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First-Order Pontryagin Maximum Principle for Risk-Averse Stochastic Optimal Control Problems

R. Bonalli* and Benoît Bonnet[†]

April 6, 2022

Abstract

In this paper, we derive a set of first-order Pontryagin optimality conditions for a risk-averse stochastic optimal control problem subject to final time inequality constraints, and whose cost is a general finite coherent risk measure. Unlike previous contributions in the literature, our analysis holds for classical stochastic differential equations driven by standard Brownian motions. Moreover, it presents the advantages of neither involving second-order adjoint equations, nor leading to the so-called weak version of the PMP, in which the maximization condition with respect to the control variable is replaced by the stationarity of the Hamiltonian.

Keywords : Risk-averse stochastic optimal control, Pontryagin Maximum Principle, Stochastic first-order necessary optimality conditions.

MSC2020 Subject Classification : 93E03, 93E20.

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1 Introduction

In the last decades, risk-averse stochastic optimal control has seen a surge of interest as a tool for designing control laws that enjoy robustness properties against uncertainties. Relevant applications of this theory range from safe financial investment to safe control of autonomous systems, as evidenced by the monographs [3, 16] and their bibliography. In this context, first-order necessary conditions for optimality in the form of the Pontryagin Maximum Principle (PMP; we will refer to these conditions as risk-averse PMP) would play a key role in characterising and numerically computing optimal control strategies, as it is known to be the case for classical stochastic optimal control problems, in which

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only expectation-based costs and constraints are considered [21]. However, extending the PMP in its general form to more involved risk-averse settings still requires substantial investigations.

To the best of our knowledge, the derivation of a risk-averse PMP was attempted firstly in [19], where appropriate adjoint equations and maximality conditions formulated in terms of the so-called G -Stochastic calculus are introduced in order to cope with the presence of risk measures. This framework was originally introduced by Peng [14], and developed by the stochastic control community later on, see e.g. [15, 18]. In this setting, the standard Brownian motion is replaced by a so-called G -Brownian motion, that is modelled as a stochastic process whose distribution is the product of a standard Gaussian with a Lipschitz map, and whose role is to transform the coherent risk measure into a standard, though non-linear expectation. While practical for some applications, this procedure requires to change the dynamics of the system, which is not always natural e.g. when the diffusion term aims at rendering an unknown uncertainty exerted on the system by the environment. Therefore, for certain classes of problems, it is still relevant to investigate optimality conditions relying on standard stochastic calculus, and which do not require to infuse additional uncertainty in the formulation of the control problem. Along this line, a risk-averse PMP for problems which are subject to stochastic differential equations stemming from classical Wiener processes is proposed in [11], though no final constraints are included therein and the underlying risk measures are assumed to be continuously Fréchet differentiable. From a different standpoint, first-order necessary optimality conditions for convex risk-averse optimization problems subject to partial differential equations and general sub-differentiable risk-measure-based costs are derived in [10], by leveraging classical tools from convex analysis. Nevertheless, final constraints are also ruled out in this work, and the necessary conditions for optimality are written down as simple Euler conditions and not as a general Karush-Kuhn-Tucker system, which would be the natural “static” counterpart of the PMP.

In this paper, we propose a first step towards bridging the aforescribed gap by establishing a first-order risk-averse PMP for a class of finite-dimensional stochastic optimal control problems. Therein, one aims at minimizing a final cost modelled as a general subdifferentiable coherent risk measures subject to final time inequality constraints, over a class of admissible trajectories driven by a controlled stochastic differential equations involving standard Wiener processes. Our proof leverages a general methodology which was first developed in [6], allowing for a natural extension of the first-order PMP for stochastic optimal control problems with expectation-based costs discussed in [9] to the risk-averse setting. Specifically, the main advantages offered by this approach over more classical needle-like variation- or Ekeland’s principle-based methods are twofold. Firstly, no additional second-order adjoint variables (nor related second-order adjoint equations) are required to establish a fully informative PMP. Secondly, it permits the derivation of the so-called strong maximum principle, in which the optimal controls are characterize as being pointwise maximizers of the Hamiltonian. This is in contrast with some reference contributions in stochastic optimal control, that establish weaker variants of the PMP in which the maximization condition is relaxed by requiring the stationarity of the Hamiltonian [7, 8]. In what follows, we propose two separate sets of optimality conditions for the class of optimal control problems at hand, depending on whether the control variable appears in the diffusion term or not. In the latter context, the variational linearization techniques subtending the proof of the maximum principle can be performed much like in the deterministic case, by considering perturbations which are tangent to the set of convexified velocities. In the former case however, it is not possible to replicate such a strategy as the Itô integral does not exhibit the nice convexifying effects of the Bochner one – a fact which is expounded by an original example in Remark 2.14 –, and one needs to impose an a priori convexity assumptions on the sets of admissible drift and diffusion pairs, similar to that considered e.g. in [9].

The paper is organized as follows. In Section 2, we recollect known concepts of stochastic calculus and set-valued analysis, which feature a counterexample to Aumann’s theorem for the Itô integral that we believe to be of independent interest. In Section 3, we expose the main contributions of this article, which are first-order Pontryagin optimality conditions for risk-averse stochastic optimal control problems. We start in Section 3.1 with the case in which the diffusion term of the driving stochastic dynamics is controlled, and expose the proof in great details in this context. We then show in Section 3.2 how the aforeproposed methodology can be used to prove the PMP under more

general assumptions when the diffusion term is control-free, and close the paper in Section 4 by briefly discussing some examples.

2 Preliminaries

In this section, we recollect some useful concepts and results of stochastic calculus, for which we mainly refer to [12, 21], and notions of set-valued analysis which are mostly excerpted from [1]. From now on, we fix $n, m, d \in \mathbb{N}$, and a finite time horizon $T > 0$, and let $\beta \in [1, +\infty)$.

2.1 Stochastic Calculus

Throughout this article, we will consider random variables defined over a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. For any sub-sigma algebra $\mathcal{S} \subset \mathcal{G}$, we denote by $L_{\mathcal{S}}^{\beta}(\Omega; \mathbb{R}^n)$ the Banach space of random variables $z : \Omega \rightarrow \mathbb{R}^n$ which are \mathcal{S} -measurable and satisfy

$$\|z\|_{L^{\beta}} \triangleq \mathbb{E}[\|z\|^{\beta}]^{1/\beta} < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm. It is a standard consequence of Riesz's theorem that $L_{\mathcal{S}}^{\beta}(\Omega; \mathbb{R})^* \cong L_{\mathcal{S}}^{\gamma}(\Omega; \mathbb{R})$ where $\gamma \in (1, +\infty]$ is the conjugate exponent of β .

Let $(W_s)_{s \in [0, T]} = (W_s^1, \dots, W_s^d)_{s \in [0, T]}$ be a d -dimensional Wiener process whose induced filtration

$$\mathcal{F} \triangleq (\mathcal{F}_t)_{t \in [0, T]} = (\sigma\{W_s : 0 \leq s \leq t\})_{t \in [0, T]},$$

is complete. We denote by $L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$ the Banach space of processes $x : (t, \omega) \in [0, T] \times \Omega \rightarrow \mathbb{R}^n$ which are progressive with respect to the filtration \mathcal{F} , and such that

$$\|x\|_{L_{\mathcal{F}}^{\beta}} \triangleq \mathbb{E} \left[\int_0^T \|x(s)\|^{\beta} ds \right]^{1/\beta} < \infty.$$

In addition, we denote by $C_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$ the Banach space of \mathcal{F} -adapted processes $x : (t, \omega) \in [0, T] \times \Omega \rightarrow \mathbb{R}^n$ which have continuous sample paths and finite sup norm

$$\|x\|_{C_{\mathcal{F}}^{\beta}} \triangleq \mathbb{E} \left[\sup_{s \in [0, T]} \|x(s)\|^{\beta} \right]^{1/\beta} < \infty.$$

In particular, $C_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n) \subset L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$. In the sequel, we will often use the standard notation $x(t) : \Omega \rightarrow \mathbb{R}^n$ for $t \in [0, T]$ to denote progressive processes.

An \mathcal{F} -adapted process $x : (t, \omega) \in [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that $x(s) \in L_{\mathcal{F}_s}^1(\Omega; \mathbb{R}^n)$ for every $s \in [0, T]$ is called a *martingale* provided that

$$\mathbb{E}[x(t)|\mathcal{F}_s] = x(s),$$

for all $0 \leq s < t \leq T$. We then say that a martingale $x : (t, \omega) \in [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is uniformly bounded in $L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\|x\|_{L_{\mathcal{F}}^{\beta}} \leq C.$$

In this setting, for every $x \in L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$ and each $i \in \{1, \dots, d\}$, we write

$$y^i : t \in [0, T] \mapsto \int_0^t x(s) dW_s^i$$

for the Itô integral of x with respect to W^i , and recall that y^i is then a martingale in $C_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$. Analogously, we introduce the notation

$$y(t) = \int_0^t x(s) dW_s \triangleq \sum_{i=1}^d \int_0^t x(s)^i dW_s^i$$

for $x \in L_{\mathcal{F}}^\beta([0, T] \times \Omega; \mathbb{R}^{n \times d})$, where $x(s) = (x(s)^1 | \dots | x(s)^d)$ and $x(s)^i \in L_{\mathcal{F}_s}^\beta(\Omega; \mathbb{R}^n)$, and recall that the latter complies with the famed *Burkholder-Davis-Gundy* inequality

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|y(t)\|^\beta \right] \leq C_\beta \mathbb{E} \left[\left(\int_0^T \|x(t)\|^2 dt \right)^{\beta/2} \right] \quad (1)$$

where $C_\beta > 0$ is a constant that only depends on $\beta \in [1, +\infty)$. The following representation theorem for martingales (see e.g. [12, Theorem 5.18]) will be crucial to derive the adjoint dynamics of the PMP in Section 3.

Theorem 2.1 (Martingale representation theorem). *Let $x : (t, \omega) \in [0, T] \times \Omega \rightarrow \mathbb{R}^n$ be a martingale which is uniformly bounded in $L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$. Then, there exist a vector $N \in \mathbb{R}^n$ and a stochastic process $\mu \in L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n \times d})$ such that*

$$x(t) = N + \int_0^t \mu(s) dW_s, \quad \text{for all } t \in [0, T].$$

In this article, we will study risk-averse stochastic optimal control problems, which involve the following class of functionals, called *finite coherent risk measures*, and studied in [16].

