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# Counting Characteristic Roots of Linear Delay-Differential Equations. Part I: Frequency-Sweeping Stability Tests and Applications

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**Abstract** This chapter addresses the stability analysis of linear dynamical systems represented by delay differential equations with a focus on the effects induced by the delay, seen as a *parameter*, on the dynamical behavior. More precisely, we propose a *frequency-sweeping framework* for treating the problem, and the stability problem is reformulated in terms of properties of frequency-sweeping curves. The presentation is teaching-oriented and focuses more on discussing the main ideas of the method and their illustration through appropriate examples and less on explicit proofs of the results. Some applications from Life Sciences complete the presentation.

## 1 Introduction

One of the important problems in the analysis of dynamical systems is to understand how changes in the systems' parameters may affect the qualitative and quantitative behavior of the systems. Such a problem becomes challenging when the system is infinite-dimensional. This chapter is devoted to such issues in the case of linear time-delay systems represented by Delay Differential Equations (DDEs). More precisely, we focus on the analysis of the *effects* induced by the *delay parameters* on the (exponential) stability of the corresponding system.

Roughly speaking, to better capture the heterogeneity of the temporal phenomena in systems' dynamics the knowledge of "past" may appear as

being essential in deriving appropriate models, and there exists a large variety of processes in nature having such characteristics. In these cases, the use of *delays*<sup>1</sup> in representing such phenomena may help in a better understanding of the underlying mechanisms or of the interactions/coupling with other (eventually spatial) phenomena. For instance, in Physics and Engineering, the delay may be used to model transport and propagation in interconnected cyber-physical systems or to represent the effect induced by the presence of humans in traffic flow models. In Life Sciences, the delays may appear as good approximations for incubation periods, maturation times or age structure in epidemic dynamics or they may be used to represent translation and transcription processes in genetic regulatory networks. Finally, in Economics, delays may appear in trade cycles, business cycles in commodity markets when defining the right balance between supply and market information-based demand.

Delay systems belong to the class of infinite-dimensional systems and there exist several ways to represent their dynamics. Functional Differential Equations (FDEs) sometimes called delay differential equations (DDEs) or differential-difference equations are, by now, a classical framework for studying the qualitative and quantitative effects induced by the delays on the systems' dynamics. Throughout this chapter, we will adopt such a model representation. For a good introduction to the theory of FDEs, we refer to Bellman and Cooke (1963); Hale and Verduyn Lunel (1993).

As pointed out in the open literature (see, e.g., Sipahi et al., 2011 and the references therein), understanding the way the delay may affect systems' dynamics is not a trivial task and we may have some dichotomic behaviors. For instance, in control, on one hand, large delays may induce instability in closed-loop even in the scalar case when controlling integrators. On the other hand, small delays in the input/output channels are useful in stabilizing oscillatory systems. There exists an abundant literature devoted to the analysis of the effects induced by the delays on the stability of DDEs and to present it in detail is out of the scope of this chapter. However, our intention is to point out some important (almost forgotten, in some cases) contributions related to the proposed methodology (i.e., *frequency-sweeping approach*), and we believe that this historical perspective is useful for a better understanding of the main ideas.

To the best of the authors' knowledge, it appears that the first complete characterization of the stability regions with respect to the system's parameters was derived by Hayes (1950) for scalar DDEs in both retarded and

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<sup>1</sup>The delays may be constant or time-varying, distributed or not over a finite or infinite time-interval, depending on the state vector or not...

neutral cases. At the end of the 50s, Pinney (1958) presented a detailed analysis for scalar and second-order DDEs by using the argument principle. Under the assumption that the delay is known (and equal to one), by introducing several notions (root plateau, root cell,  $D$ -set), Pinney proposed an algorithm to compute the number of unstable roots of a quasipolynomial of low-order as a function of its coefficients. Through this procedure, the parameter-space is divided in several regions, such that each region is characterized by a constant number of unstable roots. The boundaries separating the regions correspond to the cases when the characteristic function has at least one root on the imaginary axis. Such a procedure is “close” to the so-called  $D$ -partition method developed by Neimark (1949) at the end of the 40s. For an historical perspective on the  $D$ -partition and some of the existing results in the literature, we refer to Gryazina et al. (2008).

Introduced by Lee and Hsu (1969) at the end of the 60s, the so-called  $\tau$ -partition may be seen as the “dual” of the  $D$ -partition method. More precisely, in the case of a single and constant delay (parameter), this method allows computing the delay intervals guaranteeing that the trivial solution of the DDE is exponentially stable. As a byproduct of the analysis, for a given delay interval, one may explicitly follow how the roots of the characteristic function move with respect to the imaginary axis as a function of the delay parameter. As mentioned in the above reference, such a method has its origin in the works of Sokolov and Miasnikov in the 40s and detailed in the monograph of Popov (1962). To the best of the authors’ knowledge, close to the ideas of root locus<sup>2</sup>, it appears that the first systematic discussion on the number of unstable roots by using the continuity of the roots with respect to the delay parameter can be found in Kashiwagi (1965), where the author introduced the notion of *stability indicative function* to count the number of unstable roots for a given delay value. Finally, for a pedagogical presentation as well as some extensions of the root locus method and classical well-known Nyquist criterion to deal with linear single-input/single-output systems with one delay in the input/output channel, we refer to Krall (1968) (see also Krall, 1970 for a survey of the root-locus methods).

At the beginning of the 70s, Els’golts’ and Norkin (1973) mention three tests for checking the asymptotic stability of DDEs: the amplitude-phase method (referring to Tsytkin’s contributions), the  $D$ -partition method (discussed above) and the direct generalization of the Routh-Hurwitz method (mainly Čebotarev’s contributions), with a more detailed discussion of the first two methods by using the argument principle as well as the Rouché’s

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<sup>2</sup>The origins of root locus go back to the works of Evans at the end of the 40s (see, e.g., Evans, 1950 and the references therein).

Lemma. The so-called *amplitude-phase method* proposed by Tsytkin (1946) in Control area in the 40s is at the origin at most of the existing *frequency-sweeping tests*<sup>3</sup> in the open literature, and its principle will be briefly presented in the forthcoming sections. More precisely, in its simplest form, Tsytkin’s criterion allows concluding on the so-called *delay-independent (asymptotic) stability* of a closed-loop system, that is the stability is guaranteed for all delay values. Starting with the 80s, the development of tools and techniques in robust control allowed to reconsider some of the existing methods in the literature by interpreting the delay as an *uncertainty*, and there exists an abundant literature on the delay-independent/delay-dependent stability with a particular emphasis on the computation of the so-called *delay margin*<sup>4</sup>. For further discussions on such topics and various references, we refer to Niculescu (2001); Gu et al. (2003); Michiels and Niculescu (2014); Fridman (2014) and the references therein.

The aim of this chapter is twofold: first, to introduce a simple approach easy to understand and to apply for characterizing the exponential stability of the trivial solution of linear DDEs with respect to the delay parameter. This approach enters in the so-called  $\tau$ -partition category and it is based in the construction of some appropriate *frequency-sweeping curves*. This construction will allow “translating” the behavior of the roots of the corresponding characteristic function with respect to the (delay) parameters in terms of properties of such curves. The continuity of the roots of the characteristic function with respect to the delay is the main ingredient of the approach. Although the main results are presented in the case of retarded DDEs including commensurate delays, the underlying ideas can be extended to incommensurate or to some classes of distributed delays and/or to other classes of DDEs (neutral). Several case studies as well as a few applications are briefly presented. Second, we wish emphasizing an *invariance principle* that is essential for having a complete characterization of the (exponential) stability of the trivial solution. In fact, the proposed framework allows a simpler understanding of the asymptotic behavior of multiple characteristic roots by using the properties of frequency-sweeping curves. The presentation is teaching-oriented including discussions on the main ideas of the method as well as its illustration through appropriate examples and

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<sup>3</sup>To the best of the authors’ knowledge, the notion of “frequency-sweeping” was formally introduced by Chen and Latchman (1995); Chen (1995) into a different methodological frame: robust analysis with respect to the delay parameter, seen as an *uncertainty*, see also Niculescu (2001).

<sup>4</sup>Under the assumption that the system free of delays is asymptotically stable, the delay margin is the maximal value  $\tau_m > 0$  such that the asymptotic stability is guaranteed for all delays inside the interval  $[0, \tau_m)$ ; see also Chen et al. (1995); Chen (1995).

less on the explicit proof of the results. The book written by Li et al. (2015) includes complementary material as well as the proofs of the main results presented in this chapter. Finally, it is worth mentioning that Pólya (1954)'s pattern on logical induction and analogy-based reasoning strongly influenced the authors in the presentation style adopted.

The remaining document is organized as follows: Section 2 includes some preliminaries and prerequisites. Next, a method based on the frequency-sweeping curves is presented in Section 3, followed by Section 4 devoted to a brief discussion on the asymptotic behavior of the critical imaginary roots at some critical delay values. The *invariance property* as well as its utility in characterizing the stability with respect to the delay are presented in Section 5. All these ideas and results allow proposing a unified frequency-sweeping framework for the stability analysis of DDEs with respect to the delay parameter, approach that is summarized in Section 6. Next, Sections 7 and 8 present various extensions of the approach as well as some applications. Finally, a few notes and comments end the chapter.

*Notations:* Throughout this chapter, the following notations will be used:  $\mathbb{Z}$  denotes the set of integers,  $\mathbb{R}$  ( $\mathbb{R}_+$ ) denotes the set of (positive) real numbers,  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  and  $\mathbb{C}$  is the set of complex numbers. For  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda)$  and  $\operatorname{Im}(\lambda)$  denote the real part and imaginary part of  $\lambda$ , respectively;  $\mathbb{C}_-$  and  $\mathbb{C}_+$  denote the sets  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\}$  and  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ , respectively;  $i\mathbb{R}$  (with  $i = \sqrt{-1}$ ) is the imaginary axis and  $\partial\mathbb{D}$  is the unit circle. For  $a, b \in \mathbb{R}$ , we denote  $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ , with the convention that  $[a, b] = \emptyset$  if  $a > b$ . For  $\gamma \in \mathbb{R}$ ,  $\lceil \gamma \rceil$  denotes the smallest integer greater than or equal to  $\gamma$ . Next,  $\mathbb{N}$ , and  $\mathbb{N}_+$  are the sets of integers, non-negative integers, and positive integers ( $\geq 1$ ), respectively. For a (quasi)polynomial  $a(\lambda)$  ( $q(\lambda)$ ),  $\deg(a)$  ( $\deg(q)$ ) denotes its degree and  $\sigma_s(a)$  ( $\sigma_s(q)$ ) its spectrum. Moreover,  $\det(\cdot)$  denotes the determinant, and  $I$  the identity matrix. Finally, for a function  $\phi(x, y)$ ,  $\phi_{x^\alpha y^\beta}$  ( $\alpha, \beta \in \mathbb{N}$ ) denotes the partial derivative  $\frac{\partial^{\alpha+\beta} \phi(x, y)}{\partial x^\alpha \partial y^\beta}$ .

## 2 Preliminaries and Prerequisites

To better fix the ideas, we recall some preliminary results and prerequisites concerning the stability of linear DDEs.

### 2.1 Linear Time-Delay Systems

Consider the following linear system described by the delay differential equation (DDE) of retarded type

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau), \quad (1)$$

under appropriate initial conditions, where  $x(t) \in \mathbb{R}^r$  ( $r \in \mathbb{N}_+$ ) is the vector state at time  $t$ ,  $A_0, A_1 \in \mathbb{R}^{r \times r}$  are constant real matrices, and  $\tau$  is the delay parameter, that is assumed to be positive ( $\tau \in \mathbb{R}_+$ ).

The corresponding characteristic function  $f : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}$  is given by

$$f(\lambda, \tau) = \det(\lambda I - A_0 - A_1 e^{-\tau\lambda}), \quad (2)$$

which is a *quasipolynomial* of the form

$$f(\lambda, \tau) := a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \dots + a_q(\lambda)e^{-q\tau\lambda}, \quad (3)$$

where  $a_0(\lambda), \dots, a_q(\lambda)$  ( $q \in \mathbb{N}_+$ ) are polynomials in  $\lambda$  with real coefficients and represent the so-called *coefficient functions*. It is easy to observe that the quasipolynomial (3) includes multiple delays  $\tau_k$  but with the particular dependence  $\tau_k = k\tau$  for all  $k \in \llbracket 0, q \rrbracket$ . Such delays are called *commensurate*, and our main ideas are presented in such a setting. For a short discussion in the incommensurate delays case, see Section 7.

Using the same terminology as Bellman and Cooke (1963), a complex number  $\lambda \in \mathbb{C}$  such that  $f(\lambda, \tau) = 0$  is called a *characteristic root*. It is well-known that, for a  $\tau > 0$ , the characteristic function (2) of the DDE (1) has an *infinite number of characteristic roots*. A fundamental well-known result from the finite-dimensional case is still valid for DDEs (see, e.g., Bellman and Cooke, 1963; Michiels and Niculescu, 2014):

**Theorem 2.1.** *The trivial solution of the DDE (1) is exponentially stable if and only if all the characteristic roots of the quasipolynomial  $f$  (3) are located in the open left half-plane  $\mathbb{C}_-$ .*

Unfortunately, due to the infinite-dimensional nature of the system, it is not realistic to apply Theorem 2.1 directly. The following example illustrates the root distribution intuitively.

**Example 2.2.** Consider the following DDEs:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -9 & -1.5 \end{pmatrix} x(t - \tau),$$

with the characteristic function  $f(\lambda, \tau) = \lambda^2 - \lambda + 1 + (1.5\lambda + 9)e^{-\tau\lambda}$ . When  $\tau = 0$ , the system has two characteristic roots  $-0.2500 \pm 3.1524i$  both located in  $\mathbb{C}_-$ . As  $\tau$  increases from 0 to  $+\varepsilon$ , infinitely many new characteristic roots appear at far left of the complex plane. Fig. 1 (left) shows the case when  $\tau = 0.01$ , where the two points denote the locations of the “original” roots. Next, as  $\tau$  increases, some roots move to the selected

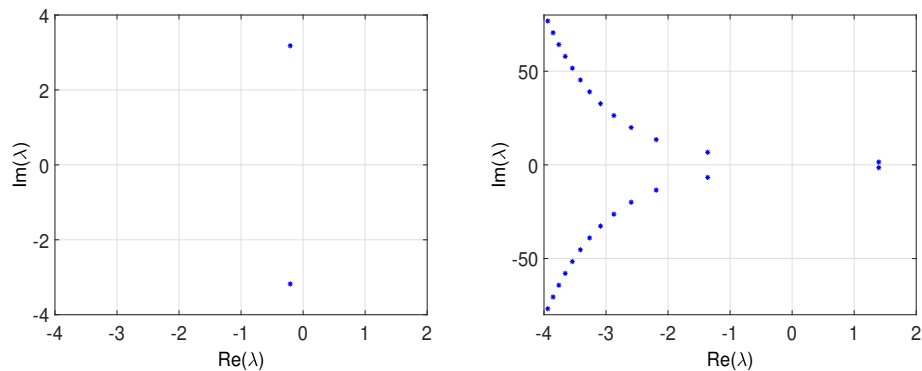


Figure 1: Example 2.2: characteristic roots location  $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$  for  $\tau = 0.01$  (left) and  $\tau = 1$  (right).

