



**HAL**  
open science

# Sliding mode control based fixed-time stabilization and synchronization of inertial neural networks with time-varying delays

Chaouki Aouiti, Qing Hui, Hediene Jallouli, Emmanuel Moulay

► **To cite this version:**

Chaouki Aouiti, Qing Hui, Hediene Jallouli, Emmanuel Moulay. Sliding mode control based fixed-time stabilization and synchronization of inertial neural networks with time-varying delays. *Neural Computing and Applications*, 2021, 33 (18), pp.11555-11572. 10.1007/s00521-021-05833-x . hal-03633832

**HAL Id: hal-03633832**

**<https://hal.science/hal-03633832>**

Submitted on 7 Apr 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Sliding mode control based fixed-time stabilization and synchronization of inertial neural networks with time-varying delays

Chaouki Aouiti\* · Qing Hui · Hediene Jallouli · Emmanuel Moulay

Received: date / Accepted: date

**Abstract** In this article, we are interested in the Fixed-Time Stabilization (FTSt) and Fixed-Time Synchronization (FTSy) of a class of Inertial Neural Networks (INNs) with time-varying and distributed delays. To obtain FTSt and FTSy, sliding mode controllers are developed based on sliding mode control (SMC) techniques and by using sliding variables. Two polynomial feedback control laws are exploited to achieve the FTSt and the FTSy but they are singular. To get rid of the singularities, the saturation function is used into the design of the controllers and the almost FTSt and almost FTSy are proved. Finally, numerical examples are presented to show the effectiveness of the theoretical results.

**Keywords** Sliding mode control · inertial neural networks · fixed-time stability and synchronization · time-varying delay

---

Chaouki Aouiti

University of Carthage, Faculty of Sciences of Bizerta, Department of Mathematics, Research Units of Mathematics and Applications UR13ES47, BP W, 7021 Zarzouna, Bizerta, Tunisia

E-mail: chaouki.aouiti@fsb.rnu.tn

Qing Hui

Department of Electrical and Computer Engineering, University of Nebraska-Lincoln, Lincoln, NE 68588-0511, USA

E-mail: qing.hui@unl.edu

Hediene Jallouli

University of Carthage, Faculty of Sciences of Bizerta, Department of Mathematics, Research Units of Mathematics and Applications UR13ES47, BP W, 7021 Zarzouna, Bizerta, Tunisia

E-mail: jalouli.hedienne@fsb.rnu.tn

Emmanuel Moulay

XLIM (UMR CNRS 7252), Université de Poitiers, 11 bd Marie et Pierre Curie, 86073 Poitiers Cedex 9, France.

E-mail: emmanuel.moulay@univ-poitiers.fr

## 1 Introduction

Artificial Neural Networks (ANNs) are a system whose design is originally inspired by the operation of biological neurons and which has subsequently moved closer to statistical methods. In the last decade, the use of ANNs has been developed in many disciplines such as economic sciences, ecology and environment or biology and medicine [32, 44, 49, 69]. They are applied in particular to solve problems of classification, prediction, categorisation, optimization, pattern recognition, associative memory or secure communication [1, 3, 5, 26, 31, 14, 22, 20, 23, 45, 30]. In data processing, ANNs are a method of approximation of complex systems particularly useful when systems are difficult to model using conventional statistical methods. In recent years, numerous outcomes on Lyapunov stability, periodicity analysis and synchronization of various types of ANNs (Hopfield, Cohen-Grossberg, Competitive, BAM, MAM) have been obtained for instance in [2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 46].

Wheeler and Schieve were the first to publish the article [60] in 1997 about INNs which are more complex than lots of kinds of NNs. INNs have been widely developed and many important results about asymptotic stability, robust stability, synchronization, stabilization and existence of periodic solutions have been established [17, 21, 29, 39, 42, 60, 67, 72]. In recent studies, the FTSt of INNs has been intensively considered where the closed-loop system solutions reach the equilibrium point in finite-time. The function giving the time for the solutions to reach the equilibrium is called the settling-time. The finite-time stability has a great importance in real applications like robotics, optimization problems, pattern recognition, vehicle system or spacecrafts [25, 28, 33, 35, 36, 53]. It is a common knowledge that the settling-time estimation in finite-time stabilization problems depends on the initial values of the dynamical systems [47]. This results in disadvantages for real applications because in many practical systems it is difficult or impossible to obtain in advance the initial conditions [16]. The initial conditions of many practical systems are difficult to obtain accurately or even impossible to know in advance [16]. To settle this problem, Polyakov has developed the concept of fixed-time stability in [51].

It is well known that there are rare and inconsistent observations in measurements with the largest part of the population of observations, called outliers. The influence of external disturbances, modeling errors, and various uncertainties in the real systems, as well as the robust and filtering techniques to cope with these problems, are best shown in the following recent papers [18, 19, 55, 71, 74]. Stabilization control of NNs has attracted more and more attention, and there are different types of stabilization: asymptotic stabilization; exponential stabilization; finite-time stabilization; fixed-time stabilization [47, 9], as well as different types of control strategies: feedback control [1, 9]; feedback tracking control [56, 57]; adaptive control [41, 40]; hybrid control [52]; intermittent control [64]; pinning control [61]; impulsive control [70], or SMC [48]. As an application of fixed-time stability, FTSt of NNs has received tremendous interest in recent years since it can make system stabilized within a bounded time even if the initial conditions are unknown in advance

[9, 38, 33]. The fixed-time controller makes it stabilized at a fixed-time and the settling-time is independent on the initial conditions.

The authors of [38] discuss the finite-time and fixed-time stabilization for delayed memristive NNs based on Filippov solutions, and discontinuous controller. The study of FTSt for a general class of NNs (Cohen-Grossberg BAM) was investigated in [37] by using differential inequalities, some comparison techniques, and fixed-time stability theorem. In [10], the authors study the FTSt for a general class of BAM with time-varying delay using the finite time and fixed time stability theory, and linear matrix inequalities. In [64], the problem of finite-time synchronisation of NNs with mixed delays was investigated by using periodically intermittent memory feedback control, finite-time stability theory, linear matrix inequalities and differential inequality techniques. In [73], the problem of finite-time stabilization for synchronization was investigated based on geometric homogeneity and integral sliding mode manifold. The robust finite-time synchronization of a class of chaotic systems via adaptive global SMC was considered in [62] based on new chattering-free control laws. The problem of improved SMC for finite-time synchronization of nonidentical delayed recurrent NNs was considered in [63] by designing a suitable SMC based on Lyapunov stability theory. The integral SMC approach is presented in [54] to investigate the projective synchronization of nonidentical chaotic NNs with mixed time delays by considering a proper sliding surface, constructing Lyapunov-Krasovskii functional and using linear matrix inequality techniques. In [27], the synchronization problems of chaotic fuzzy cellular NNs with time-varying delays were investigated based on SMC approach. In [68], the authors study the projective synchronization for different chaotic time-delayed NNs by using Lyapunov-Krasovskii functional and linear matrix inequality techniques.

In this article, the problems of FTSt and FTSSy are addressed for INNs under a new control strategy, namely SMC which is a nonlinear control technique that possesses remarkable properties of precision, robustness and easy implementation [59]. SMC has two main advantages: first the system dynamics behaviour can be adapted to the choice of the sliding surface, secondly the closed-loop system is robust to uncertainties [24, 58, 66]. Its concise design process and its ability to stabilize and synchronize the delayed INNs in a small fixed time are two advantages of the SMC.

The contributions of this article are described below:

- A novel control strategy for FTSt and FTSSy of INNs with time-varying and distributed delays is introduced. We investigate the FTSt and FTSSy of INNs with time-varying and distributed delays by using SMC with an adapted sliding surface. Moreover, by writing our system as a second-order multi-variable system, two different feedback control laws are built to obtain the FTSt and FTSSy of INNs with time-varying and distributed delays. However, these controllers have singularities.
- By using some proprieties of a well defined sliding surface, new sufficient conditions for almost FTSt and almost FTSSy of INNs with time-varying delay are obtained such that the controllers are not singular.

The paper is organized as follows. After some technical lemmas and definitions recalled in Section 2, the model description is given in Section 3. In Section 4, two sliding mode controllers are derived for the FTSt of INN with time-varying delay, while the FTSt of INN with time-varying delay is further considered in Section 5. In Section 6, illustrative examples are given to show the effectiveness of the proposed control design methods. The article ended with a conclusion in Section 7.

## 2 Preliminaries

Let  $\mathbb{R}$  denote the set of real numbers,  $\mathbb{R}_+$  the set of positive real numbers,  $\mathbb{R}^n$  the  $n$ -dimensional Euclidean space,  $\mathbb{R}^{n \times n}$  the set of squared matrices of size  $n$ ,  $\|\cdot\|$  the Euclidean norm such that  $\|x\|^2 = x^T x$ ,  $A^T$  the transpose of the matrix  $A = (a_{ij})_{1 \leq i, j \leq n}$ ,  $\text{diag}(a_1, \dots, a_n)$  the diagonal matrix,  $|A|$  the matrix  $(|a_{ij}|)_{1 \leq i, j \leq n}$  and  $I_n$  the identity matrix of size  $n$ .

Let us recall some results on finite-time stability and fixed-time stability. Consider the following ordinary differential equation

$$\begin{aligned} \dot{y}(t) &= f(y(t)), & y(t) &\in \mathbb{R}^n \\ y(0) &= y_0 \end{aligned} \quad (1)$$

with  $f$  a continuous function such that  $f(0) = 0$ .

**Definition 1** [47] System (1) is globally finite-time stable if it is Lyapunov stable and for all  $y_0 \in \mathbb{R}^n$  there exists  $T(y_0) > 0$  dependent on the initial conditions such that  $\lim_{t \rightarrow T(y_0)} \|y(t)\| = 0$ , i.e.  $\|y(t)\| \equiv 0$  for all  $t \geq T(y_0)$ . The function  $T$  is called the settling-time.

**Definition 2** [51] System (1) is globally fixed-time stable if:

- (1) it is globally finite-time stable;
- (2) the settling-time function  $T$  is upper bounded by a constant  $T^{\max} > 0$ , i.e.  $T(y_0) \leq T^{\max}$ .

**Definition 3** [75] System (1) is globally almost fixed-time stable if it is globally finite-time stable and there exists a constant  $T^{\max} > 0$  such that for all  $\epsilon > 0$  we have  $T(y_0) \leq T^{\max} + T_\epsilon$  where  $T_\epsilon \rightarrow 0$  when  $\epsilon \rightarrow 0$ .