Definition 2.2 (Finite coherent risk measure). *A mapping $\rho : L_{\mathcal{S}}^1(\Omega; \mathbb{R}) \rightarrow \mathbb{R}$ is a finite coherent risk measure if it satisfies the following properties.*

1. (Convexity) For every $Z_1, Z_2 \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$ and all $\lambda \in [0, 1]$, it holds

$$\rho(\lambda Z_1 + (1 - \lambda)Z_2) \leq \lambda \rho(Z_1) + (1 - \lambda)\rho(Z_2),$$

2. (Monotonicity) If $Z_1, Z_2 \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$ are such that $Z_1 \leq Z_2$, then

$$\rho(Z_1) \leq \rho(Z_2).$$

3. (Translation invariance) For every $Z \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$ and $\alpha \in \mathbb{R}$, it holds

$$\rho(Z + \alpha) = \rho(Z) + \alpha.$$

4. (Positive homogeneity) For every $Z \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$ and $\alpha > 0$, it holds

$$\rho(\alpha Z) = \alpha \rho(Z).$$

It is a standard fact in stochastic analysis that coherent risk measures satisfy the following fundamental properties (see e.g. [16, Chapter 6]).

Theorem 2.3 (Structure of finite coherent risk measures). *Given a finite coherent risk measure $\rho : L_{\mathcal{S}}^1(\Omega; \mathbb{R}) \rightarrow \mathbb{R}$, the following properties hold true.*

1. For every $Z \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$, the risk measure can be represented as

$$\rho(Z) = \sup_{\xi \in \partial \rho(0)} \mathbb{E}[\xi Z],$$

where $\partial \rho(0)$ denotes its convex subdifferential at $Z = 0$.

2. For every $Z \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$, the subdifferential $\partial \rho(Z) \subset L_{\mathcal{S}}^\infty(\Omega; \mathbb{R})$ is nonempty, convex, and weakly-* compact, and can be expressed as

$$\partial \rho(Z) = \arg \sup_{\xi \in \partial \rho(0)} \mathbb{E}[\xi Z].$$

3. For every $Z, H \in L_{\mathcal{S}}^1(\Omega; \mathbb{R})$, the mapping ρ has a sublinear directional derivative $D\rho(Z) \cdot H$ at Z along H , which satisfies

$$D\rho(Z) \cdot H = \sup_{\xi \in \partial \rho(Z)} \mathbb{E}[\xi H].$$

As previously mentioned in the introduction, coherent risk measures appear very naturally in a broad range of stochastic decision problems, with their most common representative being the *Average Value-at-Risk*, see e.g. [16, Section 6.2.4]) and the discussion in Section 4.

2.2 Stochastic Differential Equations

In what follows, we detail the setting in which we shall control stochastic dynamics. Let $U \subset \mathbb{R}^m$ be a compact set representing admissible control values, and consider a stochastic drift mapping $f : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ as well as a stochastic diffusion mapping $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$ which satisfy the following series of assumptions (these are quite standard, see e.g. [21, Chapter 3.3]).

Main Assumptions on the Stochastic Dynamics – (MSD).

(i) For every $(x, u) \in \mathbb{R}^n \times U$, the applications

$$f(\cdot, \cdot, x, u) : [0, T] \times \Omega \rightarrow \mathbb{R}^n, \quad \sigma(\cdot, \cdot, x, u) : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times d},$$

are progressive. Moreover, for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$, the maps

$$f(t, \omega, \cdot, \cdot) : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \quad \sigma(t, \omega, \cdot, \cdot) : \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$$

are continuous.

(ii) There exists a map $k \in L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}_+)$ such that¹

$$\|f(t, \omega, 0, u)\| + \|\sigma(t, \omega, 0, u)\| \leq k(t, \omega),$$

for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$ and each $u \in U$.

(iii) For every \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$ and all $u \in U$, the mappings

$$f(t, \omega, \cdot, u) : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \sigma(t, \omega, \cdot, u) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d},$$

are continuously Fréchet differentiable, and there exists $L > 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t, \omega, x, u) \right\| + \left\| \frac{\partial \sigma}{\partial x}(t, \omega, x, u) \right\| \leq L,$$

as well as

$$\left\| \frac{\partial f}{\partial x}(t, \omega, x, u) - \frac{\partial f}{\partial x}(t, \omega, y, u) \right\| + \left\| \frac{\partial \sigma}{\partial x}(t, \omega, x, u) - \frac{\partial \sigma}{\partial x}(t, \omega, y, u) \right\| \leq L \|x - y\|,$$

for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$, any $u \in U$ and all $x, y \in \mathbb{R}^n$.

From now on, we fix an initial condition $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$. For every progressive control $u : [0, T] \times \Omega \rightarrow U$, the stochastic differential equation

$$\begin{cases} dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW_t, \\ x(0) = x_0, \end{cases} \quad (\text{SDE})$$

has a unique solution $x_u \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ under hypotheses (MSD)-(i) and (ii), up to stochastic indistinguishability. In the following lemma, we recall a useful estimate for this class of dynamics (see e.g. [13, Proposition 2.1]).

Lemma 2.4. *Let $u : [0, T] \times \Omega \rightarrow U$ be a progressive control signal and suppose that assumptions (MSD) hold. Then, the corresponding solution $x_u \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ of (SDE) satisfies the estimate*

$$\|x_u\|_{C^1_{\mathcal{F}}} \leq C \mathbb{E} \left[\|x_0\| + \int_0^T \|f(s, 0, u(s))\| ds + \left(\int_0^T \|\sigma(s, 0, u(s))\|^2 ds \right)^{1/2} \right],$$

where the constant $C > 0$ only depends on T and L .

¹Since $U \subset \mathbb{R}^m$ is assumed to be compact, this assumption encompasses control-affine dynamics.

2.3 Set-valued Analysis

In the sequel given a closed set $K \subset \mathbb{R}^n$, we define its *closed convex hull* by

$$\overline{\text{co}}K := \overline{\bigcup_{N \geq 1} \left\{ \sum_{i=1}^N \alpha_i x_i : x_i \in K, \alpha_i \geq 0 \text{ for } i \in \{1, \dots, N\} \text{ and } \sum_{i=1}^N \alpha_i = 1 \right\}}. \quad (2)$$

If the set K is convex, we shall denote its *tangent cone* at some $x \in K$ by

$$T_K(x) := \overline{\left\{ v \in \mathbb{R}^n : \lim_{h \rightarrow 0^+} \frac{1}{h} \text{dist}_K(x + hv) = 0 \right\}}, \quad (3)$$

where $\text{dist}_K(x) := \inf_{y \in K} \|x - y\|$.

Given a filtered space of the form $([0, T] \times \Omega, \mathcal{F})$, we will write $F : [0, T] \times \Omega \rightrightarrows \mathbb{R}^n$ to denote a *set-valued map* – or *multifunction* –, that is a mapping from $[0, T] \times \Omega$ into the subsets of \mathbb{R}^n . In this context, we say that the latter has closed, compact or convex images if it is valued in closed, compact or convex subsets of \mathbb{R}^n respectively.

Definition 2.5 (Progressive set-valued maps). *A set-valued map $F : [0, T] \times \Omega \rightrightarrows \mathbb{R}^n$ is progressive if it is measurable with respect to the filtration \mathcal{F} , namely*

$$F^{-1}(\mathcal{O}) := \left\{ (t, x) \in [0, T] \times \Omega : F(t, \omega) \cap \mathcal{O} \neq \emptyset \right\}$$

is \mathcal{F} -measurable for every open set $\mathcal{O} \subset \mathbb{R}^n$

We recall in the following theorem a direct consequence of [1, Theorem 8.1.3].

Theorem 2.6 (Existence of progressive selections). *A progressive set-valued map $F : [0, T] \times \Omega \rightrightarrows \mathbb{R}^n$ with nonempty closed images admits a progressive selection, i.e. there exists a progressive map $f : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ such that $f(t, \omega) \in F(t, \omega)$ for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$.*

In the following definitions, we recall classical adaptations of the concepts of integral boundedness and Lipschitz regularity for progressive set-valued maps with compact images. The latter of these properties is expressed in terms of the so-called *Pompeiu-Hausdorff* distance, defined by

$$d_{\mathcal{H}}(A, B) := \max \left\{ \sup_{x \in A} \text{dist}_B(x), \sup_{y \in B} \text{dist}_A(y) \right\},$$

defined for any two compact sets $A, B \subset \mathbb{R}^n$.

Definition 2.7 (Integrably bounded multifunction). *A set-valued mapping $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with nonempty compact images is integrably bounded if*

$$F(t, \omega, x) \subset k(t, \omega)\mathbb{B}$$

for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$ and all $x \in \mathbb{R}^n$, where $k \in L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}_+)$ and $\mathbb{B} \subset \mathbb{R}^n$ denotes the closed unit ball centered at the origin.

Definition 2.8 (Progressively Lipschitz multifunction). *A set-valued mapping $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with compact images is progressively Lipschitz if*

$$(t, \omega) \in [0, T] \times \Omega \rightrightarrows F(t, \omega, x) \in \mathbb{R}^n,$$

is progressive for each $x \in \mathbb{R}^n$, and provided that there exists $L > 0$ such that

$$d_{\mathcal{H}}(F(t, \omega, x), F(t, \omega, y)) \leq L|x - y|,$$

for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$ and all $x, y \in \mathbb{R}^n$.

We recall in the following theorem some classical adaptations of [1, Corollary 8.2.13, Theorem 8.5.1, Corollary 8.5.2], which provide the existence of progressive selections in several useful contexts.

Theorem 2.9 (Some progressive selection results). *Let $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be progressively L -Lipschitz with nonempty compact images, fix $x, y \in C_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$ and $l \in L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}_+)$. Then, the following results hold.*

(a) *The set-valued mapping*

$$(t, \omega) \in [0, T] \times \Omega \rightrightarrows F(t, \omega, x(t, \omega)) \subset \mathbb{R}^n$$

is progressive, and thus admits a progressive selection.

(b) *Let $(t, \omega) \in [0, T] \times \Omega \mapsto f(t, \omega) \in F(t, \omega, x(t, \omega))$ be a progressive selection such that $f \in L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$. Then the set-valued mapping*

$$(t, \omega) \in [0, T] \times \Omega \rightrightarrows T_{\text{co}F(t, \omega, x(t, \omega))}(f(t, \omega)) \subset \mathbb{R}^n$$

is progressive as well, and admits selections in $L_{\mathcal{F}}^{\beta}([0, T] \times \Omega; \mathbb{R}^n)$.

(c) *If for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$, the sets*

$$F(t, \omega, x(t, \omega)) \cap \left\{ f \in \mathbb{R}^n : \|f - y(t, \omega)\| \leq l(t, \omega) \right\}$$

are nonempty, then there exists a progressive selection $(t, \omega) \mapsto f(t, \omega) \in F(t, \omega, x(t, \omega))$ such that $\|f(t, \omega) - y(t, \omega)\| \leq l(t, \omega)$.

Remark 2.10 (Shorter notation for stochastic processes). *For the sake of conciseness, we will often drop the dependence with respect to the parameter $\omega \in \Omega$ and write $t \in [0, T] \mapsto f(t) \in F(t, x(t))$ to refer to progressive selections and maps.*

We end this preliminary section by recalling an adaptation of a general min-max theorem excerpted from [17] and that is due to Sion.