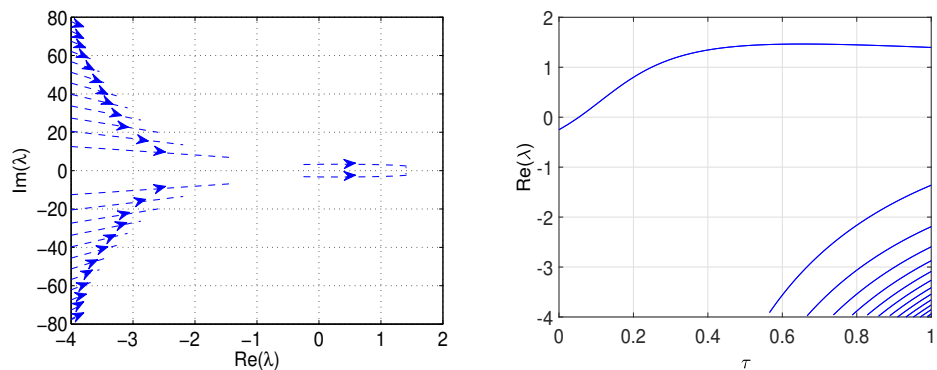


Figure 2: Example 2.2:  $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$  (left) and  $\text{Re}(\lambda)$  vs.  $\tau$  (right).

domain defined by  $\text{Re}(\lambda) \in [-4, 2]$  and  $\text{Im}(\lambda) \in [-4, 4]$ . For instance, when  $\tau = 1$ , some roots enter in the “selected” domain and the “original” roots will leave the left-half plane and will enter in the right-half plane as shown in Fig. 1 (right). For further illustration, Fig. 2 gives the corresponding root loci w.r.t. the delay parameter.<sup>5</sup>  $\square$

A simple inspection of this example suggests that it will be important to understand how the behavior of the characteristic roots is affected by the

<sup>5</sup>In in this chapter, the root loci are numerically generated by using the DDE-BIFTOOL (Engelborghs et al., 2002; Sieber et al., 2016).



delay parameter and, in particular, in the case when the delay is increased from 0 to  $0_+$ . These issues will be addressed in the sequel.

## 2.2 Characteristic Roots and Delay Parameter

In finite dimension, it has long been recognized that the roots of a polynomial are continuous functions of the coefficients as long as the leading coefficient does not vanish (see, e.g., Knopp, 1996; Marden, 1949). Furthermore, in the case of simple roots, these functions are also differentiable. Similar properties hold for quasipolynomials.

For the sake of simplicity, consider the simplest case  $q = 1$  and assume that  $a_0$  is a monic polynomial. Excepting the delay, assume that the set of parameters includes also the coefficients of the corresponding polynomials  $a_0$  and  $a_1$  and introduce the vector notation  $\vec{p}$  for representing such parameters. Let  $\mathcal{O}_p \subset \mathbb{R}^{n_p}$  be an open set, and assume that  $\deg(a_0) > \deg(a_1)$ , for all  $\vec{p} \in \mathcal{O}_p$ . Under these assumptions, the characteristic function (3) rewrites as  $f : \mathbb{C} \times \mathcal{O}_p \times \mathbb{R}_+ \mapsto \mathbb{C}$ :

$$f(\lambda, \vec{p}, \tau) := a_0(\lambda, \vec{p}) + a_1(\lambda, \vec{p})e^{-\lambda\tau}. \quad (4)$$

By using the properties of analytic functions, the quasipolynomial  $f$  given by (4) has some nice and interesting properties. For instance, the characteristic roots are isolated and only a finite number of roots lie in any compact set of the complex plane  $\mathbb{C}$ . Furthermore, any vertical strip of the complex plane includes at most a finite number of characteristic roots. Finally, there exists a real number  $\gamma$ , such that all the characteristic roots are confined to the half-plane  $\mathbb{C}_\gamma: \{\lambda \in \mathbb{C} : \Re(\lambda) < \gamma\}$ <sup>6</sup>. In conclusion, it is easy to observe that, surprisingly, despite its infinite-dimensional nature, the quasipolynomial  $f$  given by (4) has only a finite number of roots in the right-half plane  $\mathbb{C}_+$ . Finally, based on Rouché's lemma (see, e.g., Ahlfors, 1979), we have:

**Theorem 2.3.** *Under the assumptions that  $a_0$  is monic and that  $\deg(a_0) > \deg(a_1)$  for all  $\vec{p} \in \mathcal{O}_p$ , let  $\lambda_0$  be a characteristic root of  $f(\cdot, \vec{p}_0, \tau_0)$  with multiplicity  $k$ . Then there exists a constant  $\bar{\varepsilon} > 0$  such that for all  $\varepsilon > 0$  satisfying  $\varepsilon < \bar{\varepsilon}$ , there exists a  $\delta_\varepsilon > 0$  such that  $f(\lambda; \vec{p}_0 + \Delta\vec{p}_0, \tau_0 + \Delta\tau_0)$ , where  $\Delta\tau_0 \in \mathbb{R}$ ,  $|\Delta\tau_0| < \delta_\varepsilon$ ,  $\tau_0 + \Delta\tau_0 \geq 0$ ,  $\Delta\vec{p}_0 \in \mathbb{R}^{n_p}$ ,  $\|\Delta\vec{p}_0\|_2 < \delta_\varepsilon$ , has exactly  $k$  zeros (multiplicity taken into account) in the disc  $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\}$ .*

**Remark 2.4.** This result simply states that, in the retarded case, as long as the leading coefficient of the polynomial  $a_0$  is not vanishing and the delay is

<sup>6</sup>For further discussions on such topics, we refer to Michiels and Niculescu (2014) and the references therein.

positive, the characteristic roots of the quasipolynomial (4) are continuous functions of the coefficients of the polynomials  $a_0$  and  $a_1$  and of the delay  $\tau$ , seen as a parameter.  $\square$

For the stability analysis purposes, it is important to know where the rightmost characteristic root is located as well as the way it is affected by parameters change. To answer to such questions, introduce now the so-called *spectral abscissa function*  $(\vec{p}, \tau) \in \mathcal{O}_p \times \mathbb{R}_+ \mapsto \alpha_s(\vec{p}, \tau) \in \mathbb{R}$ ,

$$\alpha_s(\vec{p}, \tau) := \sup \{ \Re(\lambda) : f(\lambda, \vec{p}, \tau) = 0, \quad \vec{p} \in \mathcal{O}_p, \quad \tau \in \mathbb{R}_+ \}.$$

As a consequence of Theorem 2.3, we have two interesting properties that will be exploited in the forthcoming sections:

- (1) the function  $\alpha_s$  always *exists*, is *bounded* and *continuous*;
- (2) as the delay and/or parameters vary, the multiplicity summation of the roots of  $f$  in the open right-half plane ( $\mathbb{C}_+$ ) can change only *if a root appears on*<sup>7</sup> *or crosses the imaginary axis*.

**Remark 2.5.** The assumption  $\deg(a_0) > \deg(a_1)$  for all parameters  $\vec{p} \in \mathcal{O}_p$  is essential to guarantee the continuity of the spectral abscissa function. Concerning the second property, an elementary proof for general second-order DDEs of retarded type ( $\deg(a_0)=2$  and  $\deg(a_1) = 1$ ) can be found in Cooke and Grossman (1982).  $\square$

**Remark 2.6.** As expected, the ideas above still hold in the commensurate delays case ( $\tau_i = i\tau$  for  $i \in \llbracket 1, r \rrbracket$  and  $\tau \in \mathbb{R}_+$  in (3)). For incommensurate delays, by introducing an appropriate notion of *delay rays*  $\{r\vec{\tau} : r \in \mathbb{R}_+\}$ , Datko (1978) proved that the continuity of the spectral abscissa holds with respect to one parameter, that is  $r \in \mathbb{R}_+$ .  $\square$

**Remark 2.7.** To construct the stability charts in the scalar and second-ordered linear DDEs in the case when  $\tau = 1$ , Pinney (1958) introduced the notion of  $(x_r, k_r)$ -*root plateau set*, that is the set of parameters for which the characteristic function  $f$  has  $k_r$  and only  $k_r$  roots “ $\lambda_i$ ”,  $i \in \llbracket 0, k_r \rrbracket$ , with  $\Re(\lambda_i) > x_r$ . Thus, in the limit cases,  $(0, 0)$ -root plateau set covers the *stability regions*, and the minimal value of  $x_r$  of the  $(x_r, 0)$ -root plateau is the *spectral abscissa* notion introduced above.  $\square$

**Remark 2.8** (“Small” delays: retarded DDEs). Consider now the case when the only parameter is the delay and assume that it is sufficiently small:  $\tau = \varepsilon > 0$ . Under the assumption that  $a_1 \neq 0$ , the use of Rouché’s

<sup>7</sup>Such a case may occur in the case of neutral DDEs or if the coefficients of the quasipolynomials depend on the delay parameters.

Theorem allows to conclude that the finite characteristic roots of  $f(\cdot, \varepsilon)$  can be made arbitrarily close to the finite roots of  $f(\cdot, 0)$  and there exists an infinite number of roots whose real parts approach negatively infinite<sup>8</sup>. In particular, if the system free of delays has no roots on the imaginary axis, when increasing the delay from 0 to  $0_+$ , although the system changes its character from finite- to infinite-dimensional, the stability/instability of the delay-free system is preserved for sufficiently small delays.  $\square$

Consider now (4) and assume that the leading coefficient of the polynomial  $a_1$  is not vanishing for all  $(\vec{p}, \tau) \in \mathcal{O}_p \times \mathbb{R}_+$ . The property mentioned above (Remark 2.8) does not necessarily hold in all the cases, and there are two particular situations of interest: (i) *neutral* case ( $\deg(a_0) = \deg(a_1)$ ) and (ii) *delay-dependent coefficients* of the polynomials  $a_0$  and  $a_1$ .

**Remark 2.9** (“Small” delays: neutral DDEs). Although the individual characteristic roots behave continuously with respect to the system’s parameters (see Michiels and Niculescu, 2014), the spectral abscissa function is, in general, *not continuous*. Recall that  $a_0$  is a monic polynomial,  $\deg(a_0) = \deg(a_1)$ , and denote by  $a_{1,m_0}$  the leading coefficient of the polynomial  $a_1$  and assume that  $a_{1,m_0} \in \mathcal{O}_0$ , where  $\mathcal{O}_0$  is an open set not including the origin ( $0 \notin \mathcal{O}_0$ ). With these notations, we introduce the delay-difference equation:  $y(t) + a_{1,m_0}y(t - \tau) = 0$  associated to the neutral DDE. The corresponding characteristic function  $f_D : \mathbb{C} \times \mathcal{O}_0 \times \mathbb{R}_+ \mapsto \mathbb{C}$  writes as:

$$f_D(\lambda, a_{1,m_0}, \tau) = 1 + a_{1,m_0}e^{-\lambda\tau}.$$

As discussed in Michiels and Niculescu (2014), the delay-difference equation above and the original neutral DDE are related by an interesting property. More precisely, a necessary condition for the exponential stability of the trivial solution of the neutral DDE is the exponential stability of the trivial solution of the corresponding delay-difference equation. Now,

- (i) if  $|a_{1,m_0}| < 1$ <sup>9</sup>, we have a similar property to the one valid in the retarded DDEs. More precisely, as the delay and/or parameters vary, the multiplicity summation of the roots of the characteristic function  $f$  in the open right-half plane ( $\mathbb{C}_+$ ) can change only *if a root appears on or crosses the imaginary axis*.
- (ii) if  $|a_{1,m_0}| > 1$ , then increasing the delay from 0 to  $0_+$  generates instability even in the case when the system free of delay is stable.

<sup>8</sup>For a simple and elementary proof, we refer to Shaughnessy and Kashiwagi (1969).

<sup>9</sup>It simply guarantees the exponential stability of the trivial solution of the corresponding delay-difference equation.

For further discussions on such topics, we refer to Hale and Verduyn Lunel (1993). Finally, Section 7 includes a few illustrative examples.  $\square$

**Remark 2.10** (“Small” delays: delay-dependent coefficients). Such a case may appear in control engineering when the controller includes the derivative of a signal that is not necessarily available for measurement. For instance, in the case of classical proportional-derivative (PD) controllers, the derivative action can be implemented by using a (standard) Euler delay-difference approximation scheme. The corresponding closed-loop system may be *improperly-posed* in the sense that the implementation scheme may lead to instability for infinitesimal delay values in constructing the delay-difference approximation even if the initial PD-controller stabilizes the original system. Such a case is illustrated in Mendéz-Barrios et al. (2022), where it was shown that if the relative degree of the system is one<sup>10</sup>, then the derivative gain may be at the origin of such a lack of continuity<sup>11</sup>. For further discussions on DDEs with delay-dependent coefficients, we refer to Chi et al. (2018a,b) and the references therein.  $\square$

By taking into account all the observations and comments above, it appears that the stability analysis of DDEs whose characteristic function is given by  $f$  in (4) can be reduced to the following three steps:

- (a) detecting all the characteristic roots “ $i\omega_c$ ” (of  $f$ ) located on the imaginary axis ( $i\mathbb{R}$ ). Such roots are called *critical (characteristic) imaginary roots*. The delays associated to a critical “ $i\omega_c$ ” are called *critical delays* and we may have an infinite number of critical delays for the same imaginary root. Finally, a pair  $(\lambda, \tau)$  is called a *critical pair* if  $\lambda$  is a critical imaginary root and  $\tau$  corresponds to a critical delay;
- (b) understanding and characterizing the behavior of the characteristic roots located on  $i\mathbb{R}$  with respect to the parameters’ change;
- (c) counting the roots crossing from  $\mathbb{C}_-$  to  $\mathbb{C}_+$  and vice versa by taking into account the root multiplicity.

Due to the infinite-dimensional nature of the system, it is clear that the steps (a)-(c) are not trivial even in the case when we consider one parameter - the *delay*. When the delay  $\tau \equiv 0$ , the characteristic roots location problem reduces to the analysis of the spectrum location of a polynomial. Next, in the simplest case when the system free of delay has not roots located on  $i\mathbb{R}$ , increasing the delay from 0 to  $0_+$  will conserve the distribution of the

<sup>10</sup>The relative degree is defined by  $\deg(a_0) - \deg(a_1)$

<sup>11</sup>More precisely, in this configuration (i.e., improperly-posed approximation), a characteristic root appears on the real axis in  $\mathbb{C}_+$  from  $+\infty$  when the delay is increased from 0 to  $0_+$ .

roots located on  $\mathbb{C}_+$  as long as, there are no roots “crossing” the imaginary axis. Assume further that the system free of delays is exponentially stable. In such a situation, when increasing the delays, the stability property holds as long as there are no characteristic roots “arriving” on the imaginary axis from  $\mathbb{C}_-$ . These intuitive ideas are at the origin of a lot of theoretical developments in the open literature as briefly explained in the sequel.

For a better understanding of the main ideas of the so-called  $\tau$ -*partition method* and related *frequency-sweeping tests*, consider now the (strictly proper) linear time-invariant (LTI) single-input/single-output (SISO) system  $\Sigma(A, b, c^T)$  with the state-space representation:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t), \end{cases}$$

where the transfer function  $H_{yu}(\lambda)$  of  $\Sigma$  writes as  $H_{yu}(\lambda) = a_1(\lambda)/a_0(\lambda)$ , for some appropriate real polynomials  $a_i$ ,  $i \in \llbracket 0, 1 \rrbracket$ , whose coefficients are given by the “entries”  $(A, b, c^T)$  of  $\Sigma$ . Assume now that  $\Sigma$  is controlled by the feedback law  $u(t) = -ky(t - \tau)$  with  $k \in \mathcal{O}_0 \subset \mathbb{R}^*$ . Under the assumption that  $k$  and  $\tau$  are the *parameters* then the stability of the system in closed-loop reduces to the analysis of the location of the spectrum of the quasipolynomial  $f(\cdot, k, \tau)$  given by  $f(\lambda, k, \tau) := a_0(\lambda) + ka_1(\lambda)e^{-\lambda\tau}$ .

