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Sliding mode control based fixed-time stabilization and synchronization of inertial neural networks with time-varying delays

Chaouki Aouiti* · Qing Hui · Hediene Jallouli · Emmanuel Moulay

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Abstract In this article, we are interested in the Fixed-Time Stabilization (FTSt) and Fixed-Time Synchronization (FTSy) of a class of Inertial Neural Networks (INNs) with time-varying and distributed delays. To obtain FTSt and FTSy, sliding mode controllers are developed based on sliding mode control (SMC) techniques and by using sliding variables. Two polynomial feedback control laws are exploited to achieve the FTSt and the FTSy but they are singular. To get rid of the singularities, the saturation function is used into the design of the controllers and the almost FTSt and almost FTSy are proved. Finally, numerical examples are presented to show the effectiveness of the theoretical results.

Keywords Sliding mode control · inertial neural networks · fixed-time stability and synchronization · time-varying delay

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1 Introduction

Artificial Neural Networks (ANNs) are a system whose design is originally inspired by the operation of biological neurons and which has subsequently moved closer to statistical methods. In the last decade, the use of ANNs has been developed in many disciplines such as economic sciences, ecology and environment or biology and medicine [32, 44, 49, 69]. They are applied in particular to solve problems of classification, prediction, categorisation, optimization, pattern recognition, associative memory or secure communication [1, 3, 5, 26, 31, 14, 22, 20, 23, 45, 30]. In data processing, ANNs are a method of approximation of complex systems particularly useful when systems are difficult to model using conventional statistical methods. In recent years, numerous outcomes on Lyapunov stability, periodicity analysis and synchronization of various types of ANNs (Hopfield, Cohen-Grossberg, Competitive, BAM, MAM) have been obtained for instance in [2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 46].

Wheeler and Schieve were the first to publish the article [60] in 1997 about INNs which are more complex than lots of kinds of NNs. INNs have been widely developed and many important results about asymptotic stability, robust stability, synchronization, stabilization and existence of periodic solutions have been established [17, 21, 29, 39, 42, 60, 67, 72]. In recent studies, the FTSt of INNs has been intensively considered where the closed-loop system solutions reach the equilibrium point in finite-time. The function giving the time for the solutions to reach the equilibrium is called the settling-time. The finite-time stability has a great importance in real applications like robotics, optimization problems, pattern recognition, vehicle system or spacecrafts [25, 28, 33, 35, 36, 53]. It is a common knowledge that the settling-time estimation in finite-time stabilization problems depends on the initial values of the dynamical systems [47]. This results in disadvantages for real applications because in many practical systems it is difficult or impossible to obtain in advance the initial conditions [16]. The initial conditions of many practical systems are difficult to obtain accurately or even impossible to know in advance [16]. To settle this problem, Polyakov has developed the concept of fixed-time stability in [51].

It is well known that there are rare and inconsistent observations in measurements with the largest part of the population of observations, called outliers. The influence of external disturbances, modeling errors, and various uncertainties in the real systems, as well as the robust and filtering techniques to cope with these problems, are best shown in the following recent papers [18, 19, 55, 71, 74]. Stabilization control of NNs has attracted more and more attention, and there are different types of stabilization: asymptotic stabilization; exponential stabilization; finite-time stabilization; fixed-time stabilization [47, 9], as well as different types of control strategies: feedback control [1, 9]; feedback tracking control [56, 57]; adaptive control [41, 40]; hybrid control [52]; intermittent control [64]; pinning control [61]; impulsive control [70], or SMC [48]. As an application of fixed-time stability, FTSt of NNs has received tremendous interest in recent years since it can make system stabilized within a bounded time even if the initial conditions are unknown in advance

[9, 38, 33]. The fixed-time controller makes it stabilized at a fixed-time and the settling-time is independent on the initial conditions.

The authors of [38] discuss the finite-time and fixed-time stabilization for delayed memristive NNs based on Filippov solutions, and discontinuous controller. The study of FTSt for a general class of NNs (Cohen-Grossberg BAM) was investigated in [37] by using differential inequalities, some comparison techniques, and fixed-time stability theorem. In [10], the authors study the FTSt for a general class of BAM with time-varying delay using the finite time and fixed time stability theory, and linear matrix inequalities. In [64], the problem of finite-time synchronisation of NNs with mixed delays was investigated by using periodically intermittent memory feedback control, finite-time stability theory, linear matrix inequalities and differential inequality techniques. In [73], the problem of finite-time stabilization for synchronization was investigated based on geometric homogeneity and integral sliding mode manifold. The robust finite-time synchronization of a class of chaotic systems via adaptive global SMC was considered in [62] based on new chattering-free control laws. The problem of improved SMC for finite-time synchronization of nonidentical delayed recurrent NNs was considered in [63] by designing a suitable SMC based on Lyapunov stability theory. The integral SMC approach is presented in [54] to investigate the projective synchronization of nonidentical chaotic NNs with mixed time delays by considering a proper sliding surface, constructing Lyapunov-Krasovskii functional and using linear matrix inequality techniques. In [27], the synchronization problems of chaotic fuzzy cellular NNs with time-varying delays were investigated based on SMC approach. In [68], the authors study the projective synchronization for different chaotic time-delayed NNs by using Lyapunov-Krasovskii functional and linear matrix inequality techniques.

In this article, the problems of FTSt and FTSy are addressed for INNs under a new control strategy, namely SMC which is a nonlinear control technique that possesses remarkable properties of precision, robustness and easy implementation [59]. SMC has two main advantages: first the system dynamics behaviour can be adapted to the choice of the sliding surface, secondly the closed-loop system is robust to uncertainties [24, 58, 66]. Its concise design process and its ability to stabilize and synchronize the delayed INNs in a small fixed time are two advantages of the SMC.

The contributions of this article are described below:

- A novel control strategy for FTSt and FTSy of INNs with time-varying and distributed delays is introduced. We investigate the FTSt and FTSy of INNs with time-varying and distributed delays by using SMC with an adapted sliding surface. Moreover, by writing our system as a second-order multi-variable system, two different feedback control laws are built to obtain the FTSt and FTSy of INNs with time-varying and distributed delays. However, these controllers have singularities.
- By using some proprieties of a well defined sliding surface, new sufficient conditions for almost FTSt and almost FTSy of INNs with time-varying delay are obtained such that the controllers are not singular.

The paper is organized as follows. After some technical lemmas and definitions recalled in Section 2, the model description is given in Section 3. In Section 4, two sliding mode controllers are derived for the FTSt of INN with time-varying delay, while the FTSt of INN with time-varying delay is further considered in Section 5. In Section 6, illustrative examples are given to show the effectiveness of the proposed control design methods. The article ended with a conclusion in Section 7.

2 Preliminaries

Let \mathbb{R} denote the set of real numbers, \mathbb{R}_+ the set of positive real numbers, \mathbb{R}^n the n -dimensional Euclidean space, $\mathbb{R}^{n \times n}$ the set of squared matrices of size n , $\|\cdot\|$ the Euclidean norm such that $\|x\|^2 = x^T x$, A^T the transpose of the matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, $\text{diag}(a_1, \dots, a_n)$ the diagonal matrix, $|A|$ the matrix $(|a_{ij}|)_{1 \leq i, j \leq n}$ and I_n the identity matrix of size n .

Let us recall some results on finite-time stability and fixed-time stability. Consider the following ordinary differential equation

$$\begin{aligned} \dot{y}(t) &= f(y(t)), & y(t) &\in \mathbb{R}^n \\ y(0) &= y_0 \end{aligned} \quad (1)$$

with f a continuous function such that $f(0) = 0$.

Definition 1 [47] System (1) is globally finite-time stable if it is Lyapunov stable and for all $y_0 \in \mathbb{R}^n$ there exists $T(y_0) > 0$ dependent on the initial conditions such that $\lim_{t \rightarrow T(y_0)} \|y(t)\| = 0$, i.e. $\|y(t)\| \equiv 0$ for all $t \geq T(y_0)$. The function T is called the settling-time.