Theorem 2.11 (Sion's minimax theorem). *Let X, Y be two convex subsets of Hausdorff topological spaces with X being compact, and consider a continuous map $\varphi : X \times Y \rightarrow \mathbb{R}$ that is such that*

$$x \in X \mapsto \varphi(x, y) \in \mathbb{R} \quad \text{is convex}$$

for each $y \in Y$, and

$$y \in Y \mapsto \varphi(x, y) \in \mathbb{R} \quad \text{is concave}$$

for each $x \in X$. Then, it holds that

$$\min_{x \in X} \sup_{y \in Y} \varphi(x, y) = \sup_{y \in Y} \min_{x \in X} \varphi(x, y).$$

2.4 Stochastic Differential Inclusions

In this section, we recollect a series of definitions and results concerning set-valued stochastic dynamics. Given a progressively Lipschitz set-valued map $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^{n+d \times n}$ with nonempty compact images, we say that $x \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ solves the *stochastic differential inclusion*

$$\begin{cases} dx(t) \in F(t, x(t)) d(\lambda \times W)_t, \\ x(0) = x_0, \end{cases} \quad (\text{SDI})$$

if there exists a progressive selection $t \in [0, T] \rightrightarrows (f(t), \sigma(t)) \in F(t, x(t))$ such that

$$\begin{cases} x(t) = f(t)dt + \sigma(t)dW_t, \\ x(0) = x_0. \end{cases} \quad (4)$$

As for deterministic differential inclusion, this class of dynamics enjoys an existence result “à la Filippov”, which comprises handy a priori estimates with respect to a given process. This is the object of the following theorem, whose proof can be established via a small variation of the arguments in [5].

Theorem 2.12 (Filippov estimates). *Let $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^{n+d \times n}$ be an integrably bounded and progressively Lipschitz set-valued mapping, fix $x_0, y_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ and $(g, \zeta) \in L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$, and consider the solution $y \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ of the stochastic differential equation*

$$\begin{cases} dy(t) = g(t)dt + \zeta(t)dW_t, \\ y(0) = y_0. \end{cases}$$

Moreover, suppose that the mismatch function, defined by

$$d : t \in [0, T] \mapsto \text{dist}_{F(t, y(t))}((g, \zeta)(t)) \in \mathbb{R}_+,$$

which is progressive, is in addition an element of $L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}_+)$.

Then, there exists a solution $x \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ of (SDI) which satisfies

$$\mathbb{E}[\|x(t) - y(t)\|^2] \leq C \mathbb{E} \left[\|x_0 - y_0\|^2 + \int_0^t d^2(s) ds \right]$$

for all times $t \in [0, T]$, where the constant $C > 0$ depends only on the magnitudes of the bounding map and Lipschitz constant of $F(\cdot, \cdot, \cdot)$.

In the sequel given an integrably bounded and progressively Lipschitz set-valued mapping $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ along with a diffusion map $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the relevant parts of Assumptions (MSD), we will also work with stochastic differential inclusions of the form

$$\begin{cases} dx(t) \in F(t, x(t)) dt + \sigma(t, x(t))dW_t, \\ x(0) = x_0, \end{cases} \quad (\text{SDI}')$$

whose solutions are defined as the processes $x \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ that solve (4) for a given progressive selection $t \in [0, T] \mapsto f(t) \in F(t, x(t))$. Below, we recall a stochastic version of the well-known relaxation theorem for this class of dynamics.

Theorem 2.13 (Relaxation). *Let $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^{d \times n}$ be integrably bounded and progressively Lipschitz, fix $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ and suppose that $x \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ is a solution of the relaxed differential inclusion*

$$\begin{cases} dx(t) \in \overline{\text{co}}F(t, x(t))dt + \sigma(t, x(t))dW_t, \\ x(0) = x_0. \end{cases}$$

Then for each $\varepsilon > 0$, there exists a solution $x_\varepsilon \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ of (SDI') which satisfies

$$\|x - x_\varepsilon\|_{C^2_{\mathcal{F}}} \leq \varepsilon.$$

Proof. Although we did not find a satisfactory reference for this result in the literature, its proof is standard and can be carried out by following the procedure detailed e.g. in [20, Section 2.7]. \square

Remark 2.14 (Obstruction to relaxation for general stochastic inclusions). *The relaxation theorem for stochastic differential inclusions of the form (SDI') stems from Aumann's famed convexity theorem for the Lebesgue integral (see e.g. [1, Theorem 8.6.4]). The latter asserts in particular that given a Borel set $I \subset [0, T]$, a real number $\beta \in [1, +\infty)$, an integrably bounded progressive set-valued map $F : I \times \Omega \rightrightarrows \mathbb{R}^n$ with closed nonempty images and a progressive selection $t \in I \mapsto f(t) \in \overline{\text{co}}F(t)$, there exists for each $\varepsilon > 0$ a progressive selection $t \in I \mapsto f_\varepsilon \in F(t)$ such that*

$$\mathbb{E} \left[\left| \int_I f(t) dt - \int_I f_\varepsilon(t) dt \right|^\beta \right] \leq \varepsilon. \quad (5)$$

However as evidenced by the following example, such an identity does not hold for the Itô integral. Consider the constant set-valued map $(t, \omega) \in [0, 1] \times \Omega \rightrightarrows F \subset \mathbb{R}^2$, defined by

$$F := \left\{ (x, y) \in [0, 1]^2 : y \in [1 - 2x, 0] \text{ if } x \in [0, \frac{1}{2}] \text{ and } y \in [0, 2x - 1] \text{ if } x \in [\frac{1}{2}, 1] \right\},$$

which is clearly integrably bounded with nonempty compact images. Fixing the constant selection $t \in [0, 1] \mapsto f(t) := (\frac{1}{2}, 1) \in \overline{\text{co}}F(t)$, it follows from Itô's isometry formula (see e.g. [12, Expression (5.8)]) that

$$\mathbb{E} \left[\left| \int_0^1 f(t) dW_t - \int_0^1 f_\varepsilon(t) dW_t \right|^2 \right] = \mathbb{E} \left[\int_0^1 |f(t) - f_\varepsilon(t)|^2 dt \right] \geq \frac{5}{8}$$

for each $\varepsilon > 0$ and any possible progressive selection $t \in [0, 1] \mapsto f_\varepsilon(t) \in F(t)$. This violates (5) for each $\beta \in [2, +\infty)$ by Hölder's inequality, whereas a simple contradiction argument based on both Lebesgue's dominated and reverse dominated convergence theorems also yields the obstruction for $\beta \in [1, 2)$.

3 Risk-Averse Optimal Control and Pontryagin Maximum Principle

In the sequel, we will investigate Pontryagin optimality conditions for the following risk-averse stochastic optimal control problem

$$\begin{cases} \min_{u \in \mathcal{U}} \rho(\varphi_0(x_u(T))), \\ \text{s.t. } \mathbb{E}[\varphi_i(x_u(T))] \leq 0, \quad i \in \{1, \dots, \ell\}. \end{cases} \quad (\text{OCP})$$

Therein $\rho : L^1_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$ is a finite coherent risk measure, while the maps $\varphi_i : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ for $i \in \{0, \dots, \ell\}$ represent a cost and constraints at the final time. The minimization is taken over the set of admissible controls

$$\mathcal{U} \triangleq \left\{ u : [0, T] \times \Omega \rightarrow U : u \text{ is progressive} \right\}$$

where $x_u \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ denotes the unique solution of (SDE) driven by $u \in \mathcal{U}$.

From now on, we assume that the maps $f : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{d \times n}$ satisfy hypotheses (MSD), and posit that the cost and constraint mappings satisfy the following assumptions.

Main Assumptions on the Cost and Constraints – (MCC).

- (i) For each $i \in \{0, \dots, \ell\}$ and all $x \in \mathbb{R}^n$, the mapping $\varphi_i(\cdot, x) : \Omega \rightarrow \mathbb{R}$ is \mathcal{F}_T -measurable and such that $\varphi_i(\cdot, 0) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R}_+)$.
- (ii) For every $i \in \{0, \dots, \ell\}$ and \mathcal{F}_T -almost every $\omega \in \Omega$, the application $\varphi_i(\omega, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously Fréchet differentiable, with

$$\left\| \frac{\partial \varphi_i}{\partial x}(\omega, x) \right\| \leq L,$$

as well as

$$\left\| \frac{\partial \varphi_i}{\partial x}(\omega, x) - \frac{\partial \varphi_i}{\partial x}(\omega, y) \right\| \leq L \|x - y\|,$$

for all $x, y \in \mathbb{R}^n$, where the constant $L > 0$ is the same as in (MSD)-(iii).

Remark 3.1 (On the equivalence between Bolza and Mayer problems). *It is a standard fact in optimal control theory that every Bolza problem involving a running cost can be recast as a Mayer problem, whose criterion is simply expressed as a final cost. While we only discuss Mayer problems in this article, our result would still apply to Bolza problems. Besides, one could then relax the compactness assumption on $U \subset \mathbb{R}^m$ by simply requiring that the latter be closed, provided that the running cost satisfies a Tonelli-type growth condition.*

Throughout this article, we will refer to solutions of (OCP) using the following terminology.

Definition 3.2 (Admissible pairs and local minima for (OCP)). *We say that (x, u) is an admissible trajectory-control pair for (OCP) if $u \in \mathcal{U}$ and $x = x_u$ is a solution to (SDE) satisfying $\mathbb{E}[\varphi_i(x(T))] \leq 0$ for all $i \in \{1, \dots, \ell\}$. Moreover, an admissible pair (x^*, u^*) is a local minimum for (OCP) if there exists $\varepsilon > 0$ such that*

$$\rho(\varphi_0(x^*(T))) \leq \rho(\varphi_0(x(T))),$$

for every other admissible pair (x, u) satisfying $\|x - x^*\|_{C_{\mathcal{F}}^2} \leq \varepsilon$.

From now on, we assume the existence of a local minimum for (OCP), denoted (x^*, u^*) .

We are now ready to state and prove our main result, which are first-order necessary optimality conditions for (OCP), in the form of a Pontryagin Maximum Principle. In what follows, we denote by $H : [0, T] \times \Omega \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ the *Hamiltonian* associated with (OCP), defined by

$$H(t, \omega, x, u, p, q) \triangleq p \cdot f(t, \omega, x, u) + \sum_{i=1}^d q_i \cdot \sigma_i(t, \omega, x, u). \quad (6)$$

for all $(t, \omega, x, u, p, q) \in [0, T] \times \Omega \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$. We also consider the set of *active indices* at $x^*(T)$, given by

$$I^\circ(x^*(T)) \triangleq \left\{ i \in \{1, \dots, \ell\} : \mathbb{E}[\varphi_i(x^*(T))] = 0 \right\}.$$

Finally, for the sake of clarity in the exposition, we separate the case of controlled diffusion from the case of uncontrolled diffusion, incrementally introducing additional assumptions for each case only when required.

3.1 The PMP with Controlled Diffusion

In the case where the control variable acts on both the drift and the diffusion terms, we need to supplement hypotheses (MSD) and (MCC) with the following additional assumption.

Additional Assumptions for Controlled Diffusion – (ACD).