For the sake of brevity, assume that  $a_0$  and  $a_1$  are coprime. If the gain is  $k = 1$ ,  $f$  simply rewrites as  $f(\lambda; \tau) := a_0(\lambda) + a_1(\lambda)e^{-\lambda\tau}$ . Surprisingly, if

$$|a_1(i\omega)| < |a_0(i\omega)|, \quad (5)$$

for all  $\omega \in \mathbb{R}$ , then  $\sigma_s(f) \cap i\mathbb{R} = \emptyset$ . By using Theorem 2.3, it is easy to observe that the characteristic roots of  $f$  can not migrate from  $\mathbb{C}_-$  to  $\mathbb{C}_+$  or vice-versa if  $\tau$  is increased from 0 to  $+\infty$ . Such a system is called *hyperbolic* and it has an interesting property: *the location of the spectrum of the polynomial  $a_0 + a_1$  will define the stability/instability of the system for all delays  $\tau \in \mathbb{R}_+$* . For a deeper discussion on such topics, see, e.g., Niculescu (2001) (commensurate delays) and Hale et al. (1985) (more general setting).

If  $0 \in \sigma_s(f(\cdot, 0))$ , then  $f(0, \tau) = 0$ ,  $\forall \tau \in \mathbb{R}_+$ . Thus, the origin is an *invariant root*<sup>12</sup>. Now, if  $0 \notin \sigma_s(f(\cdot, 0))$ , checking (5) for  $\forall \omega \in \mathbb{R}_+^*$  is sufficient to guarantee system’s hyperbolicity. Assume now that  $\sigma_s(f(\cdot, 0)) \subset \mathbb{C}_-$ . As observed by Tsytkin (1946), the closed-loop system is *delay-independently stable* if and only if the condition (5) holds for all  $\omega \in \mathbb{R}_+^*$ . It can be simply checked from the plot of  $z_1$ , where the mapping  $\omega \mapsto z_1(\omega) := -a_0(i\omega)/a_1(i\omega)$ , for  $\omega \in \mathbb{R}_+^*$  defines the simplest *frequency sweeping curve*.

<sup>12</sup>The common roots  $a_0$  and  $a_1$  on  $i\mathbb{R}$  are also invariant roots w.r.t.  $\tau$ .

**Example 2.11** (Scalar case). Consider the following DDE:

$$\dot{x}(t) = -ax(t) - bx(t - \tau),$$

where  $a, b \in \mathbb{R}$ . The characteristic function is given by  $f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$  with  $a_0(\lambda) = \lambda + a$  and  $a_1(\lambda) = b$ . It is easy to see that:

$$\sup_{\omega \in \mathbb{R}_+^*} \frac{|a_1(i\omega)|}{|a_0(i\omega)|} = \sup_{\omega \in \mathbb{R}_+^*} \frac{|b|}{\sqrt{\omega^2 + a^2}} = \frac{|b|}{|a|},$$

and thus if  $|a| > |b|$ , the system is *hyperbolic*. Consider now the case  $|a| = |b|$ . If  $b = -a$  and  $b \neq 0$ , then  $f$  has a double root at the origin and the hyperbolicity is lost. Furthermore, the root at the origin is invariant with respect to the delay. Now, if  $a = b \neq 0$ , the system is still hyperbolic.

Based on the remarks above, if  $b \in \mathbb{R}^*$  and  $|a| \geq |b|$ , we have that the system is *stable (unstable) independent* of the *delay* if and only if  $a + b > 0$  ( $a + b < 0$ ). Finally, it is worth mentioning that in the delay-independent instability case, the characteristic function has one (and only one) unstable real root moving on the positive real axis as long as  $\tau$  varies.  $\square$

Consider now the case when the system above is *not hyperbolic*. Then there exists at least one value  $\omega_c \in \mathbb{R}$ , such that  $f(i\omega_c, \tau) = 0$  for some critical delay  $\tau = \tau_c \in \mathbb{R}_+$ . The real “ $\omega_c$ ” is called *crossing frequency*, and the collection of all “ $\omega_c$ ” defines the *crossing set*:

$$\Omega_c := \{\omega \in \mathbb{R} : |a_0(i\omega)| = |a_1(i\omega)|\}.$$

At this stage, there are two important remarks:

- (i) first,  $\text{card}(\Omega_c)$  is *finite*, and its computation reduces to the computation of the positive roots of an appropriate polynomial;
- (ii) second, the knowledge of a crossing frequency  $\omega_{i,c} \in \Omega_c$  allows to compute the minimal critical delay value  $\tau_{i,c}^* \in \mathbb{R}_+$ <sup>13</sup> generating the set of *critical (crossing) delays*

$$\mathcal{T}(\omega_{i,c}) := \left\{ \tau_{i,c}^* + \frac{2k\pi}{\omega_c} \geq 0, \quad k \in \mathbb{Z} \right\}.$$

Indeed, if we formally denote  $z = e^{-i\omega\tau}$ , then  $f(i\omega, \tau) = a_0(i\omega) + a_1(i\omega)e^{-i\omega\tau}$  can be interpreted as a two-variate polynomial  $f_a(\omega, z) = a_0(i\omega) + a_1(i\omega)z$  with  $z$  on the unit circle of the complex plane<sup>14</sup>. Thus, the “quantity” (if it

<sup>13</sup>Such a value always exists and it may be 0.

<sup>14</sup>It is easy to see the way the roots of  $f$  and  $f_a$  are linked. For instance, for any pair  $(i\omega_s, \tau_s) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$  satisfying  $f(i\omega_s, \tau_s) = 0$ ,  $f_a(\omega_s, z_s) = 0$ , where  $z_s = e^{-i\omega_s\tau_s}$ , etc.

exists)  $z = -a_0(i\omega)/a_1(i\omega)$  may lead to a solution of  $f_a$  at some frequency  $\omega_c$  if  $|a_1(i\omega_c)| = |a_0(i\omega_c)|$ , condition naturally related to the definition of the crossing (frequency) set above. For a deeper discussion of the remarks (i)–(ii) above, we refer to Michiels and Niculescu (2014).

Under the assumption of a simple characteristic root  $\omega_0 \in \Omega_c$  for some delay  $\tau_0 \in \mathcal{T}(\omega_0)$ , Cooke and Grossman (1982) discussed the behavior of the characteristic root  $i\omega_0$  for values close to  $\tau_0$  by using the “quantity”  $s_c := \text{sgn}(\text{Re}(d\lambda/d\tau))$  evaluated at  $\lambda = i\omega_0$  and  $\tau = \tau_0$ . Such an idea was further refined in Cooke and van den Driessche (1986) and largely used in the open literature during the last 30 years. More precisely, if the characteristic root located on the imaginary axis moves towards instability (stability), we will have a *stability switch (reversal)*<sup>15</sup>. Surprisingly, in the case of simple imaginary roots, the “quantity”  $s_c$  above does not include any information on the delay parameter. In fact, the derivative of the function  $g_a : \mathbb{R} \mapsto \mathbb{R}$  defined by  $g_a(\omega) := |a_0(i\omega)|^2 - |a_1(i\omega)|^2$  evaluated at the crossing frequency  $\omega = \omega_c$  indicates the crossing type: *switch (reversal)* if  $g'(\omega_c) > 0$  ( $< 0$ ).

### 2.3 Stability Problem and the Delay Parameter

Following the notation used in the literature (see, e.g., Lee and Hsu, 1969; Olgac and Sipahi, 2002), denote by  $NU(\tau_0) \in \mathbb{N}$  the number of the characteristic roots located in  $\mathbb{C}_+$  for the delay  $\tau = \tau_0$ . According to Theorem 2.1, the (linear) system is asymptotically stable for a delay value  $\tau = \tau_0$ , if and only if there are no characteristic roots located on the imaginary axis and  $NU(\tau_0) = 0$ . In the commensurate delays case, our objective is to obtain its exhaustive stability set for the delay parameter  $\tau$ <sup>16</sup>, which is referred to as the *complete stability problem*. Based on the root continuity property mentioned in the previous paragraphs, the complete stability analysis can be fulfilled in two steps by solving two problems:

**Problem 1:** *How to exhaustively detect the critical imaginary roots and the corresponding critical delays?*

For a critical pair  $(\lambda_\alpha, \tau_{\alpha,k})$ , denote by  $n \in \mathbb{N}_+$  the multiplicity of  $\lambda_\alpha$  at  $\tau_{\alpha,k}$ . Clearly, a critical imaginary root is called a *simple (multiple) critical imaginary root* if the corresponding index  $n = 1$  ( $n > 1$ ). In other words, the index  $n$  simply implies that for  $\lambda = \lambda_\alpha$  and  $\tau = \tau_{\alpha,k}$ ,

$$f_{\lambda^0} = \cdots = f_{\lambda^{n-1}} = 0, f_{\lambda^n} \neq 0.$$

<sup>15</sup>To the best of the authors’ knowledge, during the 70s, the notions of (*stability*) *switches/reversals* appear in Cooke’s publications.

<sup>16</sup>i.e., the whole set for  $\tau \geq 0$  such that  $NU(\tau) = 0$  excluding the possible critical points

Next, introduce the index  $g \in \mathbb{N}_+$  at  $(\lambda_\alpha, \tau_{\alpha,k})$ , by which we may *artificially* treat  $\tau_{\alpha,k}$  as a  $g$ -multiple root for  $f(\lambda, \tau) = 0$  when  $\lambda = \lambda_\alpha$ , having the property that when  $\lambda = \lambda_\alpha$  and  $\tau = \tau_{\alpha,k}$ ,

$$f_{\tau^0} = \cdots = f_{\tau^{g-1}} = 0, f_{\tau^g} \neq 0.$$

**Remark 2.12.** Unlike for the critical imaginary roots, the analytic computation for the other characteristic roots is generally very difficult.  $\square$

It is relatively simple to solve Problem 1 and various effective methods are available in the literature. In Section 3, a method based on the *frequency-sweeping curves* is discussed. Once Problem 1 is solved, we have to analyze the variation of a critical imaginary root as  $\tau$  increases near the corresponding critical delay (value), called the *asymptotic behavior* of a critical imaginary root.

**Problem 2:** *How to analyze the asymptotic behavior of the critical imaginary roots w.r.t. the corresponding critical delays?*

**Remark 2.13.** Owing to the conjugate symmetry of the spectrum, it suffices to consider only the critical imaginary roots with non-negative imaginary parts.  $\square$

**Remark 2.14.** The critical delays divide the positive  $\tau$ -axis into infinitely many subintervals and within each subinterval  $NU(\tau)$  is constant. Solving *Problem 2* allows monitoring  $NU(\tau)$  as  $\tau$  increases. For instance, consider a subinterval  $\tau \in (\tau', \tau'')$  where  $\tau'$  and  $\tau''$  are two positive critical delays such that there are no other critical delays inside this subinterval. If the value of  $NU(\tau' - \varepsilon)$  is known and the asymptotic behavior of the critical imaginary roots at  $\tau = \tau'$  is properly studied, we may precisely know the value of  $NU(\tau' + \varepsilon)$ . According to the root continuity argument, for any  $\tau \in (\tau', \tau'')$ ,  $NU(\tau) = NU(\tau' + \varepsilon)$ .  $\square$

*Problem 2* is rather involved, and is moreover divided into two sub-problems, as follows:

**Problem 2.1:** *How to analyze the asymptotic behavior of a critical imaginary root at a critical delay?*

To such an end, introduce some further notations. Suppose that  $(\alpha, \beta)$  (with  $\beta > 0$ ) is a critical pair with the index  $n$ . Near this critical pair, there exist  $n$  (characteristic) roots  $\lambda_i(\tau)$  (counted with multiplicities) continuous w.r.t.  $\tau$  satisfying  $\alpha = \lambda_i(\beta)$ ,  $i = 1, \dots, n$ . Under some perturbation  $\varepsilon$  ( $-\varepsilon$ ) on  $\beta$ , the  $n$  roots are expressed by  $\lambda_i(\beta + \varepsilon)$  ( $\lambda_i(\beta - \varepsilon)$ ),  $i = 1, \dots, n$ . Denote the number of unstable roots among  $\lambda_1(\beta + \varepsilon), \dots, \lambda_n(\beta + \varepsilon)$  ( $\lambda_1(\beta -$



$\varepsilon), \dots, \lambda_n(\beta - \varepsilon))$  by  $NU_\alpha(\beta^+)$  ( $NU_\alpha(\beta^-)$ ). With these notations, we define:

$$\frac{\Delta NU_\alpha(\beta) := \Delta}{= NU_\alpha(\beta^+) - NU_\alpha(\beta^-)}.$$

Here,  $\Delta NU_\alpha(\beta)$  stands for the change of  $NU(\tau)$  caused by the variation of the critical imaginary root  $\lambda = \alpha$  as  $\tau$  increases from  $\beta - \varepsilon$  to  $\beta + \varepsilon$ .

**Remark 2.15.** The function  $NU$  introduced above is similar to the so-called *stability indicative function* introduced in the 60s by Kashiwagi (1965). To the best of the authors' knowledge, under the assumption of simple characteristic imaginary roots, the first systematic discussion on the number of unstable roots by using the continuity of the roots with respect to the delay can be found in Kashiwagi's works.  $\square$

**Remark 2.16.** If the system (1) free of delays has original critical imaginary roots  $\lambda_\alpha = i\omega_\alpha$ , then  $\tau_{\alpha,0} = 0$ . In this case, the asymptotic behavior of the critical pair  $(\lambda_\alpha, \tau_{\alpha,0})$  refers to how the original critical imaginary root  $\lambda_\alpha$  varies as  $\tau$  increases from 0. This information is necessary for computing  $NU(+\varepsilon)$  (as it will be discussed in Theorem 6.1).  $\square$

The asymptotic behavior of a critical imaginary root at a critical delay can be properly derived if the associated Puiseux series<sup>17</sup> can be obtained (see, for instance, Section 4 below). As a critical imaginary root has an infinite number of critical delays (see Remark 3.5 below), we need to solve the second sub-problem of *Problem 2* described as follows:

**Problem 2.2:** *How to analyze the asymptotic behavior of a critical imaginary root w.r.t. all the infinitely many positive critical delays?*

To solve this problem, the *invariance property* is essential (see Section 5 below).

### 3 Frequency-Sweeping Curves

We start this section by proposing the procedure to generate the *frequency-sweeping curves*. First, the characteristic function  $f(\lambda, \tau)$  can be transformed by letting  $z = e^{-\tau\lambda}$  into a *two-variate (auxiliary) polynomial*:

$$p_a(\lambda, z) = \sum_{i=0}^q a_i(\lambda) z^i.$$

---

<sup>17</sup>For an elementary introduction to Puiseux series, we refer to Casas-Alvero (2000).

**Frequency-Sweeping Curves:** sweep  $\omega \geq 0$  and for each  $\lambda = i\omega$  we have  $q$  values of  $z$  such that  $p(i\omega, z) = 0$  (denoted by  $z_1(i\omega), \dots, z_q(i\omega)$ ). Thus, we obtain  $q$  frequency-sweeping curves  $\Gamma_i(\omega): |z_i(i\omega)|$  vs.  $\omega$ ,  $i \in \llbracket 1, q \rrbracket$ .