Definition 2 [51] System (1) is globally fixed-time stable if:

- (1) it is globally finite-time stable;
- (2) the settling-time function T is upper bounded by a constant $T^{\max} > 0$, i.e. $T(y_0) \leq T^{\max}$.

Definition 3 [75] System (1) is globally almost fixed-time stable if it is globally finite-time stable and there exists a constant $T^{\max} > 0$ such that for all $\epsilon > 0$ we have $T(y_0) \leq T^{\max} + T_\epsilon$ where $T_\epsilon \rightarrow 0$ when $\epsilon \rightarrow 0$.

Remark 1 According to Definition 1 and Definition 2, we conclude that contrary to finite-time stability, the settling-time of fixed-time stability is independent on initial conditions. In many fields, such as pattern recognition or secure communication, the initial values are mostly hard to be acquired [76]. So fixed-time stability is more suitable and applicable than finite-time stability.

Lemma 1 [50] *If there exists a continuously differentiable positive definite radially unbounded function $\mathbb{V} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ such*

$$\dot{\mathbb{V}}(y(t)) \leq -\hat{\alpha}\mathbb{V}(y(t))^{\hat{\alpha}} - \hat{\beta}\mathbb{V}(y(t))^{\hat{\beta}} \quad (2)$$

where $\hat{\alpha} > 0$, $\hat{\beta} > 0$ and $\varrho_1 > 1 > \varrho_2 > 0$ then system (1) is globally fixed-time stable and the settling-time $T(y_0)$ satisfies

$$T(y_0) \leq T_{\max} := \frac{1}{\hat{\alpha}(\varrho_1 - 1)} + \frac{1}{\hat{\beta}(1 - \varrho_2)} \quad (3)$$

The function \mathbb{V} is called a Lyapunov function for system (1).

Remark 2 Without loss of generality, if $\varrho_1 = 1 + \frac{1}{\mu}$ and $\varrho_2 = 1 - \frac{1}{\mu}$ are satisfied with $\mu > 1$, the upper bound of the settling-time function $T(y_0)$ of system (1) can be estimated by

$$T(y_0) \leq \frac{\mu\pi}{2\sqrt{\hat{\alpha}\hat{\beta}}}. \quad (4)$$

Lemma 2 [75] *The following system*

$$\dot{z}(t) = -\hat{\alpha}\|z(t)\|^{\varrho_1-1}z(t) - \hat{\beta}\|z(t)\|^{\varrho_2-1}z(t), \quad (5)$$

where $z \in \mathbb{R}^n$, $\hat{\alpha}, \hat{\beta} > 0$, $\varrho_1 > 1$ and $0 < \varrho_2 < 1$ is globally fixed-time stable and the settling-time is given by (3).

3 Model description

The following INN with time-varying and distributed delays is considered

$$\begin{aligned} \frac{d^2 r_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dr_\ell(t)}{dt} - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} g_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t - \tilde{\tau}_\ell(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(r_k(\sigma)) d\sigma + u_\ell(t), \quad t > 0 \end{aligned} \quad (6)$$

where

- $\ell = 1, \dots, n$ with n the number of neurons to consider;
- $r(\cdot) = (r_1(\cdot), \dots, r_n(\cdot))^T$ is the neuron state vector;
- $\frac{d^2 r_\ell(t)}{dt^2}$ denotes the inertial term of system (6);
- $\lambda_\ell > 0$, $a_\ell > 0$;
- \mathbf{k}_k stands for the delay kernel;
- $\tilde{\tau}(\cdot)$ stands for the time-varying transmission delay;
- $f_k(\cdot)$, $g_k(\cdot)$ denotes the activation functions with $f_k(0) = g_k(0) = 0$ for $k = 1, \dots, n$;
- $u_\ell(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ denotes the control input;
- the initial conditions of system (6) are given by

$$r_\ell(s) = \varphi_\ell(s), \quad \frac{dr_\ell(s)}{ds} = \xi_\ell(s); \quad s \in [-\infty, 0] \quad (7)$$

where $\ell = 1, \dots, n$, φ_ℓ and ξ_ℓ are bounded and continuous functions.

Letting $h_\ell(t) = \frac{dr_\ell(t)}{dt}$, then system (6) is written as

$$\begin{cases} \frac{dr_\ell(t)}{dt} = h_\ell(t) \\ \frac{dh_\ell(t)}{dt} = -\lambda_\ell h_\ell(t) - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t - \tilde{\tau}_\ell(t))) \\ \quad + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(r_k(\sigma)) d\sigma + u_\ell(t) \end{cases} \quad (8)$$

that is

$$\begin{cases} \dot{r}(t) = h(t) \\ \dot{h}(t) = -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ \quad + B \int_{-\infty}^t \mathbf{K}(t - \sigma) G(r(\sigma)) d\sigma + u(t) \end{cases} \quad (9)$$

where $r = (r_1, \dots, r_n)$, $h = (h_1, \dots, h_n)$, $A = \text{diag}(a_1, \dots, a_n)$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $C = (c_{\ell k})_{n \times n}$, $B = (b_{\ell k})_{n \times n}$, $D = (d_{\ell k})_{n \times n}$, $\mathbf{K}(\cdot) = \text{diag}(\mathbf{k}_1, \dots, \mathbf{k}_n)$, $F(\cdot) = (f_1(\cdot), \dots, f_n(\cdot))^T$, $G(\cdot) = (g_1(\cdot), \dots, g_n(\cdot))^T$ and $u(\cdot) = (u_1(\cdot), \dots, u_n(\cdot))^T$. So we can write system (6) as a multidimensional second-order system.

Definition 4 [43] System (9) is globally finite-time stabilizable if there exists a feedback control $u(r, h)$ such that the closed-loop system is finite-time stable, i.e. there exists a functional \hat{T} such that for all initial condition r_0 , the solution $r(t, r_0)$ of the closed-loop system (9) with controller $u(r, h)$ is well defined and the settling-time $\hat{T}(r_0)$ satisfy

$$\lim_{t \rightarrow \hat{T}(r_0)} r(t, r_0) = 0.$$

Moreover, system (9) is globally fixed-time stabilizable if:

- it is globally finite-time stabilizable;
- the settling-time functional $\hat{T}(r_0)$ is upper bounded by a constant $\hat{T}_{\max} \geq 0$.

Now, let us introduce some assumptions useful in the following.

- (A₁) All continuous functions f_k with $k = 1, \dots, n$ satisfy the Lipschitz condition

$$\|f_k(\nu) - f_k(\kappa)\| \leq M_k \|\nu - \kappa\|$$

where $\nu, \kappa \in \mathbb{R}^n$ and M_k is a positive constant.

We denote $M = \text{diag}(M_1, \dots, M_n)$.

- (A₂) The delay Kernel $\mathbf{k}_k(\cdot)$ with $k = 1, \dots, n$ are positive bump functions such that

$$\int_0^{+\infty} \mathbf{k}_k(\sigma) d\sigma = \mathbf{k}_k.$$

- (A₃) For all $k = 1, \dots, n$, there exists constants N_k such that

$$|g_k(\kappa)| \leq N_k, \quad \kappa \in \mathbb{R}.$$

We denote $N = \text{diag}(N_1, \dots, N_n)$.