The stochastic drift $f : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and the diffusion term $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}$ are such that the velocity sets, defined by

$$F(t, \omega, x) \triangleq \left\{ (f(t, \omega, x, u), \sigma(t, \omega, x, u)) : u \in U \right\} \subset \mathbb{R}^{n+n \times d},$$

are convex for \mathcal{F} -almost every $(t, \omega) \in [0, T] \times \Omega$ and all $x \in \mathbb{R}^n$.

Remark 3.3. *The above assumption, which has already been considered in [9] in a similar setting, is standard in deterministic optimal control, where it is very useful to guarantee the existence of optimal controls. In particular, (ACD) holds true e.g. when f and σ are affine in the control variable, and U is convex.*

Theorem 3.4 (Risk-averse PMP for (OCP) with controlled diffusion). *Suppose that hypotheses (MSD), (MCC), and (ACD) hold, and let (x^*, u^*) be a local minimum for (OCP). Then there exists an element $\xi^* \in \partial \rho(\varphi_0(x^*(T)))$, non-trivial Lagrange multipliers $(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \in \{-1, 0\} \times \mathbb{R}_-^\ell$ and a pair of stochastic processes $(p^*, q^*) \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n) \times L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{d \times n})$ such that the following holds.*

(i) *The complementary slackness conditions*

$$\mathbf{p}_i \mathbb{E}[\varphi_i(x^*(T))] = 0 \quad (7)$$

hold for each $i \in \{1, \dots, \ell\}$.

(ii) The risk parameter $\xi^* \in \partial\rho(\varphi_0(x^*(T)))$ is characterised by the condition

$$\mathbb{E}[\xi^* \varphi_0(x^*(T))] = \max_{\xi \in \partial\rho(0)} \mathbb{E}[\xi \varphi_0(x^*(T))]. \quad (8)$$

(iii) The process $p^* \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ solves the adjoint dynamics

$$\begin{cases} dp^*(t) = -\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p^*(t), q^*(t))dt + q^*(t)dW_t \\ p^*(T) = \xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)). \end{cases} \quad (9)$$

(iv) The Pontryagin maximisation condition

$$H(t, x^*(t), u^*(t), p^*(t), q^*(t)) = \max_{u \in U} H(t, x^*(t), u, p^*(t), q^*(t)) \quad (10)$$

holds \mathcal{F} -almost everywhere.

Furthermore, if there exists a solution $y_{g_1, g_2} \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ of (LSDE_{g_1, g_2}) (see below) that is such that

$$\mathbb{E}[\nabla \varphi_i(x^*(T)) \cdot y_{g_1, g_2}^*(T)] < 0$$

for every $i \in I^\circ(x^*(T))$, then the PMP is normal, i.e. $\mathbf{p}_0 = -1$.

We split the proof of Theorem 3.4 into five steps. In Step 1, we start by introducing a class of set-valued linearizations along a candidate optimal trajectory-control pair. We subsequently perform a separation argument on the reachable set of the corresponding linearized system and the linearizing cone to the constraints, first in the absence of qualification conditions in Step 2, and then when the constraints are qualified in Step 3. We further show in Step 4 that one can in fact select an optimal risk parameter for which the variational inequalities hold uniformly for the whole reachable set, and finally conclude in Step 5 by proving that these latter in conjunction with the adjoint dynamics yield the PMP.

In what follows, we will almost systematically use the convention introduced in Remark 2.10 for stochastic processes, and drop all explicit dependence in the variable $\omega \in \Omega$ unless necessary.

Step 1 – Variational linearizations along (x^*, u^*) . For every $(t, \omega, x, u) \in [0, T] \times \Omega \times \mathbb{R}^n \times U$, we introduce the notation

$$(f, \sigma)(t, \omega, x, u) \triangleq (f(t, \omega, x, u), \sigma(t, \omega, x, u))$$

and recall following hypotheses (ACD) that

$$F(t, \omega, x) = \left\{ (f, \sigma)(t, \omega, x, u) : u \in U \right\} \subset \mathbb{R}^{n+n \times d}.$$

Under hypotheses (MCC), one can easily prove that the set-valued mapping $F : [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \mathbb{R}^{n+n \times d}$ is integrably bounded as well as progressively Lipschitz with nonempty, compact convex images, following e.g. [1, Theorem 8.2.8]. In particular, using the condensed notation of Remark 2.10, it holds that

$$t \in [0, T] \mapsto (f, \sigma)(t, x^*(t), u^*(t)) \in F(t, x(t)),$$

is an element of $L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$. Moreover, it follows from Theorem 2.9 that the progressive set-valued map

$$t \in [0, T] \mapsto T_{F(t, x^*(t))}((f, \sigma)(t, x^*(t), u^*(t))),$$

has nonempty, compact, and convex images, and thus admits progressive selections

$$t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}((f, \sigma)(t, x^*(t), u^*(t)))$$

which belong to $L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$.

Given such a progressive selections (g_1, g_2) , we denote by $y_{g_1, g_2} \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ the unique (up to stochastic indistinguishability) solution to the linearized stochastic differential equation

$$\begin{cases} dy(t) = \left(A(t)y(t) + g_1(t) \right) dt + \sum_{i=1}^d \left(D_i(t)y(t) + g_2^i(t) \right) dW_t^i, \\ y(0) = 0, \end{cases} \quad (\text{LSDE}_{g_1, g_2})$$

where we used the condensed notations

$$A(t) \triangleq \frac{\partial f}{\partial x}(t, x^*(t), u^*(t)) \quad \text{and} \quad D_i(t) \triangleq \frac{\partial \sigma_i}{\partial x}(t, x^*(t), u^*(t)),$$

for almost every $t \in [0, T]$ and each $i \in \{1, \dots, d\}$. In the following lemma, we prove that y_{g_1, g_2} is continuous with respect to (g_1, g_2) for the strong $L^2_{\mathcal{F}}$ -topology. This result will be useful later on in the proof of the maximum principle.

Lemma 3.5. *There exists a constant $C > 0$ depending only on the magnitudes of $T, \|k\|_{L^2_{\mathcal{F}}}$ and L such that for any given pair of progressive selections*

$$t \in [0, T] \mapsto (g_1, g_2)(t), (\tilde{g}_1, \tilde{g}_2)(t) \in T_{F(t, x^*(t))}((f, \sigma)(t, x^*(t), u^*(t))),$$

it holds

$$\|y_{g_1, g_2} - y_{\tilde{g}_1, \tilde{g}_2}\|_{C^2_{\mathcal{F}}} \leq C \|(g_1, g_2) - (\tilde{g}_1, \tilde{g}_2)\|_{L^2_{\mathcal{F}}}.$$

Proof. Thanks to hypotheses (MSD) and a routine application of Burkholder-Davis-Gundy's and Hölder's inequalities, we obtain that for every $t \in [0, T]$, it holds

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|y_{g_1, g_2}(s) - y_{\tilde{g}_1, \tilde{g}_2}(s)\|^2 \right] \\ & \leq C \mathbb{E} \left[\left(\int_0^t \|A(s)\| \|y_{g_1, g_2}(s) - y_{\tilde{g}_1, \tilde{g}_2}(s)\| ds \right)^2 \right] \\ & \quad + C \sum_{i=1}^d \mathbb{E} \left[\int_0^t \|D_i(s)\|^2 \|y_{g_1, g_2}(s) - y_{\tilde{g}_1, \tilde{g}_2}(s)\|^2 ds \right] \\ & \quad + C \mathbb{E} \left[\left(\int_0^t \|g_1(s) - \tilde{g}_1(s)\| ds \right)^2 + \sum_{i=1}^d \int_0^t \|g_2^i(s) - \tilde{g}_2^i(s)\|^2 ds \right] \\ & \leq C \mathbb{E} \left[\int_0^t \sup_{\zeta \in [0, s]} \|y_{g_1, g_2}(\zeta) - y_{\tilde{g}_1, \tilde{g}_2}(\zeta)\|^2 ds + \int_0^t \|(g_1, g_2)(s) - (\tilde{g}_1, \tilde{g}_2)(s)\|^2 ds \right], \end{aligned}$$

where $C > 0$ denotes some overloaded constant which only depends on the magnitudes of $T, \|k\|_{L^2_{\mathcal{F}}}$ and L . We then conclude by an application of Gronwäll's inequality. \square

In this context, we have the following fundamental linearization result.

Theorem 3.6 (Variational linearization). *For any given progressive selection $t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$ and each $\varepsilon > 0$, there exists a solution $x_{g_1, g_2}^\varepsilon \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI) such that*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{t \in [0, T]} \|x_{g_1, g_2}^\varepsilon(t) - x^*(t) - \varepsilon y_{g_1, g_2}(t)\| \right] = 0, \quad (11)$$

where $y_{g_1, g_2} \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ is the unique solution of (LSDE $_{g_1, g_2}$).

Proof. Our proof is inspired from that of [2, Theorem 3.12]. We fix a progressive selection $t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$, some $\varepsilon > 0$ and consider the progressive mapping

$$t \in [0, T] \mapsto d_\varepsilon(t) := \text{dist}_{F(t, x^*(t))} \left((f, \sigma)(t, x^*(t), u^*(t)) + \varepsilon (g_1, g_2)(t) \right).$$

It can be checked that the latter satisfies

$$d_\varepsilon(t) \leq \varepsilon \|(g_1, g_2)(t)\| \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} d_\varepsilon(t)/\varepsilon = 0,$$

for almost every $t \in [0, T]$, so that in particular $d_\varepsilon \in L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R})$. Let $\bar{x}_{g_1, g_2}^\varepsilon \in C^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^n)$ be the unique (up to stochastic indistinguishability) solution to

$$\begin{cases} dx(t) = \left(f(t, x(t), u^*(t)) + \varepsilon g_1(t) \right) dt \\ \qquad \qquad \qquad + \left(\sigma(t, x(t), u^*(t)) + \varepsilon g_2(t) \right) dW_t, \\ x(0) = x_0. \end{cases} \quad (\text{SDI}_{g_1, g_2}^\varepsilon)$$

Thanks to hypotheses [\(MSD\)](#) and a routine application of Burkholder-Davis-Gundy and Hölder inequalities, we readily obtain that for every $t \in [0, T]$, it holds

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)\|^2 \right] \leq C\varepsilon^2 \mathbb{E} \left[\left(\int_0^t \|g_1(s)\| ds \right)^2 + \sum_{i=1}^d \int_0^t \|g_2(s)^i\|^2 ds \right] \\ & + C\mathbb{E} \left[\left(\int_0^t \left\| \int_0^1 \frac{\partial f}{\partial x} \left(s, x^*(s) + \theta(\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)), u^*(s) \right) (\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)) d\theta \right\|^2 ds \right)^2 \right] \\ & + C \sum_{i=1}^d \mathbb{E} \left[\int_0^t \left\| \int_0^1 \frac{\partial \sigma^i}{\partial x} \left(s, x^*(s) + \theta(\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)), u^*(s) \right) (\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)) d\theta \right\|^2 ds \right] \\ & \leq C \left(\mathbb{E} \left[\int_0^t \sup_{\zeta \in [0, s]} \|\bar{x}_{g_1, g_2}^\varepsilon(\zeta) - x^*(\zeta)\|^2 ds + \varepsilon^2 \int_0^T \|(g_1, g_2)(s)\|^2 ds \right] \right), \end{aligned}$$

where $C > 0$ denotes some overloaded constant which only depends on T and L . Then, a direct application of Gronwall's inequality leads to