Denote by  $\Im_1$  the line parallel to the abscissa axis with ordinate 1. If  $(\lambda_\alpha, \tau_{\alpha,k})$  is a critical pair with index  $g$ , then  $g$  frequency-sweeping curves intersect  $\Im_1$  at  $\omega = \omega_\alpha$  and the frequency  $\omega_\alpha$  is called a critical frequency.

Such a simple construction shows that *Problem 1* can be effectively solved by appropriately using the frequency-sweeping curves.

**Remark 3.1.** For each given  $\omega$ ,  $p(i\omega, z) = 0$  is a polynomial equation of  $z$ . It can be (numerically) solved by using the MATLAB command `roots`.  $\square$

**Remark 3.2.** Consider now the case when some curves  $\Gamma_i(\omega)$  intersect  $\Im_1$  at  $\omega = 0$ . We may have two situations:  $z = 1$  and  $z \neq 1$ . In the first case, we have an invariant root at the origin (for all  $\tau \in \mathbb{R}_+$ ). In the second case,  $\lambda = 0$  is not a characteristic root and this point should be ignored.  $\square$

**Remark 3.3.** As expected, if no critical imaginary roots are detected from the frequency-sweeping curves, the system is *hyperbolic* and the property holds independently of the delay value.  $\square$

**Example 3.4.** Consider again the scalar DDE in Example 2.11. One can easily obtain the same conclusions by observing the frequency-sweeping curves. First, the system is asymptotically stable when  $\tau = 0$  under the condition  $a + b > 0$ . Second, the frequency-sweeping curve does not intersect the line  $\Im_1$  for any  $\tau > 0$  under the condition  $a \geq |b|$ . It is worth noting that, the frequency-sweeping curve intersects the line  $\Im_1$  at  $\omega = 0$  (with the corresponding  $z = -1$ ) if  $a = b > 0$ . However, as discussed in Remark 3.2,  $\lambda = 0$  is not a critical imaginary root. For illustration, the frequency-sweeping curves in the *delay-independent stability* cases are given respectively in Fig. 3 (left: strong; right: weak).  $\square$

Consider now that the system is *not hyperbolic*. In this case, without any loss of generality, suppose there are “ $u$ ” critical pairs for  $p(\lambda, z) = 0$ :  $(\lambda_0 = i\omega_0, z_0)$ ,  $(\lambda_1 = i\omega_1, z_1)$ ,  $\dots$ ,  $(\lambda_{u-1} = i\omega_{u-1}, z_{u-1})$  where  $\omega_0 \leq \omega_1 \leq \dots \leq \omega_{u-1}$ . Notice that two critical pairs may share the same critical imaginary root. Once all the critical pairs  $(\lambda_\alpha, z_\alpha)$ ,  $\alpha = 0, \dots, u-1$ , are found, all the critical pairs  $(\lambda, \tau)$  can be obtained. For instance, for each critical imaginary root  $\lambda_\alpha$ , the corresponding critical delays are given by  $\frac{\tau_{\alpha,k} := \Delta}{= \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}}$ ,  $k \in \mathbb{N}$ , where  $\frac{\tau_{\alpha,0} := \Delta}{= \min\{\tau \geq 0: e^{-\tau\lambda_\alpha} = z_\alpha\}}$ . Thus, the pairs  $(\lambda_\alpha, \tau_{\alpha,k})$ ,  $k \in \mathbb{N}$  define a set of critical pairs associated to  $(\lambda_\alpha, z_\alpha)$ .

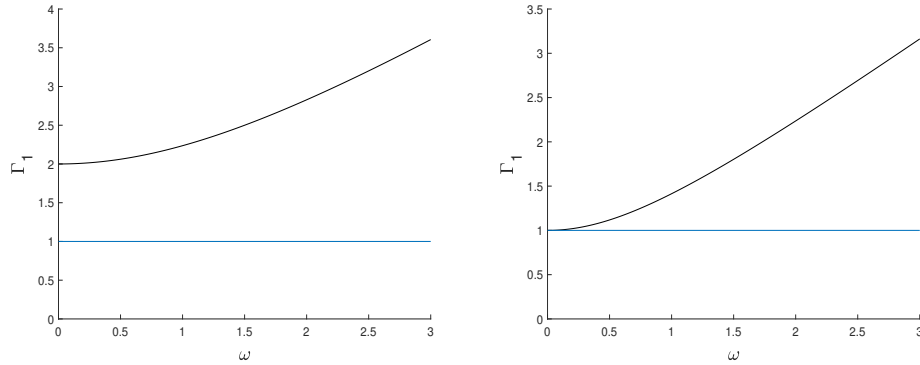


Figure 3: Example 3.4: frequency-sweeping curves  $\Gamma_1$  vs.  $\omega$  for  $b = 1$ ; left:  $a = 2$  and right:  $a = 1$ .

**Remark 3.5.** A critical imaginary root  $\lambda_\alpha$  is *invariant* with respect to the *delay shift*  $\frac{2\pi}{\omega_\alpha}$ <sup>18</sup>. However, the multiplicity of a critical imaginary root is not necessarily conserved by the delay shift.  $\square$

**Example 3.6.** Consider the system (Example 5.11 in Gu et al., 2003)

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t - \tau),$$

for which  $f(\lambda, \tau) = \lambda^2 - 0.1\lambda + 2 - e^{-\tau\lambda}$  and  $p_a(\lambda, z) = -z + \lambda^2 - 0.1\lambda + 2$ . The frequency-sweeping curve can be easily generated by using MATLAB<sup>19</sup>. For instance, in MATLAB environment, for each given  $\omega$ , we assign its value to a variable  $\mathbf{w}$ . The solution of  $z_1(i\omega)$  for  $p(i\omega, z) = 0$  can be obtained by using the command `roots([-1, (w*i)^2-0.1*w*i+2])`.

The frequency-sweeping curve is depicted in Fig. 4. Two critical pairs  $(\lambda, z)$  for  $p_a(\lambda, z) = 0$  are found from the frequency-sweeping curve:  $(\lambda_0 = 1.0025i, z_0 = 0.9950 - 0.1003i)$  and  $(\lambda_1 = 1.7277i, z_1 = -0.9850 - 0.1728i)$ . For the first critical pair, we calculate the corresponding critical delays such that  $e^{-\tau\lambda_0} = z_0 = e^{-(0.1004+2k\pi)i}$ . More precisely, one gets:  $\tau_{0,k} = 0.1002 + \frac{2k\pi}{1.0025}$ ,  $k \in \mathbb{N}$ . Similarly, for the second critical pair, the associated critical delays can be computed straightforwardly from the condition:  $e^{-\tau\lambda_1} = z_1 =$

<sup>18</sup>More precisely, if  $\lambda_\alpha$  is a critical imaginary root for  $\tau = \tau_{\alpha,0}$ , then the system has a critical imaginary root  $\lambda_\alpha$  for all  $\tau = \tau_{\alpha,0} + k\frac{2\pi}{\omega_\alpha}$ ,  $k \in \mathbb{N}$ .

<sup>19</sup>Or other software for scientific computation.

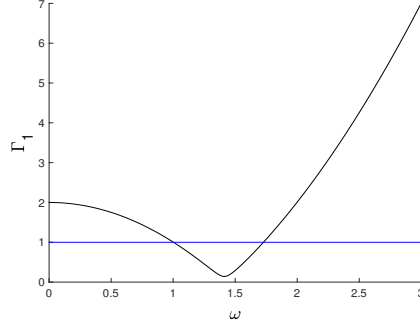


Figure 4: Example 3.6: frequency-sweeping curve  $\Gamma_1$  vs.  $\omega$ .

$e^{-(2.9680+2k\pi)i}$ . We have that  $\tau_{1,k} = 1.7178 + \frac{2k\pi}{1.72277}$ ,  $k \in \mathbb{N}$ . Thus, the  $\tau$ -axis is divided into intervals:  $[0, 0.1002)$ ,  $(0.1002, 1.7178)$ ,  $(1.7178, 5.3546)$ ,  $(5.3546, 6.3676)$ ,  $\dots$ , and  $NU(\cdot)$  is constant in each such intervals.  $\square$

#### 4 Asymptotic Behavior of a Critical Imaginary Root at a Critical Delay

As  $f(\lambda, \tau)$  (3) is a quasipolynomial, it is analytical w.r.t. the variables  $\lambda$  and  $\tau$ . Thus, in a small neighborhood of a critical pair  $(\lambda_\alpha, \tau_{\alpha,k})$ , the characteristic function  $f(\lambda, \tau)$  can be expanded as a convergent power series of the form:

$$f(\lambda, \tau) = f(\lambda_\alpha, \tau_{\alpha,k}) + (f_\lambda \Delta\lambda + f_\tau \Delta\tau) + \frac{f_{\lambda\lambda}(\Delta\lambda)^2 + 2f_{\lambda\tau} \Delta\lambda \Delta\tau + f_{\tau\tau}(\Delta\tau)^2}{2!} + \frac{f_{\lambda^3}(\Delta\lambda)^3 + 3f_{\lambda^2\tau}(\Delta\lambda)^2 \Delta\tau + 3f_{\lambda\tau^2} \Delta\lambda(\Delta\tau)^2 + f_{\tau^3}(\Delta\tau)^3}{3!} + \dots, \quad (6)$$

where  $\lambda = \lambda_\alpha + \Delta\lambda$  and  $\tau = \tau_{\alpha,k} + \Delta\tau$ . The expression (6) is a standard two-variable Taylor expansion of  $f(\lambda, \tau)$ . Next, we may reformulate (6) in a more convenient form. Since  $f(\lambda, \tau) = f(\lambda_\alpha, \tau_{\alpha,k}) = 0$ , we have:

$$0 = (f_\lambda \Delta\lambda + f_\tau \Delta\tau) + \frac{f_{\lambda\lambda}(\Delta\lambda)^2 + 2f_{\lambda\tau} \Delta\lambda \Delta\tau + f_{\tau\tau}(\Delta\tau)^2}{2!} + \frac{f_{\lambda^3}(\Delta\lambda)^3 + 3f_{\lambda^2\tau}(\Delta\lambda)^2 \Delta\tau + 3f_{\lambda\tau^2} \Delta\lambda(\Delta\tau)^2 + f_{\tau^3}(\Delta\tau)^3}{3!} + \dots. \quad (7)$$

Recall the definition of the index  $n$ :  $f_\lambda = \dots = f_{\lambda^{n-1}} = 0$  and  $f_{\lambda^n} \neq 0$ . As a result, from the right-hand side of (7), for a critical pair  $(\lambda_\alpha, \tau_{\alpha,k})$ , we now obtain a series expression  $F_{(\lambda_\alpha, \tau_{\alpha,k})}(\Delta\lambda, \Delta\tau)$  describing the relation

between  $\Delta\lambda$  and  $\Delta\tau$  as follows:

$$F_{(\lambda_\alpha, \tau_{\alpha, k})}(\Delta\lambda, \Delta\tau) = \sum_{i=n}^{\infty} L_{i0}(\Delta\lambda)^i + \sum_{i=0}^{\infty} (\Delta\lambda)^i \sum_{l=1}^{\infty} L_{il}(\Delta\tau)^l = 0, \quad (8)$$

where  $L_{il} = \frac{f_{\lambda, \tau}^{i+l}}{(i+l)!} \binom{i+l}{i}$ <sup>20</sup>. In addition, in view of the index  $g$ , we have that  $L_{01} = \dots = L_{0(g-1)} = 0$  and  $L_{0g} \neq 0$ . From the root loci, it is easy to observe that for a  $\Delta\tau$ ,  $\Delta\lambda$  must have  $n$  solutions (multiplicity taken into account) satisfying  $F_{(\lambda_\alpha, \tau_{\alpha, k})}(\Delta\lambda, \Delta\tau) = 0$  and  $\Delta\lambda \rightarrow 0$  as  $\Delta\tau \rightarrow 0$ :

**Theorem 4.1.** *Consider the DDE (1) and assume  $\lambda_\alpha \neq 0$  is an  $n$ -multiple imaginary root for  $\tau = \tau_{\alpha, k}$ . If  $\tau$  is perturbed at  $\tau_{\alpha, k}$  by  $\Delta\tau$ , the variation  $\Delta\lambda$  of  $\lambda$  at  $\lambda_\alpha$  corresponds to  $n$  Puiseux series solutions with respect to  $\Delta\tau$ . Any Puiseux series solution converges in a neighborhood of  $(\Delta\lambda = 0, \Delta\tau = 0)$ .*

**Algorithm 1** (Puiseux series computation)

Step 0: Let  $\alpha_0 = 0$  and  $\beta_0 = g$ .

Step 1: Define  $\mu = \max\{\frac{\beta_0 - \beta}{\alpha - \alpha_0} > 0 : L_{\alpha\beta} \neq 0, \alpha > \alpha_0, \beta < \beta_0\}$ , where the coefficients  $L_{\alpha\beta}$  are defined in (8).

Step 2: If there exists a  $\mu$ , go to Step 3. Otherwise, skip to Step 5.

Step 3: Collect all the non-zero  $L_{\alpha\beta}$  satisfying  $\frac{\beta_0 - \beta}{\alpha - \alpha_0} = \mu$  to form a set

$$\left\{ L_{\alpha_1\beta_1}(\Delta\lambda)^{\alpha_1}(\Delta\tau)^{\beta_1}, L_{\alpha_2\beta_2}(\Delta\lambda)^{\alpha_2}(\Delta\tau)^{\beta_2}, \dots \right\},$$

with the order  $\alpha_1 > \alpha_2 > \dots$ . We find a set of Puiseux series

$$\Delta\lambda = \tilde{C}_{\mu, l}(\Delta\tau)^\mu + o((\Delta\tau)^\mu), \quad l = 1, \dots, \alpha_1 - \alpha_0,$$

where the coefficients  $\tilde{C}_{\mu, l}$  are the solutions of the polynomial equation

$$L_{\alpha_1\beta_1}C^{\alpha_1 - \alpha_0} + L_{\alpha_2\beta_2}C^{\alpha_2 - \alpha_0} + \dots + L_{\alpha_0\beta_0} = 0.$$

Step 4: Let  $\alpha_0 = \alpha_1, \beta_0 = \beta_1$  and return to Step 1.

Step 5: The algorithm stops.

**Theorem 4.2.** *For an  $n$ -multiple non-zero critical imaginary root of the DDE (1), all the Puiseux series can be obtained by Algorithm 1.*

**Remark 4.3.** It is known that a simple critical imaginary root's asymptotic behavior corresponds to a Taylor series and we may treat them as a specific type of Puiseux series. In addition, the Puiseux series for a multiple critical imaginary root may include a Taylor series (in the case of more than one conjugacy class), see e.g., Examples 4.3 and 4.4 in Li et al. (2015).  $\square$

<sup>20</sup>Here,  $\binom{i+l}{i}$  denotes the number of  $i$ -combinations from a set of  $i+l$  elements.

**Remark 4.4.** In general, when  $\Delta\tau = \pm\varepsilon$ , the first-order terms of the Puiseux series do not contain purely imaginary numbers, and are sufficient for the asymptotic behavior analysis. However, there exists a few cases, called *degenerate*, when such a property does not hold and, to conclude, higher order terms are necessary. In this case, we may still invoke Algorithm 1 in an iterative manner to obtain them. For a deeper discussion, we refer to Subsection 4.3 of Li et al. (2015).  $\square$

For an  $n$ -multiple critical imaginary root, we may invoke  $n$  *independent* Puiseux series. Unfortunately, such expressions are not always simple to use. However, they can be expressed in a more compact form if we introduce the concept of *conjugacy class*. Roughly speaking, for  $n$  Puiseux series belonging to one conjugacy class, one expression of Puiseux series with polydromy order  $n$  will be sufficient to describe all of them (see, e.g., Subsection 4.4 of Li et al. (2015) for further details).