4 Fixed time stabilization of time-delayed INNs

In this section, the FTSt of INNs with time-varying delay is investigated. To perceive the FTSt of system (9), we start with the following sliding variable:

$$\mathbf{s}(r(t), h(t)) = h(t) + \hat{\alpha}_2 r(t) \|r(t)\|^{p_2-1} + \hat{\beta}_2 r(t) \|r(t)\|^{q_2-1} \quad (10)$$

where $\hat{\alpha}_2, \hat{\beta}_2 > 0$, $p_2 > 1$ and $\frac{1}{2} < q_2 < 1$ and the sliding surface

$$\mathbf{S} = \{(r, h) \in \mathbb{R}^{2n} : \mathbf{s}(r, h) = 0\}.$$

Moreover, we propose the following feedback control design:

$$\begin{aligned} u(r(t), h(t)) = & -\hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & - \hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2-1} h(t) \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}, \end{aligned} \quad (11)$$

where $\hat{\alpha}_1, \hat{\beta}_1 > 0$, $p_1 > 1$ and $\frac{1}{2} < q_1 < 1$, $\Phi = (\phi_{\ell k})_{n \times n}$ with $\Phi \geq A + M|C|$, $\Psi = \text{diag}(\psi_1, \dots, \psi_n)$ with $\psi_\ell \geq A_\ell$ and $Q \geq N(|B|K + |D|)$. Let us remark that $r(t)r^T(t)$ belongs to $\mathbb{R}^{n \times n}$.

Theorem 1 *Under assumptions (A₁), (A₂) and (A₃), system (9) is global fixed-time stabilizable under the proposed controller (11) and the settling-time satisfies*

$$\hat{T}(r_0) \leq \sum_{\ell=1}^2 \frac{1}{\hat{\alpha}_\ell(p_\ell - 1)} + \frac{1}{\hat{\beta}_\ell(1 - q_\ell)}. \quad (12)$$

Proof By computing the derivative of the sliding variable $\mathbf{s}(t)$ along the trajectory of system (9), we obtain

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ & + B \int_{-\infty}^t \mathbf{K}(t - \sigma) G(r(\sigma)) d\sigma + u(t) \\ & + \hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & + \hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2-1} h(t). \end{aligned} \quad (13)$$

Applying the controller (11) into (13) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (14)$$

Now consider the Lyapunov function $V(\mathbf{s}(t)) = \frac{1}{2} \mathbf{s}(t)^T \mathbf{s}(t)$. We obtain

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) = & \mathbf{s}(t)^T \left[-\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \right. \\ & + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & \left. - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|} \right] \end{aligned}$$

So, it leads to

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \Lambda \|\mathbf{s}(t)\| \|h(t)\| + A \|\mathbf{s}(t)\| \|r(t)\| + M|C| \|\mathbf{s}(t)\| \|r(t)\| + N|D| \|\mathbf{s}(t)\| \\ & + N|B| \|\mathbf{K}\| \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} - \Phi \|r(t)\| \|\mathbf{s}(t)\| \\ & - \Psi \|h(t)\| \|\mathbf{s}(t)\| - Q \|\mathbf{s}(t)\|, \end{aligned}$$

that is

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & (A + M|C| - \Phi) \|\mathbf{s}(t)\| \|r(t)\| + (\Lambda - \Psi) \|\mathbf{s}(t)\| \|h(t)\| \\ & + (N(|D| + |B|\mathbf{K}) - Q) \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 (2V(\mathbf{s}(t)))^{\frac{p_1+1}{2}} - \hat{\beta}_1 (2V(\mathbf{s}(t)))^{\frac{q_1+1}{2}}. \end{aligned} \quad (15)$$

Letting $y(t) = \sqrt{2V(t)}$ then $\dot{y}(t) = \frac{\dot{V}(t)}{\sqrt{2V(t)}}$ and we get from (15)

$$\dot{y}(t) \leq -\hat{\alpha}_1 y^{p_1}(t) - \hat{\beta}_1 y^{q_1}(t). \quad (16)$$

Then, from Lemma 1 we conclude that $\mathbf{s}(t)$ converges to zero in fixed time and the settling-time is bounded by

$$\hat{T}_1 := \frac{1}{\hat{\alpha}_1(p_2 - 1)} + \frac{1}{\hat{\beta}_1(1 - q_1)}. \quad (17)$$

It implies that $\mathbf{s}(t) = 0$ for all $t \geq \hat{T}_1$. From (10), we obtain

$$\dot{r}(t) = -\hat{\alpha}_2 r(t) \|r(t)\|^{p_2-1} - \hat{\beta}_2 r(t) \|r(t)\|^{q_2-1}. \quad (18)$$

From Lemma 2, we conclude that system (18) achieved the fixed-time stability and the settling-time is upper bounded by

$$\hat{T}_2 := \frac{1}{\hat{\alpha}_2(p_2 - 1)} + \frac{1}{\hat{\beta}_2(1 - q_2)}. \quad (19)$$

As a conclusion, $r(t) = 0$ for all $t \geq \hat{T}_1 + \hat{T}_2$, which it implies that $h(t) = \dot{r}(t) = 0$ for all $t \geq \hat{T}_1 + \hat{T}_2$. ■

The algorithm of the proposed SMC is presented as follow:

Algorithm 1:

Step 1: Choose $\hat{\alpha}_2, \hat{\beta}_2, p_2$, and q_2 . Then, we construct the sliding variable.

Step 2: Since $f_k(\cdot)$, $\mathbf{k}_k(\cdot)$, and $g_k(\cdot)$ ($k = 1, \dots, n$) are known, we calculate M_k, \mathbf{k}_k , and N_k . Then, we choose $\psi_\ell, \phi_{\ell k}$, and $q_{\ell k}$ ($\ell = 1, \dots, n$).

Step 3: Select the design parameters $\hat{\alpha}_1, \hat{\beta}_1, p_1$, and q_1 . Subsequently, we design the sliding mode controller.

It is clear that controller (11) is undefined in the set $\{(r(t), h(t)) : r(t) = 0\}$ due to the fact that $\frac{1}{2} < q_2 < 1$. To get rid of the singularity of controller (11), we use the saturation function given by

$$\text{sat}_\epsilon(z(t)) = \text{sign}(z(t)) \min\{|z(t)|, \epsilon\} \quad (20)$$

with $\epsilon > 0$ a threshold parameter and $r \in \mathbb{R}$. It results a new controller written as follows

$$\begin{aligned} u(t) = & -\hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & -\hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2} \text{sat}_\epsilon \left(\frac{h(t)}{\|r(t)\|} \right) \\ & -\hat{\alpha}_1 \|s(t)\|^{p_1-1} s(t) - \hat{\beta}_1 \|s(t)\|^{q_1-1} s(t) - \Phi \frac{\|r(t)\|}{\|s(t)\|} s(t) \\ & -\Psi \frac{\|h(t)\|}{\|s(t)\|} s(t) - Q \frac{s(t)}{\|s(t)\|}. \end{aligned} \quad (21)$$

Let us now state a new theorem with the non singular controller (21) whose proof is inspired by [75].

Theorem 2 Under assumptions (A₁), (A₂) and (A₃), system (9) is globally almost fixed-time stable under the non singular controller (21) and the settling-time verifies

$$\hat{T}(r_0) \leq \sum_{k=1}^2 \frac{1}{\hat{\alpha}_k(p_k - 1)} + \frac{1}{\hat{\beta}_k(1 - q_k)} + \hat{T}_\epsilon \quad (22)$$

where \hat{T}_ϵ denote the small time-margin accounting for the saturation function.