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t)\|^2 \right] \leq C\varepsilon^2 \|(g_1, g_2)\|_{L^2_{\mathcal{F}}}^2. \quad (12)$$

On the other hand, by introducing the notations

$$\begin{cases} A_{g_1, g_2}^\varepsilon(t) \triangleq \int_0^1 \left(\frac{\partial f}{\partial x} \left(s, x^*(s) + \theta(\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)), u^*(s) \right) - \frac{\partial f}{\partial x} \left(s, x^*(s), u^*(s) \right) \right) d\theta, \\ D_{g_1, g_2}^{\varepsilon, i}(t) \triangleq \int_0^1 \left(\frac{\partial \sigma^i}{\partial x} \left(s, x^*(s) + \theta(\bar{x}_{g_1, g_2}^\varepsilon(s) - x^*(s)), u^*(s) \right) - \frac{\partial \sigma^i}{\partial x} \left(s, x^*(s), u^*(s) \right) \right) d\theta, \end{cases}$$

for all times $t \in [0, T]$, one may easily show that the process defined by

$$r_{g_1, g_2}^\varepsilon(t) \triangleq \bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t) - \varepsilon y_{g_1, g_2}(t)$$

solves the stochastic differential equation

$$\begin{cases} dr(t) = \left(A(t) + A_{g_1, g_2}^\varepsilon(t) \right) r(t) dt + \sum_{i=1}^d \left(D_i(t) + D_{g_1, g_2}^{\varepsilon, i}(t) \right) r(t) dW_t^i \\ \qquad \qquad \qquad + \varepsilon \left(A_{g_1, g_2}^\varepsilon(t) y_{g_1, g_2}(t) dt + \sum_{i=1}^d D_{g_1, g_2}^{\varepsilon, i}(t) y_{g_1, g_2}(t) dW_t^i \right), \\ r(0) = 0. \end{cases}$$

Thanks to hypotheses [\(MSD\)](#), it then follows from Lemma 2.4 applied to this latter dynamics that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t) - \varepsilon y_{g_1, g_2}(t)\| \right] \\ & \leq \varepsilon C \left(\mathbb{E} \left[\int_0^T \|A_{g_1, g_2}^\varepsilon(s)\| \|y_{g_1, g_2}(s)\| ds \right] + \sum_{i=1}^d \mathbb{E} \left[\int_0^T \|D_{g_1, g_2}^{\varepsilon, i}(s)\|^2 \|y_{g_1, g_2}(s)\|^2 ds \right]^{1/2} \right). \end{aligned}$$

where $C > 0$ denotes some overloaded constant which only depends on the magnitudes of T , $\|k\|_{L^2_{\mathcal{F}}}$ and L . Observe that now that we may infer from (12) that

$$\sup_{t \in [0, T]} \|\bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t)\|^2 \xrightarrow{\varepsilon \rightarrow 0^+} 0,$$

almost surely. From hypothesis (MSD) and the dominated convergence, we thus have

$$\mathbb{E} \left[\int_0^T \|A_{g_1, g_2}^\varepsilon(s)\| \|y_{g_1, g_2}(s)\| ds \right] + \sum_{i=1}^d \mathbb{E} \left[\int_0^T \|D_{g_1, g_2}^{\varepsilon, i}(s)\|^2 \|y_{g_1, g_2}(s)\|^2 ds \right] \xrightarrow{\varepsilon \rightarrow 0^+} 0.$$

which allows us to conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t) - \varepsilon y_{g_1, g_2}(t)\| \right] = 0.$$

To end the proof of our claim, there remains to establish the existence of a solution $x_{g_1, g_2}^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI) which satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \left[\sup_{t \in [0, T]} \|x_{g_1, g_2}^\varepsilon(t) - \bar{x}_{g_1, g_2}^\varepsilon(t)\|^2 \right] = 0. \quad (13)$$

By Theorem 2.9, there exists for every $\varepsilon > 0$ a progressive selection $t \in [0, T] \mapsto (h_1^\varepsilon, h_2^\varepsilon)(t) \in F(t, x^*(t))$ which is such that

$$\|(f, \sigma)(t, x^*(t), u^*(t)) + \sqrt{\varepsilon}(g_1, g_2)(t) - (h_1^\varepsilon, h_2^\varepsilon)(t)\| = d_{\sqrt{\varepsilon}}(t),$$

\mathcal{F} -almost everywhere. Therefore, the progressive maps defined by

$$t \in [0, T] \mapsto (g_1^\varepsilon, g_2^\varepsilon)(t) \triangleq \frac{(h_1^\varepsilon, h_2^\varepsilon)(t) - (f, \sigma)(t, x^*(t), u^*(t))}{\sqrt{\varepsilon}},$$

are elements of $L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$ since they are bounded \mathcal{F} -almost everywhere in norm by $2\|(g_1, g_2)\| + 1$, and are such that

$$(f, \sigma)(t, x^*(t), u^*(t)) + \sqrt{\varepsilon}(g_1^\varepsilon, g_2^\varepsilon)(t) \in F(t, x^*(t)). \quad (14)$$

Moreover, it can be easily checked that since $d_{\sqrt{\varepsilon}}(t)/\sqrt{\varepsilon} \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$, one has

$$\|(g_1, g_2) - (g_1^\varepsilon, g_2^\varepsilon)\|_{L^2_{\mathcal{F}}} \xrightarrow{\varepsilon \rightarrow 0^+} 0 \quad (15)$$

by Lebesgue's dominated convergence theorem. Similarly, by Theorem 2.9 combined with (14), we can find a selection $t \in [0, T] \mapsto (\kappa_1^\varepsilon, \kappa_2^\varepsilon)(t) \in F(t, \bar{x}_{g_1, g_2}^\varepsilon(t))$ for which

$$\begin{aligned} & \|(f, \sigma)(t, x^*(t), u^*(t)) + \sqrt{\varepsilon}(g_1^\varepsilon, g_2^\varepsilon)(t) - (\kappa_1^\varepsilon, \kappa_2^\varepsilon)(t)\| \\ &= \text{dist}_{F(t, \bar{x}_{g_1, g_2}^\varepsilon(t))} \left((f, \sigma)(t, x^*(t), u^*(t)) + \sqrt{\varepsilon}(g_1^\varepsilon, g_2^\varepsilon)(t) \right) \\ &\leq d_{\mathcal{H}}(F(t, x^*(t)), F(t, \bar{x}_{g_1, g_2}^\varepsilon(t))) \\ &\leq L \|x^*(t) - \bar{x}_{g_1, g_2}^\varepsilon(t)\| \end{aligned} \quad (16)$$

holds \mathcal{F} -almost everywhere. At this stage, thanks to the convexity requirement formulated in hypothesis (ACD), one can further observe that

$$(1 - \sqrt{\varepsilon})(f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) + \sqrt{\varepsilon}(\kappa_1^\varepsilon, \kappa_2^\varepsilon)(t) \in F(t, \bar{x}_{g_1, g_2}^\varepsilon(t)), \quad (17)$$

which implies in particular that

$$\text{dist}_{F(\cdot, \bar{x}_{g_1, g_2}^\varepsilon(\cdot))} \left((f, \sigma)(\cdot, \bar{x}_{g_1, g_2}^\varepsilon(\cdot), u^*(\cdot)) + \varepsilon(g_1, g_2)(\cdot) \right) \in L^2_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R}).$$

Since $\bar{x}_{g_1, g_2}^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ solves $(\text{SDI}_{g_1, g_2}^\varepsilon)$, we can apply Theorem 2.12 to obtain the existence of a solution $x_{g_1, g_2}^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI) which satisfies

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \|x_{g_1, g_2}^\varepsilon(t) - \bar{x}_{g_1, g_2}^\varepsilon(t)\|^2 \right] \\ & \leq C \mathbb{E} \left[\int_0^T \text{dist}_{F(t, \bar{x}_{g_1, g_2}^\varepsilon(t))} \left((f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) + \varepsilon(g_1, g_2)(t) \right)^2 dt \right]. \end{aligned}$$

This last identity together with the convergence results of (13)-(15) and the constructions detailed in (16)-(17) allows us to obtain that

$$\begin{aligned} & \frac{1}{\varepsilon^2} \mathbb{E} \left[\int_0^T \text{dist}_{F(t, \bar{x}_{g_1, g_2}^\varepsilon(t))} \left((f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) + \varepsilon(g_1, g_2)(t) \right)^2 dt \right] \\ & \leq \frac{1}{\varepsilon^2} \mathbb{E} \left[\int_0^T \left\| (f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) + \varepsilon(g_1, g_2)(t) \right. \right. \\ & \quad \left. \left. - (1 - \sqrt{\varepsilon})(f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) - \sqrt{\varepsilon}(\kappa_1^\varepsilon, \kappa_2^\varepsilon)(t) \right\|^2 dt \right] \\ & \leq C \mathbb{E} \left[\int_0^T \|(g_1, g_2)(t) - (g_1^\varepsilon, g_2^\varepsilon)(t)\|^2 dt \right] \\ & \quad + \frac{C}{\varepsilon} \mathbb{E} \left[\int_0^T \|(f, \sigma)(t, \bar{x}_{g_1, g_2}^\varepsilon(t), u^*(t)) - (f, \sigma)(t, x^*(t), u^*(t))\|^2 dt \right] \\ & \quad + \frac{C}{\varepsilon} \mathbb{E} \left[\int_0^T \|(f, \sigma)(t, x^*(t), u^*(t)) + \sqrt{\varepsilon}(g_1^\varepsilon, g_2^\varepsilon)(t) - (\kappa_1^\varepsilon, \kappa_2^\varepsilon)(t)\|^2 dt \right] \\ & \leq C \mathbb{E} \left[\int_0^T \|(g_1, g_2)(t) - (g_1^\varepsilon, g_2^\varepsilon)(t)\|^2 dt \right] + \frac{C}{\varepsilon} \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{x}_{g_1, g_2}^\varepsilon(t) - x^*(t)\|^2 \right] \xrightarrow{\varepsilon \rightarrow 0^+} 0, \end{aligned}$$

thanks to (12), where $C > 0$ is some overloaded constant which only depends on the magnitudes of T , $\|k\|_{L_{\mathcal{F}}^2}$ and L , from which the thesis follows. \square

Step 2 – Separation argument without constraint qualification. From now on, up to relabeling the indices, we assume without loss of generality that there exists $j \in \{1, \dots, \ell\}$ such that $I(x^*(T)) = \{1, \dots, j\}$. In addition, we posit that

$$\nabla \varphi_0(x^*(T)) \neq 0 \quad \text{and} \quad \nabla \varphi_i(x^*(T)) \neq 0 \quad \text{for every } i \in I^\circ(x^*(T))$$

as elements of $L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$. Otherwise if $\nabla \varphi_k(x^*(T)) = 0$ for some $k \in \{0, \dots, j\}$, one can observe that the statements of Theorem 3.4 are trivially satisfied with $\mathbf{p}_k = 1$, $\mathbf{p}_i = 0$ for $i \in \{0, \dots, \ell\} \setminus \{k\}$ and ξ^*, p^*, q^* being all set to zero.