*Remark 1* According to Definition 1 and Definition 2, we conclude that contrary to finite-time stability, the settling-time of fixed-time stability is independent on initial conditions. In many fields, such as pattern recognition or secure communication, the initial values are mostly hard to be acquired [76]. So fixed-time stability is more suitable and applicable than finite-time stability.

**Lemma 1** [50] *If there exists a continuously differentiable positive definite radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  such*

$$\dot{V}(y(t)) \leq -\hat{\alpha}V(y(t))^{\hat{\alpha}} - \hat{\beta}V(y(t))^{\hat{\beta}} \quad (2)$$

where  $\hat{\alpha} > 0$ ,  $\hat{\beta} > 0$  and  $\varrho_1 > 1 > \varrho_2 > 0$  then system (1) is globally fixed-time stable and the settling-time  $T(y_0)$  satisfies

$$T(y_0) \leq T_{\max} := \frac{1}{\hat{\alpha}(\varrho_1 - 1)} + \frac{1}{\hat{\beta}(1 - \varrho_2)} \quad (3)$$

The function  $\mathbb{V}$  is called a Lyapunov function for system (1).

*Remark 2* Without loss of generality, if  $\varrho_1 = 1 + \frac{1}{\mu}$  and  $\varrho_2 = 1 - \frac{1}{\mu}$  are satisfied with  $\mu > 1$ , the upper bound of the settling-time function  $T(y_0)$  of system (1) can be estimated by

$$T(y_0) \leq \frac{\mu\pi}{2\sqrt{\hat{\alpha}\hat{\beta}}}. \quad (4)$$

**Lemma 2** [75] *The following system*

$$\dot{z}(t) = -\hat{\alpha}\|z(t)\|^{e_1-1}z(t) - \hat{\beta}\|z(t)\|^{e_2-1}z(t), \quad (5)$$

where  $z \in \mathbb{R}^n$ ,  $\hat{\alpha}, \hat{\beta} > 0$ ,  $\varrho_1 > 1$  and  $0 < \varrho_2 < 1$  is globally fixed-time stable and the settling-time is given by (3).

### 3 Model description

The following INN with time-varying and distributed delays is considered

$$\begin{aligned} \frac{d^2 r_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dr_\ell(t)}{dt} - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} g_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t - \tilde{\tau}_\ell(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(r_k(\sigma)) d\sigma + u_\ell(t), \quad t > 0 \end{aligned} \quad (6)$$

where

- $\ell = 1, \dots, n$  with  $n$  the number of neurons to consider;
- $r(\cdot) = (r_1(\cdot), \dots, r_n(\cdot))^T$  is the neuron state vector;
- $\frac{d^2 r_\ell(t)}{dt^2}$  denotes the inertial term of system (6);
- $\lambda_\ell > 0$ ,  $a_\ell > 0$ ;
- $\mathbf{k}_k$  stands for the delay kernel;
- $\tilde{\tau}(\cdot)$  stands for the time-varying transmission delay;
- $f_k(\cdot)$ ,  $g_k(\cdot)$  denotes the activation functions with  $f_k(0) = g_k(0) = 0$  for  $k = 1, \dots, n$ ;
- $u_\ell(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  denotes the control input;
- the initial conditions of system (6) are given by

$$r_\ell(s) = \varphi_\ell(s), \quad \frac{dr_\ell(s)}{ds} = \xi_\ell(s); \quad s \in [-\infty, 0] \quad (7)$$

where  $\ell = 1, \dots, n$ ,  $\varphi_\ell$  and  $\xi_\ell$  are bounded and continuous functions.

Letting  $h_\ell(t) = \frac{dr_\ell(t)}{dt}$ , then system (6) is written as

$$\begin{cases} \frac{dr_\ell(t)}{dt} = h_\ell(t) \\ \frac{dh_\ell(t)}{dt} = -\lambda_\ell h_\ell(t) - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t - \tilde{\tau}_\ell(t))) \\ \quad + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(r_k(\sigma)) d\sigma + u_\ell(t) \end{cases} \quad (8)$$

that is

$$\begin{cases} \dot{r}(t) = h(t) \\ \dot{h}(t) = -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ \quad + B \int_{-\infty}^t \mathbf{K}(t - \sigma) G(r(\sigma)) d\sigma + u(t) \end{cases} \quad (9)$$

where  $r = (r_1, \dots, r_n)$ ,  $h = (h_1, \dots, h_n)$ ,  $A = \text{diag}(a_1, \dots, a_n)$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ ,  $C = (c_{\ell k})_{n \times n}$ ,  $B = (b_{\ell k})_{n \times n}$ ,  $D = (d_{\ell k})_{n \times n}$ ,  $\mathbf{K}(\cdot) = \text{diag}(\mathbf{k}_1, \dots, \mathbf{k}_n)$ ,  $F(\cdot) = (f_1(\cdot), \dots, f_n(\cdot))^T$ ,  $G(\cdot) = (g_1(\cdot), \dots, g_n(\cdot))^T$  and  $u(\cdot) = (u_1(\cdot), \dots, u_n(\cdot))^T$ . So we can write system (6) as a multidimensional second-order system.

**Definition 4** [43] System (9) is globally finite-time stabilizable if there exists a feedback control  $u(r, h)$  such that the closed-loop system is finite-time stable, i.e. there exists a functional  $\hat{T}$  such that for all initial condition  $r_0$ , the solution  $r(t, r_0)$  of the closed-loop system (9) with controller  $u(r, h)$  is well defined and the settling-time  $\hat{T}(r_0)$  satisfy

$$\lim_{t \rightarrow \hat{T}(r_0)} r(t, r_0) = 0.$$

Moreover, system (9) is globally fixed-time stabilizable if:

- it is globally finite-time stabilizable;
- the settling-time functional  $\hat{T}(r_0)$  is upper bounded by a constant  $\hat{T}_{\max} \geq 0$ .

Now, let us introduce some assumptions useful in the following.

- (A<sub>1</sub>) All continuous functions  $f_k$  with  $k = 1, \dots, n$  satisfy the Lipschitz condition

$$\|f_k(\nu) - f_k(\kappa)\| \leq M_k \|\nu - \kappa\|$$

where  $\nu, \kappa \in \mathbb{R}^n$  and  $M_k$  is a positive constant.

We denote  $M = \text{diag}(M_1, \dots, M_n)$ .

- (A<sub>2</sub>) The delay Kernel  $\mathbf{k}_k(\cdot)$  with  $k = 1, \dots, n$  are positive bump functions such that

$$\int_0^{+\infty} \mathbf{k}_k(\sigma) d\sigma = \mathbf{k}_k.$$

- (A<sub>3</sub>) For all  $k = 1, \dots, n$ , there exists constants  $N_k$  such that

$$|g_k(\kappa)| \leq N_k, \quad \kappa \in \mathbb{R}.$$

We denote  $N = \text{diag}(N_1, \dots, N_n)$ .

#### 4 Fixed time stabilization of time-delayed INNs

In this section, the FTSt of INNs with time-varying delay is investigated. To perceive the FTSt of system (9), we start with the following sliding variable:

$$\mathbf{s}(r(t), h(t)) = h(t) + \hat{\alpha}_2 r(t) \|r(t)\|^{p_2-1} + \hat{\beta}_2 r(t) \|r(t)\|^{q_2-1} \quad (10)$$

where  $\hat{\alpha}_2, \hat{\beta}_2 > 0$ ,  $p_2 > 1$  and  $\frac{1}{2} < q_2 < 1$  and the sliding surface

$$\mathbf{S} = \{(r, h) \in \mathbb{R}^{2n} : \mathbf{s}(r, h) = 0\}.$$

Moreover, we propose the following feedback control design:

$$\begin{aligned} u(r(t), h(t)) = & -\hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & - \hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2-1} h(t) \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}, \end{aligned} \quad (11)$$

where  $\hat{\alpha}_1, \hat{\beta}_1 > 0$ ,  $p_1 > 1$  and  $\frac{1}{2} < q_1 < 1$ ,  $\Phi = (\phi_{\ell k})_{n \times n}$  with  $\Phi \geq A + M|C|$ ,  $\Psi = \text{diag}(\psi_1, \dots, \psi_n)$  with  $\psi_\ell \geq A_\ell$  and  $Q \geq N(|B|K + |D|)$ . Let us remark that  $r(t)r^T(t)$  belongs to  $\mathbb{R}^{n \times n}$ .

**Theorem 1** *Under assumptions (A<sub>1</sub>), (A<sub>2</sub>) and (A<sub>3</sub>), system (9) is global fixed-time stabilizable under the proposed controller (11) and the settling-time satisfies*

$$\hat{T}(r_0) \leq \sum_{\ell=1}^2 \frac{1}{\hat{\alpha}_\ell(p_\ell - 1)} + \frac{1}{\hat{\beta}_\ell(1 - q_\ell)}. \quad (12)$$

*Proof* By computing the derivative of the sliding variable  $\mathbf{s}(t)$  along the trajectory of system (9), we obtain

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ & + B \int_{-\infty}^t \mathbf{K}(t - \sigma) G(r(\sigma)) d\sigma + u(t) \\ & + \hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & + \hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2-1} h(t). \end{aligned} \quad (13)$$



Applying the controller (11) into (13) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (14)$$

Now consider the Lyapunov function  $V(\mathbf{s}(t)) = \frac{1}{2} \mathbf{s}(t)^T \mathbf{s}(t)$ . We obtain

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) = & \mathbf{s}(t)^T \left[ -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \right. \\ & + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & \left. - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|} \right] \end{aligned}$$

So, it leads to

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \Lambda \|\mathbf{s}(t)\| \|h(t)\| + A \|\mathbf{s}(t)\| \|r(t)\| + M|C| \|\mathbf{s}(t)\| \|r(t)\| + N|D| \|\mathbf{s}(t)\| \\ & + N|B| \|\mathbf{K}\| \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} - \Phi \|r(t)\| \|\mathbf{s}(t)\| \\ & - \Psi \|h(t)\| \|\mathbf{s}(t)\| - Q \|\mathbf{s}(t)\|, \end{aligned}$$

that is

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & (A + M|C| - \Phi) \|\mathbf{s}(t)\| \|r(t)\| + (\Lambda - \Psi) \|\mathbf{s}(t)\| \|h(t)\| \\ & + (N(|D| + |B|\mathbf{K}) - Q) \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 (2V(\mathbf{s}(t)))^{\frac{p_1+1}{2}} - \hat{\beta}_1 (2V(\mathbf{s}(t)))^{\frac{q_1+1}{2}}. \end{aligned} \quad (15)$$

