\varphi_0 + \frac{l^2}{4\sqrt{3}}\sin\left(\frac{\pi}{3} - 2\varphi_0\right)\Big|_0^{\varphi_0} + \frac{l^2}{2}\sin^2\varphi\Big|_{\varphi_0}^{\frac{\pi}{3}} + \frac{l^2\pi}{6\sqrt{3}} + \left[\frac{\sqrt{3}}{16}\sin 2\varphi\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right. -$$

$$\left.\frac{\sqrt{3}}{16}\sin\left(2\varphi - \frac{2\pi}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \frac{1}{16}\sin\left(2\varphi + \frac{2\pi}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \frac{1}{8\sqrt{3}}\sin\left(2\varphi + \frac{\pi}{3}\right)\right| +$$

$$\left.\frac{1}{8\sqrt{3}}\sin\left(2\varphi + \frac{\pi}{3}\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right] l = \frac{\sqrt{3}}{4}(A-B)^2\varphi_0 + \sqrt{3}(A-B)\,l\left[\frac{\sqrt{3}}{2} - \cos\left(\varphi_0 - \frac{\pi}{6}\right)\right] +$$

$$l^2\left[\frac{\varphi_0}{\sqrt{3}}+\frac{\pi}{6\sqrt{3}}+\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}-2\varphi_0\right)-\frac{1}{2}\sin^2\varphi_0+\frac{1}{8}\right]. \tag{15}$$

Finally the set $\Im_1(\varphi)$ if $\varphi\in[0,\varphi_0]$ is the union of three polygons: an horizontal parallelogram with sides $A-2l_2(\varphi)$ and $\frac{l}{\sqrt{3}}\sin\varphi$, havig angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, and two oblique parallelograms; of eachone of the parallelograms we eliminate an equilateral triangle of side $\frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3}-\varphi\right)-(A-B)$. The horizontal sides of this parallelogram are $l_1(\varphi)$ and $l_2(\varphi)$; The vertical height have length $B\frac{\sqrt{3}}{2}-l\sin\varphi$.

Hence for $\varphi\in[0,\varphi_0]$ we have

$$Area\ \Im_1(\varphi)=\left[A-\frac{2l}{\sqrt{3}}\sin\left(\varphi+\frac{\pi}{3}\right)\right]\frac{l}{2}\sin\varphi+$$

$$\left(B\frac{\sqrt{3}}{2}-l\sin\varphi\right)\left(\sin\left(\varphi+\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}-\varphi\right)\right)\frac{l}{\sqrt{3}}-$$

$$2\cdot\frac{\sqrt{3}}{4}\left[\frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3}-\varphi\right)-(A-B)\right]^2=\frac{Al}{2}\sin\varphi+\frac{\sqrt{3}}{2}Bl\cos\varphi-$$

$$\frac{\sqrt{3}}{2}(A-B)^2+2(A-B)l\sin\left(\frac{\pi}{3}-\varphi\right)-$$

$$l^2\left[\frac{5}{4\sqrt{3}}+\frac{1}{2\sqrt{3}}\cos\left(\frac{\pi}{3}-2\varphi\right)\right],$$

$$\int_0^{\varphi_0}Area\ \Im_1(\varphi)\,d\varphi=\frac{Al}{2}(1-\cos\varphi_0)+\frac{\sqrt{3}}{2}Bl\sin\varphi_0-\frac{\sqrt{3}}{2}(A-B)^2\varphi_0+$$

$$2(A-B)l\left[\cos\left(\frac{\pi}{3}-\varphi_0\right)-\frac{1}{2}\right]-l^2\left[\frac{5}{4\sqrt{3}}\varphi_0+\frac{1}{8}-\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}-2\varphi_0\right)\right]$$

and therefore it is had that:

$$\int_0^{\frac{\pi}{2}}Area\ \Im_1(\varphi)\,d\varphi=\frac{Al}{2}(1+\cos\varphi_0)+Bl\left(1-\frac{1}{2}\cos\varphi_0\right)-\frac{\sqrt{3}}{2}(A-B)^2\varphi_0+$$

$$2(A-B)l\left[\cos\left(\frac{\pi}{3}-\varphi_0\right)-\frac{1}{2}\right]-$$

$$l^2\left[\frac{5}{4\sqrt{3}}\varphi_0+\frac{5\pi}{24\sqrt{3}}+\cos^2\varphi_0-\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}-2\varphi_0\right)\right]. \tag{16}$$