By leveraging the notation introduced hereinabove, we define the reachable set of the linearized Cauchy problem (LSDE_{g_1, g_2}) by

$$\begin{aligned} \mathcal{R}_T \triangleq & \left\{ y_{g_1, g_2}(T) \in L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n) : y_{g_1, g_2} \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n) \text{ solves } (\text{LSDE}_{g_1, g_2}) \text{ for some} \right. \\ & \left. t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}((f, \sigma)(t, x^*(t), u^*(t))) \right\}. \end{aligned}$$

Since the images of $T_{F(\cdot, x^*(\cdot))}(f, \sigma)(\cdot, x^*(\cdot), u^*(\cdot))$ are convex cones and (LSDE_{g_1, g_2}) is linear with respect to both y_{g_1, g_2} and (g_1, g_2) , one can easily check that $\mathcal{R}_T \subset L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$ is a nonempty convex cone as well. At this stage, we introduce the set

$$\mathcal{B}_T \triangleq \left\{ \left(\mathbb{E}[\nabla \varphi_1(x^*(T)) \cdot y_{g_1, g_2}(T)], \dots, \mathbb{E}[\nabla \varphi_j(x^*(T)) \cdot y_{g_1, g_2}(T)] \right) : y_{g_1, g_2}(T) \in \mathcal{R}_T \right\},$$

which is a nonempty convex cone in \mathbb{R}^j , and assume first that

$$\mathcal{B}_T \cap (-\infty, 0)^j = \emptyset.$$

In that case, by the separation theorem, we may infer the existence of a non-trivial element $\mathbf{p} \in \mathbb{R}^j \setminus \{0\}$ such that

$$-\infty < \sup_{a \in (-\infty, 0)^j} \mathbf{p} \cdot a \leq \inf_{b \in \mathcal{B}_T} \mathbf{p} \cdot b < \infty.$$

Observing that both \mathcal{B}_T and $(-\infty, 0)^j$ are cones while using standard results of convex analysis, the latter separation inequality implies that

$$\sum_{i=1}^j \mathbf{p}_i \mathbb{E}[\nabla \varphi_i(x^*(T)) \cdot y_{g_1, g_2}(T)] \geq 0 \quad \text{and} \quad \mathbf{p}_i \geq 0 \quad \text{for each } i \in \{1, \dots, j\}. \quad (18)$$

Step 3 – Separation argument with constraint qualification. We now investigate the scenario in which $\mathcal{B}_T \cap (-\infty, 0)^j \neq \emptyset$, which requires a more involved variational analysis. For this, we define an additional nonempty convex cone of \mathbb{R}^{j+1}

$$\mathcal{A}_T \triangleq \left\{ \left(D\rho(\varphi_0(x^*(T))) \cdot (\nabla \varphi_0(x^*(T)) \cdot y_{g_1, g_2}(T)), \mathbb{E}[\nabla \varphi_1(x^*(T)) \cdot y_{g_1, g_2}(T)], \dots, \right. \right. \\ \left. \left. \mathbb{E}[\nabla \varphi_j(x^*(T)) \cdot y_{g_1, g_2}(T)] \right) : y_{g_1, g_2}(T) \in \mathcal{R}_T \right\},$$

and assume by contradiction that

$$\mathcal{A}_T \cap (-\infty, 0)^{j+1} \neq \emptyset.$$

The latter identity is tantamount to the existence a progressive selection $t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$ such that

$$D\rho(\varphi_0(x^*(T))) \cdot (\nabla \varphi_0(x^*(T)) \cdot y_{g_1, g_2}(T)) < 0, \quad (19)$$

and

$$\mathbb{E}[\nabla \varphi_i(T, x^*(T)) \cdot y_{g_1, g_2}(T)] < 0 \quad \text{for each } i \in \{1, \dots, j\}. \quad (20)$$

At this stage, thanks to Theorem 3.6, we may find for every $\varepsilon > 0$ a solution $x_{g_1, g_2}^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI) which satisfies (11). In particular, from [1, Theorem 8.1.3 and Theorem 8.2.10] we readily obtain the existence of a progressive control mapping $u_{g_1, g_2}^\varepsilon : [0, T] \times \Omega \rightarrow U$ such that $x_{g_1, g_2}^\varepsilon = x_{u_{g_1, g_2}^\varepsilon}$ is an admissible trajectory of (SDE). Besides, by leveraging the structure results of Theorem 2.3, we may write that

$$\begin{aligned} \rho(\varphi_0(x_{g_1, g_2}^\varepsilon(T))) &= \rho(\varphi_0(x^*(T))) + D\rho(\varphi_0(x^*(T))) \cdot (\varphi_0(x_{g_1, g_2}^\varepsilon(T)) - \varphi_0(x^*(T))) \\ &\quad + o\left(\|\varphi_0(x_{g_1, g_2}^\varepsilon(T)) - \varphi_0(x^*(T))\|_{L^1_{\mathcal{F}_T}}\right) \\ &= \rho(\varphi_0(x^*(T))) + D\rho(\varphi_0(x^*(T))) \cdot \left(\varepsilon \nabla \varphi_0(x^*(T)) \cdot y_{g_1, g_2}(T) + o(\varepsilon)\right) \\ &\quad + o(\|x_{g_1, g_2}^\varepsilon - x^*\|_{C^1_{\mathcal{F}}}) \\ &\leq \rho(\varphi_0(x^*(T))) + \varepsilon D\rho(\varphi_0(x^*(T))) \cdot (\nabla \varphi_0(x^*(T)) \cdot y_{g_1, g_2}(T)) + o(\varepsilon) \end{aligned} \quad (21)$$

where we used hypothesis (MCC)-(ii) along with the distance estimates of Theorem 3.6. Analogously, it holds for every $i \in \{1, \dots, j\}$ that

$$\mathbb{E}[\varphi_i(x_{g_1, g_2}^\varepsilon(T))] \leq \mathbb{E}[\varphi_i(x^*(T))] + \varepsilon \mathbb{E}[\nabla \varphi_i(x^*(T)) \cdot y_{g_1, g_2}(T)] + o(\varepsilon). \quad (22)$$

By combining (19)-(21) on the one hand and (16)-(22) on the other hand, we conclude that whenever $\varepsilon > 0$ is small enough, $(x_{g_1, g_2}^\varepsilon, u_{g_1, g_2}^\varepsilon)$ is an admissible pair for (OCP) whose cost is strictly lower than that of (x^*, u^*) , which contradicts our standing assumption. Whence, it necessarily holds that $\mathcal{A}_T \cap (-\infty, 0)^{j+1} = \emptyset$.

At this stage, by applying yet again the separation theorem we may infer the existence of a nontrivial multiplier $(\mathbf{p}_0, \mathbf{p}) \triangleq (\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_\ell) \in \mathbb{R}^{j+1} \setminus \{0\}$ such that

$$-\infty < \sup_{a \in (-\infty, 0)^{j+1}} (\mathbf{p}_0, \mathbf{p}) \cdot a \leq \inf_{b \in \mathcal{A}_T} (\mathbf{p}_0, \mathbf{p}) \cdot b < \infty.$$

First, we show that we necessarily have $\mathbf{p}_0 \neq 0$. Indeed, if by contradiction we assume that $\mathbf{p}_0 = 0$, the latter inequality becomes

$$-\infty < \sup_{a \in (-\infty, 0)^j} \mathbf{p} \cdot a \leq \inf_{b \in \mathcal{B}_T} \mathbf{p} \cdot b < \infty.$$

Now, since we assumed that there exists at least one element in $c \in \mathcal{B}_T \cap (-\infty, 0)^j$, we may select by continuity another point $a_c \in (-\infty, 0)^j$ in such a way that

$$\mathbf{p} \cdot a_c \leq \sup_{a \in (-\infty, 0)^j} \mathbf{p} \cdot a \leq \inf_{b \in \mathcal{B}_T} \mathbf{p} \cdot b \leq \mathbf{p} \cdot c < \mathbf{p} \cdot a_c,$$

which leads to a contradiction. Moreover, since $(-\infty, 0)^j$ and \mathcal{B}_T are cones, we further obtain up to a renormalization by \mathbf{p}_0 that $\mathbf{p}_i \geq 0$ for every $i = 1, \dots, j$, and

$$D\rho(\varphi_0(x^*(T))) \cdot (\nabla\varphi_0(x^*(T)) \cdot y_{g_1, g_2}(T)) + \sum_{i=1}^j \mathbf{p}_i \mathbb{E} \left[\nabla\varphi_i(x^*(T)) \cdot y_{g_1, g_2}(T) \right] \geq 0. \quad (23)$$

Up to trivially embedding \mathbf{p} into \mathbb{R}^ℓ and changing its sign and merging (23) with (18), there exists a nontrivial multiplier $(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \in \{-1, 0\} \times \mathbb{R}_-^\ell$ such that

$$(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \neq 0 \quad \text{and} \quad \mathbf{p}_i \mathbb{E}[\varphi_i(x^*(T))] = 0 \quad \text{for every } i \in \{1, \dots, \ell\},$$

and for which, thanks to Theorem 2.3, the following linearized inequality

$$\inf_{\xi \in \partial\rho(\varphi_0(x^*(T)))} \mathbb{E} \left[\left(\xi \mathbf{p}_0 \nabla\varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla\varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right] \leq 0, \quad (24)$$

holds for any selection $t \in [0, T] \mapsto (g_1(t), g_2(t)) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$. In particular, the Lagrange multiplier $(\mathbf{p}_0, \dots, \mathbf{p}_\ell)$ is non-trivial, and complies with the complementary slackness conditions (7) of the PMP.

Step 4 – Universal separation theorem. In what follows, we extract further information from (24), by observing that the latter inequality can be rewritten as

$$\sup_{(g_1, g_2)} \min_{\xi \in \partial\rho(\varphi_0(x^*(T)))} \mathbb{E} \left[\left(\xi \mathbf{p}_0 \nabla\varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla\varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right] \leq 0, \quad (25)$$

which leads us to consider the mapping

$$\mathcal{H}(\xi, (g_1, g_2)) \triangleq \mathbb{E} \left[\left(\xi \mathbf{p}_0 \nabla\varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla\varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right],$$

that is defined for each $\xi \in L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R})$ and every progressive selection $t \in [0, T] \mapsto (g_1(t), g_2(t)) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$.