Consider the critical pair  $(\lambda_\alpha, \tau_{\alpha,k})$  with  $\tau_{\alpha,k} > 0$ . Then  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$  can be accurately calculated by means of the Puiseux series. More precisely, we substitute  $\Delta\tau = +\varepsilon$  ( $\Delta\tau = -\varepsilon$ ) into the corresponding Puiseux series, and the value of  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$  can be obtained by comparing the numbers of the values of the Puiseux series in  $\mathbb{C}_+$  when  $\Delta\tau = +\varepsilon$  ( $\Delta\tau = -\varepsilon$ ).

**Example 4.5.** Consider a DDE with the characteristic function  $f(\lambda, \tau) = e^{-\tau\lambda} + \frac{3\pi}{8}\lambda^5 - \frac{\pi^2}{8}\lambda^4 + \frac{5\pi}{4}\lambda^3 - \frac{\pi^2}{4}\lambda^2 + \frac{7\pi}{8}\lambda - \frac{\pi^2}{8} + 1$ . For  $\tau = \pi$ ,  $\lambda = i$  is a triple critical imaginary root with  $g = 1$ . By invoking Algorithm 1, we have three expressions of the Puiseux series  $\Delta\lambda = (0.55 + 0.09i)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$ ,  $\Delta\lambda = (-0.36 + 0.43i)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$ , and  $\Delta\lambda = (-0.20 - 0.53i)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$ . These three expressions correspond to the same conjugacy class. Therefore, any one among them is sufficient to fully express the asymptotic behavior of the triple critical imaginary root. For instance, we choose the expression  $\Delta\lambda = (0.55 + 0.09i)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$ .

The variation of the triple critical imaginary root as the delay increases from  $\pi$  to  $\pi + \varepsilon$  ( $\pi - \varepsilon$  to  $\pi$ ) can be deduced by substituting the three values of  $(+\varepsilon)^{\frac{1}{3}}$  ( $(-\varepsilon)^{\frac{1}{3}}$ ) into  $(\Delta\tau)^{\frac{1}{3}}$  for this expression. One may notice that the value sets of the Puiseux series by the substitution of the values of  $(+\varepsilon)^{\frac{1}{3}}$  and  $(-\varepsilon)^{\frac{1}{3}}$  do not change if we choose the other two expressions of the Puiseux series. As a result, we see that the number of the characteristic roots located in  $\mathbb{C}_+$  decreases by 1 due to the asymptotic behavior of the triple critical imaginary root, that is,  $\Delta NU_i(\pi) = -1$  (see, for instance, the root loci in Fig. 5).  $\square$

To summarize, we can now properly solve *Problem 2.1*.

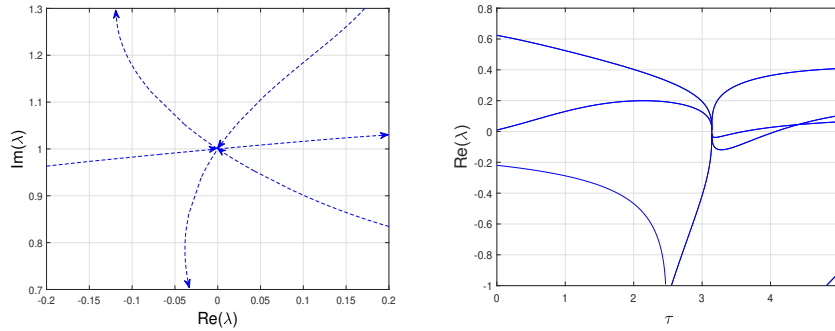


Figure 5: Example 4.5: root loci  $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$  and  $\text{Re}(\lambda)$  vs.  $\tau$ .

## 5 Invariance Property of Asymptotic Behavior

In the sequel, we introduce some necessary notations concerning the asymptotic behavior of frequency-sweeping curves. For further properties and deeper discussions, we refer to Chapter 8 of Li et al. (2015).

Under the assumption  $\lambda_\alpha \neq 0$ , suppose that  $\{(\lambda_\alpha, \tau_{\alpha,k}), k \in \mathbb{N}\}$  is a set of critical pairs with the index  $g$ . It is important to mention that  $g$  is a constant w.r.t. different  $k$  (see Property 1.2 of Li et al., 2015). Then there must exist  $g$  frequency-sweeping curves such that  $z_i(i\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$  intersecting  $\mathfrak{S}_1$  when  $\omega = \omega_\alpha$ . Among such  $g$  frequency-sweeping curves, when  $\omega = \omega_\alpha + \varepsilon$  ( $\omega = \omega_\alpha - \varepsilon$ ), we denote the number of the frequency-sweeping curves above the line  $\mathfrak{S}_1$  by  $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$  ( $NF_{z_\alpha}(\omega_\alpha - \varepsilon)$ ). Introduce now a new notation  $\Delta NF_{z_\alpha}(\omega_\alpha)$  as

$$\begin{aligned} \Delta NF_{z_\alpha}(\omega_\alpha) &:= \Delta \\ &= NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \end{aligned}$$

Such a “quantity” describes the asymptotic behavior of the frequency-sweeping curves at the critical frequency  $\omega = \omega_\alpha$ .

**Theorem 5.1.** *For a critical imaginary root  $\lambda_\alpha$  of the DDEs (1), it always holds that  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$  is a constant  $\Delta NF_{z_\alpha}(\omega_\alpha)$  for all  $\tau_{\alpha,k} > 0$ .*

The contribution of the above Theorem 5.1 is two-fold:

- (i) First, it provides a simple method (observing the frequency-sweeping curves) to compute  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ , without invoking the Puiseux series; in other words, the change of  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$  as the delay is increased from  $\tau_{\alpha,k}^-$  to  $\tau_{\alpha,k}^+$  can be expressed in terms of changes of frequency-sweeping curves w.r.t. the line  $\mathfrak{S}_1$ .

- (ii) Second, an interesting *invariance property* is claimed: for a critical imaginary root  $\lambda_\alpha$ , since  $\Delta NF_{z_\alpha}(\omega_\alpha)$  is invariant w.r.t. the delay parameter, the same property holds for  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$  for all  $\tau_{\alpha,k} > 0$ . Such a property is helpful to overcome the peculiarity that a critical imaginary root corresponds to infinitely many critical delays.

**Remark 5.2.** By using different arguments, the invariance property was addressed by Olgac and Sipahi (2002) (simple critical roots on imaginary axis:  $n = 1$ ) and by Jarlebring and Michiels (2010) (case  $n = 2, g = 1$ ).  $\square$

**Example 5.3.** Consider a DDE with the characteristic function  $f(\lambda, \tau) = \sum_{i=0}^4 a_i(\lambda)e^{-i\tau\lambda}$  where  $a_0(\lambda) = \frac{15}{8}\pi^2\lambda^6 + (\frac{11}{4}\pi - \frac{15}{8}\pi^2)\lambda^4 + \frac{9}{2}\pi\lambda^3 + (1 + \frac{1}{2}\pi - \frac{75}{8}\pi^2)\lambda^2 + (3 + \frac{9}{2}\pi)\lambda + 1 - \frac{9}{4}\pi - \frac{45}{8}\pi^2$ ,  $a_1(\lambda) = \frac{5}{4}\pi\lambda^5 + \frac{11}{2}\pi\lambda^4 + (1 + \frac{7}{2}\pi)\lambda^3 + (\pi + 7)\lambda^2 + (11 + \frac{9}{4}\pi)\lambda + 4 - \frac{9}{2}\pi$ ,  $a_2(\lambda) = \frac{5}{4}\pi\lambda^5 + \frac{11}{4}\pi\lambda^4 + (3 - \pi)\lambda^3 + (13 + \frac{1}{2}\pi)\lambda^2 + (15 - \frac{9}{4}\pi)\lambda + 6 - \frac{9}{4}\pi$ ,  $a_3(\lambda) = 3\lambda^3 + 9\lambda^2 + 9\lambda + 4$ , and  $a_4(\lambda) = \lambda^3 + 2\lambda^2 + 2\lambda + 1$ .

We study the asymptotic behavior of critical pairs  $(i, (2k+1)\pi)$ , with  $g = 2$ . The frequency-sweeping curves are given in Fig. 6. According to Theorem 5.1, it is easy to see from Fig. 6 that  $\Delta NU_i((2k+1)\pi) = 0$  for all  $k \in \mathbb{N}$ . In fact, the asymptotic behavior of critical pairs  $(i, (2k+1)\pi)$  is complex. The multiplicity  $n$  of the critical root  $\lambda = i$  is 2, 3, 4, 2, when  $\tau$  is  $\pi, 3\pi, 5\pi, 7\pi$ , respectively. The Puiseux series, all degenerate, are:

$$\begin{cases} \Delta\lambda = 0.1592i\Delta\tau + (0.5371 - 0.3138i)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = 0.0796i\Delta\tau + 0.0063i(\Delta\tau)^2 + 0.0421i(\Delta\tau)^3 \\ \quad + (0.0362 + 0.0137i)(\Delta\tau)^4 + o((\Delta\tau)^4), \\ \Delta\lambda = (0.0385 + 0.0698i)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}}), \\ \Delta\lambda = 0.1592i\Delta\tau + 0.0253i(\Delta\tau)^2 + 0.6696i(\Delta\tau)^3 \\ \quad + (1.1585 + 0.4376i)(\Delta\tau)^4 + o((\Delta\tau)^4), \\ \Delta\lambda = -0.1592i\Delta\tau + (-0.5371 + 0.3644i)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = -0.0988i(\Delta\tau)^{\frac{1}{3}} + (-0.0356 + 0.0028i)(\Delta\tau)^{\frac{2}{3}} + o((\Delta\tau)^{\frac{2}{3}}), \\ \Delta\lambda = -0.0796i\Delta\tau + (-0.0671 + 0.0487i)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = -0.1592i\Delta\tau + 0.0253i(\Delta\tau)^2 + 0.6615i(\Delta\tau)^3 \\ \quad + (-1.1585 - 0.4363i)(\Delta\tau)^4 + o((\Delta\tau)^4), \end{cases}$$

for  $k = 0, 1, 2$ , and 3, respectively. The above Puiseux series are consistent with the analysis by Theorem 5.1. Moreover, Theorem 5.1 significantly reduces the computation burden for asymptotic behavior analysis, as the Puiseux series for this system are rather involved<sup>21</sup>.  $\square$

<sup>21</sup>For instance, in our case, for each  $k$ , the Puiseux series has multiple conjugacy classes;



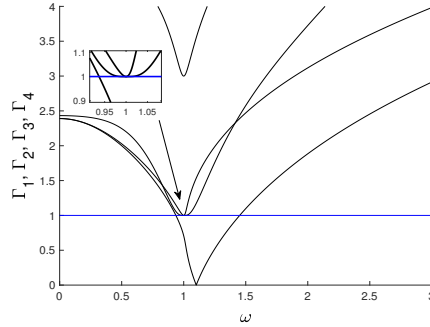


Figure 6: Example 5.3: frequency-sweeping curves  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$  vs.  $\omega$ .

Thus, *Problem 2.2* is appropriately solved.

## 6 A Unified Frequency-Sweeping Approach for Complete Stability Problem

With the results above, we can now systematically solve our *problem*.

### 6.1 Computation of $NU(+\varepsilon)$

As a first step, we keep track<sup>22</sup> of  $NU(\tau)$  from  $\tau = +\varepsilon$ .

**Theorem 6.1.** *If the system (1) has no critical imaginary roots when  $\tau = 0$ ,  $NU(+\varepsilon) = NU(0)$ . Otherwise,  $NU(+\varepsilon) - NU(0)$  equals to the number of the values in  $\mathbb{C}_+$  of the Puiseux series for all the corresponding critical imaginary roots when  $\tau = 0$  with  $\Delta\tau = +\varepsilon$ .*

**Example 6.2.** Consider the DDE with the characteristic function  $f(\lambda, \tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$ . It is easy to see that  $f(\lambda, 0)$  has four characteristic roots. More precisely,  $\lambda = i$  ( $\lambda = -i$ ) is a double critical imaginary root. We may have the Puiseux series for the critical pair  $(i, 0)$ :

$$\Delta\lambda = (0.3536 + 0.3536i)(\Delta\tau)^{\frac{3}{2}} + o((\Delta\tau)^{\frac{3}{2}}). \quad (9)$$

Substituting  $\Delta\tau = +\varepsilon$  into (9) indicates that as  $\tau$  increases from 0, the double root  $i$  splits into two branches towards  $\mathbb{C}_-$  and  $\mathbb{C}_+$  respectively, as

next, for each  $k$ , the Puiseux series involves many degenerate terms, and finally, the structure of Puiseux series is variable w.r.t. different  $k$ .

<sup>22</sup>Since some DDEs may have critical imaginary roots when  $\tau = 0$ , one needs to consider a sufficiently “small” delay value  $\varepsilon > 0$ .

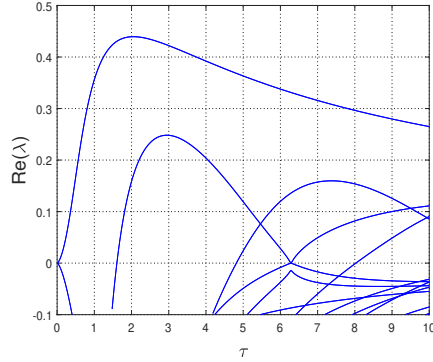


Figure 7: Example 6.2:  $\text{Re}(\lambda)$  vs.  $\tau$ .

seen in the root loci given in Fig. 7. Thus, by using the conjugate symmetry property,  $NU(+\varepsilon) = +2$ .  $\square$

## 6.2 Explicit $NU(\tau)$ Expression

The invariance property allows concluding with the following:

**Theorem 6.3.** *For any  $\tau > 0$  which is not a critical delay,  $NU(\tau)$  for the DDE (1) can be explicitly expressed as*

$$NU(\tau) = NU(+\varepsilon) + \sum_{\alpha=0}^{u-1} NU_{\alpha}(\tau),$$

where

$$NU_{\alpha}(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_{\alpha}} \left[ \frac{\tau - \tau_{\alpha,0}}{2\pi/\omega_{\alpha}} \right], \tau > \tau_{\alpha,0}, & \text{if } \tau_{\alpha,0} \neq 0, \\ \\ 0, \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_{\alpha}} \left[ \frac{\tau - \tau_{\alpha,1}}{2\pi/\omega_{\alpha}} \right], \tau > \tau_{\alpha,1}, & \text{if } \tau_{\alpha,0} = 0. \end{cases}$$

## 6.3 Further Classification

With the explicit  $NU(\tau)$  expression, we may accurately study the stability for any finitely long  $\tau$ -interval. However, in order to thoroughly solve the complete stability problem, we need to understand the way  $NU(\tau)$  changes when  $\tau \rightarrow \infty$ . To such an end, we introduce the following notions: a critical frequency  $\omega_{\alpha}$  is called a *crossing (touching) frequency* for a

frequency-sweeping curve  $\Gamma_i(\omega)$ , if  $\Gamma_i(\omega)$  crosses (touches without crossing) the line  $\Im_1$  as  $\omega$  increases near  $\omega_\alpha$ . We have the following:

**Theorem 6.4.** *If the frequency-sweeping curves have a crossing frequency, there exists some delay value  $\tau^*$  such that the time-delay system (1) is unstable for all  $\tau > \tau^*$  and  $\lim_{\tau \rightarrow \infty} NU(\tau) = \infty$ .*

**Theorem 6.5.** *The DDE (1) must fall in the following three types:*

- (i) *Type 1: Crossing frequencies exist and  $\lim_{\tau \rightarrow \infty} NU(\tau) = \infty$ .*
- (ii) *Type 2: Crossing and touching frequencies do not exist and  $NU(\tau) = NU(0)$  for all  $\tau > 0$ .*
- (iii) *Type 3: Crossing frequencies do not exist but touching frequencies exist and, with the exception of critical delays,  $NU(\tau)$  is a constant for all  $\tau \geq 0$ .*

**Remark 6.6.** A DDE of Type 2 is hyperbolic. Furthermore, if  $NU(0) = 0$ , it is *stable independent of the delay*. Discussions on other cases can be found in Section 9.1 of Li et al. (2015).  $\square$

#### 6.4 Procedure for Complete Stability Analysis

We now present a unified approach for studying our stability problem:

Step 1: Generate the frequency-sweeping curves, through which we can detect all the critical imaginary roots and the corresponding critical delays.