Form the formulas (14), (15),(16) and (4) follows:

Theorem 2 *Let s a line segment of length l, that respect to the dimension of lattice $\Re$ satisfies the relations $A-B \leq l \leq h \leq 2(A-B)$ or $A-B \leq l \leq 2(A-B) \leq h$. Let $\varphi_0 \in \left[0, \frac{\pi}{3}\right]$ is the only angle with the property $\sin\left(\frac{\pi}{3}-\varphi_0\right) = \frac{A-B}{l} \cdot \frac{\sqrt{3}}{2}$ .The line segment s is uniformly distribuited in a bounded region of the Euclidean plane, then the probability that s intersects one time, two times or three times the sides of lattice $\Re$ are respectively*

$$p_1 = \frac{4}{\pi\sqrt{3}\left(A-\frac{B}{2}\right)B}\left[\frac{Al}{2}\left(1+\cos\varphi_0\right)+Bl\left(1-\frac{1}{2}\cos\varphi_0\right)-\right.$$

$$\frac{\sqrt{3}}{2}(A-B)^2\varphi_0+2(A-B)l\left(\cos\left(\frac{\pi}{3}-\varphi_0\right)-\frac{1}{2}\right)-$$

$$\left.l^2\left(\frac{5\varphi_0}{4\sqrt{3}}+\frac{5\pi}{24\sqrt{3}}+\cos^2\varphi_0-\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}-2\varphi_0\right)\right)\right], \tag{17}$$

$$p_2 = \frac{4}{\pi\sqrt{3}\left(A-\frac{B}{2}\right)B}\left[\frac{\sqrt{3}}{4}(A-B)^2\varphi_0+\sqrt{3}(A-B)l\left(\frac{\sqrt{3}}{2}-\cos\left(\varphi_0-\frac{\pi}{6}\right)\right)\right.$$

$$\left.+l^2\left(\frac{\varphi_0}{\sqrt{3}}+\frac{\pi}{6\sqrt{3}}+\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}-2\varphi_0\right)-\frac{1}{2}\sin^2\varphi_0+\frac{1}{8}\right)\right], \tag{18}$$

$$p_3 = \frac{4}{\pi\sqrt{3}\left(A-\frac{B}{2}\right)B}\left[\frac{l(A-B)}{2}\left(\cos\varphi_0-1\right)\right.$$

$$\left.+\frac{l^2}{4}-\frac{l^2}{4\sqrt{3}}\left(\frac{\pi}{6}+\varphi_0+\sin\left(\frac{\pi}{3}-2\varphi_0\right)\right)\right]. \tag{19}$$

References

[1] Duma, A., Stoka M.I., Problems of Buffon type with multiple intersections for lattices of parallelograms, Rend. Circ. Mate. Palermo, serie II, 2006, pp. 241-248.

[2] Duma, A., Stoka M.I., Problems of Buffon Type with multiple intersections for lattices of equilateral triangles, preprint.

[3]Duma, A., Problems of Buffon type for "non-small" needles, Rend. Circ. Mat. Palermo, serie II, 1999, pp. 23-40.

[4] Stoka, M.I., Probabiltités géométrique de type Buffon dans le plan euclidean, Atti Acc. Sci. Torino, vol. 10, (1975-1976), pp. 53-59.

Giuseppe Caristi
Department D.E.S.Ma.S. "V. Pareto"
Faculty of Economics, University of Messina - ITALY
Via dei Verdi 75, 98122 Messina.
E-mail address: gcaristi@dipmat.unime.it