By Theorem 2.3, the set $\partial\rho(\varphi_0(x^*(T))) \subset L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R})$ is convex and weakly-* compact, whereas the set of all progressive selections in $t \in [0, T] \mapsto (g_1(t), g_2(t)) \in T_{F(t, x^*(t))}(f, \sigma)(t, x^*(t), u^*(t))$ is a convex subset of $L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$. Moreover, it can be checked straightforwardly that

$$\xi \mapsto \mathcal{H}(\xi, (g_1, g_2))$$

is linear and hence continuous for the weak-* topology of $L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R})$, whereas

$$(g_1, g_2) \mapsto \mathcal{H}(\xi, (g_1, g_2))$$

is continuous for the strong topology of $L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n+n \times d})$, as a consequence of Lemma 3.5. Thus by the separation result of Theorem 2.11, we can rewrite (25) as

$$\min_{\xi \in \partial \rho(\varphi_0(x^*(T)))} \sup_{(g_1, g_2)} \mathbb{E} \left[\left(\xi \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right] \leq 0.$$

In addition, because the pointwise supremum of a family of linear functions is lower semicontinuous and convex (see e.g. [4, Corollary 4.23]), the mapping

$$\xi \mapsto \sup_{(g_1, g_2)} \mathcal{H}(\xi, (g_1, g_2))$$

is weakly-* lower-semicontinuous, which along with the fact that $\partial \rho(\varphi_0(x^*(T))) \subset L_{\mathcal{F}_T}^\infty(\Omega; \mathbb{R})$ is weakly-* compact yields the existence of $\xi^* \in \partial \rho(\varphi_0(x^*(T)))$ such that

$$\sup_{(g_1, g_2)} \mathcal{H}(\xi^*, (g_1, g_2)) = \min_{\xi \in \partial \rho(\varphi_0(x^*(T)))} \sup_{(g_1, g_2)} \mathcal{H}(\xi, (g_1, g_2)) \leq 0. \quad (26)$$

In particular, this directly provides us with the maximization condition (8) of the PMP as consequence of the characterization of $\partial \rho(\varphi_0(x^*(T)))$ given in Theorem 2.3.

Step 5 – Costate dynamics and maximisation condition. In what follows, we recover the adjoint equation and derive the maximality condition from (26), which will conclude the proof of Theorem 3.4. To this end, notice first that the latter identity straightforwardly implies that

$$\mathbb{E} \left[\left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \cdot y_{g_1^u, g_2^u}(T) \right] \leq 0, \quad (27)$$

where for any $u \in \mathcal{U}$, we introduced the maps

$$g_1^u(t) \triangleq f(t, x^*(t), u(t)) - f(t, x^*(t), u^*(t))$$

and

$$g_2^u(t) \triangleq \sigma(t, x^*(t), u(t)) - \sigma(t, x^*(t), u^*(t)),$$

which are well-defined \mathcal{F} -almost everywhere. We denote by $\phi, \psi \in L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n \times n})$ the unique (up to stochastic indistinguishability) solutions of the matrix-valued stochastic differential equations

$$\begin{cases} d\phi(t) = A(t)\phi(t)dt + \sum_{i=1}^d D_i(t)\phi(t)dW_t^i \\ \phi(0) = \text{Id}, \end{cases}$$

and

$$\begin{cases} d\psi(t) = -\psi(t) \left(A(t) - \sum_{i=1}^d D_i^2(t) \right) dt - \sum_{i=1}^d \psi(t) D_i(t) dW_t^i \\ \psi(0) = \text{Id}, \end{cases}$$

whose well-posedness are guaranteed e.g. by [21, Section 1.6.3]. We list in the following lemma elementary properties of these maps, whose proofs are rather simple and rely on componentwise applications of the Itô formula in the spirit e.g. of [21, Theorem 6.14, Chapter 1].

Lemma 3.7. *The maps ϕ, ψ are elements of $C_{\mathcal{F}}^\beta([0, T] \times \Omega; \mathbb{R}^{n \times n})$ for every $\beta \in [1, +\infty)$, and satisfy the identity $\psi(t) = \phi(t)^{-1}$ for all times $t \in [0, T]$.*

Thanks to [21, Theorem 6.14, Chapter 1], any solution $y_{g_1, g_2} \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ of (LSDE $_{g_1, g_2}$) generated by a progressive selection $t \in [0, T] \mapsto (g_1, g_2)(t) \in T_{F(t, x^*(t))}((f, \sigma)(t, x^*(t), u^*(t)))$ can be expressed as

$$y_{g_1, g_2}(t) = \phi(t) \int_0^t \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds + \phi(t) \sum_{i=1}^d \int_0^t \psi(s) g_2^i(s) dW_s^i, \quad (28)$$

for all times $t \in [0, T]$. At this stage, let it be noted that the process

$$\left(\mathbb{E} \left[\phi(T)^\top \left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \middle| \mathcal{F}_t \right] \right)_{t \in [0, T]}$$

is a martingale that is uniformly bounded in $L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ as a direct consequence of Jensen's and Hölder's inequalities. Therefore, thanks to Theorem 2.1, we obtain the existence of a vector $N \in \mathbb{R}^n$ and a process $\mu \in L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n \times d})$ such that

$$\begin{aligned} N + \sum_{i=1}^d \chi_i(t) &\triangleq N + \sum_{i=1}^d \int_0^t \mu_i(s) dW_s^i \\ &= \mathbb{E} \left[\phi(T)^\top \left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \middle| \mathcal{F}_t \right], \end{aligned} \quad (29)$$

for every $t \in [0, T]$. Then, it follows from (28) and (29) that

$$\begin{aligned} &\mathbb{E} \left[\left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\phi(T)^\top \left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \middle| \mathcal{F}_T \right] \cdot \right. \\ &\quad \left. \left(\int_0^T \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds + \sum_{i=1}^d \int_0^T \psi(s) g_2^i(s) dW_s^i \right) \right] \\ &= \mathbb{E} \left[\int_0^T N \cdot \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds \right] \\ &\quad + \mathbb{E} \left[\sum_{j=1}^d \left(\int_0^T \mu_j(s) dW_s^j \right) \cdot \left(\int_0^T \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds \right) \right] \\ &\quad + \mathbb{E} \left[\sum_{i, j=1}^d \left(\int_0^T \mu_j(s) dW_s^j \right) \cdot \left(\int_0^T \psi(s) g_2^i(s) dW_s^i \right) \right]. \end{aligned}$$

At this stage, thanks to the Itô formula, it further holds that

$$\begin{aligned} &\mathbb{E} \left[\sum_{j=1}^d \left(\int_0^T \mu_j(s) dW_s^j \right) \cdot \left(\int_0^T \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds \right) \right] \\ &= \mathbb{E} \left[\sum_{j=1}^d \int_0^T \chi_j(s) \cdot \psi(s) \left(g_1(s) - \sum_{i=1}^d D_i(s) g_2^i(s) \right) ds \right], \end{aligned}$$

as well as

$$\begin{aligned} &\mathbb{E} \left[\sum_{i, j=1}^d \left(\int_0^T \mu_j(s) dW_s^j \right) \cdot \left(\int_0^T \psi(s) g_2^i(s) dW_s^i \right) \right] \\ &= \mathbb{E} \left[\sum_{i, j=1}^d \left\langle \int_0^\cdot \mu_j(s) dW_s^j, \int_0^\cdot \psi(s) g_2^i(s) dW_s^i \right\rangle_T \right], \end{aligned}$$

wherein $\langle \cdot, \cdot \rangle$ stands for standard the quadratic variation of a continuous martingale (see e.g. [12, Section 4.3]). Merging the previous computations finally leads to

$$\begin{aligned} & \mathbb{E} \left[\left(\xi^* \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \cdot y_{g_1, g_2}(T) \right] \\ &= \mathbb{E} \left[\int_0^T \left(N + \sum_{j=1}^d \chi_j(s) \right) \cdot \psi(s) g_1(s) ds \right] \\ & \quad + \mathbb{E} \left[\int_0^T \sum_{i=1}^d \left(\mu_i(s) \cdot \psi(s) g_2^i(s) - \left(N + \sum_{j=1}^d \chi_j(s) \right) \cdot \psi(s) D_i(s) g_2^i(s) \right) ds \right]. \end{aligned}$$

Notice at this point that, by defining the costate curves

$$\begin{cases} p^*(t) \triangleq \psi(t)^\top \left(N + \sum_{i=1}^d \chi_i(t) \right), \\ q^*(t) \triangleq \left[\left(\psi(s)^\top \mu_1(s) - D_1(s)^\top p^*(s) \right) \mid \dots \mid \left(\psi(s)^\top \mu_d(s) - D_d(s)^\top p^*(s) \right) \right], \end{cases} \quad (30)$$

for almost every $t \in [0, T]$, the variational inequality (27) can be rewritten as

$$\begin{aligned} & \mathbb{E} \left[\int_0^T p^*(t) \cdot \left(f(s, x^*(s), u(s)) - f(s, x^*(s), u^*(s)) \right) ds \right] \\ & \quad + \mathbb{E} \left[\int_0^T \sum_{i=1}^d q_i^*(t) \cdot \left(\sigma_i(s, x^*(s), u(s)) - \sigma_i(s, x^*(s), u^*(s)) \right) ds \right] \leq 0 \end{aligned}$$

which by the definition (6) of the Hamiltonian is tantamount to

$$\mathbb{E} \left[\int_0^T \left(H(s, x^*(s), p^*(s), q^*(s), u(s)) - H(s, x^*(s), p^*(s), q^*(s), u^*(s)) \right) ds \right] \leq 0, \quad (31)$$

for every $u \in \mathcal{U}$.

We are now going to show that (31) in fact yields the maximization condition. Given an integer $m \geq 1$, consider the closed subset of controls

$$\begin{aligned} \tilde{U}_m(t, \omega) := \left\{ u \in U : H(t, \omega, u, x^*(t, \omega), p^*(t, \omega), q^*(t, \omega)) \geq \right. \\ \left. H(t, \omega, x^*(t, \omega), u^*(t, \omega), p^*(t, \omega), q^*(t, \omega)) + \frac{1}{m} \right\}, \end{aligned} \quad (32)$$

and suppose by contradiction that the corresponding \mathcal{F} -measurable set

$$\tilde{\mathcal{F}}_m := \left\{ (t, \omega) \in [0, T] \times \Omega : \tilde{U}_m(t, \omega) \neq \emptyset \right\} \quad (33)$$

has positive measure. Then, by choosing any admissible control signal $\tilde{u}_m : [0, T] \times \Omega \rightarrow U$ such that $\tilde{u}_m(t, \omega) \in \tilde{U}_m(t, \omega)$ for \mathcal{F} -almost every $(t, \omega) \in \tilde{\mathcal{F}}_m$ and $\tilde{u}_m(t, \omega) = u^*(t, \omega)$ otherwise, it holds that

$$\mathbb{E} \left[\int_0^T \left(H(s, x^*(s), p^*(s), q^*(s), u(s)) - H(s, x^*(s), p^*(s), q^*(s), u^*(s)) \right) ds \right] \geq \frac{1}{m} dt \otimes \mathbb{P}(\tilde{\mathcal{F}}_m),$$

which contradicts (31). Whence, the set defined by

$$\tilde{\mathcal{F}}_\infty := \bigcup_{m \geq 1} \tilde{\mathcal{F}}_m$$

necessarily has zero $dt \otimes \mathbb{P}$ -measure, which together with (32)-(33) implies that the maximisation condition (10) of the PMP holds.