Step 2: For each critical imaginary root  $\lambda_\alpha$ , we may choose any positive critical delay  $\tau_{\alpha,k}$  to compute  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ . Alternatively, according to Theorem 5.1, we may directly have from the frequency-sweeping curves that  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ . Step 3: Compute  $NU(+\varepsilon)$  (by Theorem 6.1).

Step 4: Obtain the explicit expression of  $NU(\tau)$  as stated in Theorem 6.3 and have a “ $NU(\tau)$  vs.  $\tau$ ” plot.

The DDE (1) is asymptotically stable for all delay intervals satisfying the condition  $NU(\tau) = 0$ . In addition, according to Theorem 6.5, the behavior when  $\tau \rightarrow \infty$  is known.

**Example 6.7.** Consider again the system in Example 5.3. When  $\tau = 0$  there are three characteristic roots located in  $\mathbb{C}_+$  and no characteristic roots located in the imaginary axis. According to Theorem 6.1,  $NU(+\varepsilon) = NU(0) = 3$ . The frequency-sweeping curves are given in Fig. 6. Theorem 6.3 allows deriving  $NU(\tau)$  expression. The variation of  $NU(\tau)$  is shown in Fig. 8. As crossing frequencies exist, in light of Theorem 6.4,  $NU(\tau) = \infty$  as  $\tau \rightarrow \infty$ . To resume, the system is *unstable independent of the delay*, but without being hyperbolic.  $\square$

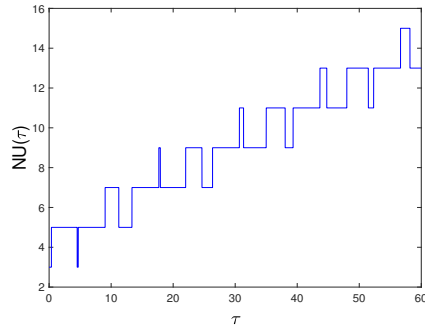


Figure 8: Example 6.7:  $NU(\tau)$  vs.  $\tau$  plot.

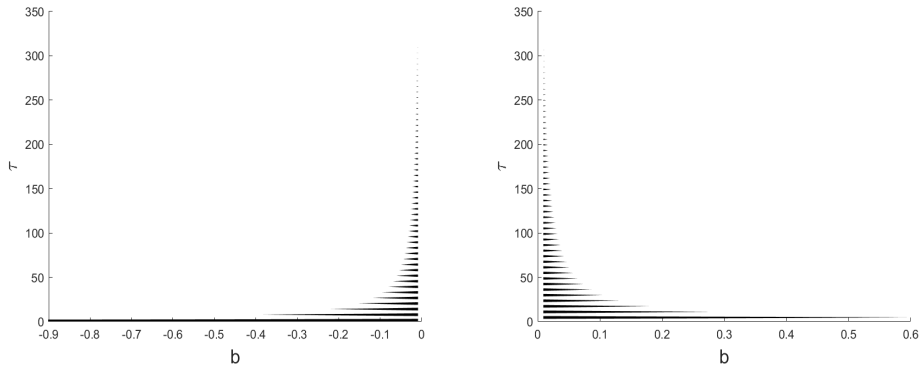


Figure 9: Example 6.8: stability regions in  $(b, \tau)$  parameter space.

**Example 6.8.** Consider the DDE with the characteristic function

$$f(\lambda, \tau) = \lambda^2 + a^2 + be^{-\tau\lambda}, \quad a > 0.$$

It is easy to see that the system is asymptotically stable if and only if one of the following cases occurs:

- (i)  $-a^2 < b < 0$  and  $\tau$  lies in the intervals  $(\frac{2k\pi}{\sqrt{a^2+b}}, \frac{(2k+1)\pi}{\sqrt{a^2-b}})$  for all  $k \in \mathbb{N}$   
such that  $\frac{2k\pi}{\sqrt{a^2+b}} < \frac{(2k+1)\pi}{\sqrt{a^2-b}}$ ;
- (ii)  $a^2 > b > 0$  and  $\tau$  lies in the intervals  $(\frac{(2k+1)\pi}{\sqrt{a^2-b}}, \frac{(2k+2)\pi}{\sqrt{a^2+b}})$  for all  $k \in \mathbb{N}$   
such that  $\frac{(2k+1)\pi}{\sqrt{a^2-b}} < \frac{(2k+2)\pi}{\sqrt{a^2+b}}$ .

For illustration of the cases (i)–(ii), take  $a = 1$ . The corresponding stability regions are depicted in Fig. 9 (left) and Fig. 9 (right), respectively.  $\square$

**Example 6.9.** [Stability reversals: further insights] Consider the time-delay system  $\dot{x}(t) = A_0x(t) + A_1x(t - \tau)$  with

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_4 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \beta_0 & \beta_1 & \cdots & \beta_4 \end{pmatrix},$$

where  $\alpha_0 = \frac{\pi}{2} - \frac{\pi^2}{8} - 1$ ,  $\alpha_1 = -2 + \frac{\pi}{2}$ ,  $\alpha_2 = -\frac{\pi^2}{4} + \pi - 10$ ,  $\alpha_3 = -3 + \frac{\pi}{2}$ ,  $\alpha_4 = -\frac{\pi^2}{8} + \frac{\pi}{2} - 8$ ,  $\beta_0 = -1$ ,  $\beta_1 = -1$ ,  $\beta_2 = -10$ ,  $\beta_3 = -1$ , and  $\beta_4 = -8$ . The characteristic function is  $f(\lambda, \tau) = \lambda^5 - \sum_{\ell=0}^4 \alpha_\ell \lambda^\ell - (\sum_{\ell=0}^4 \beta_\ell \lambda^\ell) e^{-\tau\lambda}$ . In this case, the frequency-sweeping curve is shown in Fig. 10 (left). First, we observe that three sets of critical pairs are detected:  $(0.3340i, 5.8296 + \frac{2k\pi}{0.3340})$ ,  $(i, (2k+1)\pi)$ , and  $(2.2421i, 1.2525 + \frac{2k\pi}{2.2421})$ . Second, when  $\tau = 0$ , all the characteristic roots are located in  $\mathbb{C}_-$  and, according to Theorem 6.1,  $NU(+\varepsilon) = NU(0) = 0$ . Then, according to Theorem 6.3, we have

$$NU(\tau) = \sum_{\alpha=0}^2 NU_\alpha(\tau), \text{ with:}$$

$$NU_0(\tau) = \begin{cases} 0, \tau < \tau_3, \\ 2U_{0.3340i} \left[ \frac{\tau - 5.8296}{18.8125} \right], \tau > \tau_3, \end{cases}$$

$$NU_1(\tau) = \begin{cases} 0, \tau < \tau_2, \\ 2U_i \left[ \frac{\tau - \pi}{2\pi} \right], \tau > \tau_2, \end{cases} \quad NU_2(\tau) = \begin{cases} 0, \tau < \tau_1, \\ 2U_{2.2421i} \left[ \frac{\tau - 1.2525}{2.8024} \right], \tau > \tau_1. \end{cases}$$

where  $\tau_1 = 1.2525$ ,  $\tau_2 = \pi$  and  $\tau_3 = 5.8296$ . In view of Theorem 5.1, the values for  $U_{0.3340i}$ ,  $U_i$ , and  $U_{2.2421i}$  are  $+1$ ,  $-1$ , and  $+1$ , respectively. The variation of  $NU(\tau)$  is shown in Fig. 10 (right). The system has two and only *two stability intervals* of  $\tau$ :  $[0, 1.2525)$  and  $(\pi, 4.0549)$ .

One can observe an interesting phenomenon. The asymptotic behavior of the critical pair  $(\lambda = i, \tau = \pi)$  with the indices<sup>23</sup>  $n = 2$  and  $g = 1$  corresponds to the Puiseux series:

$$\Delta\lambda = 0.1468i(\Delta\tau)^{\frac{1}{2}} + (-0.0033 - 0.1473i)(\Delta\tau)^{\frac{2}{3}} + o((\Delta\tau)^{\frac{2}{3}}).$$

In such a case, as  $\tau$  increases from  $\pi - \varepsilon$  to  $\pi$  two root loci<sup>24</sup> collide on the imaginary axis  $i\mathbb{R}$  and thereby a double critical imaginary root  $\lambda = i$  appears. As  $\tau$  further increases from  $\pi$  to  $\pi + \varepsilon$  the double root  $\lambda = i$  splits

<sup>23</sup>The critical root  $\lambda = i$  is a double critical imaginary root at  $\tau = \pi$ .

<sup>24</sup>In our case, one characteristic root is located in the right-half plane  $\mathbb{C}_+$  and the other in the left-half plane  $\mathbb{C}_-$ .

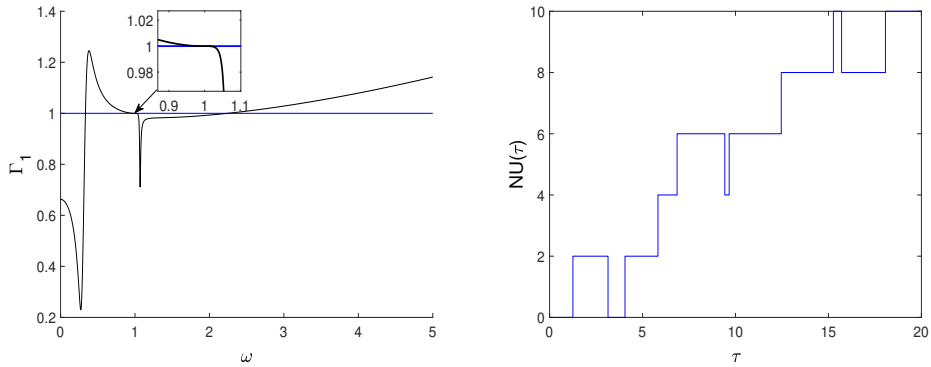


Figure 10: Example 6.9: frequency-sweeping curve  $\Gamma_1$  vs.  $\omega$  and  $NU(\tau)$  vs.  $\tau$  plot

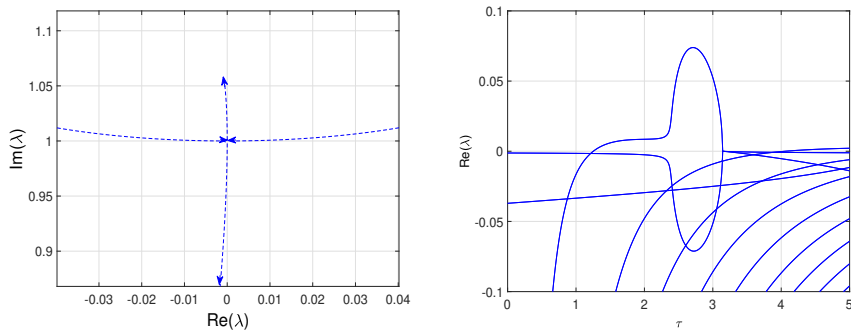


Figure 11: Example 6.9:  $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$  and  $\text{Re}(\lambda)$  vs.  $\tau$ .

into two root loci, both towards the left-half plane  $\mathbb{C}_-$ . Meanwhile, there are no other characteristic roots in  $\mathbb{C}_+$  when  $\tau = \pi - \varepsilon$ , and hence the system *regains asymptotic stability* at  $\tau = \pi + \varepsilon$ . Thus, as  $\tau$  increases near  $\pi$ , the appearance of the double critical imaginary root  $i$  brings a *stability reversal*. In order to verify the above results, the “ $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$ ” plot near  $(\lambda = i, \tau = \pi)$  and the “ $\text{Re}(\lambda)$  vs.  $\tau$ ” plot are given in Fig. 11 (left) and in Fig. 11 (right), respectively. More relevant results as well as a finer characterization of stability reversals can be found in Li et al. (2019a).  $\square$

## 7 Further Extensions of the Frequency-Sweeping Approach

The DDE (1) under consideration in the previous sections are all of *retarded* type. In the sequel, we address some extensions of the proposed methodology. In particular, we consider two classes: neutral delay differential equations (NDDE) and distributed delay differential equations (DDDEs) with uniform distributions.

### 7.1 Neutral Delay Differential Equations

Consider the following DDE of *neutral* type

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \tau), \quad (10)$$

where  $A \in \mathbb{R}^{r \times r}$ ,  $B \in \mathbb{R}^{r \times r}$ , and  $C \in \mathbb{R}^{r \times r}$ ,  $C \neq 0$  are constant matrices. The characteristic function of system (10) is given by

$$f_N(\lambda, \tau) = \det(\lambda I - A - Be^{-\tau\lambda} - \lambda Ce^{-\tau\lambda}),$$

which is a quasipolynomial of the form

$$f_N(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \dots + a_q(\lambda)e^{-q\tau\lambda}, \quad (11)$$

where  $a_i(\lambda)$ ,  $i \in \llbracket 0, q \rrbracket$  are polynomials in  $\lambda$  with real coefficients. As mentioned in the Preliminaries, compared to the retarded DDEs (1), the stability of the trivial solution of the neutral DDEs (10) has an additional *necessary* condition: the stability of the neutral delay-difference equation

$$x(t) - Cx(t - \tau) = 0. \quad (12)$$

The other issues for studying the complete stability problem are similar<sup>25</sup> and can be directly addressed by using the frequency-sweeping approach. For a comprehensive introduction to the spectral properties of linear neutral DDEs, one may refer to Gu (2012) (see also Michiels and Niculescu, 2014).

**Lemma 7.1.** *The trivial solution of the neutral delay-difference equation (12) is exponentially stable for any positive  $\tau$  if and only if*

$$\rho(C) < 1.$$

**Theorem 7.2.** *The trivial solution of the neutral delay-difference equation (12) is exponentially stable if and only if all the frequency-sweeping curves are above the line  $\Im_1$  as  $\omega \rightarrow \infty$ .*

<sup>25</sup>Both the characteristic functions (3) and (11) are standard quasipolynomials.

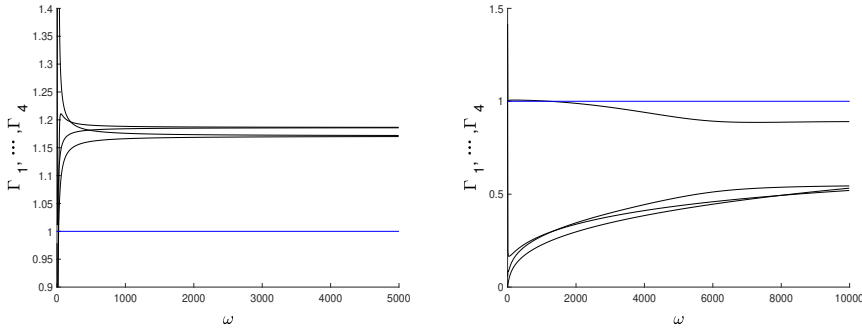


Figure 12: Examples 7.3 and 7.4: frequency-sweeping curves  $\Gamma_1, \dots, \Gamma_4$  vs.  $\omega$ .