Letting  $y(t) = \sqrt{2V(t)}$  then  $\dot{y}(t) = \frac{\dot{V}(t)}{\sqrt{2V(t)}}$  and we get from (15)

$$\dot{y}(t) \leq -\hat{\alpha}_1 y^{p_1}(t) - \hat{\beta}_1 y^{q_1}(t). \quad (16)$$

Then, from Lemma 1 we conclude that  $\mathbf{s}(t)$  converges to zero in fixed time and the settling-time is bounded by

$$\hat{T}_1 := \frac{1}{\hat{\alpha}_1(p_2 - 1)} + \frac{1}{\hat{\beta}_1(1 - q_1)}. \quad (17)$$

It implies that  $\mathbf{s}(t) = 0$  for all  $t \geq \hat{T}_1$ . From (10), we obtain

$$\dot{r}(t) = -\hat{\alpha}_2 r(t) \|r(t)\|^{p_2-1} - \hat{\beta}_2 r(t) \|r(t)\|^{q_2-1}. \quad (18)$$

From Lemma 2, we conclude that system (18) achieved the fixed-time stability and the settling-time is upper bounded by

$$\hat{T}_2 := \frac{1}{\hat{\alpha}_2(p_2 - 1)} + \frac{1}{\hat{\beta}_2(1 - q_2)}. \quad (19)$$

As a conclusion,  $r(t) = 0$  for all  $t \geq \hat{T}_1 + \hat{T}_2$ , which it implies that  $h(t) = \dot{r}(t) = 0$  for all  $t \geq \hat{T}_1 + \hat{T}_2$ . ■

The algorithm of the proposed SMC is presented as follow:

**Algorithm 1:**

**Step 1:** Choose  $\hat{\alpha}_2, \hat{\beta}_2, p_2$ , and  $q_2$ . Then, we construct the sliding variable.

**Step 2:** Since  $f_k(\cdot), \mathbf{k}_k(\cdot)$ , and  $g_k(\cdot)$  ( $k = 1, \dots, n$ ) are known, we calculate  $M_k, \mathbf{k}_k$ , and  $N_k$ . Then, we choose  $\psi_\ell, \phi_{\ell k}$ , and  $q_{\ell k}$  ( $\ell = 1, \dots, n$ ).

**Step 3:** Select the design parameters  $\hat{\alpha}_1, \hat{\beta}_1, p_1$ , and  $q_1$ . Subsequently, we design the sliding mode controller.

It is clear that controller (11) is undefined in the set  $\{(r(t), h(t)) : r(t) = 0\}$  due to the fact that  $\frac{1}{2} < q_2 < 1$ . To get rid of the singularity of controller (11), we use the saturation function given by

$$\text{sat}_\epsilon(z(t)) = \text{sign}(z(t)) \min\{|z(t)|, \epsilon\} \quad (20)$$

with  $\epsilon > 0$  a threshold parameter and  $r \in \mathbb{R}$ . It results a new controller written as follows

$$\begin{aligned} u(t) = & -\hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{p_2-1} h(t) \\ & -\hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2} \text{sat}_\epsilon \left( \frac{h(t)}{\|r(t)\|} \right) \\ & -\hat{\alpha}_1 \|s(t)\|^{p_1-1} s(t) - \hat{\beta}_1 \|s(t)\|^{q_1-1} s(t) - \Phi \frac{\|r(t)\|}{\|s(t)\|} s(t) \\ & -\Psi \frac{\|h(t)\|}{\|s(t)\|} s(t) - Q \frac{s(t)}{\|s(t)\|}. \end{aligned} \quad (21)$$

Let us now state a new theorem with the non singular controller (21) whose proof is inspired by [75].

**Theorem 2** Under assumptions  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  and  $(\mathbf{A}_3)$ , system (9) is globally almost fixed-time stable under the non singular controller (21) and the settling-time verifies

$$\hat{T}(r_0) \leq \sum_{k=1}^2 \frac{1}{\hat{\alpha}_k(p_k - 1)} + \frac{1}{\hat{\beta}_k(1 - q_k)} + \hat{T}_\epsilon \quad (22)$$

where  $\hat{T}_\epsilon$  denote the small time-margin accounting for the saturation function.

*Proof* Consider the sliding variable (10) and substituting the feedback control (21) into the time derivative of the sliding variable given by (13) leads to

$$\begin{aligned} \dot{\mathbf{s}}(r(t), h(t)) &= -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ &\quad + B \int_{-\infty}^t \mathbf{K}(t - \sigma)G(r(\sigma))d\sigma - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) \\ &\quad + \hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{r(t)r^T(t)}{\|r(t)\|} \right] \|r(t)\|^{q_2} \left[ \frac{h(t)}{\|r(t)\|} - \text{sat}_\epsilon \left( \frac{h(t)}{\|r(t)\|} \right) \right] \\ &\quad - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|r(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \Psi \frac{\|h(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (23)$$

To show that the sliding surface is globally attractive we define the following subsets:

$$\mathbf{S}_k = \left\{ (r, h) \in \mathbb{R}^{2n} : \frac{|h_k|}{\|r\|} \geq \epsilon \right\}; \quad k = 1, \dots, n \quad (24)$$

and

$$\mathbb{Y} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \dots \cup \mathbf{S}_n, \quad \mathbb{X} = \mathbb{R}^{2n} \setminus \mathbb{Y}.$$

*Case 1:* the state of system (9) is in  $\mathbb{X}$ . Then we have  $\text{sat}_\epsilon \left( \frac{h(t)}{\|r(t)\|} \right) = \frac{h(t)}{\|r(t)\|}$  and (23) rewrites as (14). Using the same technique as in Theorem 1, we obtain that  $\mathbf{s}(t) = 0$  for all  $t > \hat{T}_1$  where  $\hat{T}_1$  is given by (17), or the system trajectory enters the set  $\mathbb{Y}$  in finite time bounded by  $\hat{T}_1$ .

*Case 2:* the state of system (9) is in  $\mathbb{Y}$ . Then there exists  $k \in \{1, \dots, n\}$  such that  $\text{sat}_\epsilon \left( \frac{h_k(t)}{\|r(t)\|} \right) = \epsilon$  which implies that due to the small value of  $r(\cdot)$ , there exists at least one element of the control input  $u$  which is saturated. From (9), the solution of  $r_k$  can be calculated as follows

$$r_k(t) = r_k(0) + \int_0^t h_k(\sigma) d\sigma.$$

As on each set  $\mathbf{S}_k$  we have  $h_k > 0$  or  $h_k < 0$ , it implies that  $r_k$  will be monotonically decreasing or increasing until  $(r(t), h(t))$  leaves the set  $\mathbf{S}_k$ . Consequently, for all  $\epsilon > 0$  there exists a finite-time  $T_{\epsilon_k}$  such that the state system  $(r(t), h(t))$  leaves  $\mathbf{S}_k$  and  $\frac{|h_k(t)|}{\|r(t)\|} < \epsilon$  so that  $\text{sat}_\epsilon \left( \frac{h_k(t)}{\|r(t)\|} \right) = \frac{h_k(t)}{\|r(t)\|}$ . The same holds for all  $\text{sat}_\epsilon \left( \frac{h_\ell(t)}{\|r(t)\|} \right) = \frac{h_\ell(t)}{\|r(t)\|}$  with  $\ell = 1, \dots, n$ . Thus, for a given  $\epsilon > 0$ , there exists a finite time  $T_\epsilon \leq \sum_{k=1}^n T_{\epsilon_k}$  such that the state of system (9) reaches  $\mathbb{X}$ .

*Case 3:* the state of system (9) satisfies  $\mathbf{s}(r(t), h(t)) = 0$ . So similar to the proof of Theorem 1, system (18) achieves the fixed-time stability and the settling-time is upper bounded by (19).

As a conclusion, system (9) is globally fixed-time stable under the feedback control (21) and the settling-time is bounded by  $\hat{T}_1 + \hat{T}_2 + T_\epsilon$ . ■

*Remark 3* If a sufficiently large threshold parameter  $\epsilon > 0$  is chosen, then  $\|r\|$  is allowed to be very small and thus the domain in which the control input is saturated is very limited. This implies that the time margin  $T_\epsilon$  can be made very small.

*Remark 4* Similar to Remark 2, if  $p_k = 1 + \frac{1}{\mu_k}$  and  $q_k = 1 + \frac{1}{\mu_k}$  with  $\mu_k > 1$  for  $k = 1, 2$ , then less conservative estimates for the upper bound of the settling-time in Theorem 2 can be calculated as

$$\hat{T}(r_0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} + T_\epsilon. \quad (25)$$

## 5 Fixed-time synchronization of time-delayed INNs

In this section, the synchronization of time-delayed INNs between master and slave systems is investigated. Let us take the following systems as the master system:

$$\begin{aligned} \frac{d^2 x_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dx_\ell(t)}{dt} - a_\ell x_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(x_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(x_k(t - \tilde{\tau}(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(x_k(\sigma)) d\sigma + I_\ell \end{aligned} \quad (26)$$

and as the slave system:

$$\begin{aligned} \frac{d^2 y_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(y_k(t - \tilde{\tau}(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(y_k(\sigma)) d\sigma + I_\ell + \bar{\mathbf{u}}_\ell(t) \end{aligned} \quad (27)$$

where  $\bar{\mathbf{u}}_\ell(\cdot)$  denotes the controller.