Proof Consider the sliding variable (10) and substituting the feedback control (21) into the time derivative of the sliding variable given by (13) leads to

$$\begin{aligned} \dot{\mathbf{s}}(r(t), h(t)) &= -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ &\quad + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) \\ &\quad + \hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2} \left[\frac{h(t)}{\|r(t)\|} - \text{sat}_\epsilon \left(\frac{h(t)}{\|r(t)\|} \right) \right] \\ &\quad - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (23)$$

To show that the sliding surface is globally attractive we define the following subsets:

$$\mathbf{S}_k = \left\{ (r, h) \in \mathbb{R}^{2n} : \frac{|h_k|}{\|r\|} \geq \epsilon \right\}; \quad k = 1, \dots, n \quad (24)$$

and

$$\mathbb{Y} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \dots \cup \mathbf{S}_n, \quad \mathbb{X} = \mathbb{R}^{2n} \setminus \mathbb{Y}.$$

Case 1: the state of system (9) is in \mathbb{X} . Then we have $\text{sat}_\epsilon \left(\frac{h(t)}{\|r(t)\|} \right) = \frac{h(t)}{\|r(t)\|}$ and (23) rewrites as (14). Using the same technique as in Theorem 1, we obtain that $\mathbf{s}(t) = 0$ for all $t > \hat{T}_1$ where \hat{T}_1 is given by (17), or the system trajectory enters the set \mathbb{Y} in finite time bounded by \hat{T}_1 .

Case 2: the state of system (9) is in \mathbb{Y} . Then there exists $k \in \{1, \dots, n\}$ such that $\text{sat}_\epsilon \left(\frac{h_k(t)}{\|r(t)\|} \right) = \epsilon$ which implies that due to the small value of $r(\cdot)$, there exists at least one element of the control input u which is saturated. From (9), the solution of r_k can be calculated as follows

$$r_k(t) = r_k(0) + \int_0^t h_k(\sigma) d\sigma.$$

As on each set \mathbf{S}_k we have $h_k > 0$ or $h_k < 0$, it implies that r_k will be monotonically decreasing or increasing until $(r(t), h(t))$ leaves the set \mathbf{S}_k . Consequently, for all $\epsilon > 0$ there exists a finite-time T_{ϵ_k} such that the state system $(r(t), h(t))$ leaves \mathbf{S}_k and $\frac{|h_k(t)|}{\|r(t)\|} < \epsilon$ so that $\text{sat}_\epsilon \left(\frac{h_k(t)}{\|r(t)\|} \right) = \frac{h_k(t)}{\|r(t)\|}$. The same holds for all $\text{sat}_\epsilon \left(\frac{h_\ell(t)}{\|r(t)\|} \right) = \frac{h_\ell(t)}{\|r(t)\|}$ with $\ell = 1, \dots, n$. Thus, for a given $\epsilon > 0$, there exists a finite time $T_\epsilon \leq \sum_{k=1}^n T_{\epsilon_k}$ such that the state of system (9) reaches \mathbb{X} .

Case 3: the state of system (9) satisfies $\mathbf{s}(r(t), h(t)) = 0$. So similar to the proof of Theorem 1, system (18) achieves the fixed-time stability and the settling-time is upper bounded by (19).

As a conclusion, system (9) is globally fixed-time stable under the feedback control (21) and the settling-time is bounded by $\hat{T}_1 + \hat{T}_2 + T_\epsilon$. ■

Remark 3 If a sufficiently large threshold parameter $\epsilon > 0$ is chosen, then $\|r\|$ is allowed to be very small and thus the domain in which the control input is saturated is very limited. This implies that the time margin T_ϵ can be made very small.

Remark 4 Similar to Remark 2, if $p_k = 1 + \frac{1}{\mu_k}$ and $q_k = 1 + \frac{1}{\mu_k}$ with $\mu_k > 1$ for $k = 1, 2$, then less conservative estimates for the upper bound of the settling-time in Theorem 2 can be calculated as

$$\hat{T}(r_0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} + T_\epsilon. \quad (25)$$

5 Fixed-time synchronization of time-delayed INNs

In this section, the synchronization of time-delayed INNs between master and slave systems is investigated. Let us take the following systems as the master system:

$$\begin{aligned} \frac{d^2 x_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dx_\ell(t)}{dt} - a_\ell x_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(x_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(x_k(t - \tilde{\tau}(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(x_k(\sigma)) d\sigma + I_\ell \end{aligned} \quad (26)$$

and as the slave system:

$$\begin{aligned} \frac{d^2 y_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(y_k(t - \tilde{\tau}(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(y_k(\sigma)) d\sigma + I_\ell + \bar{\mathbf{u}}_\ell(t) \end{aligned} \quad (27)$$

where $\bar{\mathbf{u}}_\ell(\cdot)$ denotes the controller.

Now, define the error between the master system and the slave system as $e_\ell(t) = y_\ell(t) - x_\ell(t)$ for $\ell = 1, \dots, n$. Then the error system can be described as follow

$$\begin{aligned} \frac{d^2 e_\ell(t)}{dt^2} = & -\lambda_\ell \left(\frac{dy_\ell(t)}{dt} - \frac{dx_\ell(t)}{dt} \right) - a_\ell (y_\ell(t) - x_\ell(t)) \\ & + \sum_{k=1}^n c_{\ell k} (f_k(y_k(t)) - f_k(x_k(t))) \\ & + \sum_{k=1}^n d_{\ell k} (g_k(y_k(t - \tilde{\tau}(t))) - g_k(x_k(t - \tilde{\tau}(t)))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) (g_k(y_k(\sigma)) - g_k(x_k(\sigma))) d\sigma + \bar{\mathbf{u}}_\ell(t). \end{aligned} \quad (28)$$

That is

$$\begin{aligned} \frac{d^2 e_\ell(t)}{dt^2} = & -\lambda_\ell \frac{de_\ell(t)}{dt} - a_\ell e_\ell(t) + \sum_{k=1}^n c_{\ell k} \Delta f_k \\ & + \sum_{k=1}^n d_{\ell k} \Delta g_k + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t-\sigma) \Delta \bar{g}_k d\sigma + \bar{\mathbf{u}}_\ell(t). \end{aligned} \quad (29)$$

where $\Delta f_k = f_k(y_k(t)) - f_k(x_k(t))$, $\Delta g_k = g_k(y_k(t - \check{\tau}(t))) - g_k(x_k(t - \check{\tau}(t)))$, $\Delta \bar{g}_k = g_k(y_k(s)) - g_k(x_k(s))$. Letting $z_\ell(t) = \frac{de_\ell(t)}{dt}$, then system (29) is written as:

$$\begin{cases} \frac{de_\ell(t)}{dt} = z_\ell(t) \\ \frac{dz_\ell(t)}{dt} = -\lambda_\ell z_\ell(t) - a_\ell e_\ell(t) + \sum_{k=1}^n c_{\ell k} \Delta f_k + \sum_{k=1}^n d_{\ell k} \Delta g_k \\ \quad + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t-\sigma) \Delta \bar{g}_k d\sigma + \bar{\mathbf{u}}_\ell(t) \end{cases} \quad (30)$$

that is

$$\begin{cases} \frac{de(t)}{dt} = z(t) \\ \frac{dz(t)}{dt} = -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma) \Delta \bar{G} d\sigma + \bar{\mathbf{u}}(t) \end{cases} \quad (31)$$

where $e = (e_1, \dots, e_n)^T$, $z = (z_1, \dots, z_n)^T$, $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n)$, $\Delta F = (\Delta f_1, \dots, \Delta f_n)^T$, $\Delta G = (\Delta g_1, \dots, \Delta g_n)^T$, $\Delta \bar{G} = (\Delta \bar{g}_1, \dots, \Delta \bar{g}_n)^T$.