**Example 7.3.** Consider the NDDEs of Example case 2 in Olgac and Sipahi (2004), i.e., the system (10) with matrices:

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -3.346 & -2.715 & 2.075 & -2.007 \\ -4 & 0 & 2 & 0 \\ -3 & 0 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 2 & -1 \\ 3 & 3 & -2 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & -3 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.2 & -0.1 & 0.5 & -0.1 \\ -0.3 & 0.09 & -0.15 & -0.027 \\ -3.333 & 0.1 & 0.2 & 1 \\ -1 & 2 & 0.5 & 1 \end{pmatrix}.$$

The four eigenvalues of  $C$  are  $0.0881 \pm 0.8494i$  and  $0.6569 \pm 0.5284i$ . Hence,  $\rho(C) < 1$ . This result can be directly obtained from the frequency-sweeping curves shown in Fig. 12 (left), based on Theorem 7.2. We see that as  $\omega \rightarrow \infty$ ,  $|z_i(i\omega)| > 1$ ,  $i = 1, \dots, 4$ .  $\square$

**Example 7.4.** Consider the NDDEs of Example b2 in Sipahi and Olgac (2003), i.e., the system (10) with matrices:

$$A = \begin{pmatrix} 12 & 10 & -6 & 14 \\ 7 & 8 & 11 & 9 \\ -5 & 7 & 3 & 3 \\ 6 & 2 & 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -169 & -276.85 & -445.76 & -675.75 \\ -11 & -46 & -61 & -83 \\ 249 & 360.05 & 1070.43 & 1431.02 \\ 81.65 & 158.32 & 127.61 & 230.85 \end{pmatrix},$$



$$C = \begin{pmatrix} -4 & 12 & 3 & 1 \\ 0 & 1 & -2 & 6 \\ 12 & -8 & 4 & 2 \\ 1.47 & -10.09 & -4.33 & 0.03 \end{pmatrix}.$$

The four eigenvalues of  $C$  are  $0.2816 \pm 1.3641i$ ,  $-1.3469$ , and  $1.8138$ . Hence,  $\rho(C) > 1$ . In this case, from the frequency-sweeping curves, Fig. 12 (right), we observe that as  $\omega \rightarrow \infty$ ,  $|z_i(i\omega)| < 1$ ,  $i \in \llbracket 1, 4 \rrbracket$ , result consistent with Theorem 7.2.  $\square$

**Example 7.5.** Consider the following first-order NDDE (Fu et al., 2006):

$$\dot{x}(t) + \beta \dot{x}(t - \tau) = -ax(t) - bx(t - \tau),$$

where<sup>26</sup>  $|\beta| < 1$ . The characteristic function is  $f_N(\lambda, \tau) = \lambda + a + (\beta\lambda + b)e^{-\tau\lambda}$ . When  $\tau = 0$ ,  $\lambda = -\frac{a+b}{1+\beta}$ . Thus,  $NU(0) = 0$  ( $NU(0) = 1$ ) if  $a+b > 0$  ( $a+b < 0$ ) while there exists an invariant characteristic root at the origin ( $\lambda = 0$ ) if  $a+b = 0$ . As  $z = -\frac{\lambda+a}{\beta\lambda+b}$ , we have the following possible cases:

- (i) If  $|b| > |a|$ , the frequency-sweeping curve intersects the line  $\Im_1$  at one and only one critical frequency (there is one and only one critical imaginary root  $\lambda_0$  with the critical delays  $\tau_{0,k}$ ). More precisely, we have: (i.1) if  $a+b < 0$ , the NDDE is unstable for all  $\tau \geq 0$ , and (i.2) if  $a+b > 0$ , there is one and only one stability  $\tau$ -interval:  $[0, \tau_{0,0})$ .
- (ii) If  $|b| < |a|$ , the frequency-sweeping curve does not intersect the line  $\Im_1$  (there is no critical imaginary root). Thus, the system is *hyperbolic*, *delay-independent stable* (*unstable*) if  $a+b > 0$  ( $a+b < 0$ ).

If  $a = b \neq 0$ , it is easy to observe that the frequency-sweeping curve intersects the line  $\Im_1$  at  $\omega = 0$ . However, since  $z = -1$ ,  $\lambda = 0$  is not a critical imaginary root. In such a case, similarly to the case (ii) above, the NFDDE is *delay-independent stable* (*unstable*) if  $a = b > 0$  ( $a = b < 0$ ).  $\square$ .

## 7.2 Distributed Delay Differential Equations with Uniform Distribution

For the retarded- and neutral-type DDEs discussed in the previous sections, the characteristic functions  $f(\lambda, \tau)$  given by (3) include polynomials  $a_i(\lambda)$ , with  $i \in \llbracket 0, q \rrbracket$ , (called *coefficient functions*), and are “*standard*” *quasipolynomials*. It is worth mentioning that the methodology developed above works for larger classes of coefficient functions. For instance, if we

<sup>26</sup>The stability of the trivial solution of the neutral delay-difference equation is guaranteed. It is worth mentioning that the case  $\beta = 0$  corresponds to the retarded DDE and it was addressed in the previous section.

assume that the coefficient functions  $a_i(\lambda)$ ,  $i \in \llbracket 0, q \rrbracket$  of  $f(\lambda, \tau)$  given by (3) are only required to be analytic in  $\mathbb{R} \setminus \{0\}$ , we are able to cover a number of epidemiological models described by integro-differential equations including delays in their representation. In the sequel, a characteristic function (3) with this relaxed condition is called *general quasipolynomial*. Due to the particular way the frequency-sweeping approach makes use of the coefficient functions, it can also be applied to general quasipolynomials. More precisely, we have the following result:

**Theorem 7.6.** *For a critical imaginary root  $\lambda_\alpha$  of the characteristic equation  $f(\lambda, \tau) = 0$  where  $f(\lambda, \tau)$  is a general quasipolynomial,  $\Delta NU_{\lambda_\alpha}(\tau_{\alpha, k})$  is a constant  $\Delta NF_{z_\alpha}(\omega_\alpha)$  for all  $\tau_{\alpha, k} > 0$ .*

To show the effectiveness of the invariance property mentioned above, consider the following Distributed Delay Differential Equation (DDDE):

$$\dot{x}(t) = A_0 x(t) + A_1 \int_{-\infty}^t \kappa(t - \theta) x(\theta) d\theta, \quad (13)$$

where  $\kappa(\theta)$  is an appropriate kernel function. Assume that  $\kappa(\cdot)$  is a *uniform-distribution* described by:

$$\kappa(\theta) = \begin{cases} \frac{1}{d_1 + d_2}, & \text{if } \tau - d_1 < \theta < \tau + d_2, \\ 0, & \text{otherwise,} \end{cases} \quad (14)$$

where  $\tau \geq d_1 \geq 0$  and  $d_2 \geq 0$ . The characteristic function rewrites as:

$$f(\lambda, \tau) = \det(\lambda I - A_0 - A_1 \frac{e^{-(\tau-d_1)\lambda} - e^{-(\tau+d_2)\lambda}}{(d_1 + d_2)\lambda}), \quad \lambda \neq 0.$$

**Example 7.7.** Consider the DDDE (13) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\frac{\pi^4 + 3\pi^2 - 4}{\pi^2(\pi^2 + 1)} & \frac{2\pi}{\pi^2 + 1} \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ \frac{\pi^2 + 4}{\pi(\pi^2 + 1)} & \frac{-1}{\pi^2 + 1} \end{pmatrix}.$$

Let  $\kappa(\theta)$  be the uniform distribution (14) with  $d_1 = d_2 = \frac{\pi}{2}$ . Then the characteristic function is a general quasipolynomial  $f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + a_2(\lambda)e^{-2\tau\lambda}$ , with the coefficient functions  $a_0(\lambda) = \lambda^2 - \frac{4}{\pi^2} - \frac{2\pi\lambda - 6}{\pi^2 + 1} + 1$ ,  $a_1(\lambda) = \frac{(e^{-\frac{\pi\lambda}{2}} - e^{\frac{\pi\lambda}{2}})(\pi^3\lambda - \pi^2 + 4)}{\pi^2\lambda(\pi^2 + 1)}$ , and  $a_2(\lambda) = -\frac{(e^{-\frac{\pi\lambda}{2}} - e^{\frac{\pi\lambda}{2}})^2}{\pi^2\lambda^2(\pi^2 + 1)}$ . At  $\tau = (2k + 1)\pi$ ,  $\lambda = i$  is a critical imaginary root:  $\lambda = i$  is double at  $\tau = \pi$  while it is simple at all  $\tau = (2k + 1)\pi, k \in \mathbb{N}_+$ . According to Theorem 7.6, we have that  $\Delta NU_i((2k + 1)\pi) = \Delta NF_{-1}(1)$  for all  $k \in \mathbb{N}$ , where  $\Delta NF_{-1}(1) = 0$

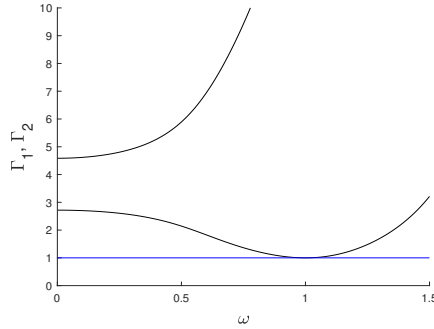


Figure 13: Example 7.7: frequency-sweeping curves  $\Gamma_1, \Gamma_2$  vs.  $\omega$ .

as observed from the frequency-sweeping curves shown in Fig. 13. Next, we verify the above result by invoking the Puiseux series for critical pairs  $(i, (2k + 1)\pi)$ ,  $k \in \llbracket 0, 2 \rrbracket$ :

$$\begin{aligned} \Delta\lambda &= (0.2290 + 0.2930i)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}}), \quad k = 0, \\ \Delta\lambda &= -0.1592i\Delta\tau + (-0.0283 + 0.0324i)(\Delta\tau)^2 + o((\Delta\tau)^2), \quad k = 1, \\ \Delta\lambda &= -0.0796i\Delta\tau + (-0.0035 + 0.0072i)(\Delta\tau)^2 + o((\Delta\tau)^2), \quad k = 0, \end{aligned}$$

and thus, we arrive at the same conclusion.  $\square$

### 7.3 Delay Differential Equations with Multiple Incommensurate Delays

Consider the following DDEs:

$$\dot{x}(t) = A_0x(t) + \sum_{\ell=1}^q A_\ell x(t - \tau_\ell), \quad (15)$$

where  $\tau_\ell \geq 0$  ( $\ell \in \llbracket 1, q \rrbracket$ ) are *independent* delays. The characteristic function for (15) is  $f(\lambda, \vec{\tau}) = \det(\lambda I - A_0 - \sum_{\ell=1}^q A_\ell e^{-\tau_\ell \lambda})$ , and we are interested to characterize the stability regions in the *delay-parameter space*.

A straightforward idea is to extend the mathematical results from the single delay case to multiple delays. However, such an extension is not trivial. As in the commensurate delays case, the core of the approach is the *invariance property*. To address the problem, we may proceed “indirectly” by fixing  $(q - 1)$  delays and leaving the remaining delay as a “free” parameter. Thus, for any combination  $\vec{\tau}^\sharp$ , we may accurately compute  $NU(\vec{\tau}^\sharp)$

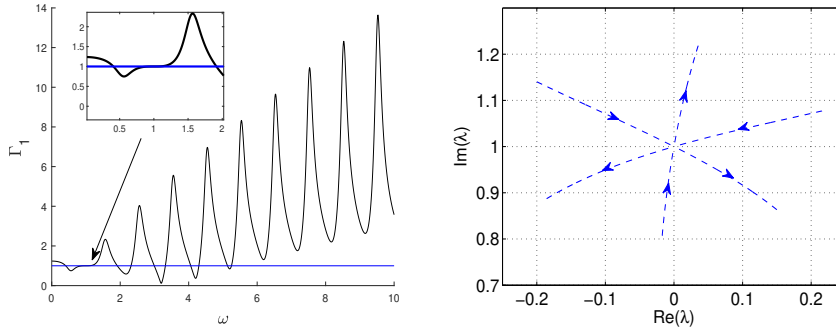


Figure 14: Example 7.8: frequency-sweeping curve  $\Gamma_1$  vs.  $\omega$  and  $\text{Re}(\lambda)$  vs.  $\text{Im}(\lambda)$ .

by using several times the frequency-sweeping tests in appropriate manner. Schematically speaking, suppose that  $\tau_k = \tau_{k,0}$  for all  $k \in \llbracket 1, q \rrbracket \setminus \{i\}$  are fixed and  $\tau_i$  is the “free” parameter for some  $i \in \llbracket 1, q \rrbracket$ . Then the corresponding characteristic function can be rewritten in the form  $f(\lambda, \tau_i)$  given by (3) where the coefficient functions  $a_h(\lambda)$  ( $h \in \llbracket 1, q \rrbracket$ ) can be seen as  $q$ -multivariate polynomials of  $\lambda$  and  $(q - 1)$  variables  $z_k = e^{-\tau_{k,0}\lambda}$ , with  $k \in \llbracket 1, q \rrbracket \setminus \{i\}$ , for which we can apply the frequency-sweeping approach.

**Example 7.8.** Consider a DDE including two delays  $\tau_1$  and  $\tau_2$ , with the characteristic function  $f(\lambda, \tau_1, \tau_2) = \frac{1}{\pi^2} + 2 + (1 - \frac{3}{\pi})\lambda + \lambda^2 + ((\frac{2}{\pi^2} + 3) + (3 - \frac{4}{\pi})\lambda)e^{-\tau_1\lambda} + ((\frac{1}{\pi^2} + 3) + (3 - \frac{1}{\pi})\lambda)e^{-\tau_2\lambda} + (1 + \lambda)e^{-(\tau_1 + \tau_2)\lambda}$ . Suppose  $\tau_2 = 2\pi$  is fixed and  $\tau_1$  is the free delay parameter denoted by  $\tau$ . Then  $f$  can be expressed by the general quasipolynomial  $f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$  with the coefficient functions  $a_0(\lambda) = \frac{1}{\pi^2} + 2 + (1 - \frac{3}{\pi})\lambda + \lambda^2 + ((\frac{1}{\pi^2} + 3) + (3 - \frac{1}{\pi})\lambda)e^{-2\pi\lambda}$  and  $a_1(\lambda) = ((\frac{2}{\pi^2} + 3) + (3 - \frac{4}{\pi})\lambda) + (1 + \lambda)e^{-2\pi\lambda}$ . At  $\tau = \pi$ ,  $\lambda = i$  is a triple critical root<sup>27</sup>. According to Theorem 7.6,  $\Delta NU_i((2k + 1)\pi) = \Delta NF_{-1}(1) = +1$  for all  $k \in \mathbb{N}$ , where  $\Delta NF_{-1}(1)$  is obtained from the frequency-sweeping curve (Fig. 14 (left)). To verify the

<sup>27</sup>It is worth mentioning that  $\lambda = i$  is simple at all  $\tau = (2k + 1)\pi$ ,  $k \in \mathbb{N}_+$ .

result, the Puiseux series for critical pairs  $(i, (2k+1)\pi)$ ,  $k \in \llbracket 0, 2 \rrbracket$  write as:

$$\begin{aligned}\Delta\lambda &= (0.3801 - 0.2846i)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}}), \quad k = 0, \\ \Delta\lambda &= -0.1592i\Delta\tau + 0.0253i(\Delta\tau)^2 + (0.0021 - 0.0096i)(\Delta\tau)^3 \\ &\quad + o((\Delta\tau)^3), \quad k = 1, \\ \Delta\lambda &= -0.0796i\Delta\tau + 0.0063i(\Delta\tau)^2 + (0.0001 - 0.0009i)(\Delta\tau)^3 \\ &\quad + o((\Delta\tau)^3), \quad k = 2.\end{aligned}$$

To further illustrate the asymptotic behavior of the triple root  $i$ , the root loci near the critical pair  $(i, \pi)$  is depicted in Fig. 14 (right).  $\square$

## 8 Applications

In this section, we present two case studies from Life Sciences: neural networks and Lotka-Volterra systems.