To conclude, we now shift our focus to the dynamics of the costate variable. First, note that p^* is adapted to the filtration \mathcal{F} by construction, and that it has continuous sample-paths. Moreover, we may infer from a straightforward use of Doob's, Jensen's, and Hölder's inequalities along with the fact that $\phi, \psi \in C^\beta([0, T] \times \Omega; \mathbb{R}^{n \times n})$ for every $\beta \in [1, \infty)$ that $p \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$. In addition we have by Itô's formula that

$$\begin{aligned} \psi(t)^\top \left(\sum_{j=1}^d \int_0^t \mu_j(s) dW_s^j \right) &= \sum_{j=1}^d \int_0^t \psi(s)^\top \mu_j(s) dW_s^j - \sum_{i=1}^d \int_0^t \psi(s)^\top D_i(s) \mu_i(s) ds \\ &\quad - \sum_{i=1}^d \int_0^t \psi(s)^\top D_i(s) \left(\sum_{j=1}^d \chi_j(s) \right) dW_s^i \\ &\quad - \int_0^t \psi(s)^\top \left(A(s) - \sum_{i=1}^d D_i(s)^2 \right) \left(\sum_{j=1}^d \chi_j(s) \right) ds, \end{aligned}$$

for all times $t \in [0, T]$. This, combined with the definition (30) of (p^*, q^*) along with that of the Hamiltonian in (6) allows us to deduce that

$$p^*(t) = N - \int_0^t \frac{\partial H}{\partial x}(s, x^*(s), u^*(s), p^*(s), q^*(s)) ds + \int_0^t q^*(s) dW_s,$$

which is precisely the adjoint dynamics posited in (9) of Theorem 3.4. Finally, by repeating the argument developed e.g. in the proof of [21, Theorem 2.2, Section 7.2], we finally obtain that $q^* \in L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{n \times d})$.

3.2 Uncontrolled Diffusion

We now turn our attention towards the simpler scenario in which the control variable does not appear in the diffusion, namely $\sigma(t, \omega, x, u) \equiv \sigma(t, \omega, x)$. Unlike the previous situation, we may relax our assumptions and obtain the PMP without hypothesis (ACD).

Theorem 3.8 (Risk-averse PMP for (OCP) with uncontrolled diffusion). *Suppose that the diffusion term is independent of the control variable, that hypotheses (MSD) and (MCC) are satisfied, and let (x^*, u^*) be a local minimum for (OCP). Then, the conclusions of Theorem 3.4 hold.*

The proof of Theorem 3.8 is almost identical to that of Theorem 3.4, and we shall thus only highlight the few key modifications needed with respect to the argument developed in Section 3.1. In this context, we will work with the set-valued map

$$F : (t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \left\{ f(t, \omega, x, u) \in \mathbb{R}^n : u \in U \right\} \subset \mathbb{R}^n.$$

Adopting the convention introduced in Remark 2.3, the set-valued maps $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows F(t, x)$ and $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^n \rightrightarrows \overline{\text{co}}F(t, x)$ have nonempty compact images, are integrably bounded and progressively L -Lipschitz under hypotheses (MSD). Thus by Theorem 2.9, the progressive multifunction $t \in [0, T] \rightrightarrows T_{\overline{\text{co}}F(t, x^*(t))}(f(t, x^*(t), u^*(t)))$ admits progressive selections

$$t \in [0, T] \mapsto g(t) \in T_{\overline{\text{co}}F(t, x^*(t))}(f(t, x^*(t), u^*(t))).$$

In what follows given such a selection, we denote by $y_g \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ the unique (up to stochastic indistinguishability) solution to the stochastic differential equation

$$\begin{cases} dy(t) = \left(A(t)y(t) + g(t) \right) + \sum_{i=1}^d D_i(t)y(t) dW_t^i, \\ y(0) = 0. \end{cases} \quad (\text{LSDE}_g)$$

where we used the condensed notations

$$A(t) \triangleq \frac{\partial f}{\partial x}(t, x^*(t), u^*(t)) \quad \text{and} \quad D_i(t) \triangleq \frac{\partial \sigma_i}{\partial x}(t, x^*(t)),$$

for almost every $t \in [0, T]$ and each $i \in \{1, \dots, d\}$. Thanks to the relaxation property of Theorem 2.13, the variational linearization studied in Theorem 3.6 can be adapted and improved as follows for stochastic dynamics with uncontrolled diffusions.

Theorem 3.9 (Variational linearization for uncontrolled diffusions). *For any progressive selection $t \in [0, T] \mapsto g(t) \in T_{\overline{\text{co}}F(t, x^*(t))} f(t, x(t), u^*(t))$ and each $\varepsilon > 0$, there exists a solution $x_g^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI') such that*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{t \in [0, T]} \|x_g^\varepsilon(t) - x^*(t) - \varepsilon y_g(t)\| \right] = 0,$$

where $y_g \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ is the unique solution of (LSDE_g).

Proof. By repeating the argument outlined earlier in the proof of Theorem 3.6, one may readily check that there exists a solution $\bar{x}_g^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to the stochastic differential inclusion

$$\begin{cases} dx(t) \in \overline{\text{co}}F(t, x(t))dt + \sigma(t, x(t))dW_t, \\ x(0) = x_0, \end{cases}$$

which satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{x}_g^\varepsilon(t) - x^*(t) - \varepsilon y_g(t)\| \right] = 0.$$

Besides by Theorem 2.13, there exists a solution $x_g^\varepsilon \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n)$ to (SDI') that is such that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \mathbb{E} \left[\sup_{t \in [0, T]} \|x_g^\varepsilon(t) - \bar{x}_g^\varepsilon(t)\|^2 \right] = 0,$$

from whence the conclusion follows. □

By repeating the arguments of Step 1 and Step 2 of Section 3.1 while using the variational linearization of Theorem 3.9 instead of Theorem 3.6, one can again recover the existence of Lagrange multipliers $(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \in \{0, -1\} \times \mathbb{R}_-^\ell$ satisfying

$$(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \neq 0 \quad \text{and} \quad \mathbf{p}_i \mathbb{E}[\varphi_i(x^*(T))] = 0 \quad \text{for every } i \in \{1, \dots, \ell\},$$

such that the variational inequality

$$\inf_{\xi \in \partial \rho(\varphi_0(x^*(T)))} \mathbb{E} \left[\left(\xi \mathbf{p}_0 \nabla \varphi_0(x^*(T)) + \sum_{i=1}^{\ell} \mathbf{p}_i \nabla \varphi_i(x^*(T)) \right) \cdot y_{g_1}(T) \right] \leq 0, \quad (34)$$

holds for any progressive selection $t \in [0, T] \mapsto g(t) \in T_{\overline{\text{co}}F(t, x^*(t))} f(t, x^*(t), u^*(t))$. From there on, one can prove the PMP by a repeting verbatim the arguments elaborated in Step 3, Step 4 and Step 5 of Section 3.1, thus details are skipped.

4 Examples of application

In this section, we briefly discuss general examples of risk-averse stochastic optimal control problems which are encompassed by our results. As the most exotic part of our contribution lies in handling the risk measure, we will mostly focus on this aspect, and place ourselves in the framework considered e.g. in Section 3. In this context, if (x^*, u^*) is a local minimum for (OCP), Theorem 3.4 provides us with the existence of non-trivial Lagrange multipliers $(\mathbf{p}_0, \dots, \mathbf{p}_\ell) \in \{-1, 0\} \times \mathbb{R}_-^\ell$ and stochastic processes $(p^*, q^*) \in C_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^n) \times L_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R}^{d \times n})$ such that (7), (9), and (10) hold.

- **First example.** Suppose that $\rho : L^1_{\mathcal{F}_T}(\Omega; \mathbb{R}) \rightarrow \mathbb{R}$ is Fréchet differentiable, as it was for instance assumed in [11]. This situation includes for instance the log-exp utility function and the mean-variance risk measures, see e.g. [16]. In that case, $\partial\rho(Z) = \{\nabla\rho(Z)\}$ for every $Z \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$, and the result of Theorem 3.4 hold with the uniquely determined risk parameter

$$\xi^* = \nabla\rho(\varphi_0(x^*(T))).$$

- **Second example.** Suppose now that $\rho : L^1_{\mathcal{F}_T}(\Omega; \mathbb{R}) \rightarrow \mathbb{R}$ is the prototypical example of subdifferentiable risk measure given by the *Average-Value-at-Risk* of a random variable $Z \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$ with level $\alpha \in (0, 1]$, namely

$$\rho_\alpha(Z) = \text{AV@R}_\alpha(Z) \triangleq \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \mathbb{E}[\max(Z - t, 0)] \right).$$

In that case, the results of Theorem 3.4 hold for some $\xi^* \in \partial\rho(\varphi_0(x^*(T)))$, which satisfies in particular (8). From [16, Example 6.16], we therefore have

$$\xi^*(\omega) \in \begin{cases} \{0\} & \text{if } \varphi_0(x^*(T, \omega)) < \text{V@R}_\alpha(\varphi_0(x^*(T))), \\ \left(0, \frac{1}{\alpha}\right) & \text{if } \varphi_0(x^*(T, \omega)) = \text{V@R}_\alpha(\varphi_0(x^*(T))), \text{ with } \mathbb{E}[\xi^*] = 1, \\ \left\{\frac{1}{\alpha}\right\} & \text{if } \varphi_0(x^*(T, \omega)) > \text{V@R}_\alpha(\varphi_0(x^*(T))), \end{cases}$$

where $\text{V@R}_\alpha(Z)$ is the *Value-at-Risk* of Z with level $\alpha \in (0, 1]$. In the generic case in which $\varphi_0(x^*(T)) \neq \text{V@R}_\alpha(\varphi_0(x^*(T)))$ almost surely, the optimal risk parameter will then be of the form

$$\xi^* = \frac{1}{\alpha} \mathbf{1}_{\Omega_\alpha},$$

for some unknown set $\Omega_\alpha \subset \Omega$ satisfying $\mathbb{P}(\Omega_\alpha) = \alpha$, and thus uniquely characterized through a *bang-bang* condition. In particular, if it holds that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \varphi_0(x^*(T, \omega)) > \text{V@R}_\alpha(\varphi_0(x^*(T)))\right\}\right) = \alpha,$$

we would then have that $\Omega_\alpha \triangleq \{\varphi_0(x^*(T)) > \text{V@R}_\alpha(\varphi_0(x^*(T)))\}$. In general however, computing the set Ω_α will be part of the resolution scheme.

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