Definition 5 The master system (26) and the slave system (27) are said to be fixed-time synchronized if there exists a functional $\hat{\mathbf{T}}$ such that for all initial conditions e_0 of system (31), the solution $e(t, e_0)$ is well defined and the settling-time $\hat{\mathbf{T}}(e_0)$ satisfies

$$\lim_{t \rightarrow \hat{\mathbf{T}}(e_0)} e(t, e_0) = 0.$$

For the FTSy of the master system (26) and the slave system (27), we define the following sliding variable

$$\mathbf{s}(t) = z(t) + \hat{\alpha}_2 e(t) \|e(t)\|^{p_2-1} + \hat{\beta}_2 e(t) \|e(t)\|^{q_2-1} \quad (32)$$

where $\hat{\alpha}_2, \hat{\beta}_2 > 0$, $p_2 > 1$ and $\frac{1}{2} < q_2 < 1$. Then, the feedback control design is given by

$$\begin{aligned} \bar{\mathbf{u}}(t) = & -\hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & -\hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2-1} z(t) \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & -\bar{\Phi} \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\Psi} \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}, \end{aligned} \quad (33)$$

where $\hat{\alpha}_1, \hat{\beta}_1 > 0$, $p_1 > 1$, $\frac{1}{2} < q_1 < 1$, $\bar{\Phi} = (\phi_{\ell k})_{n \times n}$ with $\bar{\Phi} \geq A + M|C|$, $\bar{\Psi} = \text{diag}(\psi_1, \dots, \psi_n)$ with $\psi_\ell \geq \Lambda_\ell$ and $\bar{\mathbf{Q}} \geq 2N(|B|K + |D|)$.

Theorem 3 Under assumptions (\mathbf{A}_1) , (\mathbf{A}_2) and (\mathbf{A}_3) , the master system (26) is fixed-time synchronized with the slave system (27) under the proposed controller (33) and the upper bound of the settling-time is given by (12).

Proof By computing the derivative of the sliding variable \mathbf{s} along the trajectory of system (31), we obtain

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \\ & + \bar{\mathbf{u}} + \hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & + \hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2-1} z(t). \end{aligned} \quad (34)$$

Applying the controller (33) into (34) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & - \Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (35)$$

Now, considered the Lyapunov function $V(\mathbf{s}(t)) = \frac{1}{2} \mathbf{s}(t)^T \mathbf{s}(t)$ we obtain

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \mathbf{s}(t)^T \left[-\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \right. \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & \left. - \Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|} \right] \end{aligned}$$

that is

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \Lambda \|\mathbf{s}(t)\| \|z(t)\| + A \|\mathbf{s}(t)\| \|e(t)\| + M|C| \|\mathbf{s}(t)\| \|e(t)\| + 2N|D| \|\mathbf{s}(t)\| \\ & + 2N|B|\mathbf{K} \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} - \Phi \|e(t)\| \|\mathbf{s}(t)\| \\ & - \Psi \|z(t)\| \|\mathbf{s}(t)\| - \bar{\mathbf{Q}} \|\mathbf{s}(t)\|. \end{aligned}$$

It yields

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & (A + M|C| - \Phi) \|\mathbf{s}(t)\| \|e(t)\| + (\Lambda - \Psi) \|\mathbf{s}(t)\| \|z(t)\| \\ & + (2N(|D| + |B|\mathbf{K}) - \bar{\mathbf{Q}}) \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 (2V)^{\frac{p_1+1}{2}} - \hat{\beta}_1 (2V)^{\frac{q_1+1}{2}}. \end{aligned} \quad (36)$$

Letting $x(t) = \sqrt{2V(t)}$ and $\dot{x}(t) = \frac{\dot{V}(t)}{\sqrt{2V(t)}}$, then we get from (36)

$$\dot{x}(t) \leq -\hat{\alpha}_1 x^{p_1}(t) - \hat{\beta}_1 x^{q_1}(t). \quad (37)$$

From Lemma 1 we conclude that $\mathbf{s}(t)$ converges to zero and the settling-time is bounded by (17). It implies that $\mathbf{s}(t) = 0$ for all $t \geq T_1$. From (31), we obtain

$$\dot{e}(t) = -\hat{\alpha}_2 e(t) \|e(t)\|^{p_2-1} - \hat{\beta}_2 e(t) \|e(t)\|^{q_2-1}. \quad (38)$$

From Lemma 2, we conclude that system (38) converges to zero and the settling-time is upper bounded by (19).

As a conclusion, $e(t) = 0$ for all $t \geq \hat{T}_1 + \hat{T}_2$ which implies that $z(t) = \dot{e}(t) = 0$ for all $t \geq \hat{T}_1 + \hat{T}_2$. ■

To get rid of the singularity of the controller (33), we use the saturation function and the new controller is expressed as follows:

$$\begin{aligned} \bar{\mathbf{u}}(t) = & -\hat{\alpha}_2 \left[I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & -\hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2} \left[\frac{z(t)}{\|e(t)\|} - \text{sat}_\epsilon \left(\frac{z(t)}{\|e(t)\|} \right) \right] \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & -\Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (39)$$

Theorem 4 Under assumptions (\mathbf{A}_1) , (\mathbf{A}_2) and (\mathbf{A}_3) , the master system (26) is globally almost fixed-time synchronized with the slave system (27) under the non-singular controller (39), and the settling-time verifies

$$\hat{T}(e_0) \leq \sum_{k=1}^2 \frac{1}{\hat{\alpha}_k(p_k - 1)} + \frac{1}{\hat{\beta}_k(1 - q_k)} + \hat{T}_\epsilon \quad (40)$$

where \hat{T}_ϵ denotes the small time margin accounting for the saturation function.

Proof Consider the sliding variable (32) and substituting the controller (39) into (34) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - A e(t) + C \Delta F + D \Delta G + B \int_{-\infty}^t \mathbf{K}(t - \sigma) \Delta \bar{G} d\sigma \\ & + \hat{\beta}_2 \left[I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2} \left[\frac{z(t)}{\|e(t)\|} - \text{sat}_\epsilon \left(\frac{z(t)}{\|e(t)\|} \right) \right] \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & -\Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (41)$$

We define the following subsets:

$$\mathbf{S}_k = \left\{ (e, z) \in \mathbb{R}^{2n} : \frac{|z_k(t)|}{\|e(t)\|} \geq \epsilon \right\}; \quad k = 1, \dots, n. \quad (42)$$

Let

$$\mathbb{Y} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \dots \cup \mathbf{S}_n, \quad \mathbb{X} = \mathbb{R}^{2n} \setminus \mathbb{Y}.$$

Case 1: the state of system (31) is in \mathbb{X} . Then we have $\text{sat}_\epsilon \left(\frac{z(t)}{\|e(t)\|} \right) = \frac{z(t)}{\|e(t)\|}$ and (41) rewrites as (35). Using the same technique as in Theorem 3, we obtain that $\mathbf{s}(t) = 0$ for all $t > \hat{T}_1$ where \hat{T}_1 is given by (17), or the system trajectory enters the set \mathbb{Y} in finite time bounded by \hat{T}_1 .

Case 2: the state of system (31) is in \mathbb{Y} . So there exists $k \in \{1, \dots, n\}$ such that $\text{sat}_\epsilon \left(\frac{z_k(t)}{\|e(t)\|} \right) = \epsilon$ which implies that due to the small value of $e(\cdot)$, there exists at least one element of the vector input control u which is saturated. From (31), the solution of e_k can be calculated as follows

$$e_k(t) = e_k(0) + \int_0^t z_k(s) ds.$$

As on each set \mathbf{S}_k , we have $z_k > 0$ or $z_k < 0$, it implies that e_k is monotonically decreasing or increasing until $(e(t), z(t))$ leaves the set \mathbf{S}_k . Consequently, for all $\epsilon > 0$, there exists a finite time T_{ϵ_k} such that the state system $(e(t), z(t))$ leave \mathbf{S}_k and $\frac{|z_k(t)|}{\|e(t)\|} < \epsilon$ so $\text{sat}_\epsilon \left(\frac{z_k(t)}{\|e(t)\|} \right) = \frac{z_k(t)}{\|e(t)\|}$. The same holds for all $\text{sat}_\epsilon \left(\frac{z_\ell(t)}{\|e(t)\|} \right) = \frac{z_\ell(t)}{\|e(t)\|}$ with $\ell = 1, \dots, n$. Thus, for a given $\epsilon > 0$, there exists a finite time $T_\epsilon \leq \sum_{j=1}^n T_{\epsilon_j}$ such that the state system reaches \mathbb{X} .