### 8.1 Neural Network Dynamical Systems

To illustrate our approach for the stability analysis of neural networks with delays, we present an example borrowed from Li et al. (2018b). Consider the Bidirectional Associative Memory (BAM) neural network given by:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + \sum_{i=1}^4 c_{i1} f_i(x_i(t - \tau_2)), \\ \dot{y}_{k-1}(t) = -\mu_k y_{k-1}(t) + c_{1k} f_k(x_1(t - \tau_1)), \quad k \in \llbracket 2, 4 \rrbracket. \end{cases} \quad (16)$$

where  $x_1(t)$  and  $y_1(t)$ ,  $y_2(t)$ ,  $y_3(t)$  denote respectively the state of the neurons in the  $I$ -layer and the  $K$ -layer. The functions  $f_i$ , called *activation functions*, are assumed to be  $\mathcal{C}^1$ -differentiable, such that  $f_i(0) = 0$ , for  $i \in \llbracket 1, 4 \rrbracket$ . The signal transmission delay from the  $I$ -layer to the  $K$ -layer is  $\tau_1$  while the delay from the  $K$ -layer to the  $I$ -layer is  $\tau_2$ . Next,  $c_{ki} \in \mathbb{R}$  ( $k, i \in \llbracket 1, 4 \rrbracket$ ) are the connection weights through the neurons in two layers and  $\mu_i \in \mathbb{R}_+$  ( $i \in \llbracket 1, 4 \rrbracket$ ) describe the stability of internal neuron process.

By letting  $u_1(t) = x_1(t - \tau_1)$ ,  $u_k(t) = y_{k-1}(t)$ ,  $k \in \llbracket 2, 4 \rrbracket$  and  $\tau = \tau_1 + \tau_2$ , the BAM neural network (16) rewrites as follows:

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + \sum_{i=1}^4 c_{i1} f_i(u_i(t - \tau)), \\ \dot{u}_k(t) = -\mu_k u_k(t) + c_{1k} f_k(u_1(t)), \quad k \in \llbracket 2, 4 \rrbracket. \end{cases} \quad (17)$$

The linearization of the model (17) at the origin writes as:

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + \sum_{i=1}^4 \alpha_{i1} u_i(t - \tau), \\ \dot{u}_k(t) = -\mu_k u_k(t) + \alpha_{1k} u_1(t), \quad k \in \llbracket 2, 4 \rrbracket, \end{cases}$$

where  $\alpha_{ik} = c_{ik} f'_k(0)$ ,  $i, k \in \llbracket 1, 4 \rrbracket$ , and with the characteristic function:

$$f(\lambda, \tau) = \det \begin{pmatrix} \lambda + \mu_1 & -\alpha_{21} e^{-\tau\lambda} & -\alpha_{31} e^{-\tau\lambda} & -\alpha_{41} e^{-\tau\lambda} \\ -\alpha_{12} & \lambda + \mu_2 & 0 & 0 \\ -\alpha_{13} & 0 & \lambda + \mu_3 & 0 \\ -\alpha_{14} & 0 & 0 & \lambda + \mu_4 \end{pmatrix}.$$

This characteristic function is a quasipolynomial  $a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$ , where

$$\begin{aligned} a_0(\lambda) &= \lambda^4 + (\mu_1 + \mu_2 + \mu_3 + \mu_4)\lambda^3 + (\mu_1\mu_2 + \mu_3\mu_4 + \mu_1\mu_3 + \mu_1\mu_4)\lambda^2 \\ &+ (\mu_2\mu_3 + \mu_2\mu_4)\lambda^2 + (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4)\lambda + \mu_1\mu_2\mu_3\mu_4, \\ a_1(\lambda) &= -(\alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{31} + \alpha_{14}\alpha_{41})\lambda^2 \\ &- (\alpha_{12}\alpha_{21}\mu_3 + \alpha_{12}\alpha_{21}\mu_4 + \alpha_{13}\alpha_{31}\mu_2 + \alpha_{13}\alpha_{31}\mu_4 + \alpha_{14}\alpha_{41}\mu_2 + \alpha_{14}\alpha_{41}\mu_3)\lambda \\ &- (\alpha_{12}\alpha_{21}\mu_3\mu_4 + \alpha_{13}\alpha_{31}\mu_2\mu_4 + \alpha_{14}\alpha_{41}\mu_2\mu_3). \end{aligned}$$

Under the assumption that the activation functions are hyperbolic tangent functions<sup>28</sup>, and with choice of the coefficients as:  $\mu_1 = 2.46$ ,  $\mu_2 = 4.5769$ ,  $\mu_3 = 0.8561$ ,  $\mu_4 = 0.9669$ ,  $\alpha_{12} = 4.6621$ ,  $\alpha_{13} = -0.3896$ ,  $\alpha_{14} = 2.3488$ ,  $\alpha_{21} = -4.1320$ ,  $\alpha_{31} = -2.8466$ ,  $\alpha_{41} = 0.7057$ , we study the local stability of the origin equilibrium. The frequency-sweeping curve is shown in Fig. 15 (left). We see that the frequency-sweeping curve intersects the line  $\Im_1$  at  $\omega_0 = 0.9059$  and  $\omega_1 = 1.7637$ . As  $\omega$  increases, the frequency-sweeping curve crosses the line  $\Im_1$  from above to below (from below to above) at  $\omega_0$  ( $\omega_1$ ). In this case, two critical imaginary roots are detected:  $\lambda_0 = 0.9059i$  (critical delays  $\tau_{0,k} = 3.3768 + 6.9355k$ ) and  $\lambda_1 = 1.7637i$  (critical delays  $\tau_{1,k} = 1.3947 + 3.5624k$ ). According to Theorem 5.1,  $\Delta NU_{\lambda_0}(\tau_{0,k}) = -1$  and  $\Delta NU_{\lambda_1}(\tau_{1,k}) = +1$  for all  $k \in \mathbb{N}$ . By applying Theorem 6.3, we have the expression of  $NU(\tau)$ , as plotted in Fig. 15 (right). Thus, the origin is locally asymptotically stable if  $\tau \in [0, 1.3947) \cup (3.3768, 4.9571)$ .

## 8.2 Lotka-Volterra Systems

To further illustrate the proposed approach, consider a Lotka-Volterra system with delays borrowed from Li et al. (2018a). More precisely, consider

<sup>28</sup>In this case,  $f_i(\cdot) = \tanh(\cdot)$ ,  $i \in \llbracket 1, 4 \rrbracket$  verifying  $f_i(0) = 0$  and  $f'_i(0) = 1$ ,  $i \in \llbracket 1, 4 \rrbracket$ .

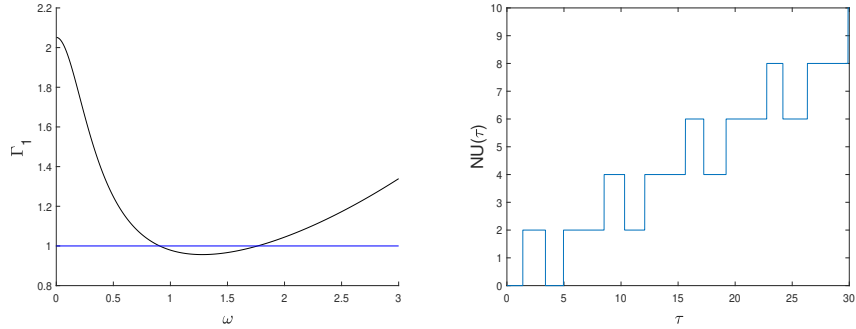


Figure 15: BAM example (frequency-sweeping curve  $\Gamma_1$  vs.  $\omega$  and  $NU(\tau)$  vs.  $\tau$  plot).

the three-species Lotka-Volterra system described by the DDEs

$$\begin{cases} \dot{x}_1(t) = x_1(t)(a_{10} + a_{11}x_1(t) + a_{12}x_2(t)), \\ \dot{x}_2(t) = x_2(t)(a_{20} + a_{21}x_1(t) + a_{23}x_3(t - \tau_{23})), \\ \dot{x}_3(t) = x_3(t)(a_{30} + a_{32}x_2(t - \tau_{32})), \end{cases} \quad (18)$$

This system (18) may model a three-species food chain dynamics, where  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  denote the population densities at time  $t$  of the lowest-level prey, the mid-level predator, and the top predator, respectively. It is assumed that the top predator species needs time  $\tau_{23}$  to possess the ability of predation and captures only the adult mid-level predator species with maturation time  $\tau_{32}$ . The unique positive equilibrium  $(x_1^*, x_2^*, x_3^*)$  is:

$$x_1^* = \frac{a_{10}a_{32} - a_{30}a_{12}}{-a_{11}a_{32}}, x_2^* = \frac{-a_{30}}{a_{32}}, x_3^* = \frac{a_{10}a_{32}a_{21} - a_{30}a_{12}a_{21} - a_{20}a_{11}a_{32}}{a_{11}a_{32}a_{23}}.$$

Let  $u_i(t) = x_i(t) - x_i^*$ ,  $i \in \llbracket 1, 3 \rrbracket$ . Then, we can rewrite (18) as

$$\begin{cases} \dot{u}_1(t) = (u_1(t) + x_1^*)(a_{11}u_1(t) + a_{12}u_2(t)), \\ \dot{u}_2(t) = (u_2(t) + x_2^*)(a_{21}u_1(t) + a_{23}u_3(t - \tau_{23})), \\ \dot{u}_3(t) = (u_3(t) + x_3^*)a_{32}u_2(t - \tau_{32}). \end{cases} \quad (19)$$

The linearized system of (19) at the origin  $(0, 0, 0)$  is

$$\begin{cases} \dot{u}_1(t) = a_{11}x_1^*u_1(t) + a_{12}x_1^*u_2(t), \\ \dot{u}_2(t) = a_{21}x_2^*u_1(t) + a_{23}x_2^*u_3(t - \tau_{23}), \\ \dot{u}_3(t) = a_{32}x_3^*u_2(t - \tau_{32}). \end{cases} \quad (20)$$

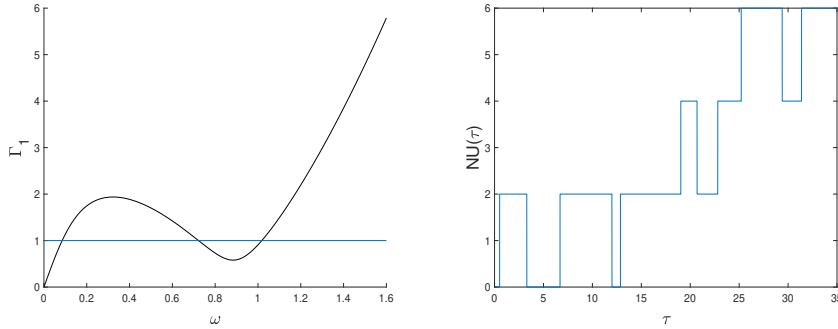


Figure 16: Lotka-Volterra DDE: Lotka-Volterra DDE (Frequency-sweeping curve  $\Gamma_1$  vs.  $\omega$  and  $NU(\tau)$  vs.  $\tau$  plot.

The characteristic function for the linear system (20) is

$$f(\lambda, \tau) = \lambda^3 - a_{11}x_1^*\lambda^2 - a_{12}a_{21}x_1^*x_2^*\lambda + (a_{11}a_{23}a_{32}x_1^*x_2^*x_3^* - a_{23}a_{32}x_2^*x_3^*\lambda)e^{-\tau\lambda},$$

where  $\tau = \tau_{23} + \tau_{32}$ . Let  $a_{10} = 0.7915$ ,  $a_{11} = -0.1358$ ,  $a_{12} = -0.9557$ ,  $a_{20} = -0.8337$ ,  $a_{21} = 0.9089$ ,  $a_{23} = -0.6429$ ,  $a_{30} = -0.5726$ ,  $a_{32} = 0.9322$ . The unique positive equilibrium is  $(x_1^* = 1.5056, x_2^* = 0.6142, x_3^* = 0.8318)$ . We analyze the local asymptotic stability. For the linearized system at the positive equilibrium, the characteristic function writes as:

$$f(\lambda, \tau) = \lambda^3 + 0.2045\lambda^2 + 0.8033\lambda + (0.3062\lambda + 0.0626)e^{-\tau\lambda}. \quad (21)$$

For  $\tau = 0$ , there are three characteristic roots (all in  $\mathbb{C}_-$ ):  $-0.0738 \pm 1.0468i$  and  $-0.0569$ . The frequency-sweeping curve is generated as shown in Fig. 16 (left). It is easy to observe that there are three critical imaginary roots  $0.0854i$  (with the critical delays  $22.8187 + 73.7657k$ ),  $0.7221i$  (with the critical delays  $3.3005 + 8.7014k$ ), and  $1.0179i$  (with the critical delays  $0.5218 + 6.1724k$ ),  $k \in \mathbb{N}$ . Next, simple computations lead to  $\Delta NU_{0.0854i}(22.8187 + 73.7657k) = +1$ ,  $\Delta NU_{0.7221i}(3.3005 + 8.7014k) = -1$ , and  $\Delta NU_{1.0179i}(0.5218 + 6.1724k) = +1$ , for all  $k \in \mathbb{N}$ . Finally, the “ $NU(\tau)$  vs.  $\tau$ ” plot is depicted in Fig. 16 (right). In conclusion, all the characteristic roots of (21) are located in  $\mathbb{C}_-$  if and only if  $\tau \in [0, 0.5218) \cup (3.3005, 6.6942) \cup (12.0019, 12.8666)$  and, thus, we have more than one stability delay interval guaranteeing the local asymptotic stability of the original system.



## 9 Notes and Comments

This chapter addressed the effects induced by the delay parameter on the (exponential) stability of linear dynamical systems represented by delay differential equations. To perform such an analysis, the authors proposed a user-friendly *frequency-sweeping framework*, and the stability problem was reformulated in terms of properties of some appropriate frequency-sweeping curves. Illustrative examples and two applications from Life Sciences show the effectiveness of the method.

The main results of this chapter devoted to single (or commensurate) linear delay systems have been reported in Li et al. (2015) (see also Li et al., 2014, 2017). For a deeper discussion of the properties of the *spectral abscissa function*, the reader is referred to Michiels and Niculescu (2014) and the references therein. Next, an extension of the frequency sweeping-approach applied to the incommensurate delay case was presented in Li et al. (2019b). Finally, a guided tour of existing methods to analyze multiple characteristic roots (including the frequency-sweeping approach) can be found in Niculescu et al. (2021).

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