Now, define the error between the master system and the slave system as  $e_\ell(t) = y_\ell(t) - x_\ell(t)$  for  $\ell = 1, \dots, n$ . Then the error system can be described as follow

$$\begin{aligned} \frac{d^2 e_\ell(t)}{dt^2} = & -\lambda_\ell \left( \frac{dy_\ell(t)}{dt} - \frac{dx_\ell(t)}{dt} \right) - a_\ell (y_\ell(t) - x_\ell(t)) \\ & + \sum_{k=1}^n c_{\ell k} (f_k(y_k(t)) - f_k(x_k(t))) \\ & + \sum_{k=1}^n d_{\ell k} (g_k(y_k(t - \tilde{\tau}(t))) - g_k(x_k(t - \tilde{\tau}(t)))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) (g_k(y_k(\sigma)) - g_k(x_k(\sigma))) d\sigma + \bar{\mathbf{u}}_\ell(t). \end{aligned} \quad (28)$$

That is

$$\begin{aligned} \frac{d^2 e_\ell(t)}{dt^2} = & -\lambda_\ell \frac{de_\ell(t)}{dt} - a_\ell e_\ell(t) + \sum_{k=1}^n c_{\ell k} \Delta f_k \\ & + \sum_{k=1}^n d_{\ell k} \Delta g_k + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t-\sigma) \Delta \bar{g}_k d\sigma + \bar{\mathbf{u}}_\ell(t). \end{aligned} \quad (29)$$

where  $\Delta f_k = f_k(y_k(t)) - f_k(x_k(t))$ ,  $\Delta g_k = g_k(y_k(t - \check{\tau}(t))) - g_k(x_k(t - \check{\tau}(t)))$ ,  $\Delta \bar{g}_k = g_k(y_k(s)) - g_k(x_k(s))$ . Letting  $z_\ell(t) = \frac{de_\ell(t)}{dt}$ , then system (29) is written as:

$$\begin{cases} \frac{de_\ell(t)}{dt} = z_\ell(t) \\ \frac{dz_\ell(t)}{dt} = -\lambda_\ell z_\ell(t) - a_\ell e_\ell(t) + \sum_{k=1}^n c_{\ell k} \Delta f_k + \sum_{k=1}^n d_{\ell k} \Delta g_k \\ \quad + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t-\sigma) \Delta \bar{g}_k d\sigma + \bar{\mathbf{u}}_\ell(t) \end{cases} \quad (30)$$

that is

$$\begin{cases} \frac{de(t)}{dt} = z(t) \\ \frac{dz(t)}{dt} = -\Lambda z(t) - A e(t) + C \Delta F + D \Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma) \Delta \bar{G} d\sigma + \bar{\mathbf{u}}(t) \end{cases} \quad (31)$$

where  $e = (e_1, \dots, e_n)^T$ ,  $z = (z_1, \dots, z_n)^T$ ,  $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n)$ ,  $\Delta F = (\Delta f_1, \dots, \Delta f_n)^T$ ,  $\Delta G = (\Delta g_1, \dots, \Delta g_n)^T$ ,  $\Delta \bar{G} = (\Delta \bar{g}_1, \dots, \Delta \bar{g}_1)^T$ .

**Definition 5** The master system (26) and the slave system (27) are said to be fixed-time synchronized if there exists a functional  $\hat{\mathbf{T}}$  such that for all initial conditions  $e_0$  of system (31), the solution  $e(t, e_0)$  is well defined and the settling-time  $\hat{\mathbf{T}}(e_0)$  satisfies

$$\lim_{t \rightarrow \hat{\mathbf{T}}(e_0)} e(t, e_0) = 0.$$

For the FTSy of the master system (26) and the slave system (27), we define the following sliding variable

$$\mathbf{s}(t) = z(t) + \hat{\alpha}_2 e(t) \|e(t)\|^{p_2-1} + \hat{\beta}_2 e(t) \|e(t)\|^{q_2-1} \quad (32)$$

where  $\hat{\alpha}_2, \hat{\beta}_2 > 0$ ,  $p_2 > 1$  and  $\frac{1}{2} < q_2 < 1$ . Then, the feedback control design is given by

$$\begin{aligned} \bar{\mathbf{u}}(t) = & -\hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & -\hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2-1} z(t) \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) \\ & -\bar{\Phi} \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\Psi} \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}, \end{aligned} \quad (33)$$

where  $\hat{\alpha}_1, \hat{\beta}_1 > 0$ ,  $p_1 > 1$ ,  $\frac{1}{2} < q_1 < 1$ ,  $\bar{\Phi} = (\phi_{\ell k})_{n \times n}$  with  $\bar{\Phi} \geq A + M|C|$ ,  $\bar{\Psi} = \text{diag}(\psi_1, \dots, \psi_n)$  with  $\psi_\ell \geq \Lambda_\ell$  and  $\bar{\mathbf{Q}} \geq 2N(|B|K + |D|)$ .

**Theorem 3** Under assumptions  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  and  $(\mathbf{A}_3)$ , the master system (26) is fixed-time synchronized with the slave system (27) under the proposed controller (33) and the upper bound of the settling-time is given by (12).

*Proof* By computing the derivative of the sliding variable  $\mathbf{s}$  along the trajectory of system (31), we obtain

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \\ & + \bar{\mathbf{u}} + \hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & + \hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2-1} z(t). \end{aligned} \quad (34)$$

Applying the controller (33) into (34) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & - \Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - Q \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (35)$$

Now, considered the Lyapunov function  $V(\mathbf{s}(t)) = \frac{1}{2} \mathbf{s}(t)^T \mathbf{s}(t)$  we obtain

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \mathbf{s}(t)^T \left[ -\Lambda z(t) - Ae(t) + C\Delta F + D\Delta G + B \int_{-\infty}^t \mathbf{K}(t-\sigma)\Delta\bar{G}d\sigma \right. \\ & - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & \left. - \Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{Q} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|} \right] \end{aligned}$$

that is

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & \Lambda \|\mathbf{s}(t)\| \|z(t)\| + A \|\mathbf{s}(t)\| \|e(t)\| + M|C| \|\mathbf{s}(t)\| \|e(t)\| + 2N|D| \|\mathbf{s}(t)\| \\ & + 2N|B|\mathbf{K} \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} - \Phi \|e(t)\| \|\mathbf{s}(t)\| \\ & - \Psi \|z(t)\| \|\mathbf{s}(t)\| - \bar{Q} \|\mathbf{s}(t)\|. \end{aligned}$$

It yields

$$\begin{aligned} \dot{V}(\mathbf{s}(t)) \leq & (A + M|C| - \Phi) \|\mathbf{s}(t)\| \|e(t)\| + (\Lambda - \Psi) \|\mathbf{s}(t)\| \|z(t)\| \\ & + (2N(|D| + |B|\mathbf{K}) - \bar{Q}) \|\mathbf{s}(t)\| - \hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1+1} - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1+1} \\ \leq & -\hat{\alpha}_1 (2V)^{\frac{p_1+1}{2}} - \hat{\beta}_1 (2V)^{\frac{q_1+1}{2}}. \end{aligned} \quad (36)$$

Letting  $x(t) = \sqrt{2V(t)}$  and  $\dot{x}(t) = \frac{\dot{V}(t)}{\sqrt{2V(t)}}$ , then we get from (36)

$$\dot{x}(t) \leq -\hat{\alpha}_1 x^{p_1}(t) - \hat{\beta}_1 x^{q_1}(t). \quad (37)$$

From Lemma 1 we conclude that  $\mathbf{s}(t)$  converges to zero and the settling-time is bounded by (17). It implies that  $\mathbf{s}(t) = 0$  for all  $t \geq T_1$ . From (31), we obtain

$$\dot{e}(t) = -\hat{\alpha}_2 e(t) \|e(t)\|^{p_2-1} - \hat{\beta}_2 e(t) \|e(t)\|^{q_2-1}. \quad (38)$$

From Lemma 2, we conclude that system (38) converges to zero and the settling-time is upper bounded by (19).

As a conclusion,  $e(t) = 0$  for all  $t \geq \hat{T}_1 + \hat{T}_2$  which implies that  $z(t) = \dot{e}(t) = 0$  for all  $t \geq \hat{T}_1 + \hat{T}_2$ . ■

To get rid of the singularity of the controller (33), we use the saturation function and the new controller is expressed as follows:

$$\begin{aligned} \bar{\mathbf{u}}(t) = & -\hat{\alpha}_2 \left[ I_n + (p_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{p_2-1} z(t) \\ & -\hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2} \left[ \frac{z(t)}{\|e(t)\|} - \text{sat}_\epsilon \left( \frac{z(t)}{\|e(t)\|} \right) \right] \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & -\Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (39)$$

**Theorem 4** Under assumptions  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  and  $(\mathbf{A}_3)$ , the master system (26) is globally almost fixed-time synchronized with the slave system (27) under the non-singular controller (39), and the settling-time verifies

$$\hat{T}(e_0) \leq \sum_{k=1}^2 \frac{1}{\hat{\alpha}_k(p_k - 1)} + \frac{1}{\hat{\beta}_k(1 - q_k)} + \hat{T}_\epsilon \quad (40)$$

where  $\hat{T}_\epsilon$  denotes the small time margin accounting for the saturation function.

*Proof* Consider the sliding variable (32) and substituting the controller (39) into (34) leads to

$$\begin{aligned} \dot{\mathbf{s}}(t) = & -\Lambda z(t) - A e(t) + C \Delta F + D \Delta G + B \int_{-\infty}^t \mathbf{K}(t - \sigma) \Delta \bar{G} d\sigma \\ & + \hat{\beta}_2 \left[ I_n + (q_2 - 1) \frac{e(t)e^T(t)}{\|e(t)\|} \right] \|e(t)\|^{q_2} \left[ \frac{z(t)}{\|e(t)\|} - \text{sat}_\epsilon \left( \frac{z(t)}{\|e(t)\|} \right) \right] \\ & -\hat{\alpha}_1 \|\mathbf{s}(t)\|^{p_1-1} \mathbf{s}(t) - \hat{\beta}_1 \|\mathbf{s}(t)\|^{q_1-1} \mathbf{s}(t) - \Phi \frac{\|e(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) \\ & -\Psi \frac{\|z(t)\|}{\|\mathbf{s}(t)\|} \mathbf{s}(t) - \bar{\mathbf{Q}} \frac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}. \end{aligned} \quad (41)$$

We define the following subsets:

$$\mathbf{S}_k = \left\{ (e, z) \in \mathbb{R}^{2n} : \frac{|z_k(t)|}{\|e(t)\|} \geq \epsilon \right\}; \quad k = 1, \dots, n. \quad (42)$$

Let

$$\mathbb{Y} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \dots \cup \mathbf{S}_n, \quad \mathbb{X} = \mathbb{R}^{2n} \setminus \mathbb{Y}.$$

*Case 1:* the state of system (31) is in  $\mathbb{X}$ . Then we have  $\text{sat}_\epsilon \left( \frac{z(t)}{\|e(t)\|} \right) = \frac{z(t)}{\|e(t)\|}$  and (41) rewrites as (35). Using the same technique as in Theorem 3, we obtain that  $\mathbf{s}(t) = 0$  for all  $t > \hat{T}_1$  where  $\hat{T}_1$  is given by (17), or the system trajectory enters the set  $\mathbb{Y}$  in finite time bounded by  $\hat{T}_1$ .