Case 3: the state of system (31) satisfies $\mathbf{s}(e(t), z(t)) = 0$. So, similar to the proof of Theorem 3, system (31) converges to zero and the settling-time is upper bounded by (19).

As a conclusion, the master system (26) is globally almost fixed-time synchronized with the slave system (27) under the feedback control (39) and the settling-time is bounded by $\hat{T}_1 + \hat{T}_1 + T_\epsilon$. ■

Remark 5 The controller (11) and (33) are discontinuous ones because they use the sign function which could produce the undesired chattering, while in some cases continuity is necessary. Under these circumstances, we can use the *sat* function in the controllers (21) and (39) to decrease this effect.

Remark 6 It should be noted that the synchronization is the stabilization of the error between the transmission and the receiver systems. The synchronization phenomenon is often encountered in nature, which is considered as an adjustment of rhythms of oscillating objects due to their internal weak couplings [15]. It has a great importance in real application. For example, fireflies in flocks give off flashes of light with the same frequency on a summer night

or the synchronous motion of space is established between the moon's orbital motion and its rotational motion. In order to achieve self-synchronization in the asynchronous communication techniques between the transmitter and the receiver, the two sides must simultaneously send and receive information over the same frequency.

Remark 7 In this work, the FTSt and the FTSy are investigated for a class of INNs with time-varying and distributed delays. For the first time, new SMC schemes are proposed in our paper for the FTSt and the FTSy of master-slave INNs with delays where the settling-time is bounded independently from the initial conditions by constructing Lyapunov functionals and by using analytical techniques. The main limitation of SMC in practical applications is the chattering problem. The chattering is the natural price that the SMC pays in order to eliminate completely unknown matched external disturbances.

Remark 8 Compared with the results of [27,54,68], the improved SMC approach of this brief provide better results for the following reasons. Firstly, the integral sliding mode surface is novel and needs less information, which has the performance that once the sliding mode variable satisfies $s(t) \rightarrow 0$, the FTSy will be guaranteed independent of the initial conditions. Secondly, the presented control approach can be conveniently verified without solving any LMIs. Finally, the FTSy between drive-response systems (26) and (27) can be obtained via the presented SMC approach, however, it cannot be guaranteed in the articles cited above.

6 Numerical Examples

In this section, we present four examples to illustrate the effectiveness of the results presented in this article.

Example 1 Consider the following INN with time-varying delay

$$\frac{d^2 r_\ell(t)}{dt^2} = -\lambda_\ell \frac{dr_\ell(t)}{dt} - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t-\tau(t))) + u_\ell(t) \quad (43)$$

and the transformation $\frac{dr_\ell(t)}{dt} = h_\ell(t)$ leading to the system:

$$\begin{cases} \frac{dr_\ell(t)}{dt} = h_\ell(t) \\ \frac{dh_\ell(t)}{dt} = -\lambda_\ell h_\ell(t) - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) \\ \quad + \sum_{k=1}^n d_{\ell k} g_k(r_k(t-\tilde{\tau}(t))) + u_\ell(t) \end{cases} \quad (44)$$

with $n = 2$, $\ell = 1, 2$, $a_1 = a_2 = 3$, $\lambda_1 = \lambda_2 = 4$, $c_{11} = 0.3$, $c_{12} = 0.15$, $c_{21} = 0.25$, $c_{22} = -0.4$, $d_{11} = 0.6$, $d_{12} = 0.2$, $d_{21} = -0.3$, $d_{22} = 0.3$, $\tilde{\tau} = 1$, the activation functions $f_\ell(s) = 0.5[|s+1| - |s-1|]$ and $g_\ell(\cdot) = \tanh(\cdot)$. We have $|f_k(\nu) - f_k(\kappa)| \leq |\nu - \kappa|$ and $|g_k(\kappa)| \leq 1$, then Assumption **(A₁)** and **(A₂)** hold with $M_k = 1$ and $N_k = 1$.

Choosing $\hat{\alpha}_\ell = \hat{\beta}_\ell = 4$. For $\mu_1 = \mu_2 = 4$, we have $p_1 = p_2 = 1 + \frac{1}{\mu_\ell} = 1.25$ and $q_1 = q_2 = 1 - \frac{1}{\mu_1} = 0.75$. The other parameters of the feedback control (11) are designed as follows: $\psi_1 = \psi_2 = 6 \geq a_1 = 3$, $\psi_2 = 6 \geq a_2 = 3$, $\phi_{11} = 4 \geq a_1 + M_1|c_{11}| = 3.3$, $\phi_{12} = 3.5 \geq a_1 + M_1|c_{12}| = 3.15$, $\phi_{21} = 3.5 \geq a_2 + M_2|c_{21}| = 3.25$, $\phi_{22} = 4 \geq a_2 + M_2|c_{22}| = 3.4$, $\psi_1 = 7 \geq \lambda_1$, $\psi_2 = 7 \geq \lambda_2$, $q_{11} = 1 \geq N|d_{11}| = 0.6$, $q_{12} = 0.6 \geq N|d_{12}| = 0.2$, $q_{21} = 1 \geq N|d_{21}| = 0.3$, $q_{22} = 0.6 \geq N|d_{22}| = 0.3$ and the initial condition are set as $r_1(0) = -0.5$, $r_2(0) = 0.5$, $h_1(0) = 1$, $h_2(0) = -1$. From Theorem 1 we obtain that system (44) is globally fixed-time stabilized under controller (11). In addition, the settling-time satisfies the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} = \frac{\pi}{2} + \frac{\pi}{2} \leq 3.1416.$$

The state trajectories of system (44) under controller (11) are plotted in Figure 1.

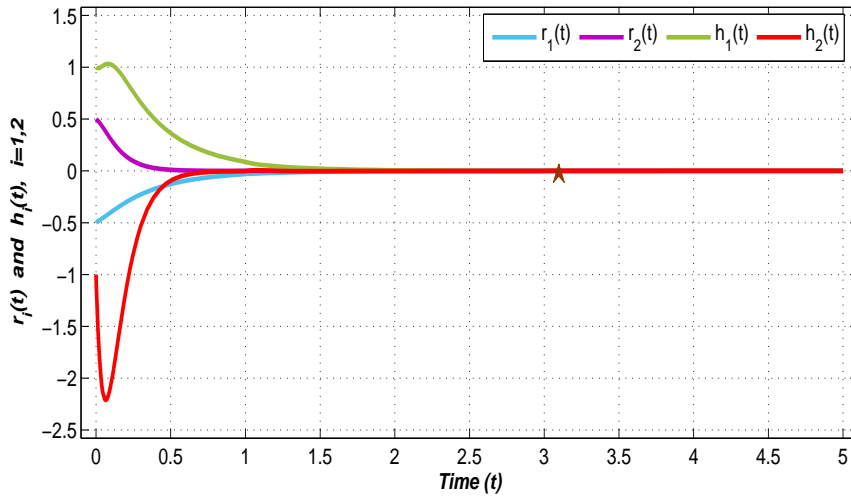


Fig. 1 Trajectories of system (44) with controller (11)

Example 2 Consider now the following two dimensional INN with mixed delays

$$\begin{aligned} \frac{d^2 r(t)}{dt^2} = & -\Lambda \frac{dr(t)}{dt} - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ & + B \int_{-\infty}^t \mathbf{K}(t-s)G(r(s))ds + u(t); \end{aligned} \quad (45)$$

and the transformation $\frac{dr(t)}{dt} = h(t)$ leading to the system

$$\begin{cases} \frac{dr(t)}{dt} = h(t) \\ \frac{dh(t)}{dt} = -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ \quad + B \int_{-\infty}^t \mathbf{K}(t-s)G(r(s))ds + u \end{cases} \quad (46)$$

where $k_1(x) = k_2(x) = e^{-x}$, $\tilde{\tau} = 2$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0.5 & 0.35 \\ -0.5 & -0.7 \end{pmatrix}$, $B = \begin{pmatrix} 0.9 & -0.2 \\ -0.4 & 0.3 \end{pmatrix}$, $D = \begin{pmatrix} 0.6 & 0.2 \\ -0.3 & 0.2 \end{pmatrix}$ and the activation functions $F(\cdot) = G(\cdot) = \tanh(\cdot)$. We have $|f_k(\nu) - f_k(\kappa)| \leq |\nu - \kappa|$ and $|g_k(\kappa)| \leq 1$, then Assumption (\mathbf{A}_1) and (\mathbf{A}_2) hold with $M_k = N_k = 1$. We have $\int_0^\infty e^{-s} ds = 1$, then assumption (\mathbf{A}_3) holds.

Taking $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\beta}_1 = \hat{\beta}_2 = 1$, $\mu_1 = 1.5$, $\mu_2 = 2.5$, $\epsilon = 0.1$, $p_1 = 1 + \frac{1}{1.5}$, $p_2 = 1 + \frac{1}{2.5}$, $q_1 = 1 - \frac{1}{1.5}$ and $q_2 = 1 - \frac{1}{2.5}$. The parameters of the feedback control (21) are designed as follow: $\phi_{11} = 3 \geq a_1 + M_1|c_{11}| = 2.5$, $\phi_{12} = 3.5 \geq a_1 + M_2|c_{12}| = 2.35$, $\phi_{21} = 3.5 \geq a_2 + M_1|c_{21}| = 2.5$, $\phi_{22} = 4 \geq a_2 + M_2|c_{22}| = 2.7$, $\psi_1 = 7 \geq \lambda_1 = 3$, $\psi_2 = 7 \geq \lambda_2 = 3$, $q_{11} = 1 \geq N_1|d_{11}| = 0.6$, $q_{12} = 0.6 \geq N_2|d_{12}| = 0.2$, $q_{21} = 1 \geq N_1|d_{21}| = 0.3$, $q_{22} = 0.6 \geq N_2|d_{22}| = 0.2$. The initial values are taken as $r_1(0) = 0.5$, $r_2(0) = -0.5$, $h_1(0) = -0.7$, $h_2(0) = 0.6$. Theorem 2 guarantees not only that system (46) is globally fixed-time stabilized under the controller (21) but also that the settling-time satisfies the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} + \hat{T}_\epsilon = \frac{1.5\pi}{2} + \frac{2.5\pi}{2} + 0.001 \leq 6.2842$$

with $\hat{T}_\epsilon = 0.001$. We see on Figure 2 that system (46) is globally almost fixed-time stable.

Example 3 We consider the following INN master system

$$\frac{d^2 x_\ell(t)}{dt^2} = -\lambda_\ell \frac{dx_\ell(t)}{dt} - a_\ell x_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(x_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(x_k(t - \tilde{\tau}(t))) \quad (47)$$

where $n = 2$, $\tilde{\tau} = 1$, $a_1 = a_2 = 12$, $\lambda_1 = \lambda_2 = 7$, $c_{11} = 0.4$, $c_{12} = 0.2$, $c_{21} = 0.5$, $c_{22} = -0.5$, $d_{11} = 0.7$, $d_{12} = 0.3$, $d_{21} = -0.4$, $d_{22} = 0.4$, the activation functions $f_\ell(s) = 0.5[|s+1| - |s-1|]$. It is clear that Assumption (\mathbf{A}_1) and (\mathbf{A}_2) hold and $M_k = N_k = 1$. The slave system is given by

$$\frac{d^2 y_\ell(t)}{dt^2} = -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} f_k(y_k(t - \tau(t))) + u_\ell(t) \quad (48)$$

where the parameters of system (48) are same as the system (47).

Choosing $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\beta}_1 = \hat{\beta}_2 = 2$, $\mu_1 = \mu_2 = 3$, $p_1 = 1 + \frac{1}{\mu_1}$, $p_2 = 1 + \frac{1}{\mu_2}$, $q_1 = 1 - \frac{1}{\mu_1}$ and $q_2 = 1 - \frac{1}{\mu_2}$. The parameters of the feedback control (33) are

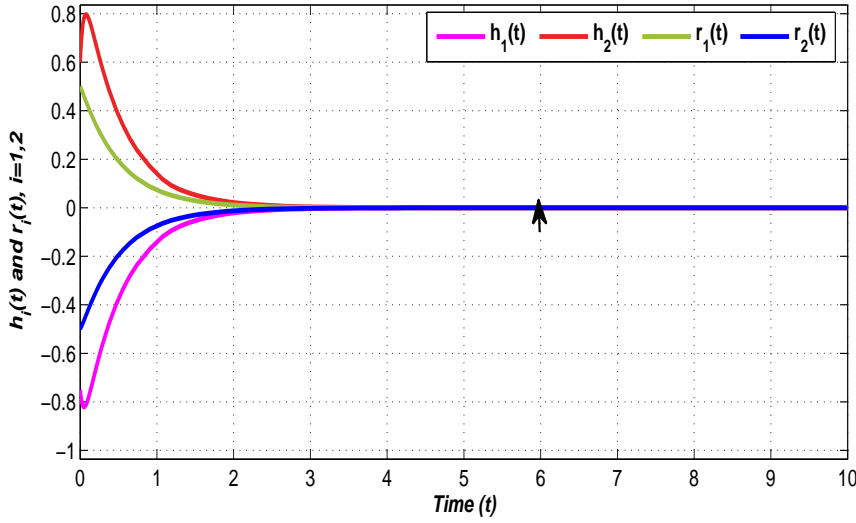


Fig. 2 Trajectories of system (46) with controller (21)

as follows: $\phi_{11} = 13 \geq a_1 + M_1|c_{11}| = 12.4$, $\phi_{12} = 13 \geq a_1 + M_1|c_{12}| = 12.2$, $\phi_{21} = 13 \geq a_2 + M_2|c_{21}| = 12.5$, $\phi_{22} = 13 \geq a_2 + M_2|c_{22}| = 12.5$, $\psi_1 = 8 \geq \lambda_1$, $\psi_2 = 8 \geq \lambda_2$, $\bar{q}_{11} = 3 \geq 2N_1|d_{11}| = 1.4$, $\bar{q}_{12} = 1.5 \geq 2N_1|d_{12}| = 0.6$, $\bar{q}_{21} = 2 \geq 2N_2|d_{21}| = 0.8$, $\bar{q}_{22} = 3 \geq 2N_2|d_{22}| = 0.8$. The initial values are taken as $x_1(0) = 1$, $x_2(0) = 0.5$, $y_1(0) = -0.5$, $y_2(0) = 0.5$. From Theorem 3, the master system (47) and the slave system (48) are globally fixed-time synchronized under controller (33). In addition, the settling-time satisfy the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_{\ell}\pi}{2\sqrt{\hat{\alpha}_{\ell}\hat{\beta}_{\ell}}} = \frac{3\pi}{2} \leq 4.7124.$$

After a short time, the master system (47) tracks the slave system (48) as illustrated in Figure 3 and Figure 4. The synchronization errors are depicted in Figure 5 and Figure 6.