*Case 2:* the state of system (31) is in  $\mathbb{Y}$ . So there exists  $k \in \{1, \dots, n\}$  such that  $\text{sat}_\epsilon \left( \frac{z_k(t)}{\|e(t)\|} \right) = \epsilon$  which implies that due to the small value of  $e(\cdot)$ , there exists at least one element of the vector input control  $u$  which is saturated. From (31), the solution of  $e_k$  can be calculated as follows

$$e_k(t) = e_k(0) + \int_0^t z_k(s) ds.$$

As on each set  $\mathbf{S}_k$ , we have  $z_k > 0$  or  $z_k < 0$ , it implies that  $e_k$  is monotonically decreasing or increasing until  $(e(t), z(t))$  leaves the set  $\mathbf{S}_k$ . Consequently, for all  $\epsilon > 0$ , there exists a finite time  $T_{\epsilon_k}$  such that the state system  $(e(t), z(t))$  leave  $\mathbf{S}_k$  and  $\frac{|z_k(t)|}{\|e(t)\|} < \epsilon$  so  $\text{sat}_\epsilon \left( \frac{z_k(t)}{\|e(t)\|} \right) = \frac{z_k(t)}{\|e(t)\|}$ . The same holds for all  $\text{sat}_\epsilon \left( \frac{z_\ell(t)}{\|e(t)\|} \right) = \frac{z_\ell(t)}{\|e(t)\|}$  with  $\ell = 1, \dots, n$ . Thus, for a given  $\epsilon > 0$ , there exists a finite time  $T_\epsilon \leq \sum_{j=1}^n T_{\epsilon_j}$  such that the state system reaches  $\mathbb{X}$ .

*Case 3:* the state of system (31) satisfies  $\mathbf{s}(e(t), z(t)) = 0$ . So, similar to the proof of Theorem 3, system (31) converges to zero and the settling-time is upper bounded by (19).

As a conclusion, the master system (26) is globally almost fixed-time synchronized with the slave system (27) under the feedback control (39) and the settling-time is bounded by  $\hat{T}_1 + \hat{T}_1 + T_\epsilon$ . ■

*Remark 5* The controller (11) and (33) are discontinuous ones because they use the sign function which could produce the undesired chattering, while in some cases continuity is necessary. Under these circumstances, we can use the *sat* function in the controllers (21) and (39) to decrease this effect.

*Remark 6* It should be noted that the synchronization is the stabilization of the error between the transmission and the receiver systems. The synchronization phenomenon is often encountered in nature, which is considered as an adjustment of rhythms of oscillating objects due to their internal weak couplings [15]. It has a great importance in real application. For example, fireflies in flocks give off flashes of light with the same frequency on a summer night



or the synchronous motion of space is established between the moon's orbital motion and its rotational motion. In order to achieve self-synchronization in the asynchronous communication techniques between the transmitter and the receiver, the two sides must simultaneously send and receive information over the same frequency.

*Remark 7* In this work, the FTSt and the FTSy are investigated for a class of INNs with time-varying and distributed delays. For the first time, new SMC schemes are proposed in our paper for the FTSt and the FTSy of master-slave INNs with delays where the settling-time is bounded independently from the initial conditions by constructing Lyapunov functionals and by using analytical techniques. The main limitation of SMC in practical applications is the chattering problem. The chattering is the natural price that the SMC pays in order to eliminate completely unknown matched external disturbances.

*Remark 8* Compared with the results of [27,54,68], the improved SMC approach of this brief provide better results for the following reasons. Firstly, the integral sliding mode surface is novel and needs less information, which has the performance that once the sliding mode variable satisfies  $s(t) \rightarrow 0$ , the FTSy will be guaranteed independent of the initial conditions. Secondly, the presented control approach can be conveniently verified without solving any LMIs. Finally, the FTSy between drive-response systems (26) and (27) can be obtained via the presented SMC approach, however, it cannot be guaranteed in the articles cited above.

## 6 Numerical Examples

In this section, we present four examples to illustrate the effectiveness of the results presented in this article.

*Example 1* Consider the following INN with time-varying delay

$$\frac{d^2 r_\ell(t)}{dt^2} = -\lambda_\ell \frac{dr_\ell(t)}{dt} - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(r_k(t-\tau(t))) + u_\ell(t) \quad (43)$$

and the transformation  $\frac{dr_\ell(t)}{dt} = h_\ell(t)$  leading to the system:

$$\begin{cases} \frac{dr_\ell(t)}{dt} = h_\ell(t) \\ \frac{dh_\ell(t)}{dt} = -\lambda_\ell h_\ell(t) - a_\ell r_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(r_k(t)) \\ \quad + \sum_{k=1}^n d_{\ell k} g_k(r_k(t-\tilde{\tau}(t))) + u_\ell(t) \end{cases} \quad (44)$$

with  $n = 2$ ,  $\ell = 1, 2$ ,  $a_1 = a_2 = 3$ ,  $\lambda_1 = \lambda_2 = 4$ ,  $c_{11} = 0.3$ ,  $c_{12} = 0.15$ ,  $c_{21} = 0.25$ ,  $c_{22} = -0.4$ ,  $d_{11} = 0.6$ ,  $d_{12} = 0.2$ ,  $d_{21} = -0.3$ ,  $d_{22} = 0.3$ ,  $\tilde{\tau} = 1$ , the activation functions  $f_\ell(s) = 0.5[|s+1| - |s-1|]$  and  $g_\ell(\cdot) = \tanh(\cdot)$ . We have  $|f_k(\nu) - f_k(\kappa)| \leq |\nu - \kappa|$  and  $|g_k(\kappa)| \leq 1$ , then Assumption **(A<sub>1</sub>)** and **(A<sub>2</sub>)** hold with  $M_k = 1$  and  $N_k = 1$ .

Choosing  $\hat{\alpha}_\ell = \hat{\beta}_\ell = 4$ . For  $\mu_1 = \mu_2 = 4$ , we have  $p_1 = p_2 = 1 + \frac{1}{\mu_\ell} = 1.25$  and  $q_1 = q_2 = 1 - \frac{1}{\mu_1} = 0.75$ . The other parameters of the feedback control (11) are designed as follows:  $\psi_1 = \psi_2 = 6 \geq a_1 = 3$ ,  $\psi_2 = 6 \geq a_2 = 3$ ,  $\phi_{11} = 4 \geq a_1 + M_1|c_{11}| = 3.3$ ,  $\phi_{12} = 3.5 \geq a_1 + M_1|c_{12}| = 3.15$ ,  $\phi_{21} = 3.5 \geq a_2 + M_2|c_{21}| = 3.25$ ,  $\phi_{22} = 4 \geq a_2 + M_2|c_{22}| = 3.4$ ,  $\psi_1 = 7 \geq \lambda_1$ ,  $\psi_2 = 7 \geq \lambda_2$ ,  $q_{11} = 1 \geq N|d_{11}| = 0.6$ ,  $q_{12} = 0.6 \geq N|d_{12}| = 0.2$ ,  $q_{21} = 1 \geq N|d_{21}| = 0.3$ ,  $q_{22} = 0.6 \geq N|d_{22}| = 0.3$  and the initial condition are set as  $r_1(0) = -0.5$ ,  $r_2(0) = 0.5$ ,  $h_1(0) = 1$ ,  $h_2(0) = -1$ . From Theorem 1 we obtain that system (44) is globally fixed-time stabilized under controller (11). In addition, the settling-time satisfies the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} = \frac{\pi}{2} + \frac{\pi}{2} \leq 3.1416.$$

The state trajectories of system (44) under controller (11) are plotted in Figure 1.

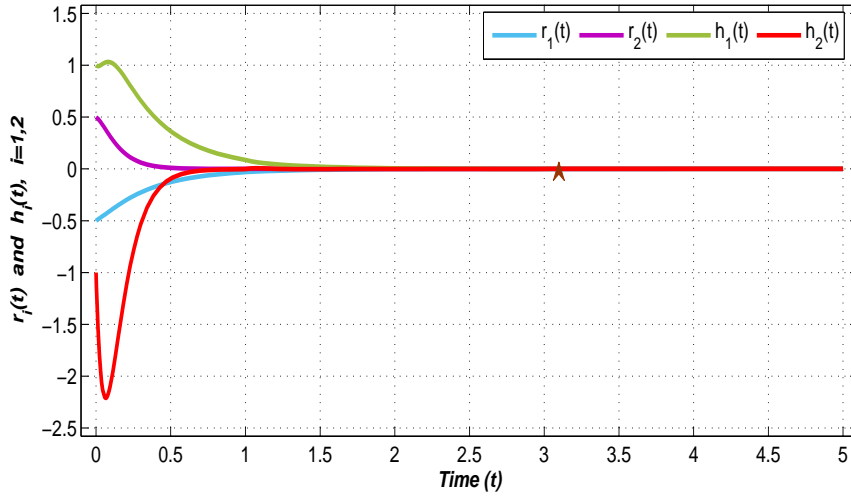


Fig. 1 Trajectories of system (44) with controller (11)

*Example 2* Consider now the following two dimensional INN with mixed delays

$$\begin{aligned} \frac{d^2 r(t)}{dt^2} = & -\Lambda \frac{dr(t)}{dt} - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ & + B \int_{-\infty}^t \mathbf{K}(t-s)G(r(s))ds + u(t); \end{aligned} \quad (45)$$

and the transformation  $\frac{dr(t)}{dt} = h(t)$  leading to the system

$$\begin{cases} \frac{dr(t)}{dt} = h(t) \\ \frac{dh(t)}{dt} = -\Lambda h(t) - Ar(t) + CF(r(t)) + DG(r(t - \tilde{\tau}(t))) \\ \quad + B \int_{-\infty}^t \mathbf{K}(t-s)G(r(s))ds + u \end{cases} \quad (46)$$

where  $k_1(x) = k_2(x) = e^{-x}$ ,  $\tilde{\tau} = 2$ ,  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0.5 & 0.35 \\ -0.5 & -0.7 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0.9 & -0.2 \\ -0.4 & 0.3 \end{pmatrix}$ ,  $D = \begin{pmatrix} 0.6 & 0.2 \\ -0.3 & 0.2 \end{pmatrix}$  and the activation functions  $F(\cdot) = G(\cdot) = \tanh(\cdot)$ . We have  $|f_k(\nu) - f_k(\kappa)| \leq |\nu - \kappa|$  and  $|g_k(\kappa)| \leq 1$ , then Assumption  $(\mathbf{A}_1)$  and  $(\mathbf{A}_2)$  hold with  $M_k = N_k = 1$ . We have  $\int_0^\infty e^{-s} ds = 1$ , then assumption  $(\mathbf{A}_3)$  holds.

Taking  $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\beta}_1 = \hat{\beta}_2 = 1$ ,  $\mu_1 = 1.5$ ,  $\mu_2 = 2.5$ ,  $\epsilon = 0.1$ ,  $p_1 = 1 + \frac{1}{1.5}$ ,  $p_2 = 1 + \frac{1}{2.5}$ ,  $q_1 = 1 - \frac{1}{1.5}$  and  $q_2 = 1 - \frac{1}{2.5}$ . The parameters of the feedback control (21) are designed as follow:  $\phi_{11} = 3 \geq a_1 + M_1|c_{11}| = 2.5$ ,  $\phi_{12} = 3.5 \geq a_1 + M_2|c_{12}| = 2.35$ ,  $\phi_{21} = 3.5 \geq a_2 + M_1|c_{21}| = 2.5$ ,  $\phi_{22} = 4 \geq a_2 + M_2|c_{22}| = 2.7$ ,  $\psi_1 = 7 \geq \lambda_1 = 3$ ,  $\psi_2 = 7 \geq \lambda_2 = 3$ ,  $q_{11} = 1 \geq N_1|d_{11}| = 0.6$ ,  $q_{12} = 0.6 \geq N_2|d_{12}| = 0.2$ ,  $q_{21} = 1 \geq N_1|d_{21}| = 0.3$ ,  $q_{22} = 0.6 \geq N_2|d_{22}| = 0.2$ . The initial values are taken as  $r_1(0) = 0.5$ ,  $r_2(0) = -0.5$ ,  $h_1(0) = -0.7$ ,  $h_2(0) = 0.6$ . Theorem 2 guarantees not only that system (46) is globally fixed-time stabilized under the controller (21) but also that the settling-time satisfies the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_\ell \pi}{2\sqrt{\hat{\alpha}_\ell \hat{\beta}_\ell}} + \hat{T}_\epsilon = \frac{1.5\pi}{2} + \frac{2.5\pi}{2} + 0.001 \leq 6.2842$$

with  $\hat{T}_\epsilon = 0.001$ . We see on Figure 2 that system (46) is globally almost fixed-time stable.

*Example 3* We consider the following INN master system

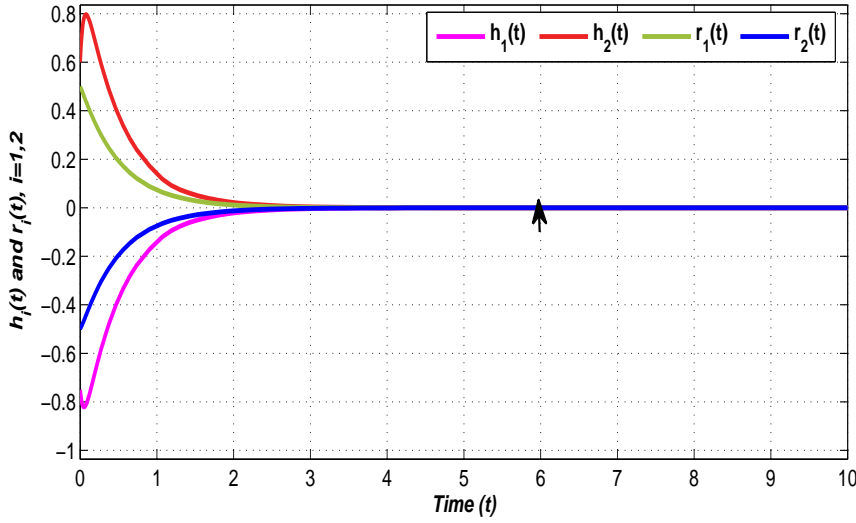
$$\frac{d^2 x_\ell(t)}{dt^2} = -\lambda_\ell \frac{dx_\ell(t)}{dt} - a_\ell x_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(x_k(t)) + \sum_{k=1}^n d_{\ell k} g_k(x_k(t - \tilde{\tau}(t))) \quad (47)$$

where  $n = 2$ ,  $\tilde{\tau} = 1$ ,  $a_1 = a_2 = 12$ ,  $\lambda_1 = \lambda_2 = 7$ ,  $c_{11} = 0.4$ ,  $c_{12} = 0.2$ ,  $c_{21} = 0.5$ ,  $c_{22} = -0.5$ ,  $d_{11} = 0.7$ ,  $d_{12} = 0.3$ ,  $d_{21} = -0.4$ ,  $d_{22} = 0.4$ , the activation functions  $f_\ell(s) = 0.5[|s+1| - |s-1|]$ . It is clear that Assumption  $(\mathbf{A}_1)$  and  $(\mathbf{A}_2)$  hold and  $M_k = N_k = 1$ . The slave system is given by

$$\frac{d^2 y_\ell(t)}{dt^2} = -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} f_k(y_k(t - \tau(t))) + u_\ell(t) \quad (48)$$

where the parameters of system (48) are same as the system (47).

Choosing  $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\beta}_1 = \hat{\beta}_2 = 2$ ,  $\mu_1 = \mu_2 = 3$ ,  $p_1 = 1 + \frac{1}{\mu_1}$ ,  $p_2 = 1 + \frac{1}{\mu_2}$ ,  $q_1 = 1 - \frac{1}{\mu_1}$  and  $q_2 = 1 - \frac{1}{\mu_2}$ . The parameters of the feedback control (33) are



**Fig. 2** Trajectories of system (46) with controller (21)

as follows:  $\phi_{11} = 13 \geq a_1 + M_1|c_{11}| = 12.4$ ,  $\phi_{12} = 13 \geq a_1 + M_1|c_{12}| = 12.2$ ,  $\phi_{21} = 13 \geq a_2 + M_2|c_{21}| = 12.5$ ,  $\phi_{22} = 13 \geq a_2 + M_2|c_{22}| = 12.5$ ,  $\psi_1 = 8 \geq \lambda_1$ ,  $\psi_2 = 8 \geq \lambda_2$ ,  $\bar{q}_{11} = 3 \geq 2N_1|d_{11}| = 1.4$ ,  $\bar{q}_{12} = 1.5 \geq 2N_1|d_{12}| = 0.6$ ,  $\bar{q}_{21} = 2 \geq 2N_2|d_{21}| = 0.8$ ,  $\bar{q}_{22} = 3 \geq 2N_2|d_{22}| = 0.8$ . The initial values are taken as  $x_1(0) = 1$ ,  $x_2(0) = 0.5$ ,  $y_1(0) = -0.5$ ,  $y_2(0) = 0.5$ . From Theorem 3, the master system (47) and the slave system (48) are globally fixed-time synchronized under controller (33). In addition, the settling-time satisfy the following inequality

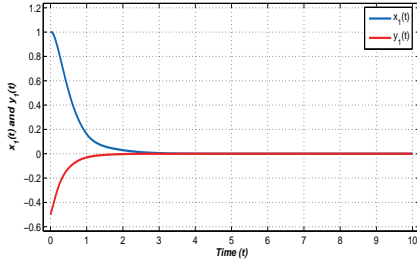
$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_{\ell}\pi}{2\sqrt{\hat{\alpha}_{\ell}\hat{\beta}_{\ell}}} = \frac{3\pi}{2} \leq 4.7124.$$

After a short time, the master system (47) tracks the slave system (48) as illustrated in Figure 3 and Figure 4. The synchronization errors are depicted in Figure 5 and Figure 6.

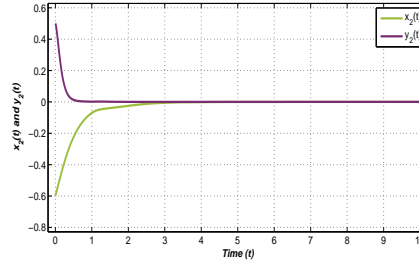
*Example 4* We consider the following INN master system:

$$\frac{d^2x(t)}{dt^2} = -A\frac{dx(t)}{dt} - Ax(t) + CG(x(t)) + DG(x(t-\tilde{\tau}(t))) + B \int_{-\infty}^t \mathbf{K}(t-s)G(x(s))ds \quad (49)$$

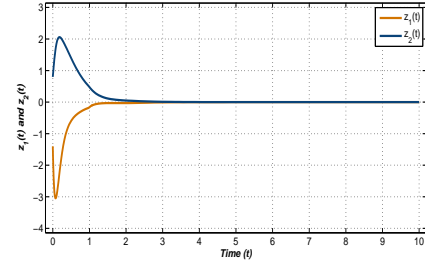
where  $k_1(x) = k_2(x) = e^{-x}$ ,  $\tilde{\tau} = 2$ ,  $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ ,  $\Lambda = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0.6 & 0.2 \\ -0.3 & 0.3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0.1 & -0.2 \\ -0.5 & -0.4 \end{pmatrix}$ ,  $D = \begin{pmatrix} 0.5 & 0.35 \\ -0.5 & -0.7 \end{pmatrix}$ , the activation function  $G(\cdot) =$



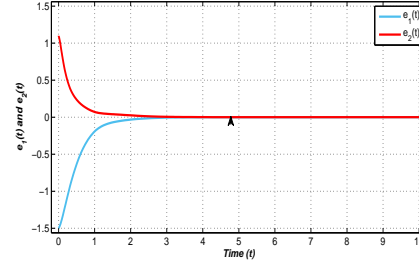
**Fig. 3** Trajectories of  $x_1(t)$  and  $y_1(t)$



**Fig. 4** Trajectories of  $x_2(t)$  and  $y_2(t)$



**Fig. 5** Trajectories of  $z_1(t)$  and  $z_2(t)$



**Fig. 6** Trajectories of  $e_1(t)$  and  $e_2(t)$

$\tanh(\cdot)$ . The salve system is given as follows:

$$\begin{aligned} \frac{d^2 y_\ell(t)}{dt^2} = & -\lambda_\ell \frac{dy_\ell(t)}{dt} - a_\ell y_\ell(t) + \sum_{k=1}^n c_{\ell k} f_k(y_k(t)) + \sum_{k=1}^n d_{\ell k} f_k(y_k(t - \tau(t))) \\ & + \sum_{k=1}^n b_{\ell k} \int_{-\infty}^t \mathbf{k}_k(t - \sigma) g_k(x_k(\sigma)) d\sigma + u_\ell(t) \end{aligned} \quad (50)$$

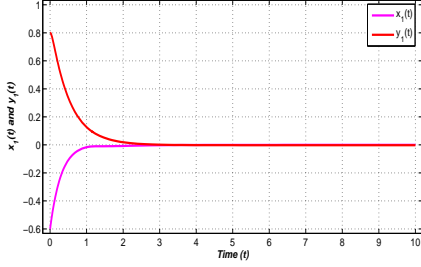
where the parameters of system (50) are same as the system(49).