Example 4 We consider the following INN master system:

$$\frac{d^2x(t)}{dt^2} = -A\frac{dx(t)}{dt} - Ax(t) + CG(x(t)) + DG(x(t-\tilde{\tau}(t))) + B \int_{-\infty}^t \mathbf{K}(t-s)G(x(s))ds \quad (49)$$

where $k_1(x) = k_2(x) = e^{-x}$, $\tilde{\tau} = 2$, $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, $C = \begin{pmatrix} 0.6 & 0.2 \\ -0.3 & 0.3 \end{pmatrix}$, $B = \begin{pmatrix} 0.1 & -0.2 \\ -0.5 & -0.4 \end{pmatrix}$, $D = \begin{pmatrix} 0.5 & 0.35 \\ -0.5 & -0.7 \end{pmatrix}$, the activation function $G(\cdot) =$

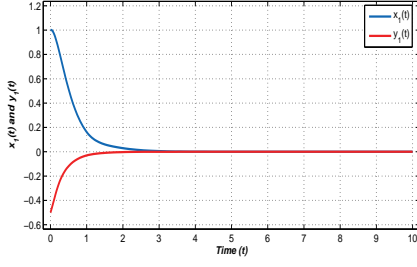


Fig. 3 Trajectories of $x_1(t)$ and $y_1(t)$

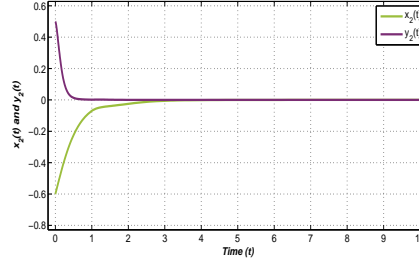


Fig. 4 Trajectories of $x_2(t)$ and $y_2(t)$

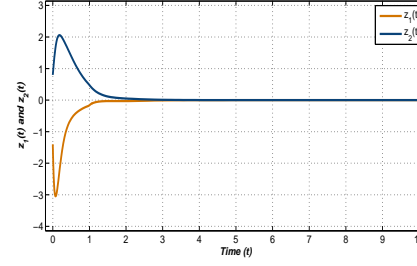


Fig. 5 Trajectories of $z_1(t)$ and $z_2(t)$

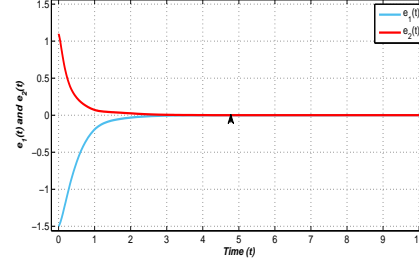


Fig. 6 Trajectories of $e_1(t)$ and $e_2(t)$

$\tanh(\cdot)$. The salve system is given as follows:

$$\begin{aligned} \frac{d^2 y_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} f_k(y_k(t - \tau(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(x_k(\sigma)) d\sigma + u_\ell(t) \end{aligned} \quad (50)$$

where the parameters of system (50) are same as the system(49).

Choosing $\mu_1 = 1.5$, $\mu_2 = 2.5$, $\hat{\alpha}_1 = \hat{\beta}_1 = \hat{\alpha}_2 = \hat{\beta}_2 = 1$, $\epsilon = 0.1$, $p_1 = 1 + \frac{1}{\mu_1} = 1.6667$, $p_2 = 1 + \frac{1}{\mu_2} = 1.4$, $q_1 = 1 - \frac{1}{\mu_1} = 0.3333$ and $q_2 = 1 - \frac{1}{\mu_2} = 0.6$. The parameters of the feedback control (39) are given by: $\phi_{11} = 9 \geq a_1 + M_1|c_{11}| = 5.6$, $\phi_{12} = 9 \geq a_1 + M_1|c_{12}| = 5.2$, $\phi_{21} = 9 \geq a_2 + M_2|c_{21}| = 5.3$, $\phi_{22} = 9 \geq a_2 + M_2|c_{22}| = 5.3$, $\psi_1 = 7 \geq \lambda_1$, $\psi_2 = 7 \geq \lambda_1$, $\bar{q}_{11} = 1.9 \geq 2N_1(|b_{11}| + |d_{11}|) = 1.2$, $\bar{q}_{12} = 1.2 \geq 2N_1(|b_{12}| + |d_{12}|) = 1.1$, $\bar{q}_{21} = 2 \geq 2N_2(|b_{21}| + |d_{21}|) = 2$, $\bar{q}_{22} = 3 \geq 2N_2(|b_{22}| + |d_{22}|) = 2.2$. The initial values are taken as $x_1(0) = -0.6$, $x_2(0) = 0.6$, $y_1(0) = 0.8$, $y_2(0) = -0.8$. From Theorem 4, the master system (49) and the slave system (50) are globally fixed-time synchronized under controller (39). After a short time, the master system (49) tracks the slave system (50) as illustrated in Figures 7 and 8. The synchronization errors are depicted in Figures 9 and 10. The settling-time

satisfy the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_{\ell}\pi}{2\sqrt{\hat{\alpha}_{\ell}\hat{\beta}_{\ell}}} + \hat{T}_{\epsilon} = \frac{1.5\pi}{2} + \frac{2.5\pi}{2} + 0.001 \leq 6.2833.$$

with $\hat{T}_{\epsilon} = 0.001$.

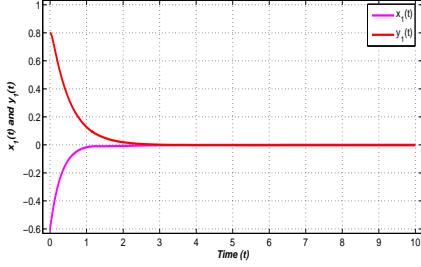


Fig. 7 Trajectories of $x_1(t)$ and $y_1(t)$

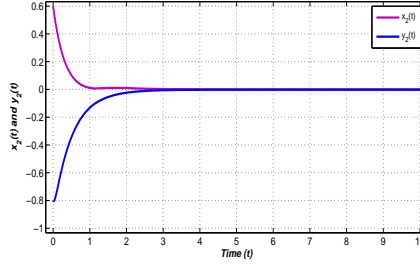


Fig. 8 Trajectories of $x_2(t)$ and $y_2(t)$

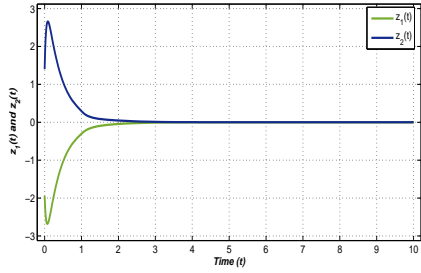


Fig. 9 Trajectories of $z_1(t)$ and $z_2(t)$

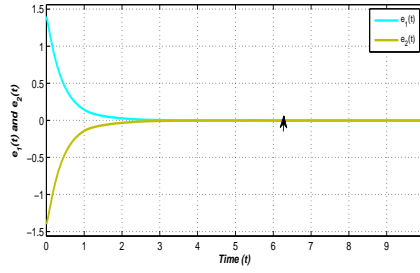


Fig. 10 Trajectories of $e_1(t)$ and $e_2(t)$

Remark 9 The study of the fixed-time stability of neural networks is of major interest for many applications such as spacecraft attitude dynamics [33] and secure communication [65]. Secure communication using synchronization between chaotic systems is a new concept of secure communication. In [1], the authors present the secure communication synchronization problem as real application of inertial neural networks.

7 Conclusion

In this article, the fixed-time stabilization and fixed-time synchronization of inertial neural networks with time-varying delay are inspected. By using sliding variables, two sliding mode controllers are obtained. Moreover, to get rid of the singularities two more sliding mode controllers are designed leading

to almost fixed-time stabilization and almost fixed-time synchronization. Our results are novel and complementary to the existing results about neural networks. There exist several ideas for future research. The first topic is related to the activation function. For processing of complex processes with delays the speed and performance of neural network in such case will depend on quality of activation function chosen. For this reason, a scaled polynomial constant unit activation function SPOCU [34] is going to be tested with respect to its performance. The second one focuses on the fixed-time stabilization and fixed-time synchronization under sliding mode control of:

1. a more general model of inertial neural networks such as inertial BAM neural networks, high-order inertial neural networks, quaternion-valued inertial Cohen-Grossberg neural networks, Clifford-valued inertial neural networks;
2. a more general model of fractional-order time-delayed inertial neural network.

Conflict of interest: The authors declare that they have no conflict of interest.

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