Choosing  $\mu_1 = 1.5$ ,  $\mu_2 = 2.5$ ,  $\hat{\alpha}_1 = \hat{\beta}_1 = \hat{\alpha}_2 = \hat{\beta}_2 = 1$ ,  $\epsilon = 0.1$ ,  $p_1 = 1 + \frac{1}{\mu_1} = 1.6667$ ,  $p_2 = 1 + \frac{1}{\mu_2} = 1.4$ ,  $q_1 = 1 - \frac{1}{\mu_1} = 0.3333$  and  $q_2 = 1 - \frac{1}{\mu_2} = 0.6$ . The parameters of the feedback control (39) are given by:  $\phi_{11} = 9 \geq a_1 + M_1|c_{11}| = 5.6$ ,  $\phi_{12} = 9 \geq a_1 + M_1|c_{12}| = 5.2$ ,  $\phi_{21} = 9 \geq a_2 + M_2|c_{21}| = 5.3$ ,  $\phi_{22} = 9 \geq a_2 + M_2|c_{22}| = 5.3$ ,  $\psi_1 = 7 \geq \lambda_1$ ,  $\psi_2 = 7 \geq \lambda_1$ ,  $\bar{q}_{11} = 1.9 \geq 2N_1(|b_{11}| + |d_{11}|) = 1.2$ ,  $\bar{q}_{12} = 1.2 \geq 2N_1(|b_{12}| + |d_{12}|) = 1.1$ ,  $\bar{q}_{21} = 2 \geq 2N_2(|b_{21}| + |d_{21}|) = 2$ ,  $\bar{q}_{22} = 3 \geq 2N_2(|b_{22}| + |d_{22}|) = 2.2$ . The initial values are taken as  $x_1(0) = -0.6$ ,  $x_2(0) = 0.6$ ,  $y_1(0) = 0.8$ ,  $y_2(0) = -0.8$ . From Theorem 4, the master system (49) and the slave system (50) are globally fixed-time synchronized under controller (39). After a short time, the master system (49) tracks the slave system (50) as illustrated in Figures 7 and 8. The synchronization errors are depicted in Figures 9 and 10. The settling-time

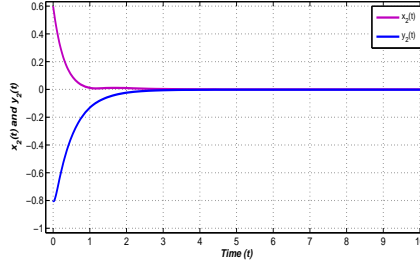
satisfy the following inequality

$$\hat{T}(0) \leq \sum_{\ell=1}^2 \frac{\mu_{\ell}\pi}{2\sqrt{\hat{\alpha}_{\ell}\hat{\beta}_{\ell}}} + \hat{T}_{\epsilon} = \frac{1.5\pi}{2} + \frac{2.5\pi}{2} + 0.001 \leq 6.2833.$$

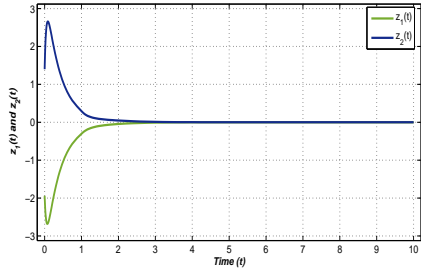
with  $\hat{T}_{\epsilon} = 0.001$ .



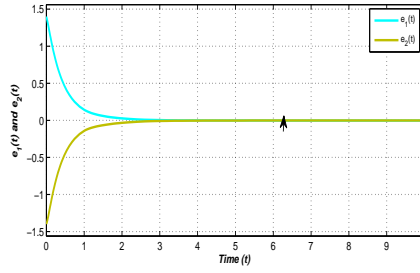
**Fig. 7** Trajectories of  $x_1(t)$  and  $y_1(t)$



**Fig. 8** Trajectories of  $x_2(t)$  and  $y_2(t)$



**Fig. 9** Trajectories of  $z_1(t)$  and  $z_2(t)$



**Fig. 10** Trajectories of  $e_1(t)$  and  $e_2(t)$

*Remark 9* The study of the fixed-time stability of neural networks is of major interest for many applications such as spacecraft attitude dynamics [33] and secure communication [65]. Secure communication using synchronization between chaotic systems is a new concept of secure communication. In [1], the authors present the secure communication synchronization problem as real application of inertial neural networks.

## 7 Conclusion

In this article, the fixed-time stabilization and fixed-time synchronization of inertial neural networks with time-varying delay are inspected. By using sliding variables, two sliding mode controllers are obtained. Moreover, to get rid of the singularities two more sliding mode controllers are designed leading

to almost fixed-time stabilization and almost fixed-time synchronization. Our results are novel and complementary to the existing results about neural networks. There exist several ideas for future research. The first topic is related to the activation function. For processing of complex processes with delays the speed and performance of neural network in such case will depend on quality of activation function chosen. For this reason, a scaled polynomial constant unit activation function SPOCU [34] is going to be tested with respect to its performance. The second one focuses on the fixed-time stabilization and fixed-time synchronization under sliding mode control of:

1. a more general model of inertial neural networks such as inertial BAM neural networks, high-order inertial neural networks, quaternion-valued inertial Cohen-Grossberg neural networks, Clifford-valued inertial neural networks;
2. a more general model of fractional-order time-delayed inertial neural network.

**Conflict of interest:** The authors declare that they have no conflict of interest.

## References

1. A. M. Alimi, C. Aouiti, and E. A. Assali. Finite-time and fixed-time synchronization of a class of inertial neural networks with multi-proportional delays and its application to secure communication. *Neurocomputing*, 332:29–43, 2019.
2. A. M. Alimi, C. Aouiti, F. Chérif, F. Dridi, and M. S. Mhamdi. Dynamics and oscillations of generalized high-order hopfield neural networks with mixed delays. *Neurocomputing*, 321:274–295, 2018.
3. C. Aouiti, A. M. Alimi, F. Karray, and A. Maalej. The design of beta basis function neural network and beta fuzzy systems by a hierarchical genetic algorithm. *Fuzzy Sets and Systems*, 154(2):251–274, 2005.
4. C. Aouiti, A. M. Alimi, and A. Maalej. A genetic-designed beta basis function neural network for multi-variable functions approximation. *Systems Analysis Modelling Simulation*, 42(7):975–1009, 2002.
5. C. Aouiti and E. A. Assali. Stability analysis for a class of impulsive high-order hopfield neural networks with leakage time-varying delays. *Neural Computing and Applications*, 31(11):7781–7803, 2019.
6. C. Aouiti, E. a. Assali, J. Cao, and A. Alsaedi. Global exponential convergence of neutral-type competitive neural networks with multi-proportional delays, distributed delays and time-varying delay in leakage delays. *International Journal of Systems Science*, 49(10):2202–2214, 2018.
7. C. Aouiti, I. Ben Gharbia, J. Cao, M. S. M’hamdi, and A. Alsaedi. Existence and global exponential stability of pseudo almost periodic solution for neutral delay BAM neural networks with time-varying delay in leakage terms. *Chaos, Solitons and Fractals*, 107:111–127, 2018.
8. C. Aouiti, P. Coirault, F. Miaadi, and E. Moulay. Finite time boundedness of neutral high-order Hopfield neural networks with time delay in the leakage term and mixed time delays. *Neurocomputing*, 260:378–392, 2017.
9. C. Aouiti, Q. Hui, H. Jallouli, and E. Moulay. Fixed-time stabilization of fuzzy neutral-type inertial neural networks with time-varying delay. *Fuzzy Sets and Systems*, 2020. <https://doi.org/10.1016/j.fss.2020.10.018>.
10. C. Aouiti, X. Li, and F. Miaadi. A New LMI Approach to Finite and Fixed Time Stabilization of High-Order Class of BAM Neural Networks with Time-Varying Delays. *Neural Processing Letters*, pages 1–24, 2018.

11. C. Aouiti, M. S. M'hamdi, and F. Chérif. New results for impulsive recurrent neural networks with time-varying coefficients and mixed delays. *Neural Processing Letters*, 46(2):487–506, 2017.
12. C. Aouiti, M. S. Mhamdi, F. Chérif, and A. M. Alimi. Impulsive generalised high-order recurrent neural networks with mixed delays: stability and periodicity. *Neurocomputing*, 2017.
13. C. Aouiti, L. Xiaodi, and F. Miaadi. Finite Time Stabilization of Uncertain Delayed Hopfield Neural Networks with a Time-varying Leakage Delay via Non-Chattering Control. *Science China Technological Sciences*, pages 1111–1122, 2018.
14. G. Aquino, J. D. J. Rubio, J. Pacheco, G. J. Gutierrez, G. Ochoa, R. Balcazar, D. R. Cruz, E. Garcia, J. F. Novoa, and A. Zacarias. Novel nonlinear hypothesis for the delta parallel robot modeling. *IEEE Access*, 8:46324–46334, 2020.
15. I. I. Blekhman, A. L. Fradkov, O. P. Tomchina, and D. Bogdanov. Self-synchronization and controlled synchronization: general definition and example design. *Mathematics and Computers in Simulation*, 58(4-6):367–384, 2002.
16. J. Cao and R. Li. Fixed-time synchronization of delayed memristor-based recurrent neural networks. *Science China Information Sciences*, 60(3):032201, 2017.
17. J. Cao and Y. Wan. Matrix measure strategies for stability and synchronization of inertial BAM neural network with time delays. *Neural Networks*, 53:165–172, 2014.
18. Z. Chen, B. Zhang, V. Stojanovic, Y. Zhang, and Z. Zhang. Event-based fuzzy control for ts fuzzy networked systems with various data missing. *Neurocomputing*, 417:322–332, 2020.
19. P. Cheng, M. Chen, V. Stojanovic, and S. He. Asynchronous fault detection filtering for piecewise homogenous markov jump linear systems via a dual hidden markov model. *Mechanical Systems and Signal Processing*, 151:107353, 2021.
20. H.-S. Chiang, M.-Y. Chen, and Y.-J. Huang. Wavelet-based eeg processing for epilepsy detection using fuzzy entropy and associative petri net. *IEEE Access*, 7:103255–103262, 2019.
21. N. Cui, H. Jiang, C. Hu, and A. Abdurahman. Global asymptotic and robust stability of inertial neural networks with proportional delays. *Neurocomputing*, 272:326–333, 2018.
22. J. de Jesús Rubio. SOFMLS: online self-organizing fuzzy modified least-squares network. *IEEE Transactions on Fuzzy Systems*, 17(6):1296–1309, 2009.
23. J. J. de Rubio. Stability analysis of the modified levenberg-marquardt algorithm for the artificial neural network training. *IEEE Transactions on Neural Networks and Learning Systems*, 2020. <https://doi.org/10.1109/TNNLS.2020.3015200>.
24. R. A. DeCarlo, S. H. Zak, and G. P. Matthews. Variable structure control of nonlinear multivariable systems: a tutorial. *Proceedings of the IEEE*, 76(3):212–232, 1988.
25. H. Du, S. Li, and C. Qian. Finite-time attitude tracking control of spacecraft with application to attitude synchronization. *IEEE Transactions on Automatic Control*, 56(11):2711–2717, 2011.
26. M. Forti, P. Nistri, and M. Quincampoix. Generalized neural network for nonsmooth nonlinear programming problems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 51(9):1741–1754, 2004.
27. Q. Gan, R. Xu, and P. Yang. Synchronization of non-identical chaotic delayed fuzzy cellular neural networks based on sliding mode control. *Communications in Nonlinear Science and Numerical Simulation*, 17(1):433–443, 2012.
28. J. H. Graham and J. M. Zurada. A neural network approach for safety and collision avoidance in robotic systems. *Reliability Engineering & System Safety*, 53(3):327–338, 1996.
29. Z. Guo, S. Gong, and T. Huang. Finite-time synchronization of inertial memristive neural networks with time delay via delay-dependent control. *Neurocomputing*, 293:100–107, 2018.
30. G. Hernández, E. Zamora, H. Sossa, G. Téllez, and F. Furlán. Hybrid neural networks for big data classification. *Neurocomputing*, 390:327–340, 2020.
31. C. Huang, J. Cao, M. Xiao, A. Alsaedi, and F. E. Alsaadi. Controlling bifurcation in a delayed fractional predator–prey system with incommensurate orders. *Applied Mathematics and Computation*, 293:293–310, 2017.



32. W. Huang, K. K. Lai, Y. Nakamori, S. Wang, and L. Yu. Neural networks in finance and economics forecasting. *International Journal of Information Technology & Decision Making*, 6(01):113–140, 2007.
33. B. Jiang, Q. Hu, and M. I. Friswell. Fixed-time attitude control for rigid spacecraft with actuator saturation and faults. *IEEE Transactions on Control Systems Technology*, 24(5):1892–1898, 2016.
34. J. Kiselák, Y. Lu, J. Švihra, P. Szépe, and M. Stehlík. "SPOCU": scaled polynomial constant unit activation function. *Neural Computing and Applications*, pages 1–17, 2020.
35. E. B. Kosmatopoulos, M. M. Polycarpou, M. A. Christodoulou, and P. A. Ioannou. High-order neural network structures for identification of dynamical systems. *IEEE transactions on Neural Networks*, 6(2):422–431, 1995.
36. S.-K. Lee and D. Jang. Translation, rotation and scale invariant pattern recognition using spectral analysis and hybrid genetic-neural-fuzzy networks. *Computers & Industrial Engineering*, 30(3):511–522, 1996.
37. H. Li, C. Li, T. Huang, and W. Zhang. Fixed-time stabilization of impulsive Cohen-Grossberg BAM neural networks. *Neural Networks*, 98:203–211, 2018.
38. R. Li and J. Cao. Finite-time and fixed-time stabilization control of delayed memristive neural networks: Robust analysis technique. *Neural Processing Letters*, 47(3):1077–1096, 2018.
39. R. Li, J. Cao, A. Alsaedi, B. Ahmad, F. E. Alsaadi, and T. Hayat. Nonlinear measure approach for the robust exponential stability analysis of interval inertial cohen-grossberg neural networks. *Complexity*, 21(S2):459–469, 2016.
40. S. Li, C. K. Ahn, and Z. Xiang. Adaptive fuzzy control of switched nonlinear time-varying delay systems with prescribed performance and unmodeled dynamics. *Fuzzy Sets and Systems*, 371:40–60, 2019.
41. S. Li, C. K. Ahn, and Z. Xiang. Command Filter Based Adaptive Fuzzy Finite-Time Control for Switched Nonlinear Systems Using State-Dependent Switching Method. *IEEE Transactions on Fuzzy Systems (2020)*, 2020. <https://doi.org/10.1109/TFUZZ.2020.2965917>.
42. Q. Liu, X. Liao, S. Guo, and Y. Wu. Stability of bifurcating periodic solutions for a single delayed inertial neuron model under periodic excitation. *Nonlinear Analysis: Real World Applications*, 10(4):2384–2395, 2009.
43. Y. Liu, H. Yue, and W. Wang. Fixed-time stabilization of second-order systems with unknown nonlinear inherent dynamics. In *12th International Conference on Control (CONTROL)*, pages 324–329. IEEE, 2018.
44. H. Malmgren, M. Borga, and L. Niklasson. *Artificial Neural Networks in Medicine and Biology: Proceedings of the ANNIMAB-1 Conference, Göteborg, Sweden, 13–16 May 2000*. Springer, 2012.
45. J. A. Meda-Campaña. On the estimation and control of nonlinear systems with parametric uncertainties and noisy outputs. *IEEE Access*, 6:31968–31973, 2018.
46. M. S. Mhamdi, C. Aouiti, A. Touati, A. M. Alimi, and V. Snasel. Weighted pseudo almost-periodic solutions of shunting inhibitory cellular neural networks with mixed delays. *Acta Mathematica Scientia*, 36(6):1662–1682, 2016.
47. E. Moulay and W. Perruquetti. Finite time stability and stabilization of a class of continuous systems. *Journal of Mathematical Analysis and Applications*, 323(2):1430–1443, 2006.
48. J. Ni, L. Liu, C. Liu, X. Hu, and S. Li. Fast fixed-time nonsingular terminal sliding mode control and its application to chaos suppression in power system. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 64(2):151–155, 2016.
49. K. Papik, B. Molnar, R. Schaefer, Z. Dombovari, Z. Tulassay, and J. Feher. Application of neural networks in medicine—a review. *Medical Science Monitor*, 4(3):MT538–MT546, 1998.
50. S. Parsegov, A. Polyakov, and P. Shcherbakov. Fixed-time consensus algorithm for multi-agent systems with integrator dynamics. *IFAC Proceedings Volumes*, 46(27):110–115, 2013.
51. A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.

52. G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut, and C. Huang. Hybrid control scheme for projective lag synchronization of Riemann–Liouville sense fractional order memristive BAM neural networks with mixed delays. *Mathematics*, 7(8):759, 2019.
53. J. Savković-Stevanović. A neural network model for analysis and optimization of processes. *Computers & Chemical Engineering*, 17:S411–S416, 1993.
54. Y. Shi, P. Zhu, and K. Qin. Projective synchronization of different chaotic neural networks with mixed time delays based on an integral sliding mode controller. *Neurocomputing*, 123:443–449, 2014.
55. V. Stojanovic, S. He, and B. Zhang. State and parameter joint estimation of linear stochastic systems in presence of faults and non-gaussian noises. *International Journal of Robust and Nonlinear Control*, 30(16):6683–6700, 2020.
56. S. Tong and Y. Li. Robust adaptive fuzzy backstepping output feedback tracking control for nonlinear system with dynamic uncertainties. *Science China Information Sciences*, 53(2):307–324, 2010.
57. S. Tong, X. Min, and Y. Li. Observer-based adaptive fuzzy tracking control for strict-feedback nonlinear systems with unknown control gain functions. *IEEE Transactions on Cybernetics (2020)*, 2020. <https://doi.org/10.1109/TCYB.2020.2977175>.
58. V. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2):212–222, 1977.
59. V. Utkin, J. Guldner, and J. Shi. *Sliding mode control in electro-mechanical systems*. CRC press, second edition, 2009.
60. D. W. Wheeler and W. Schieve. Stability and chaos in an inertial two-neuron system. *Physica D: Nonlinear Phenomena*, 105(4):267–284, 1997.
61. W. Wu, W. Zhou, and T. Chen. Cluster synchronization of linearly coupled complex networks under pinning control. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 56(4):829–839, 2008.
62. X. Xi, S. Mobayen, H. Ren, and S. Jafari. Robust finite-time synchronization of a class of chaotic systems via adaptive global sliding mode control. *Journal of Vibration and Control*, 24(17):3842–3854, 2018.
63. J.-J. Xiong, G.-B. Zhang, J.-X. Wang, and T.-H. Yan. Improved sliding mode control for finite-time synchronization of nonidentical delayed recurrent neural networks. *IEEE transactions on neural networks and learning systems*, 31(6):2209–2216, 2019.
64. F. Yang, J. Mei, and Z. Wu. Finite-time synchronisation of neural networks with discrete and distributed delays via periodically intermittent memory feedback control. *IET Control Theory & Applications*, 10(14):1630–1640, 2016.
65. T. Yang. A survey of chaotic secure communication systems. *International journal of computational cognition*, 2(2):81–130, 2004.
66. K. D. Young, V. I. Utkin, and U. Ozguner. A control engineer’s guide to sliding mode control. In *IEEE International Workshop on Variable Structure Systems (VSS’96)*, pages 1–14. IEEE, 1996.
67. K. Yunquan and M. Chunfang. Stability and existence of periodic solutions in inertial BAM neural networks with time delay. *Neural Computing and Applications*, 23(3-4):1089–1099, 2013.
68. D. Zhang and J. Xu. Projective synchronization of different chaotic time-delayed neural networks based on integral sliding mode controller. *Applied Mathematics and Computation*, 217(1):164–174, 2010.
69. W. Zhang. *Computational ecology: artificial neural networks and their applications*. World Scientific, 2010.
70. W. Zhang, Y. Tang, Q. Miao, and J.-A. Fang. Synchronization of stochastic dynamical networks under impulsive control with time delays. *IEEE transactions on neural networks and learning systems*, 25(10):1758–1768, 2013.
71. X. Zhang, H. Shuping, V. Stojanovic, X. Luan, and F. Liu. Finite-time asynchronous dissipative filtering of conic-type nonlinear markov jump systems. *SCIENCE CHINA Information Sciences*.
72. Z. Zhang and Z. Quan. Global exponential stability via inequality technique for inertial BAM neural networks with time delays. *Neurocomputing*, 151:1316–1326, 2015.
73. Z.-S. Zhao, J. Zhang, and L.-K. Sun. Sliding mode control in finite time stabilization for synchronization of chaotic systems. *International Scholarly Research Notices*, 2013, 2013. <http://dx.doi.org/10.1155/2013/320180>.

- 
74. L. Zhou, H. Tao, W. Paszke, V. Stojanovic, and H. Yang. Pd-type iterative learning control for uncertain spatially interconnected systems. *Mathematics*, 8(9):1528, 2020.
  75. Z. Zuo, M. Defoort, B. Tian, and Z. Ding. Fixed-time stabilization of second-order uncertain multivariable nonlinear systems. In *35th Chinese Control Conference (CCC)*, pages 907–912. IEEE, 2016.
  76. Z. Zuo, Q.-L. Han, B. Ning, X. Ge, and X.-M. Zhang. An overview of recent advances in fixed-time cooperative control of multiagent systems. *IEEE Transactions on Industrial Informatics*, 14(6):2322–2334, 2018.